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ORIGINAL ARTICLE

Sesquilinear forms as eigenvectors in quasi *-algebras, with an application to ladder elements

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Abstract

We consider a particular class of sesquilinear forms on a Banach quasi *-algebra $(\mathcal{A}[\|.\|], \mathcal{A}_0[\|.\|_0])$ that we call *eigenstates of an element* $a \in \mathcal{A}$, and we deduce some of their properties. We further apply our definition to a family of ladder elements, that is, elements of $\mathcal{A}$ obeying certain commutation relations physically motivated, and we discuss several results, including orthogonality and biorthogonality of the forms, via Gelfand–Naimark–Segal (GNS) representation.

KEYWORDS
eigenvectors, ladder elements, quasi *-algebras, sesquilinear forms

1 | INTRODUCTION AND PRELIMINARIES

The role of eigenstates is quite well known both in pure and in applied mathematics, and it is particularly relevant in quantum mechanics. In this case, for instance, eigenvectors of a given Hamiltonian (which usually describes the energy of a given physical system $\mathcal{G}$) are typically interpreted as the stationary states of $\mathcal{G}$. This means that, if $\mathcal{G}$ is *prepared* in one specific eigenstate at the initial time $t = 0$, then, in absence of other effects, $\mathcal{G}$ *stays in that state*. This is one of the (physical) reasons why the properties of the set of all the eigenstates of a given operator have been studied at length during the past decades. A more mathematical reason for this interest is that, in many situations, this set spans the whole Hilbert space where $\mathcal{G}$ is defined, [21, 23]. Quite generally, in the literature, eigenstates are meant to be vectors in the Hilbert space. Recently, [16], the authors considered the notion of eigenvectors in a C*-algebraic setting, replacing vectors with linear functional. C*-algebras are often used in connection with quantum systems with infinite degrees of freedom in view of the possibility of using the same underlying structure to describe different physical situations, using inequivalent representations of the abstract C*-algebra describing the system, [13, 14, 26, 31, 32].

In this paper, we further extend the notion of eigenvectors to Banach quasi *-algebra, by using sesquilinear forms, [18]. This extension can be useful in the presence of unbounded operators, which often appears when dealing with concrete physical systems, [4, 5, 30], and which are not so naturally analyzed in C*-algebras.

The paper is organized as follows: After some preliminaries, we introduce our definition of eigenstates for sesquilinear forms, and we deduce several properties. In particular, we show that from any such form it is possible to define a second form, which is again an eigenstate, but of a different operator. Section 3 is devoted to a detailed example based on a class of ladder operators known as quons, [17, 20, 24]. In particular, we show how a family of eigenstates of a quonic number-like operator n_0 can be defined, and we also discuss the *ortogonality* of these states, after explaining what orthogonality is for

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us, in our context. We both consider the cases of $n_0 = n_0^*$, and what happens if this equality does not hold. In this latter case, we show that a sort of biorthogonality between forms can be introduced, if some suitable conditions are satisfied. Section 4 contains our conclusions. To keep the paper self-contained, in the Appendix we include some basic facts on quons, relevant for what we will do in Section 3.

1.1 | Basic notions

We briefly recall here some definitions and facts needed in the sequel. Let $\mathcal{A}$ be a complex vector space and $\mathcal{A}_0$ a *-algebra contained in $\mathcal{A}$. $\mathcal{A}$ is said to be a *quasi* *-algebra with distinguished *-algebra $\mathcal{A}_0$ (or, simply, over $\mathcal{A}_0$) if

1. the left multiplication ax and the right multiplication xa of an element a of $\mathcal{A}$ and an element x of $\mathcal{A}_0$ that extend the multiplication of $\mathcal{A}_0$ are always defined and bilinear;
2. $x_1(x_2a) = (x_1x_2)a$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;
3. an involution $*$ that extends the involution of $\mathcal{A}_0$ is defined in $\mathcal{A}$ with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$.

We say that $(\mathcal{A}, \mathcal{A}_0)$ has a *unit* if there exists an element $e \in \mathcal{A}_0$ such that $ae = ea = a$, for every $a \in \mathcal{A}$.

A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to be *locally convex* if $\mathcal{A}$ is endowed with a topology τ that makes of $\mathcal{A}$ a locally convex space and such that the involution $a \mapsto a^*$ and the multiplications $\mathcal{A} \ni a \mapsto ax$, $a \mapsto xa$, $x \in \mathcal{A}_0$, are continuous. If τ is a norm topology and the involution is isometric with respect to the norm, we say that $(\mathcal{A}[\|.\|], \mathcal{A}_0[\|.\|_0])$ is a *normed quasi *-algebra* and, if it is complete, we say it is a *Banach quasi*-algebra*, [4].

For instance, let $\mathcal{A}_0$ be a C^*-algebra, with norm $\|\cdot\|_0$ and involution *. Let $\|\cdot\|$ be a norm on $\mathcal{A}_0$, weaker than $\|\cdot\|_0$ and such that, for every $a, b \in \mathcal{A}_0$

(i) $\|ab\| \leq \|a\|\|b\|_0$,
(ii) $\|a^*\| = \|a\|$.

Let $\mathcal{A}$ denote the $\|\cdot\|$-completion of $\mathcal{A}_0$; then the pair $(\mathcal{A}, \mathcal{A}_0)$ is called a (proper) *CQ*-algebra*, first introduced in [9], which is a particular example of Banach quasi*-algebra.

As an example, the pair $(L^p([0,1]), L^\infty([0,1]))$, $1 \leq p < +\infty$, may be regarded as an abelian CQ*-algebra as well as a Banach quasi *-algebra.

A second example is given by a noncommutative version of the previous one: Let $\mathcal{M}$ be a von Neumann algebra and τ a normal faithful semifinite trace defined on $\mathcal{M}_+$. For each $p \geq 1$, let

$$\mathcal{J}_p = \{X \in \mathcal{M} : \tau(|X|^p) < \infty\}.$$

Then $\mathcal{J}_p$ is a *-ideal of $\mathcal{M}$. We denote with $L^p(\tau)$ the Banach space completion of $\mathcal{J}_p$ with respect to the norm

$$\|X\|_p := \tau(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.$$

We further call, as it is usually done in the literature, $L^\infty(\tau) = \mathcal{M}$. The pair $(L^p(\tau), L^\infty(\tau) \cap L^p(\tau))$ is a noncommutative Banach quasi *-algebra, [10, 26].

