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The method of lower and upper solutions for second order periodic Stieltjes differential equations

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Abstract. As an application of the Schauder Fixed Point Theorem, an existence theory is developed for second order nonlinear differential equations with Stieltjes derivative related to a left-continuous, nondecreasing function. By the method of lower and upper solutions, under a Nagumo-type assumption we get a very general result which can be further applied to deduce the existence of solutions for second order nonlinear problems in the settings of impulsive differential equations, time scale analysis or generalized differential equations.

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1. Introduction

In the study of nonlinear differential equations, the method of lower and upper solutions (initiated in [23,24]) is a powerful tool; we refer the reader to the survey [1] for a comprehensive analysis of the topic.

When applied to the setting of second order differential equations, a growth condition with respect to the dependence on the first derivative must be imposed on the nonlinear part of the equation. The most widely used such assumption is the so-called Nagumo condition [22] and its generalizations.

At the same time, due to the numerous applications in studying real life problems (e.g., [9,12,13,26] or [27]), theories that allow one to describe the behavior of systems in whose evolution, besides the continuous dynamics, discrete perturbations appear along with stationary intervals increasingly gain

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in popularity. One of such theories is that of measure differential equations (or differential equations driven by measures, see [2–4,29,30,35]).

In the last decade, since the Stieltjes derivative (i.e., the derivative of a function with respect to another function) was reformulated in [25] in the spirit of [36], it was shown that measure differential equations can be described in terms of this concept of derivative; therefore, the theory of Stieltjes differential problems has been continuously developing (see [5,6,9,12–14,19,26,31] for the single-valued case or [16–18,32,33] for the set-valued framework).

As for second order equations with Stieltjes derivative, as far as the authors know, only the linear case with initial value boundary condition was studied so far [5,6].

In the present work, we go further and obtain the existence of solutions for nonlinear second order Stieltjes equations by assuming the existence of lower and upper solutions. The Nagumo-type assumption we impose is inspired by [10] (generalizing that in [37]), as well as the line of proof. However, many difficulties (arisen from the fact that the derivative is related to a function which generally speaking has discontinuities and also stationary intervals) have to be overcome by methods specific to the theory of Stieltjes differential problems.

We end this introduction with the remark that the obtained existence result is of wide generality since differential problems involving the Stieltjes derivative are strongly connected with generalized differential problems [11,28,34,35], to dynamic equations on time scales [4,34] as well as to impulsive differential inclusions [25]. Thus, new results could be deduced for second order periodic nonlinear equations in all the mentioned settings.

2. Notations and preliminary results

We say that a map $u : [0, T] \to \mathbb{R}$, $T > 0$, is regulated [8,21] if there exist the right and left limits $u(t+)$ and $u(s-)$ at every points $t \in [0, T)$ and $s \in (0, T]$. The class of regulated functions contains the class of bounded variation maps and also that of continuous maps. Regulated functions are bounded and their space is a Banach space when endowed with the norm

$$\|u\|_C = \sup_{t\in[0,T]} |u(t)|.$$

A family $\mathcal{A}$ of regulated $\mathbb{R}$-valued functions on $[0, T]$ is said to be equiregulated if for every $\bar{t} \in [0, T]$ and every $\varepsilon > 0$ one can find $\delta > 0$ such that for any $u \in \mathcal{A}$

$$|u(t) - u(\bar{t}-)| < \varepsilon, \text{ for every } t \in (\bar{t} - \delta, \bar{t})$$

and

$$|u(t) - u(\bar{t}+)| < \varepsilon, \text{ for every } t \in (\bar{t}, \bar{t} + \delta).$$

Let us recall an Ascoli-type result.

Lemma 2.1. ([8, Corollary 2.4]) *A set of* $\mathbb{R}$*-valued regulated functions is relatively compact if and only if it is equiregulated and pointwise bounded.*

Let $g : [0,T] \to \mathbb{R}$ be a nondecreasing left-continuous function and μ_g be the Stieltjes measure induced by g (see [7]).

Define the following sets:

$$D_g = \{t \in [0,T] : g(t+) - g(t) > 0\},$$

i.e., the set of atoms of the measure μ_g and

$$C_g = \{t \in [0,T] : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

together with

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus D_g,$$

where $C_g = \bigcup_{n\in\mathbb{N}}(u_n, v_n)$ is a disjoint decomposition of C_g. Let $N_g^- = \{u_n : n \in \mathbb{N}\}\setminus D_g$ and $N_g^+ = \{v_n : n \in \mathbb{N}\}\setminus D_g$; clearly $N_g = N_g^+ \cup N_g^-$. Since in [25] it was proved that $\mu_g(C_g) = \mu_g(N_g) = 0$ these two sets do not effect the study of differential equations. Moreover we observe that $D_g \cap C_g = \emptyset$.

Let us now recall the notion of differentiability related to Stieltjes type integrals introduced in [5] (which extends that in [25] in such a way that the points in C_g are also covered).

Definition 2.2. Let $g : [0,T] \to \mathbb{R}$ be a nondecreasing left-continuous function such that $0 \notin N_g^-$ and $T \notin C_g \cup N_g^+$. The derivative with respect to g (or the g-derivative) of a function $f : [0,T] \to \mathbb{R}$ at a point $\bar{t} \in [0,T]$ is given by

$$f_g'(\bar{t}) = \lim_{t\to\bar{t}} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \notin D_g \cup C_g,$$

$$f_g'(\bar{t}) = \lim_{t\to\bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in D_g,$$

$$f_g'(\bar{t}) = \lim_{t\to v_n+} \frac{f(t) - f(v_n)}{g(t) - g(v_n)} \quad \text{if } t \in (u_n, v_n) \subseteq C_g,$$

provided the limits exist. The points of N_g must be treated as follows:

$$f_g'(\bar{t}) = \lim_{t\to\bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^+,$$

$$f_g'(\bar{t}) = \lim_{t\to\bar{t}-} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^-.$$

Note that if $t \in D_g$, the g-derivative $f_g'(t)$ exists if and only if the sided limit $f(t+)$ exists, and in this case

$$f_g'(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)},$$

while if $t \in (u_n, v_n) \subseteq C_g$ the g-derivative $f_g'(t)$ exists if and only if there exists the right g-derivative at v_n.

We recall that the Lebesgue-Stieltjes (shortly, LS-) integrability w.r.t. g is the abstract Lebesgue integrability w.r.t. the Stieltjes measure μ_g. From now on, for the LS-integral on a measurable set A of a function f w.r.t. g the notation $\int_A f(s)\mathrm{d}g(s)$ will be used.

For $q \in [1, \infty)$, by $L^q_g([0,T])$ we denote the space of real measurable functions on $[0,T]$ with the property that $|f|^q$ is LS-integrable (w.r.t. g) with its natural topological structure given by the norm

$$\|f\|_q = \left(\int_{[0,T)} |f(t)|^q \mathrm{d}g(t)\right)^{\frac{1}{q}}.$$

The connection between Stieltjes integrals and the Stieltjes derivative is given by Fundamental Theorems of Calculus [25, Theorems 5.4, 6.2, 6.5].

