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On the spectrum of supercyclic/hypercyclic operators

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Abstract

This paper concerns the spectral structure of hypercyclic and supercyclic operators defined on Banach spaces, or defined on Hilbert spaces. We also consider the spectral properties of operators in Hilbert spaces that commute with a hypercyclic operator. A result of Herrero and Kitai (Proc Am Math Soc 116(3):873–875, 1992) is extended to Drazin invertible operators. In particular, a Drazin invertible operator is hypercyclic if and only if is invertible. An analogous result holds for supercyclic operators T in the case were the dual T^* has empty point spectrum.

Keywords Weyl spectra of hypercyclic and supercyclic operators · Drazin invertible operators

Mathematics Subject Classification 47A10 · 47A11; 47A53 · 47A55

1 Introduction

This article is devoted to the study of the spectral structure of hypercyclic or supercyclic operators, defined on Banach spaces, or on Hilbert spaces. Fundamental results regarding the spectral theory of these operators were established by C. Kitai in her thesis [18] (not published). Much more can be said, in Sect. 3 we give, for these operators, more informations on the various spectra originating from Fredholm theory. We also consider, for Hilbert space operators, some spectral properties of the operators S that commute with a hypercyclic operator. In particular we show that the spectral mapping theorem holds for some of the Weyl spectra of S. A result of Herrero and Kitai [17], concerning the hypercyclicity of the inverse T^{-1} of an invertible hypercyclic operator

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T, is extended to Drazin invertible operators. This extension is an useful tool, since it allows us to establish that various classes of operators cannot be hypercyclic. We also consider the case of a Drazin invertible supercyclic operators for which the dual T^* has empty point spectrum.

2 Definitions and preliminary results

By X we denote an infinite dimensional complex Banach space, while by H will always denote a Hilbert space. A bounded linear operator $T \in L(X)$, is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) := \dim \ker T < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) := \operatorname{codim} T(X) < \infty$. The class of *Fredholm* operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, while the class of *semi-Fredholm operators* is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$. The *essential (Fredholm) spectrum* $\sigma_{\mathrm{e}}(T)$ is defined as the set $\{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}$, while the *semi-Fredholm spectrum* $\sigma_{\mathrm{sf}}(T)$ is the set $\{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\}$. If $T \in \Phi_\pm(X)$ then the *index* is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The class of *Weyl operators* is defined by $W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\}$, while the class of *upper semi-Weyl operators* defined by $W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\}$, and the class of *lower semi-Weyl operators* defined by $W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}$. Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$, the *upper semi-Weyl spectrum* $\sigma_{\mathrm{uw}}(T)$ and the *lower semi-Weyl spectrum* $\sigma_{\mathrm{lw}}(T)$, defined similarly. It is well known that there is a duality: if T^* denotes the dual of T then $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{lw}}(T^*)$ and $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*)$.

The *ascent* of an operator T, if it exists, is the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. Analogously, the *descent* of T, if it exists, is the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 1.20]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent. The class of all *Browder operators* is defined as the set $B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}$; the class of all *upper semi-Browder operators* is defined $B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$. The class of Browder operators generate the *Browder spectrum* $\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$ and the *upper semi-Browder spectrum* $\sigma_{\mathrm{ub}}(T)$, defined in a similar way. Clearly, $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Recall that R is said *Riesz* if $\lambda I - R \in \Phi(X)$ for every $\lambda \neq 0$, or equivalently, $\lambda I - R \in B(X)$ for every $\lambda \neq 0$. Every Riesz operator has a finite spectrum, or $\sigma(R)$ consists of a sequence of eigenvalues of finite multiplicity which clusters at 0. The classical *approximate point spectrum* is denoted by $\sigma_{\mathrm{ap}}(T)$, the *surjectivity spectrum* is denoted by $\sigma_{\mathrm{s}}(T)$.

The single-valued extension property is one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces. An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$, (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, both

T and T^* have SVEP at the isolated points of the spectrum. Note that $p(\lambda I - T) < \infty$ entails that T has SVEP at λ, while $q(\lambda I - T) < \infty$ entails that the dual T^* has SVEP at λ, see [1, Chapter 2]. Furthermore, if $\lambda I - T \in \Phi_{\pm}(X)$ then T has SVEP at λ if and only if $p(\lambda I - T) < \infty$, while T^* has SVEP at λ if and only if $q(\lambda I - T) < \infty$, see [1, Theorem 2.97, Theorem 2.98]. In the case of Hilbert space operators is better to consider the Hilbert adjoint T', instead of T^*. Note that

$$T^* \quad \text{has SVEP} \quad \Leftrightarrow \quad T' \quad \text{has SVEP},$$

see [3, p.365]. Furthermore, if T has SVEP then $\sigma(T) = \sigma_{\mathrm{s}}(T)$, and dually, if T^* has SVEP then $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, see [1, Theorem 2.68].

