Journal of Pure and Applied Algebra 229 (2025) 107871

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Codimension growth of algebras with superautomorphism ☆

Antonio Ioppolo [a,*], Daniela La Mattina [b]

[a] *Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio 1, 67100, L'Aquila, Italy*
[b] *Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, 90123, Palermo, Italy*

ARTICLE INFO

Article history:
Received 12 May 2023
Received in revised form 29 April 2024
Available online 10 January 2025
Communicated by A. Solotar

MSC:
Primary: 16R10; 16R50; secondary: 16W10; 16W50

Keywords:
Polynomial identity
Superautomorphism
Codimension
Growth
Exponent

ABSTRACT

Let A be a finite dimensional algebra endowed with a superautomorphism over a field of characteristic zero. In this paper we study the asymptotic behavior of the sequence of φ-codimensions $c_n^\varphi(A)$, $n = 1, 2, \ldots$. More precisely, we shall prove that $\lim_{n\to\infty} \sqrt[n]{c_n^\varphi(A)}$ always exists and it is an integer related in an explicit way to the dimension of a suitable semisimple subalgebra of A. This result gives a positive answer to a conjecture of Amitsur in this setting. In the final part of the paper we characterize the algebras whose exponential growth is bounded by 2.

1. Introduction

Let F be a fixed field of characteristic zero. This paper is devoted to the study of finite dimensional associative superalgebras over F endowed with superautomorphisms. If $A = A_0 \oplus A_1$ is a superalgebra, a graded linear map φ of order ≤ 2 is a superautomorphism on A if, for any homogeneous elements $a, b \in A_0 \cup A_1$,

$$(ab)^\varphi = (-1)^{|a||b|} a^\varphi b^\varphi.$$

☆ The authors were partially supported by GNSAGA-INDAM.

* Corresponding author.
E-mail addresses: antonio.ioppolo@univaq.it (A. Ioppolo), daniela.lamattina@unipa.it (D. La Mattina).

https://doi.org/10.1016/j.jpaa.2025.107871

Recently it was showed (see [15,17]) that superautomorphisms represent the connection link between graded involutions, superinvolutions and pseudoinvolutions; such maps play a prominent role in the setting of Lie and Jordan superalgebras (see for instance [20,23,24]).

In [16], the authors started a study of superalgebras with superautomorphism in the context of the theory of polynomial identities. It is well-known that the study of the polynomial identities satisfied by an ordinary algebra A (with no additional structure) is equivalent to the study of the multilinear ones and an effective way to measure such identities is through the sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, of A. Recall that if P_n is the space of multilinear polynomials in the non-commutative variables $x_1, \ldots, x_n$ and $\mathrm{Id}(A)$ is the ideal of identities of A, then $c_n(A) = \dim P_n/(P_n \cap \mathrm{Id}(A))$.

The asymptotic behavior of this sequence has been extensively studied leading to classification results of several varieties of algebras. The key result in this area says that the sequence of codimensions of an algebra satisfying a non-trivial polynomial identity is exponentially bounded ([25]) and its exponential rate of growth is an integer ([9,10]).

When A is endowed with a structure of superalgebra with superautomorphism the objects to study are the φ-polynomial identities satisfied by A and one defines a corresponding sequence $c_n^\varphi(A)$ of φ-codimensions. Notice that such a sequence is bounded from above by $4^n n!$. Nevertheless when A satisfies an ordinary (non trivial) identity, $c_n^\varphi(A)$ is exponentially bounded (see [16]).

Our aim is to determine the exponential rate of growth of the sequence of φ-codimensions. More precisely, we shall prove that if A is a finite dimensional superalgebra with superautomorphism

$$\exp^\varphi(A) = \lim_{n\to\infty} \sqrt[n]{c_n^\varphi(A)}$$

exists and it is a non-negative integer, called the φ-exponent of A. Moreover $\exp^\varphi(A)$ can be explicitly computed and it turns out to be equal to the dimension of a suitable semisimple φ-subalgebra of A over an algebraically closed field. In case A is endowed with trivial grading we rediscover the results proved in [2,3,8].

The last part of the paper is devoted to the characterization of those algebras whose φ-exponent is bounded by 2 (see also [7,11,14,18]). If the φ-exponent of an algebra A is bounded by 1, it is equivalent to say that the φ-codimensions are polynomially bounded and that the variety generated by A does not contain the group algebra of $\mathbb{Z}_2$ and the algebra of 2×2 upper triangular matrices with suitable superautomorphisms (see [16]). These are the only algebras generating minimal varieties of φ-exponent 2.

Finally, new results concerning φ-algebras generating varieties of minimal φ-exponent > 2 will be obtained.

It is important to highlight that the starting point in the proof of all results of this paper is the Wedderburn-Malcev decomposition of a finite dimensional φ-algebra based on the classification of the simple ones given in [16].