Theorem 2.3. *Let $g : [0,T] \to \mathbb{R}$ be a nondecreasing, left-continuous function. Then $f : [0,T] \to \mathbb{R}$ is g-absolutely continuous if and only if it is g-differentiable μ_g-a.e., f'_g is Lebesgue-Stieltjes integrable w.r.t. g and*

$$f(t') = f(t'') + \int_{[t'',t')} f'_g(s)dg(s), \quad \text{for every } 0 \le t'' < t' \le T.$$

The g-derivative was involved in solving many interesting problems where abrupt changes (corresponding to discontinuity points of g) and stationary periods (corresponding to intervals where g is constant) occur, such as [9,26] or [27].

Let us recall [9] that a map $f : [0,T] \to \mathbb{R}$ is g-continuous at a point $t \in [0,T]$ if for every $\varepsilon > 0$ one can find $\delta_{t,\varepsilon} > 0$ such that

$$s \in [0,T], \ |g(t) - g(s)| < \delta_{t,\varepsilon} \ \Rightarrow \ |f(t) - f(s)| < \varepsilon$$

while g-continuity on $[0,T]$ means g-continuity at every $t \in [0,T]$.

Also (e.g., [25]), f is called g-absolutely continuous if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon$$

for any set $\{(a_j, b_j); j = 1, \dots, m\}$ of disjoint subintervals of $[0,T]$ satisfying

$$\sum_{j=1}^{m} (g(b_j) - g(a_j)) < \delta_\varepsilon.$$

Clearly, g-absolutely continuous functions are g-continuous and it was proved in [25, Proposition 5.3] that such functions have essentially the same properties as g: they are left-continuous everywhere, continuous at the points where g is continuous and constant on the intervals where g is constant.

Note that g-continuous functions are not necessarily bounded, this is why it is necessary to consider the space $\mathcal{BC}_g([0,T])$ of functions which are bounded and g-continuous [9].

Recently, the way to study higher order differential equations with Stieltjes derivative was open by [5]. Thus, the following space of functions was introduced: $\mathcal{BC}^1_g([0,T])$ is the space of functions $f : [0,T] \to \mathbb{R}$ with the properties that f is g-differentiable everywhere on $[0,T]$ and its g-derivative f'_g is bounded and g-continuous. It is endowed with the norm

$$\|f\|_{\mathcal{BC}^1_g([0,T])} = \|f\|_C + \|f'_g\|_C$$

and by [5, Theorem 3.15], the space $(\mathcal{BC}^1_g([0,T]), \|\cdot\|_{\mathcal{BC}^1_g([0,T])})$ is a Banach space.

[15, Remark 7.3] provides us a change of variable formula for the Lebesgue–Stieltjes integral.

Lemma 2.4. *Let $g:[0,T]\to\mathbb{R}$ be a nondecreasing, left-continuous function and $f:[0,T]\to\mathbb{R}$ be g-absolutely continuous. If $0\le a<b\le T$ and m is a real-valued map continuous on $[f(a),f(b)]$, then*

$$\int_{f(a)}^{f(b)} m(u)\mathrm{d}u = \int_{[a,b)} \left(\int_0^1 m\left(f(t)+\tau\cdot f'_g(t)\cdot\mu_g(\{t\})\right)\mathrm{d}\tau\right)\cdot f'_g(t)\mathrm{d}g(t).$$

Proof. It suffices to take in [15, Remark 7.3] $h(s)=\int_{f(a)}^s m(u)\mathrm{d}u$ (it is continuously differentiable since m is continuous) and to note that in this case $h'(s)=m(s)$ on $[f(a),f(b)]$. □

Let us next remind the reader of an existence and uniqueness result for first order linear periodic differential equations with Stieltjes derivative.

The authors of [31,32] proved such a result for the problem

$$\begin{cases} u'_g(t)+b(t)u(t)=f(t), & \mu_g\ \text{–a.e. in } [0,T],\\ u(0)=u(T), \end{cases} \tag{2.1}$$

where $b:[0,T]\to\mathbb{R}$ is a LS-integrable (w.r.t. g) function satisfying the non-resonance condition:

$$1-b(t)\mu_g(\{t\})\neq 0, \quad \text{for every } t\in[0,T]. \tag{2.2}$$

To solve the problem (2.1), the sign of $1-b(t)\mu_g(\{t\})$ was considered.

The set

$$D_g^- = \{t\in D_g : 1-b(t)\mu_g(\{t\})<0\}$$

is finite (see [9]) and if its elements are $t_1<\cdots<t_k$ (let $t_{k+1}=T$), consider

$$\sigma(t)=\begin{cases} 1, \text{ if } 0\le t\le t_1\\ (-1)^i, \text{ if } t_i<t\le t_{i+1},\ i=1,\dots,k. \end{cases}$$

Theorem 2.5. ([32, Theorem 2 and Remark 3] in the particular one-dimensional case) *Let $b:[0,T]\to\mathbb{R}$ be LS-integrable with respect to g, satisfying* (2.2) *and let $f:[0,T]\to\mathbb{R}$ be LS-integrable with respect to g.*

Denoting by

$$\tilde{b}(t)=\begin{cases} b(t), \text{ if } t\in[0,T]\setminus D_g\\ \frac{-\log|1-b(t)\mu_g(\{t\})|}{\mu_g(\{t\})}, \text{ if } t\in D_g. \end{cases}$$

and by

$$\tilde{g}(t,s)=\frac{1}{\sigma(T)e^{\int_{[0,T)}\tilde{b}(r)\mathrm{d}g(r)}-1}\begin{cases} \sigma(T)e^{\int_{[0,T)}\tilde{b}(r)\mathrm{d}g(r)-\int_{[s,t)}\tilde{b}(r)\mathrm{d}g(r)}, \text{ if } 0\le s\le t\le T\\ e^{\int_{(t,s]}\tilde{b}(r)\mathrm{d}g(r)}, \text{ if } 0\le t<s\le T, \end{cases}$$

the function $u : [0,T] \to \mathbb{R}$ *is the g-absolutely continuous solution of problem* (2.1) *if and only if*

$$u(t) = \frac{1}{\sigma(t)} \int_{[0,T)} \frac{\sigma(s)}{1 - b(s)\mu_g(\{s\})} \tilde{g}(t,s) f(s) \mathrm{d}g(s), \ \forall t \in [0,T]. \qquad (2.3)$$

Obviously, if

$$1 - b(t)\mu_g(\{t\}) > 0 \quad \text{for all } t \in [0,T],$$

then $\sigma(t) = 1$ for every $t \in [0,T]$, therefore the formulas can be simplified.