3 Spectral properties of hypercyclic and supercyclic operators

A vector $x \in X$ is said to be *hypercyclic* for $T \in L(X)$ if the orbit

$$\mathrm{orb}\,(T, x) = \{x, Tx, T^2x, \ldots\}$$

is norm-dense in X, while $x \in X$ is said to be *supercyclic* for T if the set of scalar multiples of $\mathrm{orb}\,(T, x)$ is norm-dense in X. An operator $T \in L(X)$, X a separable infinite dimensional complex Banach space, is said to be *hypercyclic* if there exists a hypercyclic vector $x \in X$, and is said to be *supercyclic* if there exists a supercyclic vector $x \in X$. Evidently, every hypercyclic operator is supercyclic. Classes of operators which satisfy a hypercyclicity or supercyclicity condition have been studied by a number of authors (see for instance [7, 12–14, 23]). The notions of hypercyclicity and supercyclicity are intimately related to the invariant subspace problem, in fact, an operator T has no non-trivial invariant closed subset if and only if each non-zero vector is hypercyclic for T.

It is known that the dual T^* of a hypercyclic operator has empty point spectrum. This implies that the range of $\lambda I - T$ is dense for every $\lambda \in \mathbb{C}$, see [18, Theorem 2.3]. If T is supercyclic there are two possibilities: T^* either has point spectrum $\sigma_{\mathrm{p}}(T^*) = \emptyset$, or $\sigma_{\mathrm{p}}(T^*)$ is a singleton $\{\alpha\}$, with $\alpha \neq 0$, see [15]. In this last case T is decomposed in the direct sum $T = S \oplus I_{\mathbb{C}}$, where $\frac{1}{\alpha} S$ is hypercyclic, see [12]. Consequently, the dual T^* of every supercyclic operator has SVEP, since an operator with countable point spectrum has SVEP.

Recall that $T \in L(X)$, T is said to be *semi-regular* if T has closed range and $\ker T \subseteq T^n(X)$ for every $n \in \mathbb{N}$. The *Kato spectrum* $\sigma_{\mathrm{k}}(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not semi-regular. Let $\sigma_{\mathrm{usf}}(T)$ denote the *upper semi-Fredholm spectrum*.

Herrero in [15, Proposition 2.2] has shown that for hypercyclic operators on Hilbert space we have $\sigma_{(\mathrm{T})} = \sigma_{\mathrm{w}}(T)$. By using the SVEP we can say much more, also in the case of Banach space operators:

Theorem 3.1 *If $T \in L(X)$ is supercyclic then*

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T) \quad \textit{and} \quad \sigma_{\mathrm{ap}}(T) = \sigma(T). \tag{3.1}$$

Furthermore, $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$.

If T *is hypercyclic then all the spectra in (3.1) coincide. Furthermore:*

$$\sigma_{\mathrm{k}}(T) = \sigma_{\mathrm{s}}(T) \quad and \quad \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{usf}}(T). \tag{3.2}$$

Proof Let T be supercyclic. The equality $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ holds since T^* has SVEP. Since $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$ for every operator, to prove the equality $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T)$ it suffices to prove the opposite inclusion. Let $\lambda \notin \sigma_{\mathrm{uw}}(T)$. Then $\lambda I - T \in \Phi_+(X)$ with ind $(\lambda I - T) \leq 0$. The SVEP for T^* entails that $q(\lambda I - T) < \infty$, hence by [1, Theorem 1.22, part (ii)], ind $(\lambda I - T) \geq 0$, so ind $(\lambda I - T) = 0$. Therefore $\lambda \notin \sigma_{\mathrm{w}}(T)$ and hence $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T)$. The equalities $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ follow since T^* has SVEP, see [1, Chapter 5].

To show the equality $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$ observe first that $\sigma_{\mathrm{sf}}(T) \subseteq \sigma_{\mathrm{lw}}(T)$ holds for every operator. Let $\lambda \notin \sigma_{\mathrm{sf}}(T)$. Then $\lambda I - T \in \Phi_\pm(X)$. Since T^* has SVEP then $q(\lambda I - T) < \infty$, so, by [1, Theorem 1.22], $\beta(\lambda I - T) \leq \alpha(\lambda I - T)$. This implies that $\beta(\lambda I - T) < \infty$, so $\lambda I - T \in \Phi_-(X)$ and ind $(\lambda I - T) \geq 0$, i.e. $\lambda \notin \sigma_{\mathrm{lw}}(T)$.

Assume now that T is hypercyclic. The equality of all the spectra in (3.1) is clear if we show that $\sigma(T) = \sigma_{\mathrm{w}}(T)$. Let $\lambda \notin \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$. Then $\alpha(\lambda I - T^*) = \beta(\lambda I - T^*) < \infty$. Since T^* has no eigenvalue we then have

$$\alpha(\lambda I - T^*) = \beta(\lambda I - T^*) = 0,$$

thus $\lambda \notin \sigma(T^*) = \sigma(T)$. Hence, $\sigma(T) = \sigma_{\mathrm{w}}(T)$.