2. On finite dimensional superalgebras with superautomorphism

Throughout this paper F will denote a field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra (an algebra graded by $\mathbb{Z}_2$, the cyclic group of order 2) over F endowed with a superautomorphism φ. We say that a linear map $\varphi\colon A \to A$ is a superautomorphism if it preserves the grading (graded map), has order ≤ 2 and for any elements $a, b \in A_0 \cup A_1$,

$$(ab)^\varphi = (-1)^{|a||b|} a^\varphi b^\varphi.$$

Here $|c| = 0$ or 1 denotes the homogeneous degree of $c \in A_0 \cup A_1$.

If A is a superalgebra with superautomorphism we shall call it simply φ-algebra.

Remark 1. If $A = A_0 \oplus A_1$ is a superalgebra such that $A_1^2 = 0$ then the superautomorphisms on A coincide with the graded automorphisms on A and, in particular, with the automorphisms on A, if $A_1 = 0$.

In case A is a finite dimensional algebra, its structure is known from a generalization of Wedderburn-Malcev's theorem proved in [16, Theorem 9]. Before stating it, recall that an ideal (subalgebra) I of A is a φ-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^\varphi = I$. The algebra A is a simple φ-algebra if $A^2 \neq 0$ and A has no non-trivial φ-ideals.

Theorem 2. *Let A be a finite dimensional φ-algebra over a field F of characteristic 0. Then there exists a semisimple φ-subalgebra B such that*

$$A = B + J = B_1 \oplus \cdots \oplus B_k + J,$$

where J, the Jacobson radical of A, is a φ-ideal of A and $B_1, \ldots, B_k$ are simple φ-algebras.

The above result can be refined further as the classification of the simple φ-algebras is known.

We start by exhibiting the simple superalgebras which are involved in such classification:

- $Q(n) = M_n(F) \oplus cM_n(F)$, where $M_n(F)$ is the algebra of $n \times n$ matrices over F and $c^2 = 1$;
- $M_{k,h}(F)$, the algebra of $n \times n$ matrices, $n = k + h$, $k \geq h \geq 0$, with the following $\mathbb{Z}_2$-grading

$$M_{k,h}(F) = \left\{ \begin{pmatrix} K & 0 \\ 0 & H \end{pmatrix} \mid K \in M_k(F),\ H \in M_h(F) \right\} \oplus \left\{ \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \mid R \in M_{k\times h}(F),\ S \in M_{h\times k}(F) \right\}.$$

Given a superalgebra B, let $\bar{B}$ denote the superalgebra with the same graded vector space structure as B but with product $\circ$ given on homogeneous elements a, b by the formula

$$a \circ b := (-1)^{|a||b|} ab.$$

Now we are ready to state the theorem giving the classification of the finite dimensional simple φ-algebras over an algebraically closed field F (see [16, Theorem 20]). First recall that two φ-algebras (A, φ) and (C, ψ) are isomorphic if there exists a graded isomorphism of algebras $\tau\colon A \to C$ such that $\tau(a^\varphi) = \tau(a)^\psi$, for any $a \in A$.

Theorem 3. *Let A be a finite dimensional simple φ-algebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

(1) *$M_{k,h}(F)$, $k \geq h \geq 0$ with superautomorphism φ defined as*

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^\varphi = \begin{pmatrix} PKP & PRQ \\ -QSP & QHQ \end{pmatrix},$$

where $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, $I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2}$, are the identity matrices of orders k_1, k_2, h_1, h_2, respectively, $k = k_1 + k_2$, $k_1 \geq k_2$, $h = h_1 + h_2, h_1 \geq h_2$;

(2) *$M_{k,h}(F) \oplus \overline{M_{k,h}(F)}$, $k \geq h \geq 0$, with the exchange superautomorphism ex;*

(3) *$Q(n) \oplus \overline{Q(n)}$ with the exchange superautomorphism ex.*

3. Codimension growth: the existence of the φ-exponent

Now we are interested in studying the φ-algebras in the context of the theory of polynomial identities. Let us start by recalling the basic definitions we need.

If A is a φ-algebra, since char $F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where $A_i^+ = \{a \in A_i \mid a^\varphi = a\}$ and $A_i^- = \{a \in A_i \mid a^\varphi = -a\}$ denote the sets of symmetric and skew elements of A_i, $i = 0, 1$, respectively.

One can define in a natural way a superautomorphism on the free associative algebra $F\langle X\rangle$ on a countable set X over F. We write X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1, respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements, respectively. The free algebra with superautomorphism is denoted $F\langle Y \cup Z, \varphi\rangle$ and we write

$$F\langle Y \cup Z, \varphi\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

A polynomial $f \in F\langle Y \cup Z, \varphi\rangle$ is a φ-polynomial identity of A (or simply a φ-identity), and we write $f \equiv 0$, if it vanishes for all substitutions $y^\pm \mapsto a^\pm \in A_0^\pm$, $z^\pm \mapsto b^\pm \in A_1^\pm$.

Let $\mathrm{Id}^\varphi(A)$ denote the set of all φ-identities of A. It is clear that it is an ideal of $F\langle Y \cup Z, \varphi\rangle$ invariant under all graded endomorphisms of the free algebra commuting with the superautomorphism φ. It is called the T_2^φ-ideal of A.

