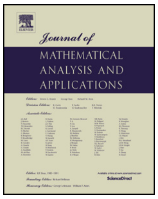
J. Math. Anal. Appl. 548 (2025) 129484

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa



Regular Articles

Limits of hypercyclic operators on Hilbert spaces

Pietro Aiena, Fabio Burderi, Salvatore Triolo [*]

Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy

A R T I C L E   I N F O

Article history:
Received 23 October 2024
Available online 14 March 2025
Submitted by M. Mathieu

Keywords:
Hypercyclic and supercyclic operators on Hilbert spaces
Shift operators on $\ell_2(\mathbb{N})$

A B S T R A C T

This article concerns the operators $T \in L(H)$, defined on a separable Hilbert space H, that belong to the norm closure $\overline{HC(H)}$ in $L(H)$ of the set $HC(H)$ of all hypercyclic operators. Starting from a Herrero's characterization of these operators [11] we deduce some criteria that are very useful in many concrete cases. We also show that if $T \in L(H)$ is invertible then $T \in \overline{HC(H)}$ if and only if $T^{-1} \in \overline{HC(H)}$. This result extends to $\overline{HC(H)}$ a known result of Kitai and Herrero established for hypercyclic operators, ([13]).

1. Introduction

Let H be a separable Hilbert space. A vector $x \in H$ is said to be *hypercyclic* for $T \in L(H)$ if the orbit

$$\operatorname{orb}(T, x) = \{x, Tx, T^2x, \ldots\}$$

is norm-dense in H, while $x \in H$ is said to be *supercyclic* for T if the set of scalar multiples of $\operatorname{orb}(T, x)$ is norm-dense in H. An operator $T \in L(H)$, is said to be *hypercyclic*, $T \in HC(H)$, if there exists a hypercyclic vector $x \in H$, and is said to be *supercyclic* if there exists a supercyclic vector $x \in H$. Evidently, every hypercyclic operator is supercyclic. Classes of operators which satisfy a hypercyclicity or supercyclicity condition have been studied by a conspicuous number of authors (see for instance, [8], [9], [19], [10]). The notions of hypercyclicity and supercyclicity are intimately related to the invariant subspace problem.

In [11] D. A. Herrero gave a spectral characterization of the operators that belong to the norm closure $\overline{HC(H)}$ of the class of all hypercyclic operators in $L(H)$, where H is a separable Hilbert space. This characterization concerns certain Fredholm properties of the spectrum. There are very few applications of the Herrero' result. In this paper we simplify the conditions given by Herrero, under some additional

---

[*] Corresponding author.
E-mail addresses: pietro.aiena@unipa.it (P. Aiena), fabio.burderi@unipa.it (F. Burderi), salvatore.triolo@unipa.it (S. Triolo).

https://doi.org/10.1016/j.jmaa.2025.129484

hypothesis on T. As an application, these semplifications lead to a characterization of the operators that belong to $\overline{HC(H)}$ is many concrete cases.

By a well known result of Kitai and Herrero ([11]) if T is invertible then T is hypercyclic if and only if T^{-1} is hypercyclic. We obtain an analogous result for operators $T \in \overline{HC(H)}$: if T is invertible, then $T \in \overline{HC(H)}$ if and only if $T^{-1} \in \overline{HC(H)}$. This an immediate consequence of a more general fact: if $T \in L(H)$ is Drazin invertible with Drazin inverse S then $T \in \overline{HC(H)}$ if and only if $S \in \overline{HC(H)}$.

2. Definitions and preliminary results on Fredholm theory

If $T \in L(H)$ we denote by $\alpha(T)$ and $\beta(T)$, the dimension of the kernel $\ker T$ and the codimension of the range $R(T) := T(H)$, respectively. Recall that $T \in L(H)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(H)$, if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in L(H)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(H)$ if $\beta(T) < \infty$. The class of *Fredholm* operators is defined by $\Phi(H) := \Phi_+(H) \cap \Phi_-(H)$, while the class of semi-Fredholm operators is defined by $\Phi_\pm(H) := \Phi_+(H) \cup \Phi_-(H)$. If $T \in \Phi_\pm(H)$ the index is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The set of *Weyl operators* is defined by $W(H) := \{T \in \Phi(H) : \operatorname{ind} T = 0\}$, the class of *upper semi-Weyl operators* is defined by $W_+(H) := \{T \in \Phi_+(H) : \operatorname{ind} T \leq 0\}$, and class of *lower semi-Weyl operators* is defined by $W_-(H) := \{T \in \Phi_-(H) : \operatorname{ind} T \geq 0\}$. Clearly, $W(H) = W_+(H) \cap W_-(H)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(H)\}$, the *upper semi-Weyl spectrum*, defined by $\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(H)\}$, and the *lower semi-Weyl spectrum*, defined by $\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(H)\}$. Let $\sigma_{\mathrm{sf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(H)\}$ the *semi-Fredholm spectrum* and denote by $p := p(T)$ the *ascent* of an operator T; i.e. the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q := q(T)$ be the *descent* of T; i.e. the smallest non-negative integer q such that $T^q(H) = T^{q+1}(H)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [15, Proposition 50.2].

The class of all *Browder operators* is defined as the set $B(H) := \{T \in \Phi(H) : p(T), q(T) < \infty\}$; the class of all *upper semi-Browder operators* is defined $B_+(H) := \{T \in \Phi_+(H) : p(T) < \infty\}$, and the class of all *lower semi-Browder operators* is defined $B_-(H) := \{T \in \Phi_-(H) : q(T) < \infty\}$. Obviously, $B(H) \subseteq W(H)$, $B_+(H) \subseteq W_+(H)$ and $B_-(H) \subseteq W_-(H)$.