To show (3.2), observe first that the inclusions $\sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{usf}}(T) \subseteq \sigma_{\mathrm{e}}(T)$ hold for every operator. Hence, to prove the equalities (3.2) we need only to prove the reverse inclusions. To do that, let $\lambda \notin \sigma_{\mathrm{k}}(T)$. Then $\lambda I - T$ has closed range. On the other hand, hypercyclicity entails that $\lambda I - T$ has dense range, consequently $(\lambda I - T)(X) = X$, so $\lambda \notin \sigma_{\mathrm{s}}(T)$.

The inclusion of $\sigma_{\mathrm{e}}(T) \subseteq \sigma_{\mathrm{usf}}(T)$, is proved in a similar way: if $\lambda \notin \sigma_{\mathrm{usf}}(T)$ then $\alpha(\lambda I - T) < \infty$ and $(\lambda I - T)(X)$ is closed. As before, $\lambda I - T$ has dense range, so $(\lambda I - T)(X) = X$, and consequently $\beta(\lambda I - T) = 0$, hence $\lambda I - T$ is Fredholm and $\lambda \notin \sigma_{\mathrm{e}}(T)$.

Remark 3.2 All the equalities observed for a hypercyclic operator in Theorem 3.1 holds for every supercyclic operator T for which $\sigma_{\mathrm{p}}(T^*) = \emptyset$. The equality $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{usf}}(T)$ for a hypercyclic operator defined on a separable Hilbert space has been noted in [19, Proposition 3.1].

Lemma 3.3 *Let* $T \in L(H)$ *be a hypercyclic operator. Then we have:*

(i) T *has no nontrivial finite-codimensional invariant subspace.*
(ii) *If* $S \in L(H)$ *commutes with* T, *then the adjoint* S' *has no eigenvalues of finite multiplicity.*

Proof The proof of part (i) may be found in [20]. Part (ii) follows easily from part (i). Indeed, let λ be an eigenvalue of S' having finite multiplicity. Denote $M :=$ ker $(\lambda I - S')$. If dim ker $(\lambda I - S') > 0$ then $M^\perp$ is a finite-codimensional, nontrivial invariant subspace of T, and this is impossible, from part (i).

Part (i) of Lemma 3.3 shows that an operator $T \in \Phi_-(H)$ that is not onto cannot be hypercyclic. Indeed, if $\beta(T) < \infty$ then, since $T(X)$ is a finite-codimensional invariant subspace, we must have $\beta(T) = 0$, hence T is onto. Analogously, if $T \in \Phi_+(H)$ that is not injective, i.e. T is not bounded below, then by duality its adjoint $T' \in \Phi_-(H)$ is not onto, so T' cannot be hypercyclic.

The relationships between the Weyl spectra of the adjoint S' of a Hilbert space operator S are the following:

$$\sigma_{\mathrm{lw}}(S) = \overline{\sigma_{\mathrm{uw}}(S')}, \quad \sigma_{\mathrm{uw}}(S) = \overline{\sigma_{\mathrm{lw}}(S')}, \quad \overline{\sigma_{\mathrm{w}}(S)} = \sigma_{\mathrm{w}}(S'),$$

and analogously $\sigma_{\mathrm{s}}(S) = \overline{\sigma_{\mathrm{ap}}(S')}$, $\sigma_{\mathrm{ap}}(S) = \overline{\sigma_{\mathrm{s}}(S')}$, see [1, Theorem 3.1].

Hypercyclicity of a Hilbert space operator T has also consequences on some spectra of the operators that commute with it:

Theorem 3.4 *Let $T \in L(H)$ be hypercyclic and suppose that $S \in L(H)$ commutes with T. Then*

$$\sigma(S) = \sigma_{\mathrm{w}}(S),$$

and

$$\sigma_{\mathrm{lw}}(S) = \sigma_{\mathrm{s}}(S), \quad \sigma_{\mathrm{uw}}(S') = \sigma_{\mathrm{ap}}(S').$$

Proof We show first the equality $\sigma(S) = \sigma_{\mathrm{w}}(S)$. Let $\lambda \notin \sigma_{\mathrm{w}}(S)$. Then $\lambda I - S$ is Weyl, and hence also its adjoint $\overline{\lambda} I - S'$, so ker $(\overline{\lambda} I - S')$ is finite-dimensional. By Lemma 3.3 we obtain that ker $(\overline{\lambda} I - S') = 0$, and being $\overline{\lambda} I - S'$ Weyl it then follows that also $\beta(\overline{\lambda} - S') = 0$. Therefore, $\lambda \notin \overline{\sigma(S')} = \sigma(S)$, so $\sigma_{\mathrm{w}}(S) = \sigma_{(\mathrm{S})}$.