2 | EIGENVECTORS

Let $(\mathcal{A}[\|.\|], \mathcal{A}_0[\|.\|_0])$ be a Banach quasi *-algebra with identity $e \in \mathcal{A}_0$, and let $\varphi(.,.)$ be a sesquilinear form on it satisfying the following conditions:

$$\begin{cases} \varphi(c, \alpha a + \beta b) = \alpha\varphi(c, a) + \beta\varphi(c, b), \\ \varphi(a, a) \geq 0, \\ \varphi(ax, y) = \varphi(x, a^*y), \end{cases} \tag{2.1}$$

where $a, b, c \in \mathcal{A}$, $x, y \in \mathcal{A}_0$, and $\alpha, \beta \in \mathbb{C}$. Because of the positivity of φ, we automatically have

$$\begin{cases} \varphi(a,b) = \overline{\varphi(b,a)}, \\ |\varphi(a,b)|^2 \leq \varphi(a,a)\varphi(b,b), \end{cases} \tag{2.2}$$

$\forall a, b \in \mathcal{A}$. We call $\mathcal{S}$ the set of sesquilinear forms satisfying (2.1), and therefore (2.2). It is clear that we can always assume that $\varphi(e,e) = 1$. Indeed, if $\varphi(e,e) \neq 1$, we can still define a new sesquilinear form, φ_e, as $\varphi_e(a,b) = \frac{\varphi(a,b)}{\varphi(e,e)}$, $a, b \in \mathcal{A}$, at least if $\varphi(e,e) \neq 0$. Hence $\varphi_e(e,e) = 1$, and $\varphi_e \in \mathcal{S}$. We refer to [18] for several details on sesquilinear forms on quasi *-algebras.

In what follows, we will often make use of the following continuity condition:

$$|\varphi(a,b)| \leq \gamma_\varphi \|a\| \|b\| \tag{2.3}$$

$\forall a, b \in \mathcal{A}$. Here γ_φ is a strictly positive constant independent of a and b. We call $\mathcal{S}_c$ the set of all the elements of $\mathcal{S}$ satisfying (2.3), and we further introduce the set $\mathcal{S}_{c_0} = \{\varphi \in \mathcal{S}_c \, : \, \gamma_\varphi = 1\}$.

Proposition 1. *Let $\varphi \in \mathcal{S}_c$, and $x \in \mathcal{A}_0$. Then calling*

$$\varphi_x(a,b) = \varphi(ax, bx), \tag{2.4}$$

$a, b \in \mathcal{A}$, we have that $\varphi_x \in \mathcal{S}_c$. In particular, if $\varphi \in \mathcal{S}_{c_0}$ and $\|x\|_0 \leq 1$, then $\varphi_x \in \mathcal{S}_{c_0}$.

Proof. We only prove the continuity of φ_x. For instance, we have, using (2.3) and the inequalities $\|ax\| \leq \|a\|\|x\|_0$ and $\|bx\| \leq \|b\|\|x\|_0$,

$$|\varphi_x(a,b)| = |\varphi(ax,bx)| \leq \gamma_\varphi \|ax\| \|bx\| = (\gamma_\varphi \|x\|_0^2) \|a\| \|b\|,$$

for all $a, b \in \mathcal{A}$. It is clear that, in particular, $|\varphi_x(a,b)| \leq \|a\|\|b\|$ if $\gamma_\varphi = 1$ and $\|x\|_0 \leq 1$. ☐

Example 2. If we consider the *Banach quasi *-algebra* $(L^p(\tau), L^\infty(\tau) \cap L^p(\tau))$, it is not hard to produce examples of sesquilinear forms satisfying (2.1) and (2.3). For that we start introducing the set

$$\mathcal{B}_+^p = \{X \in L^{p/(p-2)}(\tau) \, : \, X \geq 0 \, , \ \| X \|_{p/(p-2)} \leq 1\}$$

where $p > 2$, with the agreement that $p/(p-2) = \infty$ if $p = 2$.

For each $W \in \mathcal{B}_+^p$, we consider the right multiplication operator

$$R_W \, : \, L^p(\tau) \to L^{p'}(\tau); \qquad R_W X = XW, \quad X \in L^p(\tau),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. In this conditions for every $p \geq 2$ and $W \in \mathcal{B}_+^p$, the sesquilinear form $\varphi(X,Y) = \tau[Y(R_W X)^*]$ is an element of $\mathcal{S}$. In fact:

for every $X \in L^p(\tau)$, we have

$$\varphi(X,X) = \tau[X(R_W X)^*] = \tau[X(XW)^*] = \tau[(XW)^* X] = \tau[W|X|^2] \geq 0.$$

For every $X \in L^p(\tau)$, $A, B \in L^\infty(\tau) \cap L^p(\tau)$, we get

$$\varphi(XA,B) = \tau(B(XAW)^*) = \tau(BW^*A^*X^*) = \tau(X^*BW^*A^*) = \tau(X^*B(AW)^*) = \varphi(A, X^*B).$$

Finally, for every $X, Y \in L^p(\tau)$,

$$|\varphi(X,Y)| \leq \| X \|_p \| Y \|_p \| W \|_{p/p-2} \leq \| X \|_p \| Y \|_p \, .$$

Then $\varphi \in \mathcal{S}_{c_0}$, in particular. Of course, if we relax condition $\| W \|_{p/(p-2)} \leq 1$, we can still check that $\varphi \in \mathcal{S}_c$.

Definition 3. Let $\varphi \in \mathcal{S}$ and let $a \in \mathcal{A}$. We say that φ is an eigenstate of a with eigenvalue $\lambda \in \mathbb{C}$, if

$$\varphi(b,a) = \lambda\varphi(b,e), \tag{2.5}$$

$\forall b \in \mathcal{A}$.

It follows that $\varphi(e,a) = \lambda\varphi(e,e) = \lambda$, and therefore $\varphi(b,a) = \varphi(b,e)\varphi(e,a)$, $\forall b \in \mathcal{A}$. This could be seen as a sort of decomposition property of φ, which however only holds when φ is an eigenstate of a, but not in general. Indeed we can easily check the following result:

Lemma 4. *Let $\varphi \in \mathcal{S}$ and let $a \in \mathcal{A}$. Then φ is an eigenstate of a if and only if $\varphi(b,a) = \varphi(b,e)\varphi(e,a)$, $\forall b \in \mathcal{A}$.*

Going back to Example 2 with $p = 2$, it is easy to construct an example of a sesquilinear form satisfying Definition 3: *In the Banach quasi *-algebra* $(L^2(\tau), L^\infty(\tau) \cap L^2(\tau))$, if $W \in L^2(\tau) \cap \mathcal{B}^2_+$, is a projection then $\varphi(X,Y) = \tau[Y(R_W X)^*] \in \mathcal{S}$. For each $\alpha \in \mathbb{C}$, we now consider the operator $A = \alpha W$

$$\varphi(X,A) = \tau[\alpha W(R_W X)^*] = \tau[(\alpha R_W X)^*] = \alpha\varphi(X,e).$$

Thus, φ is an eigenstate of A with eigenvalue α.