In the sequel we denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum*, defined by $\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$, and denote by $\sigma_{\mathrm{s}}(T)$ *surjectivity spectrum*. We set $\rho_{\mathrm{ap}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$ and $\rho_{\mathrm{uw}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{uw}}(T)$. Recall that an operator $T \in L(X)$, X a Banach space, is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$, (abbreviated SVEP at λ_0), if for every open "neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and the dual T^* have SVEP at the isolated points of the spectrum.

In the case of Hilbert space operators is better to consider the Hilbert adjoint T', instead of T^*. The following equivalence is well known:

$$T^* \quad \text{has SVEP} \quad \Leftrightarrow \quad T' \quad \text{has SVEP.}$$

3. Hypercyclic or supercyclic operators

It is known that the adjoint T' of a hypercyclic operator has empty point spectrum. This implies that the range of $\lambda I - T$ is dense for every $\lambda \in \mathbb{C}$, see [17, Theorem 2.3]. If T is supercyclic then T' either has point spectrum $\sigma_{\mathrm{p}}(T') = \emptyset$ or $\sigma_{\mathrm{p}}(T') = \{\alpha\}$. Consequently, the dual T' of every supercyclic operator has SVEP, since an operator with countable point spectrum has SVEP.

Set

$$\sigma_{\mathrm{sf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_{\pm}(H)\}.$$

Lemma 3.1. *$\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$ if and only if ind$(\lambda I - T) \geq 0$ for every $\lambda \notin \sigma_{\mathrm{sf}}(T)$. In particular, if T' has SVEP then $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$.*

Proof. Trivially if $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$ and $\lambda \notin \sigma_{\mathrm{sf}}(T)$ then $\lambda I - T$ is lower Weyl, so ind $(\lambda I - T) \geq 0$. Conversely, suppose that ind $(\lambda I - T) \geq 0$ for every $\lambda \notin \sigma_{\mathrm{sf}}(T)$. To show the equality $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$ it suffices to prove that $\sigma_{\mathrm{lw}}(T) \subseteq \sigma_{\mathrm{sf}}(T)$, since the opposite inclusion holds for every operator. Let $\lambda \notin \sigma_{\mathrm{lw}}(T)$. Then $\lambda I - T \in \Phi_{-}(H)$, thus $\lambda \notin \sigma_{\mathrm{sf}}(T)$.

Suppose now that T' has SVEP and $\lambda \notin \sigma_{\mathrm{sf}}(T)$. Since $\lambda I - T \in \Phi_{\pm}(H)$, so $\lambda \in \Phi_{+}(H)$ or $\lambda \in \Phi_{-}(H)$. The SVEP of T' at λ implies that $q(\lambda I - T) < \infty$. By [2, Theorem 1.22] then $\beta(\lambda I - T) \leq \alpha(\lambda I - T)$. The last inequality obviously implies that $\lambda I - T \in \Phi_{-}(H)$. Finally, since ind $(\lambda I - T) = \alpha(\lambda I - T) - \beta(\lambda I - T) \geq 0$ then $\lambda \notin \sigma_{\mathrm{lw}}(T)$. ■

In [4] it has been proved that if $T \in L(X)$, X a Banach space, is hypercyclic then

$$\sigma(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \quad \text{and} \quad \sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T). \tag{1}$$

Denote by $\partial\mathbf{D}$ the boundary of the unit closed disc $\mathbf{D}$ in $\mathbb{C}$. In [11] D. A. Herrero gave the following spectral characterization of the operators $T \in \overline{HC(H)}$ through some Fredholm properties of the spectrum of T:

Theorem 3.2. *$\overline{HC(H)}$ is the class of all $T \in L(H)$ that satisfy the following conditions:*

(i) *$\sigma_{\mathrm{w}}(T) \cup \partial\mathbf{D}$ is connected;*

(ii) *$\sigma(T) = \sigma_{\mathrm{b}}(T)$;*

(iii) *ind$(\lambda I - T) \geq 0$ for every $\lambda \notin \sigma_{\mathrm{sf}}(T)$, or equivalently $\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{lw}}(T)$.*

Observe that the conditions (i) and (ii) of Theorem 3.2 imply that $\sigma(T) \cup \partial\mathbf{D}$ is connected. It is interesting to note that $T \in \overline{HC(H)}$ if and only if for every $\varepsilon > 0$ there exists a compact operator $K \in L(H)$ such that $\|K\| < \varepsilon$ and $T + K \in HC(H)$, see [12].

Remark 3.3. Recall that if $T \in HC(H)$ then T has no nontrivial finite-codimensional invariant subspace, see [18].

For every $F \subseteq \mathbb{C}$ set $\overline{F} := \{\overline{\lambda} : \lambda \in F\}$. For Hilbert space operators T the relationship between the dual T^*, in the sense of Banach space operators, and the adjoint T' is the following

$$\overline{\lambda} I - T' = U^{-1}(\lambda I - T^*)U$$

where $U : H \to H^*$, H^* the dual of H, is a conjugated-linear isometry that associates to every $y \in H$ the linear form $f_y(x) := \langle x, y\rangle$. From this easily follows, taking into account that $\lambda I - T \in \Phi_{\pm}(H)$ if and only if $\lambda I - T^* \in \Phi_{\pm}(H)$, that $\sigma_{\mathrm{sf}}(T') = \overline{\sigma_{\mathrm{sf}}(T^*)} = \overline{\sigma_{\mathrm{sf}}(T)}$. Analogously we have:

$$\sigma_{\mathrm{lw}}(T) = \overline{\sigma_{\mathrm{uw}}(T')} \quad \text{and} \quad \sigma_{\mathrm{uw}}(T) = \overline{\sigma_{\mathrm{lw}}(T')},$$

and $\sigma_{\mathrm{w}}(T') = \overline{\sigma_{\mathrm{w}}(T)}$. Analogously,

$$\sigma_{\mathrm{ap}}(T) = \overline{\sigma_{\mathrm{s}}(T')} \quad \text{and} \quad \sigma_{\mathrm{s}}(T) = \overline{\sigma_{\mathrm{ap}}(T')}.$$