In 1950 Specht conjectured that the ideal of all polynomial identities satisfied by an algebra was finitely generated as a T-ideal. Such conjecture was proved in the affirmative by Kemer in ([21,22]) for associative algebras, provided that the ground field has characteristic zero. Similar results have been obtained in the setting of algebras with additional structure (e.g. graded algebras, involutions), see [1,4]. We expect a similar outcome for superalgebras with superautomorphism. We remark that counterexamples arise in case the ground field has positive characteristic [5,6,12,26].

As in the ordinary case, it is easily seen that in characteristic zero, every φ-identity is equivalent to a system of multilinear φ-identities. Hence if we denote by

$$P_n^\varphi = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \dots, n\}$$

the space of multilinear polynomials of degree n in the variables y_i^+, y_i^-, z_i^+, z_i^-, $i = 1, \dots, n$, the study of $\mathrm{Id}^\varphi(A)$ is equivalent to the study of $P_n^\varphi \cap \mathrm{Id}^\varphi(A)$, for all $n \geq 1$. The n-th φ-codimension of A is the non-negative integer

$$c_n^\varphi(A) = \dim_F \frac{P_n^\varphi}{P_n^\varphi \cap \mathrm{Id}^\varphi(A)},\ n \geq 1.$$

If A satisfies an ordinary polynomial identity, it was proved that $c_n^\varphi(A)$, $n = 1, 2, \dots$, is exponentially bounded ([16]).

Our goal is to determine the exponential rate of growth of the sequence of φ-codimensions of a finite dimensional φ-algebra, by giving a positive answer to the Amitsur's conjecture in this setting. We make the following definition.

Definition 4. Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional φ-algebra and let $C_1, \dots, C_h$ be distinct simple φ-subalgebras of A from $\{B_1, \dots, B_k\}$. The φ-algebra $C = C_1 \oplus \cdots \oplus C_h$ is called φ-admissible if $C_1 J \cdots J C_{h-1} J C_h \neq 0$.

Define

$$d = d(A) = \max\,(\dim C),\ \text{ where } C \text{ runs over all admissible } \varphi\text{-subalgebras of } A. \tag{1}$$

In the next theorem we shall explain the role of such an integer in the description of the asymptotic behavior of the φ-codimensions.

Theorem 5. *Let A be a finite dimensional φ-algebra over a field F of characteristic zero and consider the integer d defined in (1). Then there exist constants $a_1 > 0$ and a_2, b_1, b_2 such that*

$$a_1 n^{b_1} d^n \leq c_n^{\varphi}(A) \leq a_2 n^{b_2} d^n.$$

Hence the φ-exponent of A, $\exp^{\varphi}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\varphi}(A)}$ exists and it is a non-negative integer.

Proof. Since the φ-codimensions do not change by extending the ground field, we may assume that the field F is algebraically closed. Now the result can be proved, with the necessary changes, by following word by word the proof given in [13] in the setting of superalgebras with superinvolution. $\square$

As an immediate consequence of the above theorem we get the following.

Corollary 6. *Under the hypotheses of Theorem 5, the sequence $c_n^{\varphi}(A)$, $n = 1, 2, \ldots$, either is polynomially bounded (i.e., $\exp^{\varphi}(A) \leq 1$) or it grows exponentially (i.e., $\exp^{\varphi}(A) \geq 2$).*

In [16], the authors described the varieties of φ-algebras of polynomial growth by giving a finite list of φ-algebras to be excluded from the variety. Recall that the growth of a variety $\mathcal{V}$ of φ-algebras is defined as the growth of the sequence of φ-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^{\varphi}(A)$. Then we say that $\mathcal{V}$ has polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Let $UT_2 = UT_2(F)$ be the algebra of 2×2 upper triangular matrices over F. We can see it as a superalgebra with trivial grading $UT_2 = UT_2 \oplus \{0\}$ or natural grading $UT_2^{gr} = (Fe_{11} \oplus Fe_{22}) \oplus Fe_{12}$. Here the e_{ij}s denote the elementary matrices. On both these superalgebras it is possible to define two superautomorphisms:

- the trivial superautomorphism $a \mapsto a$;
- the natural superautomorphism $ae_{11} + be_{22} + ce_{12} \mapsto ae_{11} + be_{22} - ce_{12}$.

Hence we have four φ-algebras:

- UT_2 and UT_2^{sup} with trivial grading and trivial and natural superautomorphism, respectively;
- UT_2^{gr} and $UT_2^{gr,sup}$ with natural grading and trivial and natural superautomorphism, respectively.

The above φ-algebras together with the two dimensional algebra $F \oplus F$ (trivial grading and exchange superautomorphism $(a,b)^{ex} = (b,a)$), characterize the varieties of polynomial growth as stated in the following result proved in [16, Theorem 23].

Theorem 7. *Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then the sequence $c_n^{\varphi}(A)$, $n = 1, 2, \ldots$, is polynomially bounded (i.e., $\exp^{\varphi}(A) \leq 1$) if and only if UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup}$, $F \oplus F \notin \mathrm{var}^{\varphi}(A)$.*

Given two φ-algebras A and B we say that they are T_2^{φ}-equivalent if $\mathrm{Id}^{\varphi}(A) = \mathrm{Id}^{\varphi}(B)$.

