Entanglement Percolation in Random State Quantum Networks

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February 19th, 2025

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A. Romancino et al., **“Random entanglement percolation”,** [Manuscript in preparation] (2024)

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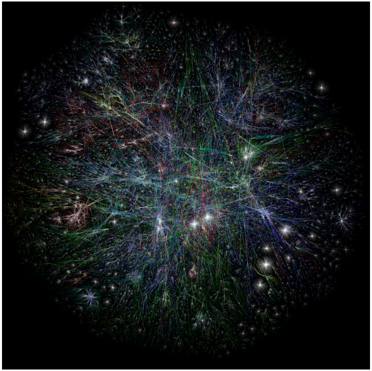
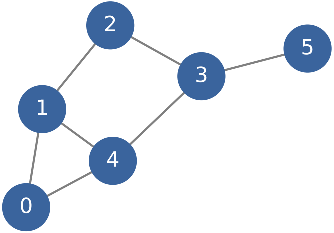
Some Interesting Concepts from Network Theory

What is a Network?



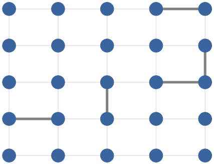


M. Newman, ***Networks (2nd edition),*** (Oxford University Press, 2018)

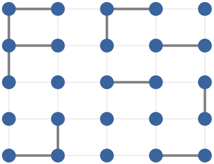
V. Latora et al., ***Complex networks: principles, methods and applications,*** (Cambridge University Press, 2017)

Percolation Theory

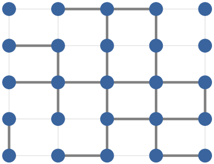
Percolation Theory



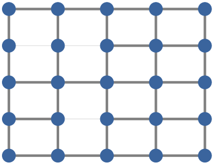
$p = 0.10$



$p = 0.35$

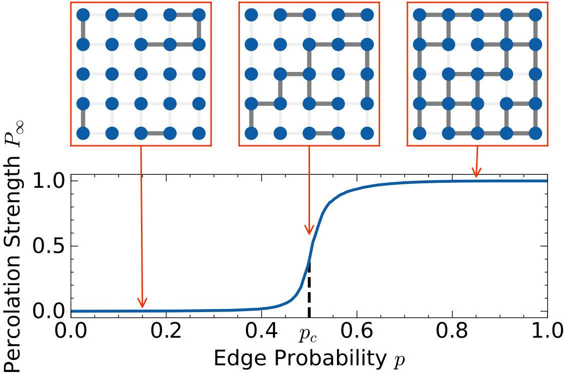


$p = 0.65$



$p = 0.90$

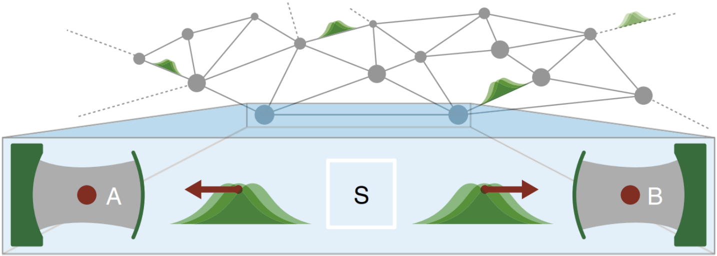
Percolation Theory



R. Albert and A.-L. Barabási, **“Statistical mechanics of complex networks”,** Reviews of Modern Physics **74**, 47–97 (2002)

Introduction to Entanglement Percolation

Quantum Internet



S. Pirandola et al., **“Fundamental limits of repeaterless quantum communications”,** Nature communications **8**, 1–15 (2017)

H. J. Kimble, **“The quantum internet”,** Nature **453**, 1023–1030 (2008)

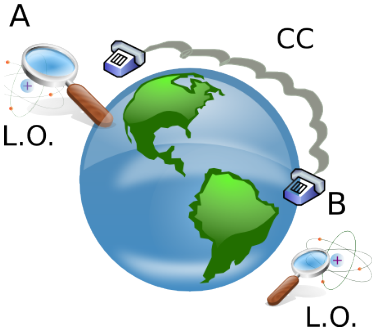
The Schmidt Decomposition

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |i_A\rangle \otimes |i_B\rangle$$

Where $\sqrt{\lambda_i}$ are the *Schmidt coefficients* and $\sum_i \lambda_i = 1$.

We will assume $\lambda_1^\psi \geq ... \geq \lambda_d^\psi$.