Corollary 2.6. *If*

$$1 - b(t)\mu_g(\{t\}) > 0 \quad \text{for all } t \in [0,T],$$

the solution of (2.1) *is given by*

$$u(t) = \int_{[0,T)} \frac{1}{1 - b(s)\mu_g(\{s\})} \tilde{\tilde{g}}(t,s) f(s) \mathrm{d}g(s), \ \forall t \in [0,T],$$

where

$$\tilde{\tilde{g}}(t,s) = \frac{1}{e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1} \begin{cases} e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r) - \int_{[s,t)} \tilde{b}(r)\mathrm{d}g(r)}, & \text{if } 0 \le s \le t \le T \\ e^{\int_{(t,s]} \tilde{b}(r)\mathrm{d}g(r)}, & \text{if } 0 \le t < s \le T. \end{cases}$$

Remark 2.7. (i) From the formula (2.3) giving the solution of (2.1), since (see [32, page 5]) there is a positive γ such that

$$|1 - b(t)\mu_g(\{t\})| > \gamma, \quad \forall \ t \in [0.T]$$

and

$$|\tilde{g}(t,s)| \le \frac{\max(M, M^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1|}, \ \forall s,t \in [0,T]$$

where $M = \sup_{s' < s'' \in [0,T]} e^{\int_{[s',s'')} \tilde{b}(r)\mathrm{d}g(r)}$ ([32, Remark 4]), if $\mathcal{A} \subset L^1_g([0,T])$ is a set satisfying, for some $\phi \in L^1_g([0,T])$, the condition

$$|f(t)| \le \phi(t), \ \mu_g - a.e., \quad \text{for every } f \in \mathcal{A}$$

it can be proved (as in [32, Theorem 4]) that the set of corresponding solutions of (2.1) is equiregulated.

If besides $(f_n)_n$ is a sequence satisfying the same inequality which converges μ_g-a.e. to $f \in L^1_g([0,T])$, it is easy to see by the dominated convergence theorem that the corresponding sequence of solutions $(u_n)_n$ tends uniformly to u.

(ii) If in addition f and b are g-continuous (thus, continuous at all the points in $[0,T] \setminus D_g$), then [31, Proposition 8] implies that the solution is g-differentiable on $[0,T]$ and its g-derivative is g-continuous (so, the solution belongs to $\mathcal{BC}^1_g([0,T])$).

(iii) Moreover, if $(f_n)_n \subset \mathcal{BC}_g([0,T])$ uniformly converges to $f \in \mathcal{BC}_g([0,T])$ then, by [32, Corollary 1] (applied in the particular single-valued case) the corresponding solutions $(u_n)_n \subset \mathcal{BC}^1_g([0,T])$ uniformly converge to the solution u corresponding to f.

What is more, writing their g-derivatives using (2.1) we infer that the convergence holds in the topology of $\mathcal{BC}^1_g([0,T])$.

(iv) Finally, from (2.3) it follows that the solution u satisfies the estimate

$$\|u\|_C \leq \max\left(1, \frac{1}{\gamma}\right) \frac{\max(M, M^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1|} \cdot \|f\|_1. \tag{2.4}$$

3. Main results

3.1. Existence and uniqueness result for second order linear periodic problems

We focus in this subsection on the second order linear Stieltjes differential problem with periodic boundary condition

$$\begin{cases} u''_g(t) + Pu'_g(t) + Qu(t) = f(t), & \mu_g \text{ −a.e. in } [0,T], \\ u(0) = u(T), \quad u'_g(0) = u'_g(T). \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $f : [0,T] \to \mathbb{R}$ be LS-integrable w.r.t. g and $P, Q \in \mathbb{R}$ be such that the equation*

$$\lambda^2 + P\lambda + Q = 0$$

has real solutions λ_1, λ_2 and the nonresonance condition

$$1 - \lambda_1\mu_g(\{t\}) \neq 0 \quad \text{and} \quad 1 - \lambda_2\mu_g(\{t\}) \neq 0 \quad \text{for every } t \in [0,T]$$

is satisfied.

Then the problem (3.1) has a unique solution in $\mathcal{BC}^1_g([0,T])$ with g-absolutely continuous derivative.

Proof. By Theorem 2.5, the periodic first order Stieltjes differential problem

$$\begin{cases} u'_g(t) - \lambda_1 u(t) = f(t), & \mu_g \text{ −a.e. in } [0,T], \\ u(0) = u(T) \end{cases} \tag{3.2}$$

has a unique g-absolutely continuous solution, which we will denote by u_1.

Then, again by Theorem 2.5, the periodic first order Stieltjes differential problem

$$\begin{cases} u'_g(t) - \lambda_2 u(t) = u_1(t), & \mu_g \text{ −a.e. in } [0,T], \\ u(0) = u(T) \end{cases} \tag{3.3}$$

has a unique g-absolutely continuous solution, say u_2.

We state that u_2 is a solution of the considered second order problem (3.1).

Indeed,

$$\begin{aligned} (u_2)''_g(t) + P(u_2)'_g(t) + Qu_2(t) &= (u_2)''_g(t) - (\lambda_1 + \lambda_2)(u_2)'_g(t) + \lambda_1\lambda_2 u_2(t) \\ &= ((u_2)'_g - \lambda_2 u_2)'_g(t) - \lambda_1((u_2)'_g(t) - \lambda_2 u_2(t)) \\ &= (u_1)'_g(t) - \lambda_1 u_1(t) = f(t), \quad \mu_g \text{ −a.e. in } [0,T]. \end{aligned}$$

Besides,

$$u_2(0) = u_2(T)$$

and

$$(u_2)'_g(0) = \lambda_2 u_2(0) + u_1(0) = \lambda_2 u_2(T) + u_1(T) = (u_2)'_g(T).$$

To get the uniqueness, it suffices to note that once u is a solution of (3.1), it satisfies (3.3) with u_1 the solution of (3.2). To see this, denote by $u_1(t) = u'_g(t) - \lambda_2 u(t)$ and easily check that u_1 satisfies

$$\begin{aligned}(u_1)'_g(t) - \lambda_1 u_1(t) &= u''_g(t) - \lambda_2 u'_g(t) - \lambda_1(u'_g(t) - \lambda_2 u(t)) \\ &= u''_g(t) + Pu'_g(t) + Qu(t) = f(t)\end{aligned}$$

together with $u_1(0) = u'_g(0) - \lambda_2 u(0) = u'_g(T) - \lambda_2 u(T) = u_1(T)$.