To show the equality $\sigma_{\mathrm{lw}}(S) = \sigma_{\mathrm{s}}(S)$, it suffices to prove the inclusion $\sigma_{\mathrm{s}}(S) \subseteq \sigma_{\mathrm{lw}}(S)$, since the opposite inclusion is true for every operator. Let $\lambda \notin \sigma_{\mathrm{lw}}(S) = \overline{\sigma_{\mathrm{uw}}(S')}$. Then $\overline{\lambda} \notin \sigma_{\mathrm{uw}}(S')$, so $\overline{\lambda} I - S' \in W_+(H)$, hence $\alpha(\overline{\lambda} I - S') < \infty$. Again by Lemma 3.3 we have $\alpha(\overline{\lambda} I - S') = 0$, and since $\overline{\lambda} I - S'$ has closed range it then follows that $\overline{\lambda} \notin \sigma_{\mathrm{ap}}(S')$, hence $\lambda \notin \overline{\sigma_{\mathrm{ap}}(S')} = \sigma_{\mathrm{s}}(S)$.

The equality $\sigma_{\mathrm{uw}}(S') = \sigma_{\mathrm{ap}}(S')$ is clear, since $\sigma_{\mathrm{uw}}(S') = \overline{\sigma_{\mathrm{lw}}(S)}$ and $\sigma_{\mathrm{ap}}(S') = \overline{\sigma_{\mathrm{s}}(S)}$.

Let $\mathcal{H}(\sigma(T))$ denote the set of all analytic functions on a open disc U containing $\sigma(T)$ and let $f(T)$ be defined by the classical functional calculus:

$$f(T) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda I - T)^{-1} \, \mathrm{d}\lambda,$$

where Γ is a contour that surrounds $\sigma(T)$ in U. It is known that in general the spectral mapping theorem does not hold for the Weyl spectra, see [2] for examples of Toeplitz operators for which the Weyl spectra do not satisfy the spectral mapping theorem.

Theorem 3.5 *Let $T \in L(H)$ be hypercyclic and suppose that $S \in L(H)$ commutes with T. Then the spectral mapping theorem holds for $\sigma_{\mathrm{lw}}(S)$ and $\sigma_{\mathrm{w}}(S)$, i.e.,*

$$\sigma_{\mathrm{lw}}(f(S)) = f(\sigma_{\mathrm{lw}}(S)) \quad \textit{for all } f \in \mathcal{H}(\sigma(S)),$$

and

$$\sigma_{\mathrm{w}}(f(S)) = f(\sigma_{\mathrm{w}}(S)) \quad \textit{for all } f \in \mathcal{H}(\sigma(S)).$$

Futhermore, for supercyclic Banach space operators T, the spectral mapping theorem holds for all the Weyl spectra of T.

Proof By Theorem 3.4 we know that $\sigma_{\mathrm{w}}(S) = \sigma(S)$. The spectral mapping theorem holds for the spectrum, so

$$f(\sigma_{\mathrm{w}}(S)) = f(\sigma(S)) = \sigma(f(S)),$$

Evidently, if $\lambda \in \rho(S) := \mathbb{C} \setminus \sigma(S)$, then

$$T(\lambda I - S)^{-1} = (\lambda I - S)^{-1} T,$$

and consequently, T commutes with $f(S)$. Therefore, always by Theorem 3.4, we have $\sigma_{(\mathrm{f(S)})} = \sigma_{\mathrm{w}}(f(S))$, so $f(\sigma_{\mathrm{w}}(S)) = \sigma_{\mathrm{w}}(f(S))$.

The spectral mapping theorem for the lower semi-Weyl spectrum of S follows arguing similarly. Indeed, by Theorem 3.4 we have $\sigma_{\mathrm{lw}}(S) = \sigma_{\mathrm{s}}(S)$ and it is well known that the spectral mapping theorem holds for $\sigma_{\mathrm{s}}(S)$. Hence,

$$f(\sigma_{\mathrm{lw}}(S)) = f(\sigma_{\mathrm{s}}(S)) = \sigma_{\mathrm{s}}(f(S)).$$

Since T commutes with $f(S)$, again by Theorem 3.4 we have $\sigma_{\mathrm{lw}}(f(S)) = \sigma_{\mathrm{s}}(f(S))$, so $f(\sigma_{\mathrm{lw}}(S)) = \sigma_{\mathrm{lw}}(f(S))$.

The spectral mapping theorem for all Weyl spectra of a Banach space supercyclic operator T holds since T^* has SVEP, see [1, Corollary 3.120].