An operator $T \in L(X)$ is said to be *Drazin invertible* (with a finite index), $T \in D(X)$ if $p(T) = q(T) < \infty$. Clearly, $\lambda I - T \in L(X)$ is Drazin invertible if and only if $\lambda I - T$ is invertible or λ is a pole of the resolvent. Note that $T \in L(X)$ is Drazin invertible (with a finite index) if and only if there exists an operator $S \in L(X)$ and $n \in \mathbb{N}$ such that

$$TS = ST, STS = S, T^n ST = T^n. \tag{2}$$

The operator S is called the *Drazin inverse* of T. The Drazin inverse S of an operator, if it exists, is uniquely determined, see [2, Chapter 2]. T is Drazin invertible if and only if there exists a decomposition $X = M \oplus N$, where M and N are closed invariant subspaces such that

$$T = T_1 \oplus T_2, \quad \text{where} \quad T_1 := T|M \text{ nilpotent}, \quad T_2 := T|N \text{ invertible}, \tag{3}$$

see [2, Theorem 1.132]. The Drazin inverse S of T is also Drazin invertible, since with respect to the decomposition $X = M \oplus N$, we have

$$S = 0 \oplus S_2, \quad \text{where} \quad S_2 := (T_2)^{-1}$$

see [2, p. 86]. It should be noted that

$$\sigma(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma(T) \setminus \{0\}\}, \tag{4}$$

and

$$\sigma_{\mathrm{b}}(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{b}}(T) \setminus \{0\}\}, \tag{5}$$

see [3].

Lemma 3.4. *If $T \in L(H)$ is Drazin invertible with Drazin inverse S then*

$$\sigma_{\mathrm{lw}}(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{lw}}(T) \setminus \{0\}\}, \tag{6}$$

and

$$\sigma_{\mathrm{sf}}(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{sf}}(T) \setminus \{0\}\}. \tag{7}$$

Proof. To show the equality (6) decompose T as in (3) and let $\lambda \neq 0$ such that $\frac{1}{\lambda} \notin \sigma_{\mathrm{lw}}(T)$, i.e. $\frac{1}{\lambda} I - T \in \Phi_-(H)$ and $\operatorname{ind}(\frac{1}{\lambda} I - T) \geq 0$. By [3, Theorem 2.8, part (iii)] then $\beta(\lambda I - S)(H) = \beta(\frac{1}{\lambda} I - T)(H) < \infty$, hence $\lambda I - S \in \Phi_-(H)$. It remains only to prove that $\operatorname{ind}(\lambda I - S) \geq 0$. Again by [3, Theorem 2.8, part (ii)] we have $\alpha(\frac{1}{\lambda} I - T) = \alpha(\lambda I - S)$, hence

$$\operatorname{ind}(\lambda I - S) = \operatorname{ind}(\frac{1}{\lambda} I - T) \geq 0.$$

Therefore, $\lambda I - S \in W_-(H)$ and hence $\lambda \notin \sigma_{\mathrm{lw}}(S)$.

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{lw}}(S)$. Then $\lambda I - S \in W_-(H)$, and as above we have $\alpha(\frac{1}{\lambda} I - T) = \alpha(\lambda I - S)$, and $(\beta(\frac{1}{\lambda} I - T)) = \beta(\lambda I - S)$, thus $\frac{1}{\lambda} I - T \in \Phi_-(H)$ and $\operatorname{ind}(\frac{1}{\lambda} I - T) = \operatorname{ind}(\lambda I - S) \geq 0$, so $\frac{1}{\lambda} I - T \in W_-(H)$. Hence $\frac{1}{\lambda} \notin \sigma_{\mathrm{lw}}(T)$.

The equality (7) may be proved in a similar way, always by using [3, Theorem 2.8]. ■

Theorem 3.5. *Let $T \in L(H)$ be Drazin invertible with Drazin inverse S. Then $T \in \overline{HC(H)}$ if and only if $S \in \overline{HC(H)}$.*

Proof. Suppose that $T \in \overline{HC(H)}$. First we show that the condition (iii) holds for S. To prove that $\sigma_{\mathrm{sf}}(S) = \sigma_{\mathrm{lw}}(S)$ it suffices to show the inclusion $\sigma_{\mathrm{lw}}(S) \subseteq \sigma_{\mathrm{sf}}(S)$. Let $\lambda \notin \sigma_{\mathrm{sf}}(S)$. If $\lambda = 0$, then S is semi-Fredholm, and being S Drazin invertible we have $p(S) = q(S) < \infty$. By [2, Theorem 1.22] we then have $\alpha(S) = \beta(S) < \infty$, so $0 \notin \sigma_{\mathrm{lw}}(S)$, since $\lambda I - S$ is closed.