By Theorem 2.5, the solution u_1 of (3.2) is g-absolutely continuous (therefore, g-continuous). Using Remark 2.7(ii) we can see that the solution u_2 of (3.3) (thus, the solution u of (3.1)) belongs to $\mathcal{BC}^1_g([0,T])$ and its g-derivative is g-absolutely continuous. □

Remark 3.2. (i) Using Remark 2.7(iv) we obtain that the solution u_1 of (3.2) satisfies for each $t \in [0,T]$ the inequality

$$\|u_1\|_C \leq \max\left(1, \frac{1}{\gamma_1}\right) \frac{\max(M_1, M_1^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1|} \cdot \|f\|_1,$$

where γ_1, M_1 are given by Remark 2.7(i) for $b(t) = -\lambda_1$, for all $t \in [0,T]$. In the same way, the solution u_2 of (3.3) (i.e., the solution u of our problem (3.1)) satisfies for each $t \in [0,T]$ the inequality

$$\begin{aligned}\|u\|_C &\leq \max\left(1, \frac{1}{\gamma_2}\right) \frac{\max(M_2, M_2^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b_2}(r)\mathrm{d}g(r)} - 1|} \cdot \|u_1\|_1 \\ &\leq \max\left(1, \frac{1}{\gamma_2}\right) \cdot \frac{\max(M_2, M_2^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b_2}(r)\mathrm{d}g(r)} - 1|} \\ &\cdot \max\left(1, \frac{1}{\gamma_1}\right) \frac{\max(M_1, M_1^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1|} \cdot \|f\|_1 \cdot \mu_g([0,T]), \qquad (3.4)\end{aligned}$$

where γ_2, M_2 are given by Remark 2.7(i) for $b(t) = -\lambda_2$, for all $t \in [0,T]$.

(ii) By Remark 2.7(iv) as well we deduce that given a bounded set $\mathcal{A} \subset L^1_g([0,T])$, the set of corresponding solutions $\{u_1^f : f \in \mathcal{A}\}$ of (3.2) satisfies for each $t \in [0,T]$ the inequality

$$\|u_1^f\|_C \leq \max\left(1, \frac{1}{\gamma_1}\right) \frac{\max(M_1, M_1^2)}{|\sigma(T)e^{\int_{[0,T)} \tilde{b}(r)\mathrm{d}g(r)} - 1|} \cdot \sup_{f\in\mathcal{A}} \|f\|_1,$$

thus the set of corresponding solutions of (3.3) is, due to Remark 2.7(i), equi-regulated. The same can be proved the set of their first g-derivatives.

(iii) If $(f_n)_n \subset L^1_g([0,T])$ is a sequence converging μ_g-a.e. to $f \in L^1_g([0,T])$ satisfying for some $\phi \in L^1_g([0,T])$ the condition

$$|f_n(t)| \leq \phi(t),\ \mu_g - a.e., \text{ for all } n \in \mathbb{N}$$

it comes from Remark 2.7.i) that the corresponding sequence of solutions $(u_n^1)_n$ of (3.2) tends uniformly to u. Next, Remark 2.7(iii) yields that the sequence of solutions of the second order problem (3.1) converges in the topology of $\mathcal{BC}^1_g([0,T])$ to the solution corresponding to f.

3.2. Existence of solutions for nonlinear second order periodic problems

The main result of the paper concerns the existence of solutions for the nonlinear second order periodic problem

$$\begin{cases} -u''_g(t) = f(t, u(t), u'_g(t)), \quad \mu_g \text{ -a.e. in } [0,T], \\ u(0) = u(T), u'_g(0) = u'_g(T) \end{cases} \tag{3.5}$$

involving the Stieltjes derivative.

Let us clarify the notion of solution to be considered.

Definition 3.3. A function $u : [0,T] \to \mathbb{R}$ is a solution of (3.5) if $u \in \mathcal{BC}^1_g([0,T])$, its first derivative u'_g is g-absolutely continuous and its second derivative (which is well defined μ_g-a.e. by Theorem 2.3) satisfies

$$-u''_g(t) = f(t, u(t), u'_g(t)), \quad \mu_g \text{ -a.e. in } [0,T]$$

together with the boundary conditions $u(0) = u(T)$ and $u'_g(0) = u'_g(T)$.

We use the method of upper and lower solutions. We proceed to describe the needed assumptions.

A function $\alpha \in \mathcal{BC}^1_g([0,T])$ with α'_g being g-absolutely continuous is a lower solution of (3.5) if

$$\begin{cases} -\alpha''_g(t) \le f(t, \alpha(t), \alpha'_g(t)), \quad \mu_g \text{ -a.e. in } [0,T], \\ \\ \alpha(0) = \alpha(T), \;\; \alpha'_g(0) \ge \alpha'_g(T) \end{cases}$$

Analogously, $\beta \in \mathcal{BC}^1_g([0,T])$ with β'_g being g-absolutely continuous is an upper solution of (3.5) if

$$\begin{cases} -\beta''_g(t) \ge f(t, \beta(t), \beta'_g(t)), \quad \mu_g \text{ -a.e. in } [0,T], \\ \\ \beta(0) = \beta(T), \;\; \beta'_g(0) \le \beta'_g(T) \end{cases}$$

For the function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ the following conditions are imposed:

(H1) $t \mapsto f(t,u,v)$ is measurable for every $u, v \in \mathbb{R}$;

(H2) $(u,v) \mapsto f(t,u,v)$ is continuous for μ_g-a.e. $t \in [0,T]$;

(H3) f satisfies a Nagumo-type condition on the set

$$\{(t,u,v) : \; t \in [0,T], \;\; \alpha(t) \le u \le \beta(t), \; v \in \mathbb{R}\},$$

that is: there exists $h \in L^q_g([0,T])$, $1 \le q < \infty$ and a continuous function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,u,v)| \le h(t)\chi(|w|)$$

on $\{(t,u,v) : \; t \in [0,T], \; \alpha(t) \le u \le \beta(t), \; v \in \mathbb{R}\}$, for every $w \in \mathbb{R}$ with $|w - v| \le |f(t,u,v)| \cdot \mu_g(\{t\})$ and

$$\limsup_{N\to\infty} \left[\int_{\eta_N}^{N} \frac{\tau^{\frac{q-1}{q}}}{\chi(\tau)} \mathrm{d}\tau - 2 \left(\sup_{t\in[0,T]} \beta(t) - \inf_{t\in[0,T]} \alpha(t) + \frac{\mu_g([0,T))\eta_N}{2} \right)^{\frac{q-1}{q}} \cdot \|h\|_q \right] > 0$$

where

$$\eta_N = \|h\|_1 \cdot \sup_{v\in[-N,N]} \chi(|v|) \tag{3.6}$$

(H4) there exist lower, respectively upper solutions of (3.5), α and β with $\alpha(t) \leq \beta(t)$ on $[0, T]$.

We begin with a basic, yet very important auxiliary result.

Lemma 3.4. *Let $f : [0, T] \to \mathbb{R}$ be g-absolutely continuous. If $f'_g(t) > 0$ for μ_g-a.e. $t \in [0, T)$, then f is increasing on $[0, T]$.*

Proof. The result immediately follows from the Fundamental Theorem of Calculus which asserts that for every $0 \leq t_1 < t_2 \leq T$,

$$f(t_2) = f(t_1) + \int_{[t_1, t_2)} f'_g(t) \mathrm{d}g(t).$$

□

We recall the following results.

Proposition 3.5. ([14, Proposition 3.29]) *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left-continuous function and $f_1, f_2 : [0, T] \to \mathbb{R}$ be g-absolutely continuous. Then, the maps $F, F_{max}, F_{min} : [0, T] \to \mathbb{R}$ defined as*

$$\begin{aligned} F(t) &= f_1(t) f_2(t), \\ F_{max}(t) &= \max\{f_1(t), f_2(t)\}, \\ F_{min}(t) &= \min\{f_1(t), f_2(t)\} \end{aligned}$$

are g-absolutely continuous on $[0, T]$.