Corollary 8. *[16] The algebras UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup}$, $F \oplus F$ are the only finite dimensional φ-algebras, up to T_2^{φ}-equivalence, generating varieties of almost polynomial growth.*

Now we recall that a variety $\mathcal{V}$ of φ-algebras is minimal with respect to the φ-exponent if for any proper subvariety $\mathcal{U}$, we have that $\exp^{\varphi}(\mathcal{V}) > \exp^{\varphi}(\mathcal{U})$. Here the φ-exponent of a variety is the φ-exponent of a generating algebra. Since the above algebras have φ-exponent equal to 2, by using this definition we get the following result.

Corollary 9. *The algebras $F \oplus F$, UT_2, UT_2^{sup}, UT_2^{gr} and $UT_2^{gr,sup}$ are the only finite dimensional φ-algebras, up to T_2^{φ}-equivalence, generating minimal varieties of φ-exponent 2.*

4. On varieties with φ-exponent bounded by 2

In the first part of this section we shall construct a finite list of φ-algebras generating minimal varieties of φ-exponent greater than 2. Let us consider the following simple φ-algebras:

- C_1, the superalgebra $M_{2,0}(F)$ with trivial superautomorphism *id*: $a \mapsto a$;
- C_2, the superalgebra $M_{2,0}(F)$ with superautomorphism φ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\varphi} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix};$$

- C_3, the superalgebra $M_{1,1}(F)$ with superautomorphism ψ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\psi} = \begin{pmatrix} a & b \\ -c & d \end{pmatrix};$$

- C_4, the superalgebra $Q(1) \oplus \overline{Q(1)}$ with exchange superautomorphism *ex*: $(a,b)^{ex} = (b,a)$.

By Theorem 5 we get the following result.

Remark 10. For any $i = 1, \dots, 4$, $\exp^{\varphi}(C_i) = 4$.

The above φ-algebras allow us to prove the following lemma.

Lemma 11. *Let B be a simple φ-algebra with $\dim_F B \geq 4$. Then $C_i \in \mathrm{var}^{\varphi}(B)$, for some $i \in \{1, \dots, 4\}$.*

Proof. We shall prove the lemma by constructing a φ-subalgebra of B isomorphic to C_i for some $i \in \{1, \dots, 4\}$.

Case 1. $B = (M_{k,0}(F), \varphi)$, $k = k_1 + k_2$, $k_1 \geq k_2$ and $\begin{pmatrix} K & R \\ S & H \end{pmatrix}^{\varphi} = \begin{pmatrix} K & -R \\ -S & H \end{pmatrix}$.

Suppose first $k_2 = 0$. In this case the superautomorphism φ is the identity map and it is immediate to see that the φ-subalgebra of B generated by the elements $a_1 = e_{11}$, $a_2 = e_{1k_1}$, $a_3 = e_{k_1 1}$, $a_4 = e_{k_1 k_1}$ is isomorphic to C_1.

Now assume $k_2 > 0$. In this case we easily get that the φ-subalgebra C' of B generated by the elements $a_1 = e_{11}$, $a_2 = e_{k_1+1\ k_1+1}$, $a_3 = e_{1\ k_1+1}$, $a_4 = e_{k_1+1\ 1}$ is isomorphic to C_2 through the isomorphism $f\colon C' \to C_2$, given by

$$f(a_1) = e_{11}, \quad f(a_2) = e_{22}, \quad f(a_3) = e_{12}, \quad f(a_4) = e_{21}.$$

Case 2. $B = (M_{k,h}(F), \varphi)$, $h > 0$, and $\begin{pmatrix} K & R \\ S & H \end{pmatrix}^{\varphi} = \begin{pmatrix} PKP & PRQ \\ -QSP & QHQ \end{pmatrix}$.

Let C' be the φ-subalgebra of B generated by the elements $a_1 = e_{11}$, $a_2 = e_{k+1\ k+1}$, $a_3 = e_{1\ k+1}$, $a_4 = e_{k+1\ 1}$. The linear map $f \colon C' \to C_3$, given by

$$f(a_1) = e_{11}, \quad f(a_2) = e_{22}, \quad f(a_3) = e_{12}, \quad f(a_4) = e_{21},$$

is an isomorphism of φ-algebras, and we are done also in this case.

Case 3. $B = \left(M_{k,h}(F) \oplus \overline{M_{k,h}(F)}, ex\right)$, $k \geq h \geq 0$.