Suppose that $\lambda \neq 0$. Then $\frac{1}{\lambda} \notin \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{sf}}(T)$, by the condition (iii) of Theorem 3.1, and this entails, by Lemma 3.4, that $\lambda \notin \sigma_{\mathrm{sf}}(S)$. Hence $\sigma_{\mathrm{sf}}(S) = \sigma_{\mathrm{lw}}(S)$. Since $\sigma(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, from (4) and (5) we see that $\sigma(S) \setminus \{0\} = \sigma_{\mathrm{b}}(S) \setminus \{0\}$. Suppose that $0 \notin \sigma_{\mathrm{b}}(S)$. Then S is Browder and from [3, Theorem 3.2] it then follows that T is Browder, so $0 \notin \sigma_{\mathrm{b}}(T) = \sigma(T)$, i.e. T is invertible, thus $S = T^{-1}$, and hence $0 \notin \sigma(S)$. This shows that the condition (ii) of Theorem 3.2 is satisfied. The condition (i) is clear, since $\sigma_{\mathrm{w}}(T) \cup \partial\mathbf{D}$ connected entails that there exists $\lambda \in \sigma_{\mathrm{w}}(T)$ such that $|\lambda| = 1$. Since $|\frac{1}{\lambda}| = 1$ then $\frac{1}{\lambda} \in \sigma_{\mathrm{w}}(S) \cup \partial\mathbf{D}$, so the last set is connected. By Theorem 3.2 then $S \in \overline{HC(H)}$.

To show that $S \in \overline{HC(H)}$ entails $T \in \overline{HC(H)}$, observe that S is Drazin invertible and if U is the Drazin inverse of S then the Drazin inverse of U is T, see [2, Remark 1.23]. Therefore, if $S \in \overline{HC(H)}$ then $U \in \overline{HC(H)}$, by the first part of the proof, and this implies $T \in \overline{HC(H)}$. ■

Herrero proved in [13] that if $T \in L(H)$ is invertible, then T is hypercyclic if and only if T^{-1} is hypercyclic. A similar result holds for invertible operators $T \in \overline{HC(H)}$:

Corollary 3.6. *Let $T \in L(H)$ invertible. Then $T \in \overline{HC(H)}$ if and only if $T^{-1} \in \overline{HC(H)}$.*

Proof. The Drazin inverse of T is T^{-1}. ■

We now consider operators that commute with a hypercyclic operator.

Theorem 3.7. *Suppose that $S \in L(H)$ commutes with a hypercyclic operator T. Then $S \in \overline{HC(H)}$ if and only if the following hold:*

(a) $\sigma(S) \cup \partial\mathbf{D}$ *is connected;*

(b) $\lambda I - S$ *is onto for every* $\lambda \notin \sigma_{\mathrm{sf}}(S)$.

Proof. Observe first that $\sigma_{\mathrm{w}}(S) = \sigma(S)$, see [4, Theorem 3.4]. Suppose that (a) and (b) hold. We show first that

$$\sigma(S) = \sigma_{\mathrm{b}}(S) = \sigma_{\mathrm{w}}(S). \tag{8}$$

Note first that the adjoint S' has no eigenvalues of finite multiplicity. Indeed, let λ be an eigenvalue of S' having finite multiplicity. Denote $M := \ker(\lambda I - S')$. If $\dim \ker(\lambda I - S') > 0$ then $M^{\perp}$ is a finite-codimensional, nontrivial invariant subspace under T, and this is impossible by Remark 3.3.

To show the equality (8) it suffices to prove that $\sigma(S) = \sigma_{\mathrm{w}}(S)$, since $\sigma_{\mathrm{w}}(S) \subseteq \sigma_{\mathrm{b}}(S) \subseteq \sigma(S)$. Let $\lambda \notin \sigma_{\mathrm{w}}(S)$. Then $\lambda I - S$ is Weyl, and hence also its adjoint $\overline{\lambda} I - S'$ is Weyl, so $\ker(\overline{\lambda} I - S')$ is finite-dimensional. As observed above this yields $\ker(\overline{\lambda} I - S') = 0$, and being $\overline{\lambda} I - S'$ Weyl it then follows that also $\beta(\overline{\lambda} - S') = 0$, i.e. $\lambda \notin \overline{\sigma(S')} = \sigma(S)$. Thus $\sigma_{\mathrm{w}}(S) = \sigma(S)$, as desired. This shows that the conditions (i) and (ii) of Theorem 3.2 hold. Obviously, the condition (b) entails $\beta(\lambda I - S) = 0$ for all $\lambda \notin \sigma_{\mathrm{sf}}(S)$, so $\operatorname{ind}(\lambda I - S) = \alpha(\lambda I - S) \geq 0$ for all $\lambda \notin \sigma_{\mathrm{sf}}(S)$, and hence the condition (iii) of Theorem 3.2 holds. Therefore, if (a) and (b) are satisfied then $S \in \overline{HC(H)}$.

Conversely, suppose that $S \in \overline{HC(H)}$. We need only to show that property (b). Let $\lambda \notin \sigma_{\mathrm{sf}}(S)$. By Lemma 3.1 we have $\sigma_{\mathrm{sf}}(S) = \sigma_{\mathrm{lw}}(S)$, so $\lambda I - S$ is lower semi-Weyl, and hence $\overline{\lambda} I - S'$ is upper semi-Weyl, consequently $\alpha(\overline{\lambda}I - S') < \infty$. This implies that $\alpha(\overline{\lambda}I - S') = 0$, as noted in first part of the proof, and since $\overline{\lambda}I - S'$ has closed range it then follows that $\overline{\lambda} \notin \sigma_{\mathrm{ap}}(S')$. Therefore, $\lambda \notin \overline{\sigma_{\mathrm{ap}}(S')} = \sigma_{\mathrm{s}}(S)$, i.e. $\lambda I - S$ is onto. ■