Nielsen's Theorem



$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \iff \lambda^{\psi} \prec \lambda^{\phi}$$

$$\lambda^{\psi} \prec \lambda^{\phi} \quad \overset{def}{\iff} \quad \sum_{i=1}^{k} \lambda_i^{\psi} \leq \sum_{i=1}^{k} \lambda_i^{\phi} \quad \text{for each } k = 1, \ldots, d$$

M. A. Nielsen, **"Conditions for a class of entanglement transformations",** Physical Review Letters **83**, 436–439 (1999)

Vidal's Formula (SLOCC)

The optimal probability to convert $|\psi\rangle \to |\phi\rangle$ with LOCC is:

$$p(\psi \to \phi) = \min_{k=1,..,d} \frac{\sum_{i=k}^{d} \lambda_i^{\psi}}{\sum_{i=k}^{d} \lambda_i^{\phi}}$$

For example, we can calculate the probability of getting a singlet for 2 qubits (Singlet Conversion Probability):

$$p_\omega = p(\omega \to \phi^+) = \min\left\{1, 2\lambda_2^{\omega}\right\}$$

G. Vidal, **"Entanglement of pure states for a single copy",** Physical Review Letters **83**, 1046–1049 (1999)

Example: Procrustean Method

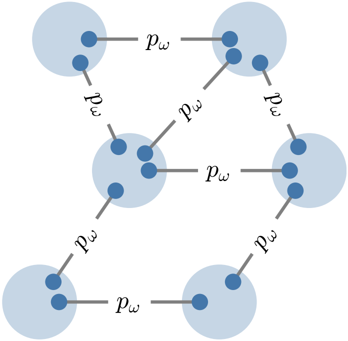
Given $|\omega\rangle = \sqrt{\lambda_1^\omega}\,|00\rangle + \sqrt{\lambda_2^\omega}\,|11\rangle$ we can implement $|\omega\rangle \rightarrow |\phi^+\rangle$ with:

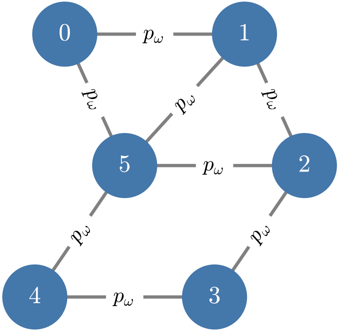
$$M_1 = \begin{pmatrix} \sqrt{\frac{\lambda_2^\omega}{\lambda_1^\omega}} & 0 \\ 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} \sqrt{1 - \frac{\lambda_2^\omega}{\lambda_1^\omega}} & 0 \\ 0 & 0 \end{pmatrix}$$

$$p(|\phi^+\rangle) = 2\lambda_2^\omega$$
$$p(|00\rangle) = 1 - 2\lambda_2^\omega$$

C. H. Bennett et al., **“Concentrating partial entanglement by local operations”,** Physical Review A **53**, 2046–2052 (1996)

Quantum Networks

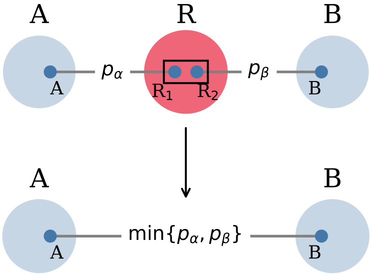




S. Perseguers et al., **“Distribution of entanglement in large-scale quantum networks”,** Reports on Progress in Physics **76**, 096001 (2013)

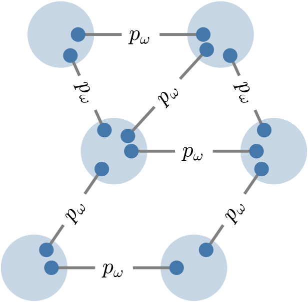
Allowed Operations

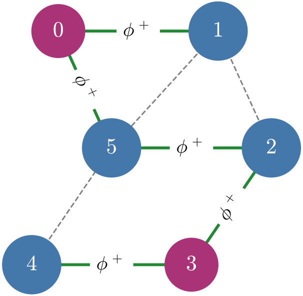
- Stochastic LOCC purification
- Entanglement swapping



J. Nokkala et al., **“Complex quantum networks: a topical review”,** Journal of Physics A: Mathematical and Theoretical **57**, 233001 (2024)

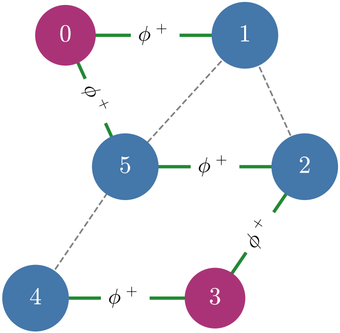
Classical Entanglement Percolation

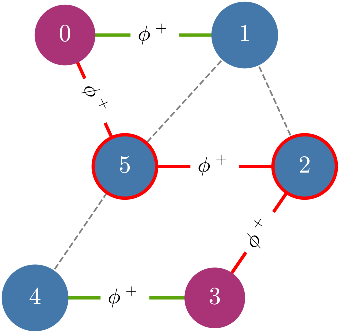




- Optimal SLOCC purification operation is applied to every link in the network.
- If the starting SCP was high enough a chain of singlets will be present $\Rightarrow$ entanglement swapping.