Suppose first $k = h = 1$. The φ-subalgebra C' generated by the elements $a_1 = (e_{11} + e_{22}, 0)$, $a_2 = (e_{12} + e_{21}, 0)$, $a_3 = (0, e_{11} + e_{22})$, $a_4 = (0, e_{12} + e_{21})$ is isomorphic to $C_4 = (F \oplus cF) \oplus \overline{(F \oplus cF)}$ through the isomorphism of φ-algebras: $f \colon C' \to C_4$, given by

$$f(a_1) = (1, 0), \quad f(a_2) = (c1, 0), \quad f(a_3) = (0, 1), \quad f(a_4) = (0, c1).$$

Now assume $k > 1$. In this case, we get that the φ-subalgebra of B generated by $a_1 = (e_{11}, e_{11})$, $a_2 = (e_{22}, e_{22})$, $a_3 = (e_{12}, e_{12})$, $a_4 = (e_{21}, e_{21})$ is isomorphic to C_1.

Case 4. $B = \left(Q(n) \oplus \overline{Q(n)}, ex\right)$.

If $n = 1$, then $B = C_4$ and there is nothing to prove. Now, let $n > 1$. Then C_1 is isomorphic to the φ-subalgebra of B generated by the elements $a_1 = (e_{11}, e_{11})$, $a_2 = (e_{22}, e_{22})$, $a_3 = (e_{12}, e_{12})$ and $a_4 = (e_{21}, e_{21})$. □

Next we need to consider some suitable $\mathbb{Z}_2$-gradings and superautomorphisms on the algebra UT_3 of 3×3 upper triangular matrices. Recall that an arbitrary triple (g_1, g_2, g_3) of elements of $\mathbb{Z}_2$ defines an elementary $\mathbb{Z}_2$-grading on UT_3 by setting:

$$(UT_3)_0 = \text{span}\{e_{ij} \mid g_i + g_j = 0 \ (\text{mod } 2)\} \text{ and } (UT_3)_1 = \text{span}\{e_{ij} \mid g_i + g_j = 1 \ (\text{mod } 2)\}.$$

On UT_3 we can define the following automorphisms (of order ≤ 2):

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{id} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \qquad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\varphi_1} = \begin{pmatrix} a & -b & -c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix},$$

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\varphi_2} = \begin{pmatrix} a & b & -c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}, \qquad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\varphi_3} = \begin{pmatrix} a & -b & c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}.$$

If we consider on UT_3 an elementary grading such that $(UT_3)_1^2 = 0$, by Remark 1, any of the above automorphisms can be viewed as a superautomorphism. Hence we get the following φ-algebras:

- C_5, with trivial grading and trivial superautomorphism id;
- C_6, with trivial grading and superautomorphism φ_1;
- C_7, with trivial grading and superautomorphism φ_2;
- C_8, with trivial grading and superautomorphism φ_3;
- C_9, with grading induced by $(0, 0, 1)$ and trivial superautomorphism id;
- C_{10}, with grading induced by $(0, 0, 1)$ and superautomorphism φ_1;
- C_{11}, with grading induced by $(0, 0, 1)$ and superautomorphism φ_2;
- C_{12}, with grading induced by $(0, 0, 1)$ and superautomorphism φ_3;
- C_{13}, with grading induced by $(0, 1, 1)$ and trivial superautomorphism id;

- C_{14}, with grading induced by $(0,1,1)$ and superautomorphism φ_1;
- C_{15}, with grading induced by $(0,1,1)$ and superautomorphism φ_2;
- C_{16}, with grading induced by $(0,1,1)$ and superautomorphism φ_3.

Also we have:

- C_{17}, with grading induced by $(0,1,0)$ and superautomorphism ψ_1 given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\psi_1} = \begin{pmatrix} a & b & -c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix};$$

- C_{18}, with grading induced by $(0,1,0)$ and superautomorphism ψ_2 given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\psi_2} = \begin{pmatrix} a & -b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix};$$

- C_{19}, with grading induced by $(0,1,0)$ and superautomorphism ψ_3 given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\psi_3} = \begin{pmatrix} a & b & c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix};$$

- C_{20}, with grading induced by $(0,1,0)$ and superautomorphism ψ_4 given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\psi_4} = \begin{pmatrix} a & -b & -c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}.$$

Remark 12. For $i = 5, \dots, 20$, we have that $\exp^{\varphi}(C_i) = 3$.

Proof. All the φ-algebras C_i have the same Wedderburn-Malcev decomposition:

$$C_i = A_1 \oplus A_2 \oplus A_3 + J,$$

where $A_1 = Fe_{11}$, $A_2 = Fe_{22}$, $A_3 = Fe_{33}$ and $J = Fe_{12} \oplus Fe_{13} \oplus Fe_{23}$. Since $A_1 J A_2 J A_3 \neq 0$, $A_1 \oplus A_2 \oplus A_3$ is a maximal dimensional φ-admissible subalgebra and the result follows by Theorem 5. □

The above φ-algebras allow us to prove the following lemma.