Theorem 3.8. *If T' is hypercyclic and $ST = TS$, $S' \in \overline{HC(H)}$ if and only if the following hold:*
(a') *$\sigma(S) \cup \partial\mathbf{D}$ is connected;*
(b') *$\lambda I - S$ is bounded below for every $\lambda \notin \sigma_{\mathrm{sf}}(S)$.*

Proof. From $\sigma(S') = \overline{\sigma(S)}$, it easily follows that $\sigma(S') \cup \partial\mathbf{D}$ is connected if and only if

$$\overline{\sigma(S) \cup \partial\mathbf{D}} = \overline{\sigma(S)} \cup \partial\mathbf{D},$$

is connected, and this obviously is true if and only if $\sigma(S) \cup \partial\mathbf{D}$ is connected. Suppose that $S' \in \overline{HC(H)}$. By Theorem 3.7 then

$$\lambda I - S' \quad \text{is onto for every } \lambda \notin \sigma_{\mathrm{sf}}(S'), \tag{9}$$

so $\overline{\lambda}I - S$ is bounded below for every $\overline{\lambda} \in \sigma_{\mathrm{sf}}(S)$, i.e. $\lambda I - S$ is bounded below for every $\lambda \notin \sigma_{\mathrm{sf}}(S)$. The converse follows in a similar way: if (a') and (b') are satisfied then $S' \in \overline{HC(H)}$. ■

It should be noted that H. N. Salas in [20] has proved that there exist hypercyclic operators $T \in L(H)$ (actually, a bilateral weighted shift on a Hilbert space) whose adjoint T' is also hypercyclic.

Under some other additional hypothesis the conditions (i)-(iii) of Theorem 3.2 may be reduced:

Theorem 3.9. *Suppose that T' SVEP. Then $T \in \overline{HC(H)}$ if and only if the following conditions hold:*
(c) *$\sigma(T) \cup \partial\mathbf{D}$ is connected;*
(d) *$\sigma(T) = \sigma_{\mathrm{w}}(T)$.*
If T has SVEP then $T' \in \overline{HC(H)}$ if and only if both conditions (c) *and* (d) *hold.*

Proof. If T' (or equivalently, the dual T^*) has SVEP then Browder's theorem holds for T, i.e. $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$. Hence the equivalence $T \in \overline{HC(H)} \Leftrightarrow$ (c) and (d) holds, follows from Theorem 3.2 and Lemma 3.1.

Suppose that T has SVEP. Since T is the adjoint of T', by the first part $T' \in \overline{HC(H)}$ if and only if

$$\sigma(T') \cup \partial\mathbf{D} \text{ is connected and } \sigma(T') = \sigma_{\mathrm{w}}(T').$$

From the equalities $\sigma(T') = \overline{\sigma(T)}$ and $\sigma_{\mathrm{w}}(T') = \overline{\sigma_{\mathrm{w}}(T)}$ it then follows that the condition (d) above is equivalent to saying that $\sigma(T') = \sigma_{\mathrm{w}}(T')$. It is easily seen that $\sigma_{\mathrm{w}}(T') \cup \partial\mathbf{D} = \overline{\sigma(T)} \cup \partial\mathbf{D}$ is connected if and only if the condition (c) above is satisfied. ■

A sufficient condition for an operator T to have a connected spectrum is that the hyper-range, defined by

$$T^{\infty}(H) := \bigcap_{n=1}^{\infty} T^{n}(H)$$

is $\{0\}$, see [14, Proposition 2]. This condition may be viewed, in some sense, as an abstract shift condition, since it is satisfied by every weighted forward shift on $\ell^p(\mathbb{N})$. Evidently, the condition $T^{\infty}(H) = \{0\}$ entails that T is not onto, hence $0 \in \sigma(T)$.

Theorem 3.10. *Suppose that $T^{\infty}(H) = \{0\}$. Then $T' \in \overline{HC(H)}$ if and only if $\sigma(T)$ intersects $\partial\mathbf{D}$.*

Proof. Since, $\ker(\lambda I - T) \subseteq T^{\infty}(H)$ for every $\lambda \neq 0$, the condition $T^{\infty}(H) = \{0\}$ entails that $\alpha(\lambda I - T) = 0$ for every $\lambda \neq 0$. Therefore, if $T^{\infty}(H) = \{0\}$, the point spectrum $\sigma_{\mathrm{p}}(T)$ is empty or is equal to $\{0\}$, and this implies that T has SVEP. Consequently Browder's theorem holds for T, i.e. $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, see [2, Chapter 5]. Observe that $\sigma(T)$ is connected, by [1, Theorem 2.82].

We show now that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T)$. For this it suffices to prove $\sigma(T) \subseteq \sigma_{\mathrm{b}}(T)$. Suppose first that T is quasi-nilpotent. Then $\sigma_{\mathrm{b}}(T) = \sigma(T) = \{0\}$, since both spectra are non-empty if H is infinite-dimensional.

Suppose that T is not quasi-nilpotent and choose $0 \neq \lambda \in \sigma(T)$. Since $\sigma(T)$ is connected and $0 \in \sigma(T)$, being T not onto, it then follows that λ cannot be an isolated point of $\sigma(T)$. This implies that $\lambda \in \sigma_{\mathrm{b}}(T)$, since if $\lambda I - T$ were Browder then the condition $p(\lambda I - T) = q(\lambda I - T) < \infty$ would imply that either $\lambda \notin \sigma(T)$ or λ is a pole (and hence an isolated point of $\sigma(T)$). Therefore, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$.