Proposition 3.6. ([14, Proposition 3.15]) *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing left-continuous function. Let f be a real valued function defined on a neighborhood of $t \notin D_g$ and h a function defined on a neighborhood of $f(t)$. If $h'(f(t))$ and $f'_g(t)$ exist then so does $(h \circ f)'_g(t)$ and*

$$(h \circ f)'_g(t) = h'(f(t)) f'_g(t).$$

The following technical result will also be useful later.

Lemma 3.7. *Let $\gamma, u : [0, T] \to \mathbb{R}$ be g-absolutely continuous and*

$$q(t, u) := \max\{u(t), \gamma(t)\} \quad \text{for } t \in [0, T].$$

The following properties hold:

(1) *$q'_g(t, u)$ exists for μ_g-a.e. $t \in [0, T]$;*
(2) *if $u_n : [0, T] \to \mathbb{R}, n \in \mathbb{N}$ are g-absolutely continuous and $(u_n)_{n \in \mathbb{N}}$ converges to u uniformly, then*

$$q(t, u_n) \to q(t, u) \quad \text{uniformly} \quad \text{on } [0, T]$$

and

$$q'_g(t, u_n) \to q'_g(t, u) \text{ for } \mu_g - \ a.e. \ t \in [0, T].$$

Proof. Since u, γ are g-absolutely continuous, by Proposition 3.5 the same is true for $q(t, u) := \max\{u(t), \gamma(t)\}$ and so $q(t, u)$ is g-differentiable μ_g-a.e.

Now assume that $u_n \to u$ uniformly; then also

$$\max\{u_n(t), \gamma(t)\} \quad \text{converges uniformly to} \quad \max\{u(t), \gamma(t)\}$$

and it follows at once that

$$q(t,u_n) \to q(t,u) \ \text{ uniformly } \ w.r.t. \ \ t\in[0,T].$$

To prove that $q'_g(t,u_n) \to q'_g(t,u)$ for $\mu_g{-}a.e.\ t\in[0,T]$ we consider separately the cases in which $t\notin D_g\cup C_g$, respectively $t\in D_g$, $t\in C_g$.

CASE $t\notin D_g\cup C_g$: assume that $u(t)>\gamma(t)$, then $q'_g(t,u)=u'_g(t)$. Indeed, since t is a point of g-continuity there exists $\delta>0$ such that for $s\in(t-\delta,t+\delta)$, $u(s)>\gamma(s)$. Since $u_n\to u$ uniformly there is $N\in\mathbb{N}$ such that for $n>N$: $u_n(t)>\gamma(t)$ and as before for each $n\in\mathbb{N}$ one can find $\delta_n>0$ such that for $s\in(t-\delta_n,t+\delta_n)$, $u_n(s)>\gamma(s)$. Therefore $q'_g(t,u_n)=(u_n)'_g(t)$, thus

$$q'_g(t,u_n)=(u_n)'_g(t)\to u'_g(t)=q'_g(t,u).$$

If $u(t)<\gamma(t)$, then $q'_g(t,u)=\gamma'_g(t)$ and as before $u_n(s)<\gamma(s)$ and so $q'_g(t,u_n)=\gamma'_g(t)$

Assume now that $u(t)=\gamma(t)$. By Proposition 3.6, at the points $\bar t$ where $u(\bar t)\neq\gamma(\bar t)$:

$$q(\bar t,u)'_g=\frac{u'_g(\bar t)+\gamma'_g(\bar t)+\operatorname{sgn}(u(\bar t)-\gamma(\bar t))(u'_g(\bar t)-\gamma'_g(\bar t))}{2}$$

Remark that if $q'_g(t,u)$ exists, then the function $\max\{u(t),\gamma(t)\}$ is g-differentiable at t, therefore necessarily $u'_g(t)=\gamma'_g(t)$. It implies that $q'_g(t,u)=u'_g(t)=\gamma'_g(t)$.

Again since $u_n\to u$ uniformly there is $N\in\mathbb{N}$ such that for $n>N$ one can find $\delta_n>0$ satisfying that on $(t-\delta_n,t+\delta_n)$:

$$q(t,u_n)=\max\{u_n(t),\gamma(t)\}=\frac{u_n(t)+\gamma(t)+|u_n(t)-\gamma(t)|}{2}.$$

By Proposition 3.6, if $u_n(t)\neq\gamma(t)$ and if $q'_g(t,u_n)$ exists then

$$q'_g(t,u_n)=\frac{(u_n)'_g(t)+\gamma'_g(t)+\operatorname{sgn}(u_n(t)-\gamma(t))\left((u_n)'_g(t)-\gamma'_g(t)\right)}{2},$$

and so

$$q'_g(t,u_n)=\begin{cases}(u_n)'_g(t) \text{ if } u_n(t)\geq\gamma(t)\\ \\ \gamma'_g(t) \text{ if } u_n(t)<\gamma(t).\end{cases}$$

Therefore

$$q'_g(t,u_n)\to q'_g(t,u).$$

CASE $t\in D_g$: if both $\gamma(t)<u(t)$ and $\gamma(t+)<u(t+)$ hold, then

$$q'_g(t,u)=\frac{q(t+,u)-q(t,u)}{\mu_g(\{t\})}=\frac{u(t+)-u(t)}{\mu_g(\{t\})}=u'_g(t).$$

Since $u_n\to u$ uniformly there is $N\in\mathbb{N}$ such that for $n>N$ both $\gamma(t)<u_n(t)$ and $\gamma(t+)<u_n(t+)$ are satisfied. Thus

$$q'_g(t,u_n)=\frac{u_n(t+)-u_n(t)}{\mu_g(\{t\})}=(u_n)'_g(t)$$

and so,

$$q'_g(t,u_n) = (u_n)'_g(t) \to u'_g(t) = q'_g(t,u).$$

Analogously if $\gamma(t) > u(t)$ and $\gamma(t+) > u(t+)$ and in this case $q'_g(t,u_n) = \gamma'_g(t) = q'_g(t,u)$.

Now suppose $u(t) = \gamma(t)$ and $u(t+) > \gamma(t+)$.