Next we have the following:

Proposition 5. *Let $\varphi \in \mathcal{S}$ and let $a \in \mathcal{A}$. Then φ is an eigenstate of a with eigenvalue $\lambda \in \mathbb{C}$ if and only if*

$$\varphi(a - \lambda e, a - \lambda e) = 0. \tag{2.6}$$

Proof. Let φ be an eigenstate of a with eigenvalue $\lambda \in \mathbb{C}$. Hence, from (2.5), $\varphi(b, a - \lambda e) = 0$, $\forall b \in \mathcal{A}$. In particular, this must be true if $b = a - \lambda e$, so that (2.6) follows.

Vice versa, if (2.6) holds true, (2.2) implies that

$$|\varphi(b, a - \lambda e)| \le \varphi(b,b)\varphi(a - \lambda e, a - \lambda e) = 0,$$

so that $\varphi(b, a - \lambda e) = 0$, $\forall b \in \mathcal{A}$. Hence φ is an eigenstate of a with eigenvalue $\lambda \in \mathbb{C}$. □

Suppose now that we have two different sesquilinear forms $\varphi_1, \varphi_2 \in \mathcal{S}$ which are both eigenvectors of a given $a \in \mathcal{A}$, with the same eigenvalue λ. Then the new form $\varphi(b,c) = q\varphi_1(b,c) + (1-q)\varphi_2(b,c)$, $q \in [0,1]$, and $b, c \in \mathcal{A}$, is still an element of $\mathcal{S}$, and it is also an eigenvector of a with the same eigenvalue λ. Moreover, if $\varphi_1, \varphi_2 \in \mathcal{S}_c$, then $\varphi \in \mathcal{S}_c$ and, in particular, if $\varphi_1, \varphi_2 \in \mathcal{S}_{c_0}$, then $\varphi \in \mathcal{S}_{c_0}$. Indeed, if φ_1, φ_2 both satisfy (2.1), then φ satisfies the same properties. Moreover, since $|\varphi_j(b,c)| \le \gamma_{\varphi_j}\|b\|\|c\|$, $j = 1, 2$, we have

$$|\varphi(b,c)| \le q|\varphi_1(b,c)| + (1-q)|\varphi_2(b,c)| \le \left(q\gamma_{\varphi_1} + (1-q)\gamma_{\varphi_2}\right)\|b\|\|c\|,$$

which is of the form (2.3). If we further assume that $\varphi_1, \varphi_2 \in \mathcal{S}_{c_0}$, then $\gamma_{\varphi_1} = \gamma_{\varphi_2} = 1$, so that $q\gamma_{\varphi_1} + (1-q)\gamma_{\varphi_2} = 1$. Hence $\varphi \in \mathcal{S}_{c_0}$. Finally, since both φ_1 and φ_2 satisfy (2.5), we conclude that

$$\varphi(b,a) = q\varphi_1(b,a) + (1-q)\varphi_2(b,a) = \lambda(q\varphi_1(b,e) + (1-q)\varphi_2(b,e)) = \lambda\varphi(b,e),$$

which shows that, as stated, φ is again an eigenstate of a with eigenvalue $\lambda \in \mathbb{C}$. It is clear that these results can be extended to an arbitrary number of elements of, for instance, $\mathcal{S}_{c_0}$: if $\varphi_j \in \mathcal{S}_{c_0}$, $j = 1, 2, \dots, N$, and if we have $\{q_j \in [0,1],\ j = 1, 2, \dots, n\}$, with $\sum_{j=1}^n q_j = 1$, then $\varphi = \sum_{j=1}^n q_j\varphi_j \in \mathcal{S}_{c_0}$. Moreover, if each φ_j is an eigenstate of $a \in \mathcal{A}$ with eigenvalue λ, then φ is an eigenstate of $a \in \mathcal{A}$ with eigenvalue λ as well.[1]

Definition 6. Let us consider $\varphi_j \in \mathcal{S}$, $j = 1, 2, \dots, n$. They are linearly independent if $\sum_{j=1}^{n} \alpha_j \varphi_j = 0$ if and only if $\alpha_j = 0$, $\forall j$.

It might be useful to recall that $\sum_{j=1}^{n} \alpha_j \varphi_j = 0$ means that $\sum_{j=1}^{n} \alpha_j \varphi_j(a, b) = 0$, $\forall a, b \in \mathcal{A}$. Following [16], we prove the following:

Proposition 7. *Let us consider $\varphi_j \in \mathcal{S}$, $j = 1, 2, \dots, n$, different eigenstates of $a \in \mathcal{A}$, with n different eigenvalues λ_j. Then these forms are linearly independent.*

The proof is not very different from the one in [16] for states on C*-algebras and will not be given here.

Our Definition 3 implies the following *natural* result:

Lemma 8. *If $a = a^*$, then $\lambda \in \mathbb{R}$ in (2.5).*

Proof. Indeed we have

$$\lambda = \varphi(e, a) = \varphi(a^*, e) = \varphi(a, e) = \overline{\varphi(e, a)} = \overline{\lambda},$$

so that the reality of λ follows. □

If $a^* \neq a$, then λ is not necessarily real. In this general case, it is possible to proceed as follows.

Proposition 9. *Let $\varphi \in \mathcal{S}$. Then, defining*

$$\psi(a, b) = \varphi(b^*, a^*), \tag{2.7}$$

$a, b \in \mathcal{A}$, ψ is also a sesquilinear positive form. Moreover

$$\psi(xa, y) = \psi(x, ya^*), \tag{2.8}$$

$\forall a \in \mathcal{A}$, $x, y \in \mathcal{A}_0$. Furthermore, if φ satisfies (2.3), ψ satisfies also

$$|\psi(a, b)| \leq \gamma_\varphi \|a\| \|b\|, \tag{2.9}$$

$a, b \in \mathcal{A}$.

Proof. The proof that ψ is sesquilinear and positive is trivial and will not be given here. As for (2.8) we have, using (2.7) and $(2.1)_3$

$$\psi(xa, y) = \varphi(y^*, (xa)^*) = \varphi(y^*, a^* x^*) = \varphi(a y^*, x^*) = \psi(x, (a y^*)^*) = \psi(x, y a^*).$$

Because of (2.3) and (2.7), we also have

$$|\psi(a, b)| = |\varphi(b^*, a^*)| \leq \gamma_\varphi \|b^*\| \|a^*\|,$$

so that (2.9) follows. □

We notice that the constant appearing in (2.3) and (2.9) is the same. Notice also that, even if $\varphi \in \mathcal{S}$, the form ψ does not belong to $\mathcal{S}$, if $(\mathcal{A}, \mathcal{A}_0)$ is not abelian, since $(2.1)_3$ is replaced here by (2.8). In this case, we could introduce a set $\tilde{\mathcal{S}}$, with (2.8) rather than $(2.1)_3$ and the other two in (2.1) unchanged. Then $\psi \in \tilde{\mathcal{S}}$ if and only if $\varphi \in \mathcal{S}$.