It is interesting to note a certain analogy between the spectral properties of operators that commute with a hypercyclic operator and operators that commutes with an injective quasi-nilpotent operator. Indeed, some of the equalities observed in Theorems 3.4, 3.1 and 3.5 are true also for operators that commutes with an injective quasi-nilpotent operator, see [5]. A typical example of these operators is given by the operators that commutes with the classical Volterra operator V on $L^2[0, 1]$, that is quasi-nilpotent and injective. Recall that the Volterra operator V is not hypercyclic (not even supercyclic), see [11]. Recall that the *quasi-nilpotent part* of an operator $T \in L(X)$ is defined:

$$H_0(T) := \{x \in X : \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Every quasi-nilpotent operator is not hypercyclic, this is known but we give here a short proof involving the quasi-nilpotent part.

Theorem 3.6 *If $T \in L(X)$ is quasi-nilpotent then T is not hypercyclic.*

Proof If T is quasi-nilpotent then $H_0(T) = X$, see [1, Theorem 2.35], so

$$\limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \quad \text{for every } x \in X.$$

Given $0 < \varepsilon < 1$ then $\|T^n x\|^{\frac{1}{n}} < \varepsilon$ for all n, except a finite set of naturals $n_1, n_2, \dots n_k$. Hence, $\|T^n x\| < \varepsilon^n < \varepsilon$ for all $n \neq n_j$, $j = 1, 2, \dots k$. On the other hand, we have $\|T^{n_j} x\| < M$, for a certain $M > 0$, so $\sup \|T^n x\| < \infty$ for all $n \in \mathbb{N}$ and $x \in X$. This implies that $\sup \|T^n\| < \infty$, i.e. T is power bounded. Consequently, all orbits are bounded, and hence T is not hypercyclic.

Note that a quasi-nilpotent operator may be supercyclic, see [25, Example 3.4]. Since the Volterra operator in $L_2[0, 1]$ is not supercyclic [11] quasi-nilpotent and injective, one may conjecture that an injective quasi-nilpotent operator is always not supercyclic. The answer to this question is negative. The example [6, Example 3.4] is a bilateral weighted shift T, that is injective and quasi-nilpotent:

Question: Under which conditions an injective quasi-nilpotent operator is not supercyclic?

A Riesz operator that commutes with an injective quasi-nilpotent operator is quasi-nilpotent, see [5]. A similar result holds for Riesz operators that commute with a hypercyclic operator:

Theorem 3.7 *Suppose that a Riesz operator $K \in L(H)$ commutes with a hypercyclic operator. Then K is quasi-nilpotent.*

Proof Observe first that for a Riesz operator K we have $\sigma_{\mathrm{w}}(K) = \{0\}$. If the Riesz operator K commutes with a hypercyclic operator then, by Theorem 3.4, $\{0\} = \sigma_{\mathrm{w}}(K) = \sigma(K)$, thus K is quasi-nilpotent.

By Theorem 3.7 a non quasi-nilpotent Riesz operator on a Hilbert space cannot commute with a hypercyclic operator, as well as it cannot commutes with an injective quasi-nilpotent operator. Hypercyclic operators can arise as perturbations of the identity by compact operators, see [10, 16]. The next corollary generalizes [10, Proposition 4.3], where the result was proved for $I + T$. Clearly, T commutes with $p(T)$ for every polynomial p.

Corollary 3.8 *If $T \in L(H)$ is compact and for some polynomial $p(T)$ is hypercyclic then T is quasi-nilpotent.*

An operator $T \in L(X)$ is said to be *Drazin invertible* (with a finite index), $T \in D(X)$ if $p(T) = q(T) < \infty$. Clearly, $\lambda I - T \in L(X)$ is Drazin invertible if and only if $\lambda I - T$ is invertible or λ is a pole of the resolvent. Drazin invertibility suggests the following definition: an operator $T \in L(X)$, is said to be *left Drazin invertible*,

$T \in LD(X)$, if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$, is said to be *right Drazin invertible*, $T \in RD(X)$ if $q := q(T) < \infty$ and $T^q(X)$ is closed. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{ap}}(T)$ then λ is said to be a *left pole* If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a *right pole*. It should be noted that $T \in L(X)$ is Drazin invertible if and only if T is both left Drazin invertible and right Drazin invertible. The *Drazin spectrum* is defined as $\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin D(X)\}$, and, analogously, may be defined the *left Drazin spectrum* $\sigma_{\mathrm{ld}}(T)$, and the *right Drazin spectrum* $\sigma_{\mathrm{rd}}(T)$. Evidently, $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{ld}}(T) \cup \sigma_{\mathrm{rd}}(T)$. The proof of the following lemma may be found in [4, Lemma 2.4]. Let $\Pi(T) := \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$ denote the set of all poles of T and $\Pi_a(T) := \sigma(T) \setminus \sigma_{\mathrm{ld}}(T)$ denote the set of all left poles of T.

Lemma 3.9 *If $T \in L(X)$ then*

$$\sigma_{\mathrm{ap}}(T) = \sigma_{(\mathrm{T})} \Leftrightarrow \sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T). \tag{3.3}$$

If T is supercyclic then $\Pi_a(T) = \Pi(T)$.