Lemma 13. *Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional φ-algebra over an algebraically closed field F of characteristic zero. If there exist three distinct simple components $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$ such that $B_{i_1} J B_{i_2} J B_{i_3} \neq 0$, then $C_i \in \mathrm{var}^{\varphi}(A)$, for some $i \in \{5, \dots, 20\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of $B_{i_1}, B_{i_2}, B_{i_3}$, respectively. Then $e_l^2 = e_l^{\varphi} = e_l \in (B_{i_l})_0$ and $e_r e_s = \delta_{rs} e_r$, for $r, s, l = 1, 2, 3$. Since $B_{i_1} J B_{i_2} J B_{i_3} \neq 0$ then $e_1 J e_2 J e_3 \neq 0$. So we may assume that there exist homogeneous symmetric or skew elements $j, j' \in J$ such that

$$e_1 j e_2 j' e_3 \neq 0.$$

Consider the φ-subalgebra U of A linearly generated by

$$e_1, \quad e_2, \quad e_3, \quad e_1 j e_2, \quad e_2 j' e_3, \quad e_1 j e_2 j' e_3.$$

The linear map $f\colon U \to UT_3$, defined by

$$f(e_1) = e_{11}, \quad f(e_2) = e_{22}, \quad f(e_3) = e_{33}, \quad f(e_1 j e_2) = e_{12}, \quad f(e_2 j' e_3) = e_{23}, \quad f(e_1 j e_2 j' e_3) = e_{13},$$

is an isomorphism of algebras. Now, by taking into account the homogeneous degrees of j and j' and their symmetry with respect to the superautomorphism φ, we get an isomorphism of φ-algebras between U and C_i, for some $i = 5, \dots, 20$. $\square$

Now let us consider the subalgebra $M = Fe_{11} \oplus Fe_{22} \oplus Fe_{33} \oplus Fe_{44} \oplus Fe_{12} \oplus Fe_{43}$ of $M_4(F)$ endowed with the following automorphism

$$\begin{pmatrix} a & e & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & f & d \end{pmatrix}^{\dagger} = \begin{pmatrix} d & f & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & e & a \end{pmatrix}.$$

If we endow $(M, \dagger)$ with trivial grading or elementary grading induced by $(0, 1, 1, 0)$, we obtain two φ-algebras M_1 and M_2, respectively. We denote by

- $C_{21} = Fe_{11} \oplus F(e_{22} + e_{33}) \oplus Fe_{44} \oplus Fe_{12} \oplus Fe_{43}$, the φ-subalgebra of M_1;
- $C_{22} = Fe_{11} \oplus F(e_{22} + e_{33}) \oplus Fe_{44} \oplus Fe_{12} \oplus Fe_{43}$, the φ-subalgebra of M_2;
- $C_{23} = Fe_{22} \oplus F(e_{11} + e_{44}) \oplus Fe_{33} \oplus Fe_{12} \oplus Fe_{43}$, the φ-subalgebra of M_1;
- $C_{24} = Fe_{22} \oplus F(e_{11} + e_{44}) \oplus Fe_{33} \oplus Fe_{12} \oplus Fe_{43}$, the φ-subalgebra of M_2.

Using the same approach as in Remark 12, we get the following result.

Remark 14. For $i = 21, \dots, 24$, we have that $\exp^{\varphi}(C_i) = 3$.

Lemma 15. *Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional φ-algebra over an algebraically closed field F of characteristic zero such that $B_i J B_l \neq 0$ with $(B_i, B_l) \in \{(F, F \oplus F), (F \oplus F, F), (F \oplus F, F \oplus F)\}$, $i \neq l$. Then $C_i \in \mathrm{var}^{\varphi}(A)$, for some $i \in \{21, \dots, 24\}$.*

Proof. Suppose first that $(B_i, B_l) = (F \oplus F, F \oplus F)$. Let $e_i = e_1 + e_4$ and $e_l = e_2 + e_3$ be the unit elements of B_i and B_l, respectively. Clearly $e_1^{\varphi} = e_4$ and $e_2^{\varphi} = e_3$. Since $B_i J B_l \neq 0$, there exists an homogeneous element $j \in J$ such that

$$e_i j e_l = (e_1 + e_4) j (e_2 + e_3) \neq 0.$$

Without loss of generality, we may assume that $e_1 j e_2 \neq 0$. Let U be the φ-algebra linearly generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_4, \quad e_1 j e_2, \quad e_4 j^{\varphi} e_3.$$

According to the homogeneous degree of j, the map f defined by

$$f(e_1) = e_{11}, \quad f(e_2) = e_{22}, \quad f(e_3) = e_{33}, \quad f(e_4) = e_{44}, \quad f(e_1 j e_2) = e_{12}, \quad f(e_4 j^{\varphi} e_3) = e_{43},$$

is an isomorphism of φ-algebras between U and M_1 or M_2. Since C_i, $i = 21, \dots, 24$ are subalgebras of M_1 or M_2, we are done in this case.

The cases $(B_i, B_l) = (F, F \oplus F)$ and $(B_i, B_l) = (F \oplus F, F)$ can be proved using a similar approach. $\square$

The following proposition proves that the list of φ-algebras $C_1, \dots, C_{24}$ cannot be reduced.

Proposition 16. *For all* $i, j \in \{1, \dots, 24\}$, $i \neq j$, $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$.

Proof. By the classification of superautomorphisms on UT_n given in [19], we have that the φ-algebras C_i, $i = 5, \dots, 20$ are not pairwise T_2^φ-equivalent. Now, let y denote an even variable, z an odd one and x any variable. The proof is completed by putting together the following facts.

- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 1, 3$, $j \in \{2, 4 \dots, 24\}$: in fact $y^- \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 1, 2, 21, 23$, $j = 3, 4, 22, 24$: in fact $z \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 2, 3$, $j \in \{1, 4, \dots, 21, 23\}$: in fact $[y_1^+, y_2^+] \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 4$, $j = 1, 2, 3, 5, \dots, 24$: in fact $[x_1, x_2] \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 21$, $j = 23$: in fact $[y_1^-, y_2^-]y_3^- \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 23$, $j = 21$: in fact $y_3^-[y_1^-, y_2^-] \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 22, 24$, $j = 21, 23$: in fact $[y_1^-, y_2^-] \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 22$, $j = 24$: in fact $z^+ y^- \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 24$, $j = 22$: in fact $y^- z^+ \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 5, \dots, 24$, $j = 1, 2, 3, 4$: in fact $\exp^\varphi(C_i) < \exp^\varphi(C_j)$.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 21, 22, 23, 24$, $j = 1, \dots, 20$: in fact $[x_1, x_2][x_3, x_4] \equiv 0$ on C_i but not on C_j.
- $\mathrm{Id}^\varphi(C_i) \not\subseteq \mathrm{Id}^\varphi(C_j)$, $i = 5, \dots, 20$, $j = 21, \dots, 24$: in fact $y_1^- y_2^- y_3^- \equiv 0$ on C_i but not on C_j. $\square$

Now we are in a position to characterize the φ-algebras A with $\exp^\varphi(A) \leq 2$.

Theorem 17. *Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then* $\exp^\varphi(A) \leq 2$ *if and only if* $C_i \notin \mathrm{var}^\varphi(A)$, *for any* $i \in \{1, \dots, 24\}$.

Proof. Since we are dealing with φ-codimensions that do not change by extending the base field, in what follows we may assume that the field F is algebraically closed.

First let $\exp^\varphi(A) \leq 2$. Since $\exp^\varphi(C_i) > 2$, by Remarks 10, 12 and 14, we get $C_i \notin \mathrm{var}^\varphi(A)$, $i \in \{1, \dots, 24\}$.

Conversely, let $C_i \notin \mathrm{var}^\varphi(A)$, for any $i \in \{1, \dots, 24\}$. Hence by Theorem 2 we can write $A = B_1 \oplus \cdots \oplus B_m + J$, where the B_j's are simple φ-algebras isomorphic to those ones given in Theorem 3. Since $C_1, \dots, C_4 \notin \mathrm{var}^\varphi(A)$, according to Lemma 11, we have that $\dim_F B_l < 4$, for any l.

Suppose by contradiction that $\exp^\varphi(A) > 2$. Then by Theorem 5, one of the following possibilities occurs:

1. there exist distinct $B_{i_1}, B_{i_2}, B_{i_3}$ such that $B_{i_1} J B_{i_2} J B_{i_3} \neq 0$ and $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$,
2. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong F$ and $B_{i_2} \cong F \oplus F$,
3. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong F \oplus F$ and $B_{i_2} \cong F$.
4. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong F \oplus F$ and $B_{i_2} \cong F \oplus F$.

If 1. holds, then, by Lemma 13, $C_i \in \mathrm{var}^\varphi(A)$, for some $i \in \{5, \dots, 20\}$, a contradiction. We reach a contradiction also in all the other cases, since by Lemma 15, we should have that $C_i \in \mathrm{var}^\varphi(A)$, for some $i \in \{21, \dots, 24\}$. $\square$

In light of Theorems 7 and 17, we get the characterization of φ-algebras with φ-exponent equal to two.

Corollary 18. *Let A be a finite dimensional φ-algebra over F, char$F = 0$. Then* $\exp^\varphi(A) = 2$ *if and only if*

- *$C_i \notin \mathrm{var}^{\varphi}(A)$, for all $i \in \{1, \ldots, 24\}$ and*
- *either UT_2 or UT_2^{sup} or UT_2^{gr} or $UT_2^{gr,sup}$ or $F \oplus F \in \mathrm{var}^{\varphi}(A)$.*

Now let us slightly change the definition of minimal varieties given at the end of Section 3: a variety $\mathcal{V}$ of φ-algebras is minimal with respect to the φ-exponent if for any proper subvariety $\mathcal{U}$, generated by a finite dimensional φ-algebra, we have that $\exp^{\varphi}(\mathcal{V}) > \exp^{\varphi}(\mathcal{U})$. By using this definition we get the following.

Corollary 19.

1. *The φ-algebras C_i, $i = 1, \ldots, 4$, generate minimal varieties of φ-exponent 4.*
2. *The φ-algebras C_i, $i = 5, \ldots, 24$, are the only finite dimensional algebras, up to T_2^{φ}-equivalence, generating minimal varieties of φ-exponent 3.*

Proof. We prove just item 2. (the proof of 1. is similar).