Proposition 10. *Let $\varphi \in \mathcal{S}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, and ψ the form in (2.7). The following statements are equivalent:*

$$\begin{aligned} &(i) && \varphi(b,a) = \lambda\,\varphi(b,e),\\ &(ii) && \psi(b,a^*) = \overline{\lambda}\,\psi(b,e),\\ &(iii) && \varphi(a,b) = \overline{\lambda}\,\varphi(e,b),\\ &(iv) && \psi(a^*,b) = \lambda\,\psi(e,b), \end{aligned} \tag{2.10}$$

$\forall b \in \mathcal{A}$. Moreover, if any of the above equalities is satisfied, then

$$\varphi(a,a) = \psi(a,a) = |\lambda|^2. \tag{2.11}$$

Proof. We only show here that (i) implies (ii). The proofs of all these claims are similar, and will not be repeated. Let us assume that (i) is satisfied: φ is an eigenstate of a with eigenvalue λ. Then we have

$$\psi(b,a^*) = \varphi(a,b^*) = \overline{\varphi(b^*,a)} = \overline{\lambda\,\varphi(b^*,e)} = \overline{\lambda}\varphi(e,b^*) = \overline{\lambda}\psi(b,e),$$

because of (i) and (2.7). The others implications can be proved in a similar way.

As for (2.11), we have for instance $\varphi(a,a) = \lambda\varphi(a,e)$, using (i). But, using (iii) with $b = e$ we also have $\varphi(a,e) = \overline{\lambda}\varphi(e,e) = \overline{\lambda}$, so that $\varphi(a,a) = |\lambda|^2$. Similarly we can prove that $\psi(a,a) = |\lambda|^2$. □

This proposition shows, in particular, that, if φ is an eigenstate of a with eigenvalue λ, then ψ is an eigenstate of a^* with eigenvalue $\overline{\lambda}$. Incidentally, formulas (iii) and (iv) suggest to introduce the notion of *left* and *right* eigenvectors, depending on the position of a inside the sesquilinear forms φ or ψ. For instance, from (i), we could say that φ is a right eigenstate of a with eigenvalue λ, while from (iii), φ can be called a left eigenstate of a with eigenvalue $\overline{\lambda}$. Similarly, see (ii) and (iv), ψ is a right eigenstate of a^* with eigenvalue $\overline{\lambda}$ and a left eigenstate of a^* with eigenvalue λ.

It is possible to introduce a norm on our sesquilinear forms in $\mathcal{S}_c$. Indeed we can put

$$\|\varphi\| = \sup_{\|a\|=1} \varphi(a,a) = \sup_{\|a\|=\|b\|=1} |\varphi(a,b)|, \tag{2.12}$$

where the equivalence of the two expressions follow from $(2.2)_2$. It is clear that, $\forall \varphi \in \mathcal{S}_{c_0}$, $\|\varphi\| = 1$. In fact we have, recalling that our quasi *-algebra has an identity e, $\|\varphi\| = \sup_{\|a\|=1} \varphi(a,a) \le \sup_{\|a\|=1} \|a\|^2 = 1$, and $\|\varphi\| = \sup_{\|a\|=1} \varphi(a,a) \ge \varphi(e,e) = 1$.

We conclude this section by noticing that the results of Proposition 10 can be enriched if φ is an eigenstate of an element a_0 in $\mathcal{A}_0$, $a_0 \in \mathcal{A}_0$. Hence we have, for instance, $\varphi(b,a_0) = \lambda\,\varphi(b,e)$, $\forall b \in \mathcal{A}$. Then we also have $\varphi(b,a_0^n) = \lambda^n\,\varphi(b,e)$, $\forall b \in \mathcal{A}$, $n \ge 0$. In fact, first of all we notice that, since $a_0 \in \mathcal{A}_0$, its powers are all well defined in $\mathcal{A}_0$. Then we have, in particular

$$\varphi(b,a_0^2) = \varphi(a_0^* b, a_0) = \lambda\varphi(a_0^* b, e) = \lambda\varphi(b,a_0) = \lambda^2\varphi(b,e),$$

$\forall b \in \mathcal{A}$. And so on. Of course, a similar result can be extended for all polynomials: Let $p(a_0)$ be a polynomial in a_0, then $\varphi(b,p(a_0)) = p(\lambda)\varphi(b,e)$. Similar results can be deduced out of the other equalities in (2.10).

3 | LADDER ELEMENTS

In this section, we will discuss in some details an application of what we have done in Section 2, and in particular we will use ladder operators of a specific kind to construct a family of sesquilinear forms which are eigenvectors of a single element of $\mathcal{A}_0$, with different eigenvalues. More explicitly, we will use elements of $\mathcal{A}_0$ satisfying a specific commutation rule that is motivated by the so-called *quons*, which are operators acting on some Hilbert space and which depend on a real parameter $q \in [-1,1]$, see [17, 20, 24]. Few facts on these operators are listed in the Appendix, where it is also discussed that they are bounded operators, if $q \in [-1,1[$.

With this in mind, we consider here two elements $x_0, y_0 \in \mathcal{A}_0$ satisfying the following *q-mutator*:

$$[x_0, y_0]_q = x_0 y_0 - q y_0 x_0 = e. \tag{3.1}$$

We call $n_0 = y_0 x_0$, which is still an element of $\mathcal{A}_0$. It is clear that, if $x_0 = y_0^*$, then $n_0 = n_0^*$. We assume that an element $\varphi_0 \in \mathcal{S}_c$ exists such that

$$\varphi_0(b, x_0) = 0, \tag{3.2}$$

$\forall b \in \mathcal{A}$. This means that φ_0 is an eigenstate of x_0 with eigenvalue $\lambda_0 = 0$. Notice that (3.2) implies that $\varphi_0(b, c_0 x_0) = 0$ for all $b \in \mathcal{A}$ and $c_0 \in \mathcal{A}_0$. Indeed we have $\varphi_0(b, c_0 x_0) = \varphi_0(c_0^* b, x_0) = 0$ since $c_0^* b \in \mathcal{A}$. Further, the continuity of φ_0 implies that $\varphi_0(b, c x_0) = 0$ for all $b, c \in \mathcal{A}$. This is because there surely exists a sequence $\{c_n \in \mathcal{A}_0\}$ such that $\|c - c_n\| \to 0$. Then we also have $\|c x_0 - c_n x_0\| \to 0$, so that $\varphi_0(b, c x_0) = \lim_n \varphi_0(b, c_n x_0) = 0$. Summarizing:

$$\varphi_0(b, c x_0) = 0, \tag{3.3}$$

$\forall b, c \in \mathcal{A}$.