Proof The equivalence (3.9) has been proved in [4]. If T is supercyclic then T^* has SVEP, so $\sigma_{\mathrm{ap}}(T) = \sigma_{(\mathrm{T})}$, and from the equivalence (3.3) we obtain $\Pi_a(T) = \Pi(T)$. □

Note that $T \in L(X)$ is Drazin invertible(with a finite index) if and only if there exists an operator $S \in L(X)$ and $n \in \mathbb{N}$ such that

$$TS = ST, STS = S, T^n ST = T^n. \tag{3.4}$$

The operator S is called the *Drazin inverse* of T. Note that the Drazin inverse S of an operator, if it exists, is uniquely determined, see [9].

Remark 3.10 If T is Drazin invertible, with Drazin inverse S, then S is Drazin invertible. If U is the Drazin inverse of S then the Drazin inverse of U is T, see [1, Remark 1.23].

In [18, 22, Corollary 2.21] Kitai has proved that if a hypercyclic operator T is invertible then its inverse is also hypercyclic, see also Herrero and Kitai [17]. We show now that a Drazin invertible operator is hypercyclic if and only if is invertible:

Theorem 3.11 *If $T \in L(X)$ the following statements are equivalent:*

(i) *T is hypercyclic and Drazin invertible;*
(ii) *T is Drazin invertible and its Drazin inverse S is hypercyclic;*
(iii) *T is hypercyclic and invertible;*
(iv) *T is invertible and T^{-1} is hypercyclic;*
(v) *$T \in B_+(X)$ and T is hypercyclic;*
(vi) *T is hypercyclic and left Drazin invertible.*

Furthermore, an operator $T \in B_-(X)$ is hypercyclic if and only if T is onto.

Proof The equivalence (iii) $\Leftrightarrow$ (iv) has been proved in [17]. (i) $\Leftrightarrow$ (iii) Suppose that T is Drazin invertible. Let $z \in X$ such that $\overline{\text{Orb}(z, T)} = X$ and suppose that T is not invertible. Then 0 is a pole of the resolvent, i.e. $0 < p(T) = q(T) < \infty$. Now, T is Drazin invertible if and only if there exists a decomposition $X = M \oplus N$, where M and N are closed invariant subspaces such that

$$T = T_M \oplus T_N, \quad T_M := T|M \text{ invertible}, \quad T_N := T|N \text{ nilpotent}, \tag{3.5}$$

see [1, Theorem 1.132]. Being T not invertible then $N \neq \{0\}$. Write

$$z = u + v \quad \text{where } u \in M \text{ and } v \in N,$$

and let $\nu \in \mathbb{N}$ be such that $T_N^\nu = 0$. Then, for $n \geq \nu$,

$$T^n z = (T_M)^n u + (T_N)^n v = T_M^n u + 0 = T^n u. \tag{3.6}$$

As pointed in [13, Remark (iii)], if z is hypercyclic for T then also $T^n z$ is hypercyclic for T, for every $n \in \mathbb{N}$. Hence, from (3.6), we see that $T^n u$ is hypercyclic for T for every $n \geq \nu$, i.e.,

$$\overline{\text{Orb}(T, T^n u)} = \overline{\{T^n u, T(T^n u), T^2(T^n u), ..\}} = X \quad \text{for } n \geq \nu.$$

But $T^k u \in M$ for every k, so

$$\overline{\text{Orb}(T, T^n u)} = M = X,$$

and this is impossible since $N \neq \{0\}$. Therefore, (i) $\Rightarrow$ (iii). The converse implication is obviously true.

(iv) $\Rightarrow$ (ii) The Drazin inverse of an invertible operator T is $S := T^{-1}$.

(ii) $\Rightarrow$ (iii) The Drazin inverse S of T is also Drazin invertible, since with respect to the decomposition $X = M \oplus N$, we have

$$S = (T_M)^{-1} \oplus 0,$$

see [1, p. 86]. By assumption S is hypercyclic, so, by the equivalence (i) $\Leftrightarrow$ (iii), S is invertible and its Drazin inverse is S^{-1}. The Drazin inverse of S^{-1} is S, thus $S = T$, by Remark 3.10.

(v) $\Leftrightarrow$ (i) By Theorem 3.1 $\sigma_{\text{ub}}(T) = \sigma_{(\text{T})}$, so $0 \notin \sigma_{\text{ub}}(T)$ if and only if T is invertible.

(vi) $\Leftrightarrow$ (i) Suppose that T is left Drazin invertible and hypercyclic. Since T^* has SVEP then $\sigma_{\text{ap}}(T) = \sigma(T)$, hence, by Lemma 3.9, T is Drazin invertible. The implication (i) $\Rightarrow$ is obvious.