Classical Entanglement Percolation

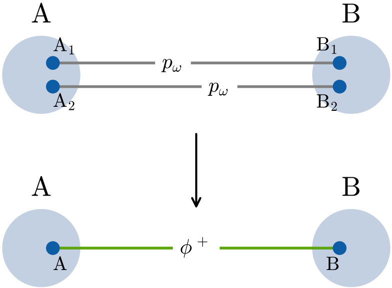




$$\boxed{p^{\text{CEP}} = p_c}$$

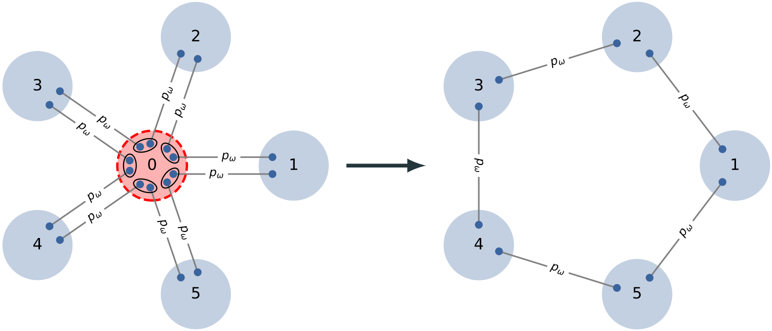
A. Acín et al., **“Entanglement percolation in quantum networks”,** Nature Physics **3**, 256–259 (2007)

Multigraph Quantum Networks



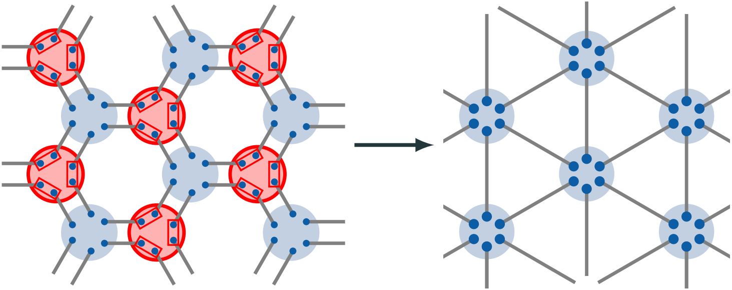
$$p(\omega^{\otimes 2} \to \phi^+) = \min\left\{1, 2\left(1 - (\lambda_1^\omega)^2\right)\right\}$$

q-Swap Operation



M. Cuquet and J. Calsamiglia, **“Entanglement percolation in quantum complex networks”,** Physical Review Letters **103**, 240503 (2009)