Remark. The existence of the sesquilinear form φ_0 satisfying (3.2) is not a trivial requirement. Indeed it is not so different from what is done in the literature for bosons, or for pseudo-bosons: In both cases, one asks for the existence of a *vacuum state*. But, while this state exists, and belongs to the Hilbert space (e.g., to $\mathcal{L}^2(\mathbb{R})$) in the bosonic case, this existence is not guaranteed for pseudo-bosons.[2] In this case, even when the vacuum exists, it could be an element not in $\mathcal{L}^2(\mathbb{R})$. We refer to [6] where the role of the distributions is discussed in details.

The following result holds true:

Proposition 11. *Calling*

$$\varphi_l(a, b) = \varphi_{l-1}(a y_0, b y_0), \tag{3.4}$$

$a, b \in \mathcal{A}$, $l = 1, 2, 3, \ldots$, *then*

$$\varphi_l(a, n_0) = \beta_l \varphi_l(b, e), \tag{3.5}$$

$\forall b \in \mathcal{A}$, *where*

$$\beta_l = \begin{cases} 0, & \text{if } l = 0, \\ 1 + q\beta_{l-1}, & \text{if } l \geq 1. \end{cases} \tag{3.6}$$

Proof. We prove our claim by induction: The statement is true for $l = 0$ because of (3.2). Let us now suppose that $\varphi_{l-1}(a, n_0) = \beta_{l-1}\varphi_{l-1}(b, e)$ for a fixed l. We want to deduce then that $\varphi_l(a, n_0) = \beta_l \varphi_l(b, e)$. Indeed we have, since $n_0 y_0 = y_0 x_0 y_0 = y_0(e + q y_0 x_0) = y_0 + q y_0^2 x_0$,

$$\varphi_l(a, n_0) = \varphi_{l-1}(a y_0, n_0 y_0) = \varphi_{l-1}(a y_0, y_0) + q\varphi_{l-1}(a y_0, y_0^2 x_0) = \varphi_l(a, y) + q\varphi_{l-1}(a y_0, y_0^2 x_0).$$

Now, because of the induction hypothesis and of (3.4), we have

$$\varphi_{l-1}(a y_0, y_0^2 x_0) = \varphi_{l-1}(y_0^* a y_0, n_0) = \beta_{l-1}\varphi_{l-1}(y_0^* a y_0, e) = \beta_{l-1}\varphi_{l-1}(a y_0, y_0) = \beta_{l-1}\varphi_l(a, e),$$

which implies that

$$\varphi_l(a, n_0) = (1 + q\beta_{l-1})\varphi_l(a, e),$$

as we had to prove. □

Incidentally we observe that, if $-1 < q < 1$, then $\beta_l = 1 + q + q^2 + \cdots + q^l = \frac{1-q^{l+1}}{1-q}$, $\forall l \geq 1$.

What this proposition shows is that, for elements of $\mathcal{A}_0$ satisfying (3.1), if we know the vacuum of the operator x_0, we can construct a full sequence of forms as in (3.4), which are all eigenvectors of a single element n_0, see (3.5). In particular, since y_0 allows us to move from φ_{l-1} to φ_l, we will call y_0 a *raising operator*. Similarly, x_0 is a *lowering operator*, as Proposition 12 below shows.

Proposition 12. *For all $a, b \in \mathcal{A}$, we have*

$$\varphi_l(ax_0, bx_0) = \beta_l^2 \varphi_{l-1}(a, b), \tag{3.7}$$

$\forall l \geq 1$.

Proof. We start recalling that $x_0 y_0 = e + q y_0 x_0 = e + q n_0$.

Now, let us suppose first that $a \in \mathcal{A}_0$. Hence we have, because of (3.4),

$$\begin{aligned}\varphi_l(ax_0, bx_0) &= \varphi_{l-1}(ax_0y_0, bx_0y_0) = \varphi_{l-1}(a(e + qn_0), b(e + qn_0)) \\ &= \varphi_{l-1}(a, b) + q\varphi_{l-1}(a, bn_0) + q\varphi_{l-1}(an_0, b) + q^2\varphi_{l-1}(an_0, bn_0).\end{aligned}$$

Now we have

$$\varphi_{l-1}(a, bn_0) = \varphi_{l-1}(b^*a, n_0) = \beta_{l-1}\varphi_{l-1}(b^*a, e) = \beta_{l-1}\varphi_{l-1}(a, b),$$

because of (3.5) and recalling that, under our assumption on a, $b^*a \in \mathcal{A}$ is well defined. Similarly we deduce that $\varphi_{l-1}(an_0, b) = \beta_{l-1}\varphi_{l-1}(a, b)$ and $\varphi_{l-1}(an_0, bn_0) = \beta_{l-1}^2\varphi_{l-1}(a, b)$. Hence

$$\varphi_l(ax_0, bx_0) = (1 + q\beta_{l-1})^2 \varphi_{l-1}(a, b),$$

which is exactly the (3.7).

If we now take $a \in \mathcal{A}$, then there exists a sequence $\{a_n \in \mathcal{A}_0\}$ such that $\|a - a_n\| \to 0$, which, in turns, implies that $\|ax_0 - a_nx_0\| \to 0$. Then, using the continuity of each φ_k,

$$\varphi_l(ax_0, bx_0) = \lim_n \varphi_l(a_nx_0, bx_0) = \beta_l^2 \lim_n \varphi_{l-1}(a_n, b) = \beta_l^2\varphi_{l-1}(a, b),$$

$a, b \in \mathcal{A}$. □

It is interesting to stress that Proposition 7 and Proposition 11, together, imply that the various φ_l are linearly independent.

3.1 | Toward coherent forms

In the literature, ladder operators acting on Hilbert spaces are usually associated to particular vectors depending on a variable $z \in \mathbb{C}$, which are called coherent states, and which, among other properties, are eigenstates of the relevant lowering operator one is considering. Many mathematical results and several physical applications exist for coherent states, and for their possible generalizations. We refer to [1–3, 6, 15, 19, 22, 27, 29] for a first reading.

We will now show that ladder elements of the kind considered in this section can also be used to define what we will call *almost coherent forms*, that is, forms which are close to be eigenstates of the lowering operator x_0 in (3.1). Our procedure reflects the one usually adopted in Hilbert spaces. We define (formally, for the moment), the following form:

$$\Phi_z(a, b) = \sum_{l=0}^{\infty} \frac{z^l}{(\beta_l!)^2} \varphi_l(a, b), \tag{3.8}$$

$\forall a,b\in\mathcal{A}$ and for $z\in\mathcal{D}\subseteq\mathbb{C}$, a subset of $\mathbb{C}$ to be defined. In other words, $\mathcal{D}$ is the set of all the complex zs for which the series in (3.8) does converge for all a and b in $\mathcal{A}$. Notice that, in principle, $\mathcal{D}$ could consist of the single point $z=0$. However, we will show that this is not the case. In (3.8), we have $\beta_0!=1$ and $\beta_l!=\beta_1\beta_2\cdots\beta_l$, $l\ge 1$.