Then

$$q'_g(t,u) = \frac{q(t+,u) - q(t,u)}{\mu_g(\{t\})} = \frac{u(t+) - \gamma(t)}{\mu_g(\{t\})} = \frac{u(t+) - u(t)}{\mu_g(\{t\})} = u'_g(t).$$

Let $\varepsilon > 0$. Since $u_n(t+) \to u(t+)$ there is $N_1 \in \mathbb{N}$ such that for $n > N_1$,

$$|u_n(t+) - u(t+)| < \varepsilon \cdot \mu_g(\{t\}) \quad \text{and} \quad u_n(t+) > \gamma(t+).$$

There is also $N_2 \in \mathbb{N}$ such that if $n > N_2$, then

$$|u_n(t) - u(t)| < \varepsilon \cdot \mu_g(\{t\}).$$

If $n > \max\{N_1, N_2\}$, then

$$\left|\frac{u_n(t+) - u_n(t)}{\mu_g(\{t\})} - \frac{u(t+) - u(t)}{\mu_g(\{t\})}\right| \le \left|\frac{u_n(t+) - u(t+)}{\mu_g(\{t\})}\right| + \left|\frac{u_n(t) - u(t)}{\mu_g(\{t\})}\right| < 2\varepsilon,$$

which implies $q'_g(t,u_n) \to q'_g(t,u)$.

Suppose finally $u(t) < \gamma(t)$ and $u(t+) > \gamma(t+)$. As before, fixed $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for $n > N$,

$$|u_n(t+) - u(t+)| < \varepsilon \cdot \mu_g(\{t\}), \quad u_n(t+) > \gamma(t+) \quad \text{and} \quad u_n(t) < \gamma(t).$$

One can see that

$$q'_g(t,u_n) = \frac{u_n(t+) - \gamma(t)}{\mu_g(\{t\})}$$

and

$$q'_g(t,u) = \frac{u(t+) - \gamma(t)}{\mu_g(\{t\})},$$

therefore

$$\left|q'_g(t,u_n) - q'_g(t,u)\right| = \left|\frac{u_n(t+) - u(t+)}{\mu_g(\{t\})}\right| < \varepsilon.$$

CASE $t \in C_g$: Observe that if $\bar{t} \in (u_n, v_n) \subseteq C_g$, since $v_n \notin D_g$ then the thesis follows reasoning as for $t \notin D_g$ for the right g-derivative at v_n. □

Consider the map p defined for every $t \in [0,T]$ and every g-absolutely continuous function $u : [0,T] \to \mathbb{R}$ by

$$p(t,u) = \begin{cases} \alpha(t), \text{ if } u(t) < \alpha(t) \\ u(t), \text{ if } \alpha(t) \le u(t) \le \beta(t) \\ \beta(t), \text{ if } u(t) > \beta(t) \end{cases}$$

Lemma 3.8. *For $u : [0,T] \to \mathbb{R}$ g-absolutely continuous the following properties hold:*

(1) *$p'_g(t,u)$ exists for μ_g-a.e. $t \in [0,T]$;*

(2) *if the sequence* $(u_n)_{n\in\mathbb{N}} : [0,T] \to \mathbb{R}$ *of g-absolutely continuous functions converges to u uniformly, then*

$$p(t,u_n) \to p(t,u) \quad \text{uniformly} \quad \text{on } [0,T]$$

and

$$p'_g(t,u_n) \to p'_g(t,u) \text{ for } \mu_g - \ a.e.\ t \in [0,T].$$

Proof. Observe that

$$p(t,u) = \max\{-u(t) + \alpha(t), 0\} - \max\{u(t) - \beta(t), 0\} + u(t),$$

therefore the assertion follows by Lemma 3.7. □

We have one more step to get the main result. We prove that the following modified problem

$$\begin{cases} -u''_g(t) + A^2 u(t) = \tilde{f}(t, p(t,u), p'_g(t,u)) + A^2 p(t,u), & \mu_g \text{ −a.e. in } [0,T], \\ u(0) = u(T), u'_g(0) = u'_g(T) \end{cases} \tag{3.7}$$

with

$$A = \frac{1}{\mu_g([0,T]) + 1}$$

and $\tilde{f} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$,

$$\tilde{f}(t,w,v) = \begin{cases} f(t,w,N), \text{ if } v > N, \\ f(t,w,v), \text{ if } -N \le v \le N, \\ f(t,w,-N), \text{ if } v < -N \end{cases}$$

has at least one solution $u \in \mathcal{BC}^1_g([0,T])$ such that u'_g is g-absolutely continuous.

The constant N is chosen such that

$$N > \max\{\|\alpha'_g\|_C, \|\beta'_g\|_C\}$$

and

$$\int_{\eta_N}^{N} \frac{\tau^{\frac{q-1}{q}}}{\chi(\tau)} \mathrm{d}\tau > 2\left(\sup_{t\in[0,T]} \beta(t) - \inf_{t\in[0,T]} \alpha(t) + \frac{\mu_g([0,T))\eta_N}{2}\right)^{\frac{q-1}{q}} \cdot \|h\|_q$$

(the choice is possible thanks to the Nagumo condition (H3)).

Lemma 3.9. *Let* $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfy the hypotheses* (H1)–(H4).

Then there exists at least one solution $u \in \mathcal{BC}^1_g([0,T])$ *of* (3.7) *with* u'_g *being g-absolutely continuous.*

Proof. To this aim, consider the operator $T : \mathcal{BC}^1_g([0,T]) \to \mathcal{BC}^1_g([0,T])$ given by

$$T(x) = u$$

where u is the solution of

$$\begin{cases} -u''_g(t) + A^2 u(t) = \tilde{f}(t, p(t,x), p'_g(t,x)) + A^2 p(t,x), & \mu_g \text{ −a.e. in } [0,T], \\ u(0) = u(T), u'_g(0) = u'_g(T). \end{cases}$$

Theorem 3.1 yields (for $\lambda_1 = A$ and $\lambda_2 = -A$) that T is well-defined and $T(x) \in \mathcal{BC}^1_g$.

Let us remark that, by Lemma 3.8, $p'_g(t,x)$ is known to exist μ_g-almost everywhere on $[0,T]$; moreover $p(t,x)$ is g-absolutely continuous so the g-derivative is LS-integrable w.r.t. g on $[0,T]$ by the Fundamental Theorem of Calculus.

We prove that the operator T is compact.

Let $(x_m)_m \subset \mathcal{BC}^1_g([0,T])$ be convergent (in the topology of $\mathcal{BC}^1_g([0,T])$) to $x \in \mathcal{BC}^1_g([0,T])$. Lemma 3.8 implies that

$$p(t,x_m) \to p(t,x) \text{ uniformly on } [0,T]$$

and

$$p'_g(t,x_m) \to p'_g(t,x) \text{ for } \mu_g - \ a.e.\ t \in [0,T].$$

The hypothesis $(H2)$ implies that $\left(\tilde{f}(t,p(t,x_m),p'_g(t,x_m)) + A^2 p(t,x_m)\right)_m$ converges μ_g-a.e. to $\tilde{f}(t,p(t,x),p'_g(t,x)) + A^2 p(t,x)$.

Note that for every $m \in \mathbb{N}$ and $t \in [0,T]$,

$$\begin{aligned} &\left|\tilde{f}(t,p(t,x_m),p'_g(t,x_m)) + A^2 p(t,x_m)\right| \\ &\le A^2 \max\{|\alpha(t)|,|\beta(t)|\} + \begin{cases} h(t)\chi(N), \text{ if } |p'_g(t,x_m)| > N, \\ h(t)\chi(|p'_g(t,x_m)|), \text{ if } -N \le p'_g(t,x_m) \le N \end{cases}, \end{aligned} \tag{3.8}$$

χ is bounded on $[-N,N]$, $h \in L^q_g([0,T]) \subset L^1_g([0,T])$ and the map $t \to \max\{|\alpha(t)|,|\beta(t)|\}$ is bounded.