Finally, since $\partial \mathbf{D}$ is connected, $\sigma(T) \cup \partial \mathbf{D} = \sigma_{\mathrm{w}}(T) \cup \partial \mathbf{D}$ is connected if and only if $\sigma(T) \cap \partial \mathbf{D} \neq \emptyset$. Hence, by Theorem 3.9 the assertion is proved. ■

Let $\mathcal{N}^{\infty}(T) := \bigcup_{n=1}^{\infty} \ker T^n$ be the *hyper-kernel* of T. According [9] an operator $T \in L(H)$ is said to be a *generalized backward shift* if $\mathcal{N}^{\infty}(T)$ is dense in X and $\dim \ker T = 1$. If T is a backward shift on a Hilbert space, relative to an orthonormal basis $\{e_j\}$, then

$$\ker T^n = \operatorname{span}\{e_1, e_2, \dots, e_{n-1}\},$$

thus T is generalized backward shift.

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions defined on an open neighborhood containing the spectrum, such that f is non-constant on each of the components of its domain.

Corollary 3.11. *Suppose that $T \in L(H)$ is a surjective generalized backward shift and $f \in \mathcal{H}_{nc}(\sigma(T))$. If there exists $\lambda \in \sigma(T)$ such that $|f(\lambda)| = 1$ then $f(T) \in \overline{HC(H)}$.*

Proof. Since $T \in L(H)$ is surjective (in particular lower semi-Fredholm), the condition $H = \overline{\mathcal{N}^{\infty}(T)}$ entails that $\sigma(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$ is connected, see [14, Theorem 3]. Furthermore, by [9, Theorem 4.11], $f(T)$ is supercyclic, hence $f(T)' = f(T')$ has SVEP. By [2, Corollary 2.89] then also T' has SVEP and hence the spectral mapping theorem holds for $\sigma_{\mathrm{w}}(T)$, see [2, Chapter 5], so

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T)),$$

and the condition (d) of Theorem 3.9 holds for $f(T)$. Since $|f(\lambda)| = 1$ then $f(\lambda) \in \partial \mathbf{D}$, and $f(\lambda) \in f(\sigma(T)) = \sigma(f(T))$. From the continuity of f we know that $\sigma(f(T))$ is connected, $\partial \mathbf{D}$ is also connected, hence $\sigma(f(T)) \cup \partial \mathbf{D}$ is connected. By Theorem 3.9 then $f(T) \in \overline{HC(H)}$. ■

Let $(\omega_n)_{n\in\mathbb{N}}$ a weight sequence, i.e. a bounded sequence of strictly positive real numbers. The *unilateral backward weighted shift* on the Banach space $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, is the operator defined by

$$T(\{x_1, x_2, \dots\}) = \{\omega_2 x_2, \omega_3 x_3, \dots\} \quad \text{for all } \{x_n\} \in \ell^p(\mathbb{N}).$$

The *unilateral forward weighted shift* on $\ell^p(\mathbb{N})$ is the operator defined by

$$T(\{x_1, x_2, \dots\}) = \{0, \omega_1 x_1, \omega_2 x_2, \dots\} \quad \text{for all } \{x_n\} \in \ell^p(\mathbb{N}).$$

If $T \in L(H)$ is a forward weighted shift then the hyperrange of T is $\{0\}$. If $T \in L(H)$ is a backward weighted shift then its adjoint T' is a forward weighted shift, so the hyperrange of T' is $\{0\}$, and this implies that $\sigma(T') = \sigma_{\mathrm{w}}(T')$, see [1, Theorem 3.116]. Denote by $r(T)$ the spectral radius of T. We have:

$$\sigma(T) = \sigma_{\mathrm{w}}(T) = \mathbf{D}(0, r(T)),$$

where $\mathbf{D}(0, r(T))$ is the closed disc centered at 0 and radius

$$r(T) = \lim_{n\to\infty} \sup\,(\omega_k \cdots \omega_{k+n-1})^{\frac{1}{n}}.$$

Hilden and Wallen in [16] proved that non quasi-nilpotent unilateral backward weighted shifts operators are always supercyclic. Generally, a backward weighted shift is not hypercyclic. However, we have:

Theorem 3.12. *Let T be a backward weighted shift on $H := \ell^2(\mathbb{N})$ with weight sequence (ω_n). Then $T \in \overline{HC(H)}$ if and only if*

$$r(T) = \lim_{n\to\infty} sup\,(\omega_k \cdots \omega_{k+n-1})^{\frac{1}{n}} \geq 1. \tag{10}$$

Proof. The adjoint T' is a forward weighted shift, and it is known that T' has SVEP, and $\sigma(T) = \sigma_{\mathrm{w}}(T) = \mathbf{D}(0, r(T))$, see [1, Corollary 3.118]. If (10) holds then $\partial\mathbf{D} \subseteq \sigma(T)$ and $\sigma(T)$ is connected. Hence the conditions (1') and (2') of Theorem 3.9 are satisfied, so $T \in \overline{HC(H)}$.