To study the convergence of $\Phi_z(a,b)$, we start stressing that, since $\varphi_0\in\mathcal{S}_c$, each φ_l belongs to $\mathcal{S}_c$ as well, and in particular

$$|\varphi_l(a,b)|\le\gamma_{\varphi_0}\|y_0\|_0^{2l}\|a\|\|b\|, \tag{3.9}$$

$a,b\in\mathcal{A}$, $l=0,1,2,3,\ldots$. This is exactly of the kind (2.3), with $\gamma_\varphi=\gamma_{\varphi_0}\|y_0\|_0^{2l}$.

Then, going back to (3.8), we have

$$|\Phi_z(a,b)|\le\sum_{l=0}^{\infty}\frac{|z|^l}{(\beta_l!)^2}\,|\varphi_l(c,b)|\le\gamma_{\varphi_0}\|a\|\|b\|\sum_{l=0}^{\infty}\frac{(|z|\|y_0\|_0^2)^l}{(\beta_l!)^2},$$

which is clearly a power series with radius of convergence

$$\rho'=\lim_l\beta_{l+1}^2=\begin{cases}\infty, & \text{if } q=1,\\ \dfrac{1}{1-q}, & \text{if } q\in]-1,1[.\end{cases} \tag{3.10}$$

Hence the series (3.8) exists for $|z|<\rho=\frac{\rho'}{\|y_0\|_0^2}$, that is for all $z\in C_\rho(0)$, the circle centered in the origin of the complex plane and of radius ρ, which is exactly the set $\mathcal{D}$ above we were trying to identify. It is now a simple exercise to check that

$$\Phi_z(ax_0,bx_0)=z\Phi_z(a,b), \tag{3.11}$$

$\forall z\in C_\rho(0)$ and for all $a,b\in\mathcal{A}$. Of course, this is not as saying that $\Phi_z(a,b)$ is an eigenstate of x_0 with eigenvalue z, but it is not so far away, and in any case, it is an interesting feature of the form Φ_z. We hope to be able to define proper coherent forms soon, and to study their properties, including the possibility of deducing some sort of resolution of the identity, as one usually expects from ordinary coherent states, [19].

3.2 | Back to the φ_ls

In [16], the authors prove, in a different context with respect to the one we are considering here, that the states they call eigenvectors become, after constructing their GNS[3] representations, *ordinary*[4] eigenvectors of the operator which represents the original element of the C*-algebra in the Hilbert space arising from the GNS construction. The same result can be proven in our framework, and this will have interesting consequences, as we will see. To start our analysis, it is worth to recall briefly how the GNS construction works in the present case.

Given a Banach quasi $*$-algebra $(\mathcal{A}[\|\cdot\|],\mathcal{A}_0[\|\cdot\|_0])$ with identity $e\in\mathcal{A}_0$ and $\varphi\in\mathcal{S}$, we put

$$\mathfrak{N}_\varphi=\{a\in\mathcal{A}\,:\,\varphi(a,a)=0\}=\{a\in\mathcal{A}\,:\,\varphi(a,b)=0,\ \forall b\in\mathcal{A}\}.$$

Let $\mathcal{D}_\varphi^0:=\mathcal{A}_0/\mathfrak{N}_\varphi=\{\lambda_\varphi(x):=x+\mathfrak{N}_\varphi\,:\,x\in\mathcal{A}_0\}$ and $\mathcal{D}_\varphi:=\mathcal{A}/\mathfrak{N}_\varphi=\{\lambda_\varphi(a):=a+\mathfrak{N}_\varphi\,:\,a\in\mathcal{A}\}$ be the usual quotient sets.

For any $a\in\mathcal{A}$, we put

$$\pi_\varphi(a)\lambda_\varphi(x)=\lambda_\varphi(ax),\ x\in\mathcal{A}_0.$$

Then by the third condition in (2.1) $\pi_\varphi(a)$ is an element of $\mathcal{L}^\dagger(\mathcal{D}_\varphi^0,\mathcal{D}_\varphi)$ (the set of all linear mappings X from $\mathcal{D}_\varphi^0$ into $\mathcal{D}_\varphi$), and $\pi_\varphi(a)^\dagger:=\pi_\varphi(a)^*\lceil_{\mathcal{D}_\varphi}=\pi_\varphi(a^*)$ and $\pi_\varphi(ax)=\pi_\varphi(a)\pi_\varphi(x)$ for all $a\in\mathcal{A}$ and $x\in\mathcal{A}_0$.

Moreover if $\xi_\varphi := \lambda_\varphi(e)$ we define in $\pi_\varphi(\mathcal{A})\xi_\varphi$ a inner product by

$$< \pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi >= \varphi(a, b). \tag{3.12}$$

We denote by $\mathcal{H}_\varphi$ the completion of $\pi_\varphi(\mathcal{A})\xi_\varphi$. Then, ξ_φ is a cyclic vector for π_φ, that is, $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in $\mathcal{H}_\varphi$. We call $(\mathcal{H}_\varphi, \xi_\varphi, \pi_\varphi)$ the GNS-representation of the Banach quasi $*$-algebra $(\mathcal{A}[\|\cdot\|], \mathcal{A}_0[\|\cdot\|_0])$ for φ.

With this in mind, it is possible to prove the following.

Proposition 13. *Let $\varphi \in \mathcal{S}_{c_0}$ be an eigenstate of $a \in \mathcal{A}$ with eigenvalue λ. Then ξ_φ satisfies $\pi_\varphi(a)\xi_\varphi = \lambda\xi_\varphi$, and vice versa.*

Proof. Indeed we have, using Proposition 5, that $\varphi(b, a) = \lambda\varphi(b, e)$ for all $b \in \mathcal{A}$, if and only if $\varphi(a - \lambda e, a - \lambda e) = 0$. But this, using (3.12), is equivalent to

$$0 = \langle \pi_\varphi(a - \lambda e)\xi_\varphi, \pi_\varphi(a - \lambda e)\xi_\varphi \rangle = \|\pi_\varphi(a - \lambda e)\xi_\varphi\|^2,$$

so that our claim easily follows. □

With this in mind, we consider now what happens for our family of forms φ_l in (3.2) and (3.5). In particular we will show that all these forms produce a single *-representation (and therefore a single Hilbert space $\mathcal{H}$ and a single cyclic vector) and, even more interesting (and expected!) that it is natural to associate the various φ_l to different vectors of $\mathcal{H}$ which, under natural hypotheses on x_0 and y_0, are orthogonal.