Suppose now that $T \in B_-(X)$. By [1, Theorem 3.34, part (ii)], $T \in B_-(X)$ if and only if there exists a decomposition $X = M \oplus N$, M and N invariant closed subspaces, such that $T|M$ is onto and $T|N$ is nilpotent, see [1, Theorem 3.34, part (ii)]. If T is hypercyclic, arguing as above we deduce that $X = M$, and $T(X) = T(M) = M = X$, so T is onto.

Recall that, in general if a supercyclic operator is invertible, then its inverse need not to be supercyclic. The following theorem shows that the result of Theorem 3.11 may be partially extended to supercyclic operators for which $\sigma_{\mathrm{p}}(T^*) = \emptyset$.

Theorem 3.12 *Let $T \in L(X)$ be such that $\sigma_{\mathrm{p}}(T^*) = \emptyset$. Then the following statements are equivalent:*

(i) *T is Drazin invertible and supercyclic;*
(ii) *T is invertible and supercyclic;*
(iii) *$T \in B_+(X)$ and is supercyclic;*
(iv) *T left Drazin invertible and supercyclic.*

Furthermore, an operator $T \in B_-(X)$ is supercyclic if and only if T is onto.

Proof (i) $\Leftrightarrow$ (ii): the implication (ii) $\Rightarrow$ (i) is clear. To show (i) $\Rightarrow$ (ii), suppose that T is supercyclic and $\sigma(T^*) = \emptyset$. If z is supercyclic for T then $p(T)z$ is supercyclic for T for every polynomial $p(\lambda)$, see the proof of [6, Theorem 2]. In particular, $T^n z$ is supercyclic for T for every $n \in \mathbb{N}$, i.e the set of scalar multiples of Orb$(T, T^n z)$ is norm dense in X. Now, if T is not invertible and T is Drazin invertible, we may decompose T as in (3.9). Write $z = u + v$, with $u \in M$ and $v \in N \neq \emptyset$. Arguing as in the proof of Theorem 3.11 then $T^n z = T^n u$ for n large enough. From that it follows that the set of scalar multiples Orb$(T, T^n u)$ is norm dense in X for n large enough, i.e. $T^n u$ is supercyclic for T for n large enough. But $T^k u \in M$ for every k, from which we obtain $X = M$, and this is impossible since $N \neq \{0\}$. Hence $T = T|M$ is invertible.

(iii) $\Leftrightarrow$ (i). We need only to prove (v) $\Rightarrow$ (ii), since the opposite implication is obvious. Suppose now that $T \in B_+(X)$. By [1, Theorem 3.34, part (i)], $T \in B_+(X)$ if and only if there exists a decomposition $X = M \oplus N$, M and N invariant closed subspaces, such that $T|M$ is bounded below and $T|N$ is nilpotent, see [1, Theorem 3.34, part (ii)]. If T is supercyclic, arguing as above we deduce that $X = M$, and so T is bounded below. But T^* has SVEP, so $\sigma_{\mathrm{ap}}(T) = \sigma(T)$, hence $0 \notin \sigma_{\mathrm{ap}}(T) = \sigma(T)$, so T is invertible.

The equivalence (iv) $\Leftrightarrow$ (i) is clear since, for every supercyclic operator T^* has SVEP, hence $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ or equivalently, by Lemma 3.9, T is left Drazin invertible exactly when T is Drazin invertible.

The last assertion may be proved as in Theorem 3.11. □

Remark 3.13 The condition $\sigma_{\mathrm{p}}(T^*) = \emptyset$ is satisfied by the operators that satisfy the Supercyclicity Criterion, see [21] for details. In particular, the adjoints of supercyclic unilateral (or bilateral) weighted shifts have empty point spectrum.

Corollary 3.14 *If $T \in L(X)$ satisfies the Supercyclicity Criterion then T is Drazin invertible if and only if T is invertible.*

We do not know if the equivalences of Theorem 3.12 holds for supercyclic operators for which $\sigma_{\mathrm{p}}(T^*) \neq \emptyset$ (except the equivalence (i) $\Leftrightarrow$ (iv), that holds since depends only from the SVEP of T^*).

Recall that an operator $K \in L(X)$ is said to be *algebraic* if there exists a complex polynomial h such that $h(K) = 0$. Examples of algebraic operators are nilpotent operators and operators K for which K^n is a finite-dimensional operator for some $n \in \mathbb{N}$.

Corollary 3.15 *If $T \in L(X)$ and 0 is a pole of T then T is not hypercyclic. Any algebraic operator T for which $0 \in \sigma(T)$ is not hypercyclic.*

Proof If 0 is a pole then T is Drazin invertible. If were T hypercyclic then T would be invertible by Theorem 3.11, and this is impossible, since 0 is an eigenvalue. Every algebraic operator has a finite spectrum, see [1, Section 3.5]. Moreover, every $\lambda \in \sigma(K)$ is a pole, see [1, Theorem 3.93], so, if $0 \in \sigma(K)$, 0 is a pole.