We may thus apply Remark 3.2(iii) to infer that $(T(x_m))_m$ converges to $T(x)$ in the topology of $\mathcal{BC}^1_g([0,T])$ and so, T is continuous.

Remark 3.2(i) immediately involves that T maps any bounded subset $\mathcal{K} \subset \mathcal{BC}^1_g([0,T])$ into a bounded set.

Moreover, it can be seen by Lemma 2.1 that $T(\mathcal{K})$ is relatively compact. Indeed, both for $T(\mathcal{K})$ and the set of the g-derivatives of the elements of $T(\mathcal{K})$ the pointwise boundedness is obvious, while the equiregulatedness comes from Remark 3.2(ii) since for any $x \in \mathcal{K}$ and $t \in [0,T]$, (3.8) holds.

Schauder's Fixed Point Theorem yields that T possesses fixed points, thus there exists $u \in \mathcal{BC}^1_g([0,T])$ such that $u = T(u)$.

Obviously, u is a solution of (3.7). □

We arrive at the main result.

Theorem 3.10. *Let the function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the hypotheses* (H1)–(H4).

Moreover, assume that for any solution u of (3.7) *the following conditions hold at each point $t \in D_g$:*

(i) *if $u(t) \le \beta(t)$ and $u'_g(t) \ge \beta'_g(t)$, then $u'_g(t+) \ge \beta'_g(t+)$;*
(ii) *if $u(t) \ge \alpha(t)$ and $u'_g(t) \le \alpha'_g(t)$, then $u'_g(t+) \le \alpha'_g(t+)$.*

Then there exists a solution u of (3.5) *such that $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in [0,T]$.*

Proof. Step I. Let us prove that for every $t \in [0,T]$, any solution of the modified problem (3.7) satisfies $\alpha(t) \le u(t) \le \beta(t)$.

Let u be such a solution and suppose $u(t) > \beta(t)$ on $[0,T]$. Then $p(t,u) = \beta(t)$ and u satisfies the equation

$$-u_g''(t) + A^2 u(t) = f(t,\beta,\beta_g') + A^2\beta(t) \le -\beta_g''(t) + A^2\beta(t),\ \mu_g - a.e. \text{ on } [0,T].$$

So $u_g''(t) - \beta_g''(t) > 0$, whence $u_g' - \beta_g'$ is strictly increasing on the entire interval (see Lemma 3.4), which is impossible since $u_g'(0) = u_g'(T)$ and $\beta_g'(0) \le \beta_g'(T)$.

Thus there exists $\tau \in [0,T]$ such that $u(\tau) \le \beta(\tau)$.

Suppose $u(T) > \beta(T)$. Then, as u is left-continuous, it satisfies the same inequality on a nonempty interval $(T-\delta, T)$ and denoting by

$$t^* = \inf\{t \in [0,T]; u(t) > \beta(t) \text{ on } [t,T]\},$$

we can obviously say that $t^* \in (0,T)$ and $u(t^*) \le \beta(t^*)$. It follows by

$$-u_g''(t) + A^2 u(t) = f(t,\beta,\beta_g') + A^2\beta(t) \le -\beta_g''(t) + A^2\beta(t),\ \mu_g - a.e. \text{ on } (t^*,T]$$

that $u_g'' > \beta_g''$ on $(t^*,T]$, whence $u_g' - \beta_g'$ is strictly increasing on this interval.

In the case $t^* \notin D_g$, this implies

$$u_g'(t^*) - \beta_g'(t^*) < u_g'(T) - \beta_g'(T); \tag{3.9}$$

otherwise (i.e., $t \in D_g$), we get

$$u_g'(t^*+) - \beta_g'(t^*+) \le u_g'(T) - \beta_g'(T). \tag{3.10}$$

As $0 \notin D_g$ and $u(0) = u(T)$, $\beta(0) = \beta(T)$, we have $u(0) > \beta(0)$ and obviously it satisfies the same inequality on a nonempty interval $(0,\overline{\delta})$; denoting by

$$t^{**} = \sup\{t \in [0,T]; u(t) > \beta(t) \text{ on } [0,t]\},$$

we can see that $t^{**} \in (0,T)$ and $u(t^{**}) \le \beta(t^{**})$. As before, $u_g'' > \beta_g''$ on $[0,t^{**})$ involving $u_g' - \beta_g'$ strictly increasing on this interval and

$$u_g'(0) - \beta_g'(0) < u_g'(t^{**}) - \beta_g'(t^{**}). \tag{3.11}$$

In the case where $t^* \notin D_g$, combining (3.9) and (3.11),

$$u_g'(t^*) - \beta_g'(t^*) < u_g'(T) - \beta_g'(T) \le u_g'(0) - \beta_g'(0) < u_g'(t^{**}) - \beta_g'(t^{**})$$

and by the facts that the definition of the g-derivative yields

$$u_g'(t^*) - \beta_g'(t^*) \ge 0 \quad \text{and} \quad u_g'(t^{**}) - \beta_g'(t^{**}) \le 0$$

we get a contradiction.

In the case $t^* \in D_g$, a contradiction is again achieved combining (3.10) and (3.11):

$$u_g'(t^*+) - \beta_g'(t^*+) \le u_g'(T) - \beta_g'(T) \le u_g'(0) - \beta_g'(0) < u_g'(t^{**}) - \beta_g'(t^{**})$$

with the fact that, as a consequence of our hypothesis,

$$u_g'(t^*+) - \beta_g'(t^*+) \ge 0. \tag{3.12}$$

From all these considerations we conclude that $u(T) \le \beta(T)$. Suppose in the sequel that one can find $t_0 \in (0,T)$ satisfying $u(t_0) > \beta(t_0)$; then as before by the left-continuity, it satisfies the same inequality on a nonempty interval $(t_0-\delta,t_0)$ and denoting by

$$\tilde{t}^* = \inf\{t\in[0,t_0]; u(t)>\beta(t) \text{ on } [t,t_0]\},$$

we have that $\tilde{t}^*>0$ and $u\left(\tilde{t}^*\right)\le\beta\left(\tilde{t}^*\right)$. It follows that $u''_g>\beta''_g$ on $\left(\tilde{t}^*,t_0\right]$, whence $u'_g-\beta'_g$ is strictly increasing on this interval and so,

$$u'_g\left(\tilde{t}^*\right)-\beta'_g\left(\tilde{t}^*\right)<u'_g(t_0)-\beta'_g(t_0) \tag{3.13}$$

in the case $\tilde{t}^*\notin D_g$, respectively

$$u'_g\left(\tilde{t}^*+\right)-\beta'_g\left(\tilde{t}^*+\right)\le u'_g(t_0)-\beta'_g(t_0) \tag{3.14}$$

in the case $\tilde{t}^*\in D_g$.