Let us consider first the form φ_0 in (3.2), and let us assume, to simplify the notation that $\varphi_0 \in \mathcal{S}_{c_0}$. This is not a big requirement: If $\gamma_{\varphi_0} \neq 1$, we could replace φ_0 with $\varphi_0' = \frac{1}{\gamma_{\varphi_0}}\varphi_0$. Using (3.12), there is a triple $(\mathcal{H}_0, \xi_0, \pi_0)$ such that

$$\varphi_0(a, b) = \langle \pi_0(a)\xi_0, \pi_0(b)\xi_0 \rangle_0, \tag{3.13}$$

$\forall a, b \in \mathcal{A}$ and where $\langle \cdot, \cdot \rangle_0$ is the scalar product in $\mathcal{H}_0$. Now, what is interesting for us is that φ_1 does not necessarily produce another triple $(\mathcal{H}_1, \xi_1, \pi_1)$, as we will now show. The reason is in formula (3.5), which shows that there is a relation between φ_1 and φ_0:

$$\varphi_1(a, b) = \varphi_0(ay_0, by_0) = \langle \pi_0(ay_0)\xi_0, \pi_0(by_0)\xi_0 \rangle_0 = \langle \pi_0(a)\pi_0(y_0)\xi_0, \pi_0(b)\pi_0(y_0)\xi_0 \rangle_0 =$$

$$= \langle \pi_0(a)\xi_1, \pi_0(b)\xi_1 \rangle_0,$$

where we have defined $\xi_1 = \pi_0(y_0)\xi_0 \in \mathcal{H}_0$. Similarly we have

$$\varphi_2(a, b) = \varphi_0(ay_0^2, by_0^3) = \langle \pi_0(ay_0^2)\xi_0, \pi_0(by_0^2)\xi_0 \rangle_0 = \langle \pi_0(a)\xi_2, \pi_0(b)\xi_2 \rangle_0,$$

where $\xi_2 = \pi_0(y_0^2)\xi_0 = \pi_0(y_0)^2\xi_0 \in \mathcal{H}_0$. And so on. This shows that the triple $(\mathcal{H}_0, \xi_0, \pi_0)$ is sufficient to deal with all the forms φ_l, $l = 0, 1, 2, 3, \ldots$, and that, in general

$$\varphi_l(a, b) = \langle \pi_0(a)\xi_l, \pi_0(b)\xi_l \rangle_0, \tag{3.14}$$

$\forall a, b \in \mathcal{A}$, $l = 0, 1, 2, 3, \ldots$. Here $\xi_l = \pi_0(y_0)^l\xi_0 \in \mathcal{H}_0$. It is interesting to observe now that the various ξ_l satisfy an orthogonality condition. To show this aspect, we start rewriting

$$\langle \xi_k, \xi_l \rangle_0 = \langle \pi_0(y_0)^k\xi_0, \pi_0(y_0)^l\xi_0 \rangle_0 = \varphi_0(y_0^k, y_0^l), \tag{3.15}$$

where we have used, in particular, (3.13), and the fact that $\pi_0(y_0)^n = \pi_0(y_0^n)$, for all $n \geq 0$. Hence, for instance, $\langle \xi_1, \xi_0 \rangle_0 = \varphi_0(y_0, e) = \varphi_0(e, y_0^*)$. Now, if $y_0^* = x_0$, we have $\varphi_0(e, y_0^*) = \varphi_0(e, x_0) = 0$ because of (3.2). Therefore, $\langle \xi_1, \xi_0 \rangle_0 = 0$. Similarly we could check that $\langle \xi_l, \xi_0 \rangle_0 = 0$, $\forall l \geq 1$, or that $\langle \xi_2, \xi_1 \rangle_0 = 0$. In this case, we have, assuming again that $y_0 = x_0^*$ and recalling that $x_0 y_0 = e + q y_0 x_0$,

$$\begin{aligned}\langle \xi_2, \xi_1 \rangle_0 &= \varphi_0(y_0^2, y_0) = \varphi_0(y_0, x_0 y_0) = \varphi_0(y_0, e) + q\varphi_0(y_0, y_0 x_0)\\ &= \varphi_0(e, x_0) + q\varphi_0(x_0 y_0, x_0) = 0.\end{aligned}$$

These results can be extended:

Proposition 14. *If* $y_0 = x_0^*$, *then*

$$\langle \xi_k, \xi_l \rangle_0 = 0 \tag{3.16}$$

for all $k \neq l$.

Proof. It is enough to observe that, if $y_0 = x_0^*$, then $n_0 = n_0^*$. Then, because of Proposition 13 and of (3.5), we have $\pi_0(n_0)\xi_l = \beta_l \xi_l$. Then our claim follows with standard steps from the facts that π_0 is a *-representation, and that $\beta_l \neq \beta_k$ whenever $l \neq k$. ☐

Remark.

(1) This proposition suggests a possible way to define a notion of orthogonality of the forms φ_l: We could say that these are *orthogonal* because their corresponding vectors ξ_l are orthogonal in $\mathcal{H}_0$. However, this definition is not really easy to extend to arbitrary forms Ω and $\tilde{\Omega}$, since there is no reason a priori that guarantees that they produce a single Hilbert space when constructing their related GNS-like representations, which was a key feature in our construction.
(2) The condition $y_0 = x_0^*$ is rather important, since it implies that $n_0 = y_0 x_0 = n_0^*$, which is used in the proof of Proposition 14. If $y_0 \neq x_0^*$, then there is no reason to expect that the orthogonality between the various ξ_l has to be true. However, in this case, we could rather imagine that a second family $\eta_k \in \mathcal{H}_0$ can be defined, which is biorthogonal to the ξ_l: $\langle \xi_l, \eta_k \rangle_0 = 0$, if $k \neq l$. This is a rather common feature in non-Hermitian quantum mechanics, [6–8, 11, 12, 25], and it is exactly what we will briefly consider below.
(3) The possibility of using the norm of sesquilinear forms to check their mutual orthogonality, as proposed in a different context in [16, 28], remains open. This approach does not look so natural for us, but a deeper understanding of this possibility is part of our future work.

In Proposition 14, we have assumed that $y_0 = x_0^*$. Let us briefly consider what happens when $y_0 \neq x_0^*$. This is interesting, and can be seen as going from, say, bosons (or fermions, or quons) to pseudo-bosons (or pseudo-fermions, or pseudo-quons), [6].