Corollary 3.16 *Suppose that an operator $T \in L(X)$ has SVEP at 0 and is supercyclic. Then $T \in \Phi_{\pm}(X) \Leftrightarrow T$ is invertible.*

Proof If T has SVEP at 0 and $T \in \Phi_{\pm}(X)$ then $p(T) < \infty$. For every hypercyclic operator T its dual T^* has SVEP, hence, if $T \in \Phi_{\pm}(X)$, then $q(T) < \infty$. Therefore, T is Drazin invertible and consequently, by Theorem 3.11, T is invertible. The opposite implication is clear.

Finally, in [24] Salas proved that that there exist hypercyclic operators $T \in L(H)$ (a bilateral weighted shift on a Hilbert space) whose adjoint T' is also hypercyclic. For these operators $(T')' = T$ has SVEP. Hence, by the previous corollary, these operators are semi-Fredholm if and only if they are invertible.

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Declarations

Conflict of interest We have no conflict of interest.

References

1. Aiena, P.: Fredholm and local spectral theory II, with application to Weyl-type theorems. Springer Lecture Notes of Math no. 2235 (2018)

2. Aiena, P.: On the spectral mapping theorem for Weyl spectra of Toeplitz operators. Adv. Oper. Theory **5**, 1618–1634 (2020)
3. Aiena, P.: Normal Operators and their Generalizations. In: Aron, R.M., et al. (eds.) Operator and Norm Inequalities and Related Topics, Trends in Mathematics, pp. 355–396. Birkhäuser, Basel (2022)
4. Aiena, P., Aponte, E., Guillén, J.: The Zariouh's property (gaz) through localized SVEP. Matematicki Vesnik **72**(4), 314–326 (2020)
5. Aiena, P., Burderi, F., Triolo, S.: On commuting quasi-nilpotent operators that are injective. Math. Proc. R. Irish Acad. **122A**(2), 101–116 (2022)
6. Ansari, S.I.: Hypercyclic and cyclic vectors. J. Funct. Anal. **128**, 374–383 (1995)
7. Bayart, F., Matheron, E.: Dynamics of Linear Operators. Cambridge Tracts in Maths, vol. 179 (2009)
8. Bourdon, P.: Orbits of hyponormal operators. Michigan Math. J. **44**(2), 345–353 (1997)
9. Caradus, S.R.: *Operator Theory of the Pseudo-Inverse*, Queen's Papers in Pure and Appl. Math., vol.38, Queen's University, Kingston, Ontario, (1974)
10. Chan, K.C., Shapiro, J.H.: The Cyclic Behavior of Translation Operators on Hilbert Spaces of Entire Functions. Indiana Univ. Math. J. **40**(4), 1421–1449 (1991)
11. Gallardo-Gutiíerrez, E.A., Montes-Rodríguez, A.: The volterra operator is not supercyclic. Integr. Equ. Oper. Theory **50**, 211–216 (2004)
12. González, M., León, F., Montes-Rodríguez, A.: Semi-Fredholm theory: Hypercyclic and supercyclic subspaces. Proc. Lond. Math. Soc. **81**, 169–189 (2000)
13. Godefroy, G., Shapiro, J.H.: Operators with dense, invariant, cyclic vector manifolds. J. Funct. Anal. **98**, 229–269 (1991)
14. Grosse-Erdmann, K.-G., Manguillot, A.P.: Linear Chaos. Universitext, Springer, New York (2011)
15. Herrero, D.A.: Limits of hypercyclic operators. J. Funct. Anal. **99**, 179–190 (1991)
16. Herrero, D.A., Wang, Z.: Compact perturbations of hypercyclic and supercyclic operators. Indiana Univ. Math. J. **39**, 819–830 (1990)
17. Herrero, D.A., Kitai, C.: On invertible hypercyclic operators. Proc. Am. Math. Soc. **116**(3), 873–875 (1992)
18. Kitai, C.: Invariant Closed Sets for Linear Operators. Thesis. Univ, Toronto (1982)
19. León, F.L., Montes-Rodríguez, A.:Spectral Theory and Hypercyclic Subspaces. Trans. Am. Math. Soc. V. **353**(1), 247–267
20. Matache, V.: Notes on Hypercyclic Operators. Acta. Sci. Mat. (Szeged) **58**, 397–406 (1993)
21. Montes-Rodrígue, A., Salas, H.: Supercyclic subspaces: spectral theory and weighted shifts. Adv. Math. **163**, 74–134 (2001)
22. Nordgren, E.A.: Composition operators. Can. J. Math. **20**, 442–449 (1968)
23. Rolewicz, S.: On orbits of elements. Stud. Math. **32**, 17–22 (1969)
24. Salas, H.N.: A hypercyclic operator whose adjoint is also hypercyclic. Proc. Am. Math. Soc. **112**, 765–770 (1991)
25. Salas, H.N.: Supercyclicity and weighted shift. Stud. Math. **135**(1), 55–73 (1999). Soc. **112**, 765–770 (1991)