Let $\mathcal{V}$ be a proper subvariety of $\mathrm{var}^{\varphi}(C_i)$, $i = 5, \ldots, 24$. Clearly $C_i \notin \mathcal{V}$. Also, by Proposition 16, we get that $C_j \notin \mathcal{V}$, for any $j = 1, \ldots, 24$. Then, from Theorem 17, $\exp^{\varphi}(\mathcal{V}) \leq 2$ and we are done.

Now suppose that there exists a minimal variety $\mathcal{U}$ of φ-exponent 3 which is not generated by any of the algebras in 2. Since $\mathcal{U}$ is minimal and its φ-exponent is 3, $C_i \notin \mathcal{U}$, for any i. Then by Theorem 17 we should have $\exp^{\varphi}(\mathcal{U}) \leq 2$, a contradiction. □

CRediT authorship contribution statement

Antonio Ioppolo: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Daniela La Mattina:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Antonio Ioppolo reports financial support was provided by Francesco Severi National Institute of Higher Mathematics National Group of Algebra and Geometric Structures and their Applications. Daniela La Mattina reports financial support was provided by Francesco Severi National Institute of Higher Mathematics National Group of Algebra and Geometric Structures and their Applications.

References

[1] E. Aljadeff, A. Giambruno, Y. Karasik, Polynomial identities with involution, superinvolutions and the Grassmann envelope, Proc. Am. Math. Soc. 145 (5) (2017) 1843–1857.

[2] E. Aljadeff, A. Giambruno, Multialternating graded polynomials and growth of polynomial identities, Proc. Am. Math. Soc. 141 (9) (2013) 3055–3065.

[3] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011) 83–100.

[4] E. Aljadeff, A. Kanel-Belov, Representability and Specht problem for G-graded algebras, Adv. Math. 225 (5) (2010) 2391–2428.

[5] A.Ya. Belov, On non-Specht varieties, Fundam. Prikl. Mat. 5 (1) (1999) 47–66.

[6] L. Centrone, D.J. Gonçalves, D. Couto Silva, Identities and central polynomials with involution for the Grassmann algebra, J. Algebra 560 (2020) 219–240.

[7] A. Giambruno, A. Ioppolo, D. La Mattina, Trace codimensions of algebras and their exponential growth, Isr. J. Math. (2022) 1–29, https://doi.org/10.1007/s11856-022-2414-3.

[8] A. Giambruno, D. La Mattina, Graded polynomial identities and codimensions: computing the exponential growth, Adv. Math. 225 (2) (2010) 859–881.

[9] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998) 145–155.
[10] A. Giambruno, M. Zaicev, Exponential codimension growth of PI-algebras: an exact estimate, Adv. Math. 142 (1999) 221–243.
[11] A. Giambruno, M. Zaicev, A characterization of varieties of associative algebras of exponent two, Serdica Math. J. 26 (2000) 245–252.
[12] L.F. Gonçalves Fonseca, T. Castilho de Mello, Graded polynomial identities for matrices with the transpose involution over an infinite field, Commun. Algebra 46 (4) (2018) 1630–1640.
[13] A. Ioppolo, The exponent for superalgebras with superinvolution, Linear Algebra Appl. 555 (2018) 1–20.
[14] A. Ioppolo, A characterization of superalgebras with pseudoinvolution of exponent 2, Algebr. Represent. Theory 24 (6) (2021) 1415–1429.
[15] A. Ioppolo, Graded linear maps on superalgebras, J. Algebra 605 (2022) 377–393.
[16] A. Ioppolo, D. La Mattina, Algebras with superautomorphism: simple algebras and codimension growth, Isr. J. Math. (2024), https://doi.org/10.1007/s11856-024-2663-4.
[17] A. Ioppolo, F. Martino, Superinvolutions on upper-triangular matrix algebras, J. Pure Appl. Algebra 222 (8) (2018) 2022–2039.
[18] A. Ioppolo, F. Martino, Classifying G-graded algebras of exponent two, Isr. J. Math. 229 (2019) 341–356.
[19] A. Ioppolo, F. Martino, Gradings and graded linear maps on algebras, preprint.
[20] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1) (1977) 8–96.
[21] A.R. Kemer, T-ideals with power growth of the codimensions are Specht, Sib. Mat. Zh. 19 (1978) 54–69 (in Russian); English translation: Sib. Math. J. 19 (1978) 37–48.
[22] A.R. Kemer, Varieties of finite rank, in: Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, vol. 2, 1979, p. 73 (in Russian).
[23] C. Martinez, E. Zelmanov, Representation theory of Jordan superalgebras. I, Trans. Am. Math. Soc. 362 (2) (2010) 815–846.
[24] M.L. Racine, E.I. Zelmanov, Simple Jordan superalgebras with semisimple even part, J. Algebra 270 (2) (2003) 374–444.
[25] A. Regev, Existence of identities in $A \otimes B$, Isr. J. Math. 11 (1972) 131–152.
[26] V.V. Shchigolev, Examples of infinitely based T-ideals, Fundam. Prikl. Mat. 5 (1) (1999) 307–312.