Quantum Entanglement Percolation



$p_c^{\hexagon} \approx 0.653$

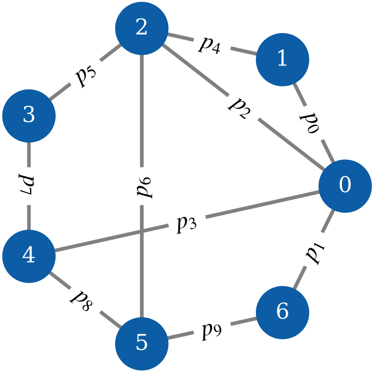
$p^{\mathrm{CEP}} \approx 0.358$

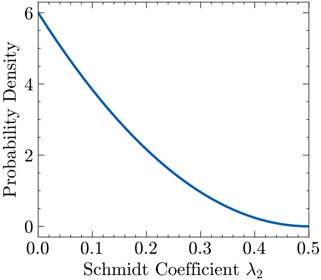
$p_c^{\triangle} \approx 0.347$

$p^{\mathrm{QEP}} \approx 0.347$

Entanglement Percolation in Random State Networks

Random State Networks





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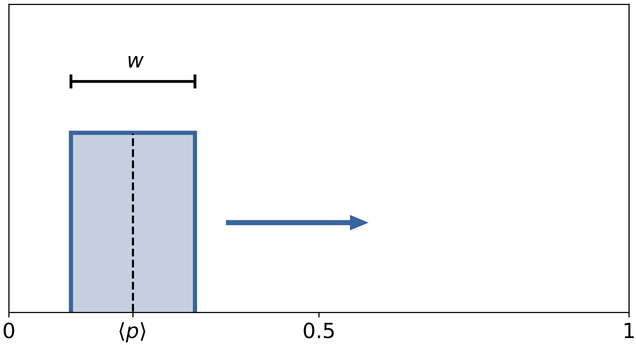
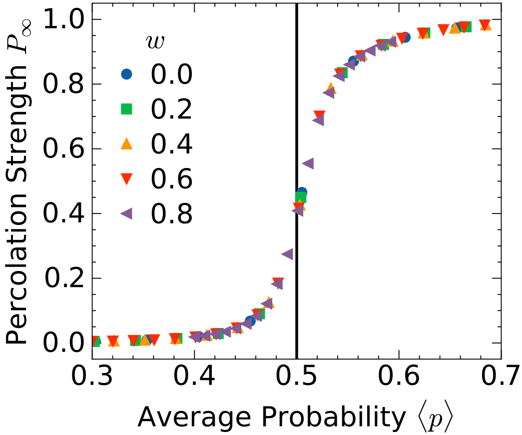
Classical Random Entanglement Percolation



Classical Random Entanglement Percolation



$$\boxed{p_{\text{rand}}^{\text{CEP}} = \langle p_\omega \rangle = p_c}$$

Random State Multiedges

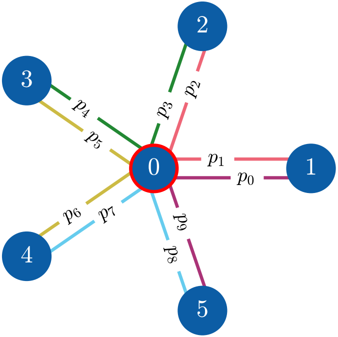
$$p_{k^{\otimes 2}} = 2\left[1 - \left(\lambda_1^{\psi_k}\right)^2\right] = 2p_k - \frac{p_k^2}{2}$$

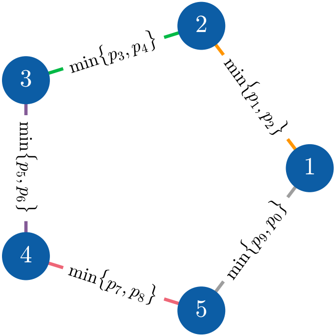
$$p_{k_1 \otimes k_2} = 2\left(1 - \lambda_1^{\psi_{k_1}} \lambda_1^{\psi_{k_2}}\right) = p_{k_1} + p_{k_2} - \frac{p_{k_1} p_{k_2}}{2}$$

$$\langle p_{k^{\otimes 2}} \rangle = 2\langle p \rangle - \frac{\langle p^2 \rangle}{2}$$

$$\langle p_{k_1 \otimes k_2} \rangle = 2\langle p \rangle - \frac{\langle p \rangle^2}{2}$$

Quantum Advantages and Disadvantages

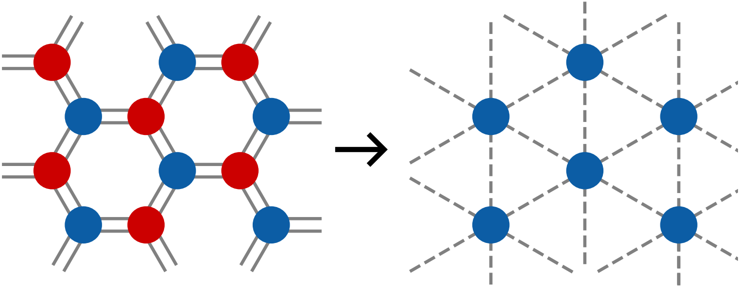




$$f_{min}(x) = 2f(x)[1 - F(x)]$$

$$\langle p_{min} \rangle = \langle p \rangle - \frac{w}{6}$$

Quantum Advantages and Disadvantages



$$\begin{cases} 0 \leq w \lesssim 0.067 & \text{advantage} \\ 0.067 \lesssim w \leq 1 & \text{disadvantage} \end{cases}$$

Conclusions

Conclusions

1. **Look into network and graph theory for your complicated problems!**
2. Percolation threshold: does not depend on the width of the distribution but only on the average.
3. With this more realistic scenario, classical is *often* better than quantum.
4. Still an open question wheter a lower bound of the initial entanglement exist and how to achieve it.

References i

[1] A. Romancino, G. M. Palma, and A. Sanpera, **“Random entanglement percolation”,** [Manuscript in preparation] (2024).

[2] M. Newman, ***Networks (2nd edition)*,** (Oxford University Press, 2018).

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