Conversely, if $T \in \overline{HC(H)}$ then $r(T)$ must be greater or equal to 1, otherwise if were $r(T) < 1$, $\sigma(T) \cup \mathbf{D}$ would be disconnected. ■

Define

$$c(T) := \lim_{n\to\infty} \inf\,(\omega_k \cdots \omega_{k+n-1})^{\frac{1}{n}} \tag{11}$$

Theorem 3.13. *Let T be a forward weighted shift on $H := \ell^2(\mathbb{N})$ with weight sequence (ω_n). If $c(T) = 0$ then $T \in \overline{HC(H)}$ if and only if*

$$\lim_{n\to\infty} sup\,(\omega_k \cdots \omega_{k+n-1})^{\frac{1}{n}} \geq 1.$$

Proof. We show first that for a forward weighted shift T, its adjoint T' has SVEP at every point $|\lambda| \leq c(T)$. If $\{e_n\}$ is the canonical basis of $\ell^2(\mathbb{N})$, by the classical formula for the radius of convergence of a vector-valued power series, the series

$$\sum_{n=1}^{\infty} \frac{\lambda^{n-1} e_n}{\omega_1 \cdot \omega_2 \cdots \omega_n}$$

converges in $\ell^2(\mathbb{N})$ for every $|\lambda| < c(T)$. Moreover, this series defines an analytic function f on the open disc $\mathbb{D}(0, c(T))$. Evidently,

$$(\lambda I - T')f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}(0, c(T)).$$

On the other hand, it is easily seen that T' has no eigenvalues outside the closed disc $\mathbf{D}(0, c(T))$. This implies that T' has SVEP at every point for which $|\lambda| \leq c(T)$. Since by assumption $c(T) = 0$ then T' has SVEP. Hence Browder's theorem holds for T, so $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Since $T^\infty(H) = \{0\}$, then, by [1, Theorem 2.82], $\sigma(T)$ is connected and, as noted in the proof of Theorem 3.10, $\sigma_{\mathrm{b}}(T) = \sigma(T)$. The condition $r(T) = \lim_{n\to\infty} \sup\,(\omega_k \cdots \omega_{k+n-1})^{\frac{1}{n}} \geq 1$ entails that $\sigma(T)$ contains the boundary $\partial\mathbf{D}$. ■

For every operator $T \in L(H)$ denote

$$i(T) := \lim_{n\to\infty} k(T^n)^{1/n},$$

where the *lower bound* $k(T)$ is defined by

$$k(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

Theorem 3.14. *Suppose that* $i(T) = r(T) = 1$ *and* T *invertible. Then* $T \in \overline{HC(H)}$ *if and only if* $\sigma(T) = \sigma_{\mathrm{w}}(T)$.

Proof. By [2, Corollary 4.71] T' has SVEP. Furthermore, $\sigma(T) \subseteq \partial\mathbf{D}$, by [2, Theorem 4.68], hence $\sigma(T) \cup \partial\mathbf{D} = \partial\mathbf{D}$ is connected. By Theorem 3.9 then $T \in \overline{HC(H)}$ if and only if $\sigma(T) = \sigma_{\mathrm{w}}(T)$. ■

Theorem 3.15. *If* $T \in L(H)$ *is non-invertible and* $i(T) = r(T) \geq 1$*. Then* $T' \in \overline{HC(H)}$.

Proof. For every non-invertible operator we have $\sigma(T) = \mathbf{D}(0, r(T))$ and $\sigma_{\mathrm{ap}}(T) = \partial\mathbf{D}(0, r(T))$, see [2, Theorem 4.68]. The last equality implies that T has SVEP. Indeed, T has SVEP at every point $\lambda \notin \sigma_{\mathrm{ap}}(T)$. Let $\lambda_0 \in \sigma_{\mathrm{ap}}(T) = \partial\mathbf{D}(0, r(T))$ and let $f : \mathbb{D}_{\lambda_0} \to H$ be an analytic function defined on a open disc centered at λ_0 such that

$$(\lambda I - T)f(\lambda) = 0, \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}.$$

Since λ belongs to the boundary of $\sigma_{\mathrm{ap}}(T)$ we can choose an open disc $\mathbb{D}_\mu$, centered at a point μ outside the spectrum, such that $\mathbb{D}_\mu \subseteq \mathbb{D}_{\lambda_0}$. Obviously T has SVEP at μ, so $f \equiv 0$ on $\mathcal{D}_\mu$. By the identity theorem of analytic functions it then follows that $f \equiv 0$ on $\mathcal{D}_{\lambda_0}$. Hence, T has SVEP at every $\lambda \in \mathbb{C}$. By [1, Theorem 3.117]

$$\sigma(T) = \sigma_{\mathrm{w}}(T) = \mathbf{D}(0, r(T)),$$

and hence $\sigma_{\mathrm{w}}(T) \cup \mathbf{D}(0, r(T)) = \sigma_{\mathrm{w}}(T)$ is connected. By Theorem 3.9 then the assertion follows. ■

Theorem 3.15 applies to non-invertible isometries, since for these operators we have $i(T) = r(T) = 1$.

Corollary 3.16. *If* T *is an non-invertible isometry then* $T' \in \overline{HC(H)}$.

Examples of non invertible isometries are the *semishifts* (i.e., those isometries for which the hyper-range $T^\infty(H) = \{0\}$). Every *right translation operator* on $L^p([0, \infty])$, with $1 \leq p < \infty$, as well as every *multiplication operator* T_f on the disc algebra $\mathcal{A}(\mathbb{D})$, is a semishift. It should be noted that for Hilbert space operators the semi-shifts coincide with the isometries for which none of the restrictions to a non-trivial reducing subspace is unitary, see Chapter 1 of Conway [7].

Let us consider a Cesàro operator defined on the Hardy space $\mathcal{H}^p(\mathbb{D})$, $1 \leq p < \infty$, defined by means of

$$(C_p f)(\lambda) := \frac{1}{\lambda} \int_0^\lambda \frac{f(\mu)}{1 - \mu} \mathrm{d}\mu \quad \text{for all } f \in H^p(\mathbb{D}),\ \lambda \in \mathbb{D}.$$

In [6] proved that C_2 on $\mathcal{H}^2(\mathbb{D})$ is hyponormal, i.e. $C_2' C_2 - C_2 C_2' \geq 0$, and this implies that C_2 is not supercyclic, see [5].