In this case, in complete analogy with what we have done in (3.2) and after, we assume that a second element $\eta_0 \in \mathcal{S}_c$ exists, which is an eigenstate of y_0^* with zero eigenvalue:

$$\eta_0(b, y_0^*) = 0, \tag{3.17}$$

$\forall b \in \mathcal{A}$. It is clear that, if $x_0 = y_0^*$, then η_0 and φ_0 collapse, while in general, they are expected to be different forms. Then, calling

$$\eta_l(a, b) = \varphi_{l-1}(a x_0^*, b x_0^*), \tag{3.18}$$

$a, b \in \mathcal{A}$, $l = 1, 2, 3, \ldots$, we have

$$\eta_l(a, n_0^*) = \beta_l \eta_l(b, e), \tag{3.19}$$

$\forall b \in \mathcal{A}$. The counterpart of (3.7) is now

$$\eta_l(a y_0^*, b y_0^*) = \beta_l^2 \eta_{l-1}(a, b), \tag{3.20}$$

$\forall l \geq 1$ and $a, b \in \mathcal{A}$. We can consider a GNS-like construction for each η_l. However, they are all related to a single GNS, the one we construct out of $\eta_0, \pi_0^{(\eta)}$, analogously to (3.14):

$$\eta_0(a,b) = \langle \pi_0^{(\eta)}(a)\nu_0, \pi_0^{(\eta)}(b)\nu_0\rangle_{\eta_0}, \tag{3.21}$$

$a, b \in \mathcal{A}$, and the scalar product is the one in $\mathcal{H}_{\eta_0}$. Moreover, $\nu_l = \pi_0^{(\eta)}(x_0^*)^l \nu_0$. It is now a trivial exercise to show that, if $(\mathcal{H}_0, \xi_0, \pi_0) = (\mathcal{H}_{\eta_0}, \nu_0, \pi_0^{(\eta)})$, then

$$\langle \xi_k, \nu_l \rangle = 0, \tag{3.22}$$

for all $k \neq l$. Here $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathcal{H}_0 = \mathcal{H}_{\eta_0}$. Hence the two sets of vectors are biorthogonal, and then we say that also the sesquilinear forms they are derived from are biorthogonal. However, it should be stressed that requiring that $(\mathcal{H}_0, \xi_0, \pi_0) = (\mathcal{H}_{\eta_0}, \nu_0, \pi_0^{(\eta)})$ is a strong assumption, and it is not guaranteed at all, in general.

4 | CONCLUSIONS

In this paper, we have proposed a possible extension of the notion of eigenvectors in the context of Banach quasi *-algebra, and we have deduced several properties of these specific sesquilinear forms, our eigenstates. An interesting result is that, if φ is an eigenstate of a certain $a \in \mathcal{A}$ with eigenvalue λ, it is possible to define another form, ψ, as in (2.10), which is eigenstate of a^* with eigenvalue $\overline{\lambda}$.

As an explicit application, we have constructed a family of eigenstates of a number-like operator defined in terms of certain ladder elements that obey suitable commutation relations of the q-uonic type. This particular application allows us to introduce the notion of orthogonality and of biorthogonality for our sesquilinear forms, by using the GNS representation defined by these forms.

Many other aspects are still open and interesting in our research, both from a mathematical side and for possible physical applications. The possibility of defining coherent forms is just one of these: We have already done some steps here, see (3.8), but there is still a long way to go. The orthogonality, and the biorthogonality, at the level of forms, rather than in representation, are also quite intriguing aspects. And, possible physical applications, possibly to ladder elements of different nature, look interesting. So, our analysis is in progress.

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DATA AVAILABILITY STATEMENT

This work does not have any experimental data.

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ENDNOTES

[1] Of course this latter property does not depend on condition $\sum_{j=1}^{n} q_j = 1$.

[2] In what we are doing here, the bosonic case corresponds to the case $y_0 = x_0^*$, while the pseudo-bosonic (or pseudo-quonic, to be more precise) to $y_0 \neq x_0^*$.

[3] GNS stands for Gelfand-Naimark-Segal.

[4] That is, in a purely Hilbertian sense.

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APPENDIX A: QUONS IN $\mathcal{H}$

This appendix is devoted to a short review on the standard results on quons, which are taken mainly from Refs. [17, 20, 24].

The standard commutation rule for a single mode quon is

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1\!\!1, \qquad q \in [-1, 1]. \tag{A.1}$$

We see that this relation interpolates between bosons ($q = 1$) and fermions ($q = -1$).

If Φ_0 is the vector of the Hilbert space $\mathcal{H}$ annihiled by the annihilation quon operator a_q, $a_q\Phi_0 = 0$, then the set of the vectors defined recursively by

$$\Phi_{n+1} = \frac{1}{\gamma_n} a_q^\dagger \Phi_n, \tag{A.2}$$

is an orthonormal basis for $\mathcal{H}$. The value of the normalization constant γ_n depends on q and n through the expression

$$\gamma_n^2 = \begin{cases} \dfrac{1-q^{n+1}}{1-q}, & \text{if } q \neq 1, \\ n+1, & \text{if } q = 1. \end{cases} \tag{A.3}$$

Defining the self-adjoint operator $N_q = a_q^\dagger a_q$, it is easy to see that

$$N_q\Phi_n = \gamma_{n-1}^2 \Phi_n,$$

where $\gamma_{-1} := 0$.

The operator a_q is bounded for any q in $[-1, 1[$. Indeed we have

$$\|a_q\|^2 = \sup_{\Psi\in\mathcal{H}:\|\Psi\|\le 1} < a_q\Psi, a_q\Psi >= \sup_{\Psi\in\mathcal{H}:\|\Psi\|\le 1} < \Psi, N_q\Psi >= \sup_{d\in l^2:\sum|d_k|^2\le 1} \sum_k |d_k|^2\gamma_{k-1}^2.$$

Here we have written $\Psi = \sum_k d_k\Phi_k$, using the fact that the vectors Φ_ks form an orthonormal basis of $\mathcal{H}$. With l^2 we indicate the usual Hilbert space of the square-integrable sequences. Therefore we have:

(1) $q = 1 \Rightarrow \gamma_{k-1}^2 = k$. This implies that $\|a_q\| = \infty$.

(2) $q = -1 \Rightarrow \gamma_{k-1}^2 = 0, 1$ depending on whether the index n is even or odd. However, in both cases, $\gamma_{k-1}^2 \le 1$, and therefore

$$\|a_q\|^2 \le \sup_{d\in l^2:\sum|d_k|^2\le 1} \sum_k |d_k|^2 = 1.$$

Of course for fermions, we have in fact $\|a_q\| = 1$.

(3) For general $q \in]-1, 1[$, we see that $\gamma_{k-1}^2 \le \frac{2}{1-q}$, independently of n. As a consequence, it can be shown that $\|a_q\|$ is bounded by $\frac{2}{1-q}$.

Of course, if a_q is bounded, then $a_q^\dagger$ and $N_q = a_q^\dagger a_q$ are bounded as well.

More information on quons can be found in [17, 20, 24]. What is relevant for us here is to notice that quons give rise to bounded ladder operators acting on an infinite-dimensional Hilbert space.