Theorem 3.17. *The adjoint* C_2' *of the Cesàro operator* C_2 *on* $\mathcal{H}^2(\mathbb{D})$ *is a limit of hypercyclic operators.*

Proof. The spectrum of C_2 is:

$$\sigma(C_2) := \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\},$$

C_2 has SVEP, since is hyponormal, and $1 \in \sigma(C_2)$, so $\sigma(C_2) \cup \partial\mathbf{D}$ is connected. Furthermore, $\sigma_{\mathrm{w}}(C_2) = \sigma(C_2)$. Indeed, if $\lambda \notin \sigma_{\mathrm{w}}(C_2)$ then $\lambda I - C_2$ is Weyl, and the SVEP implies that $\lambda I - C_2$ is Browder. If were $\lambda \in \sigma(C_2)$ then λ would be an isolated point of the spectrum and this is impossible. Therefore $\lambda \notin \sigma(C_2)$, and hence $\sigma_{\mathrm{w}}(C_2) = \sigma(C_2)$. Theorem 3.9 then applies. ■

The *discrete Cesàro operator* is defined in $\ell^p(\mathbb{N})$ by

$$C_0(x_1, x_2, x_3, \dots) = (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \dots) \quad \text{for all } \{x_n\} \in \ell^p(\mathbb{N}).$$

It should be noted that C_0 in $\ell^2(\mathbb{N})$ is not supercyclic, since the adjoint of C_0 has point spectrum

$$\sigma_{\mathrm{p}}(C_0) = \{\lambda \in \mathbb{C} : |1 - \lambda| < 1\},$$

see [6, Theorem 2], and as noted before the adjoint of a supercyclic operator must have empty point spectrum or is a singleton.

Theorem 3.18. *The adjoint C_0' of the discrete Cesàro operator C_0 on $\ell^2(\mathbb{N})$ is a limit of hypercyclic operators.*

Proof. The proof is analogous to that of Theorem 3.17. We have

$$\sigma(C_0) := \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}.$$

Furthermore C_0 has SVEP, since is hyponormal, see [6], $\sigma(C_0)$ intersects $\partial\mathbf{D}$, hence $\sigma(C_0) \cup \partial\mathbf{D}$ is connected. Furthermore, arguing as in the proof of Theorem 3.17 we obtain that $\sigma_{\mathrm{w}}(C_0) = \sigma(C_0)$. ■

Acknowledgment

The authors wish to thank the referee for a careful reading and comments for the original draft.

Data availability

This work does not have any experimental data.

References

[1] P. Aiena, Fredholm and Local Spectral Theory II, with Application to Multipliers, Kluwer Academic Publishers, Dordrecht, 2004.
[2] P. Aiena, Fredholm and Local Spectral Theory II, with Application to Weyl-Type Theorems, Springer Lecture Notes of Math., vol. 2235, 2018.
[3] P. Aiena, S. Triolo, Fredholm spectra and Weyl type theorems for Drazin invertible operators, Mediterr. J. Math. 13 (6) (2016) 4385–4400.
[4] P. Aiena, F. Burderi, S. Triolo, On the spectrum of supercyclic/ hypercyclic operators, Adv. Oper. Theory (2024), in press.
[5] P. Bourdon, Orbits of hyponormal operators, Mich. Math. J. 44 (2) (1997) 345–353.
[6] A. Brown, P.R. Halmos, A.L. Shields, Cesáro operators, Acta Sci. Math. 26 (1–2) (1965) 125–137.
[7] J.B. Conway, The Theory of Subnormal Operators, Mathematical Survey and Monographs, vol. 36, American Mathematical Society/Springer, Providence/New York, 1992.
[8] M. González, F.L. León-Saavedra, A. Montes-Rodríguez, Semi-Fredholm theory: hypercyclic and supercyclic subspaces, Proc. Lond. Math. Soc. 81 (2000) 169–189.
[9] G. Godefroy, J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991) 229–269.
[10] Karl-G. Grosse-Erdmann, A.P. Manguillot, Linear Chaos, Universitext, Springer, 2011.
[11] D.A. Herrero, Limits of hypercyclic and supercyclic operators, J. Funct. Anal. 99 (1) (1991) 179–190.
[12] D.A. Herrero, Z.Y. Wang, Compact perturbations of hypercyclic and supercyclic operators, Indiana Univ. Math. J. 39 (3) (1990) 819–829 (Fall, 1990).
[13] D.A. Herrero, C. Kitai, On invertible hypercyclic operators, Proc. Am. Math. Soc. 116 (3) (1992) 873–875.

[14] G. Herzog, C. Schmoeger, On operators T such that $f(T)$ is hypercyclic, Stud. Math. 108 (3) (1994) 209–216.
[15] H. Heuser, Functional Analysis, Marcel Dekker, New York, 1982.
[16] H.M. Hilden, L.J. Wallen, Some cyclic and non-cyclic vectors of certain operators, Indiana Univ. Math. J. 23 (1974) 557–565.
[17] C. Kitai, Invariant Closed Sets for Linear Operators, Thesis, Univ. Toronto, 1982.
[18] V. Matache, Notes on hypercyclic operators, Acta Sci. Math. 58 (1993) 397–406.
[19] S. Rolewicz, On orbits of elements, Stud. Math. 32 (1969) 17–22.
[20] H.N. Salas, A hypercyclic operator whose adjoint is also hypercyclic, Proc. Am. Math. Soc. 112 (1991) 765–770.