If $t_0\notin D_g$ then we can do the same reasoning at the right of t_0 and denoting by

$$\tilde{t}^{**} = \sup\{t\in[t_0,T]; u(t)>\beta(t) \text{ on } [t_0,t]\},$$

we can see that $\tilde{t}^{**}<T$, $u\left(\tilde{t}^{**}\right)\le\beta\left(\tilde{t}^{**}\right)$, that $u''_g>\beta''_g$ on $\left[t_0,\tilde{t}^{**}\right)$ (whence $u'_g-\beta'_g$ is strictly increasing on this interval) and

$$u'_g(t_0)-\beta'_g(t_0)<u'_g\left(\tilde{t}^{**}\right)-\beta'_g\left(\tilde{t}^{**}\right).$$

But since $u'_g\left(\tilde{t}^*\right)$, $u'_g\left(\tilde{t}^{**}\right)$, $\beta'_g\left(\tilde{t}^*\right)$ and $\beta'_g\left(\tilde{t}^{**}\right)$ exist, by definition

$$u'_g\left(\tilde{t}^*\right)-\beta'_g\left(\tilde{t}^*\right)\ge 0 \quad\text{and}\quad u'_g\left(\tilde{t}^{**}\right)-\beta'_g\left(\tilde{t}^{**}\right)\le 0.$$

In the case $\tilde{t}^*\notin D_g$, this together with (3.13) contradicts the hypothesis.

In the case $\tilde{t}^*\in D_g$, as before, as a consequence of the hypothesis (i), it follows that

$$u'_g\left(\tilde{t}^*+\right)-\beta'_g\left(\tilde{t}^*+\right)\ge 0$$

and this together with (3.14) give us a contradiction.

If instead $t_0\in D_g$, then again (from (3.13) when $\tilde{t}^*\notin D_g$, respectively from (3.14) when $\tilde{t}^*\in D_g$), $u'_g(t_0)\ge\beta'_g(t_0)$ and since

$$u'_g(t_0)=\frac{u(t_0+)-u(t_0)}{g(t_0+)-g(t_0)} \quad\text{and}\quad \beta'_g(t_0)=\frac{\beta(t_0+)-\beta(t_0)}{g(t_0+)-g(t_0)}$$

it follows that $u(t_0+)-\beta(t_0+)\ge u(t_0)-\beta(t_0)>0$.

Denoting by

$$\tilde{\tilde{t}}^{**} = \sup\{t\in[t_0,T]; u(t)>\beta(t) \text{ on } [t_0,t]\},$$

we note that $\tilde{\tilde{t}}^{**}<T$, $u\left(\tilde{\tilde{t}}^{**}\right)\le\beta\left(\tilde{\tilde{t}}^{**}\right)$, that $u''_g>\beta''_g$ on $\left[t_0,\tilde{\tilde{t}}^{**}\right)$ (whence $u'_g-\beta'_g$ is strictly increasing on this interval) and

$$u'_g(t_0)-\beta'_g(t_0)<u'_g\left(\tilde{\tilde{t}}^{**}\right)-\beta'_g\left(\tilde{\tilde{t}}^{**}\right).$$

Again, when $\tilde{t}^*\notin D_g$ this together with (3.13) leads to a contradiction coming from the remark that $u'_g\left(\tilde{t}^*\right)-\beta'_g\left(\tilde{t}^*\right)\ge 0$ and $u'_g\left(\tilde{\tilde{t}}^{**}\right)-\beta'_g\left(\tilde{\tilde{t}}^{**}\right)\le 0$.

When $\tilde{t}^*\in D_g$ we act similarly using (3.14).

In conclusion, $u(t)\le\beta(t)$ on the whole interval $[0,T]$ (and, obviously, in the same way it can be proved that $u(t)\ge\alpha(t)$ on $[0,T]$).

Step II. Let us check that $|u_g'(t)| \leq N$ on the whole interval.

Suppose there is $t_2 \in [0,T]$ such that $u_g'(t_2) > N$ (the case $u_g'(t_2) < -N$ is similar). We claim that one can find $t_1 \in [0,T]$ satisfying $u_g'(t_1) \in [0, \eta_N]$.

Indeed, if it were supposed that $u_g' > \eta_N$ on $[0,T]$ or that $u_g' < 0$ on $[0,T]$ then we would contradict

$$\int_{[0,T)} u_g'(s)\mathrm{d}g(s) = u(T) - u(0) = 0.$$

So, there is a point $t' \in [0,T]$ where $u_g'(t') > \eta_N$ and there are points where $u_g' < 0$; considering (by the left-continuity of u_g')

$$t'' = \inf\{t \in [0,t'] : u_g' > \eta_N \text{ on } [t,t']\},$$

in the case $t'' > 0$ we will have that $u_g'(t'') \leq \eta_N$. If $u_g'(t'') \in [0,\eta_N]$, it is done (let $t_1 = t''$). If not, then $t'' \in D_g$ and by (3.6)

$$\begin{aligned} u_g''(t'') &= \frac{u_g'(t''+) - u_g'(t'')}{\mu_g(\{t''\})} > \frac{\eta_N}{\mu_g(\{t''\})} \\ &= \frac{\|h\|_1 \sup_{v\in[-N,N]} \chi(|v|)}{\mu_g(\{t''\})} \geq h(t'') \sup_{v\in[-N,N]} \chi(|v|) \end{aligned}$$

which contradicts the facts that u is a solution of (3.7) and on the whole $[0,T]$, $u(t) \in [\alpha(t), \beta(t)]$ (which give us $|u_g''(t)| \leq h(t) \sup\{|\chi(v)| : v \in [-N,N]\}$).

In the case $t'' = 0$, it means that on $[0,t']$ the function satisfies $u_g' > \eta_N$, so there is a point $\tilde{t}' > t'$ where $u_g'(\tilde{t}') < 0$ and we arrive at a contradiction in the same way, since u_g' is left-continuous.

So we found $t_1, t_2 \in [0,T]$ with $u_g'(t_2) > N$ and $u_g'(t_1) \in [0,\eta_N]$; suppose $t_1 < t_2$ and from their construction we see that $u_g' > 0$ on $[t_1,t_2]$.

Considering now

$$\bar{t}_2 = \inf\{t \in [0,t_2] : u_g' > N \text{ on } [t,t_2]\},$$

obviously $u_g'(\bar{t}_2) \leq N$.

Case I. Suppose $u_g'(\bar{t}_2) = N$. If $q > 1$, then by Lemma 2.4,

$$\begin{aligned} \int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u &\leq \int_{u_g'(t_1)}^{u_g'(\bar{t}_2)} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u \\ &= \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{|u_g'(t) + \tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u_g'(t) + \tau u_g''(t)\mu_g(\{t\})|)}\mathrm{d}\tau \right) \cdot u_g''(t)\mathrm{d}g(t) \\ &\leq \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{|u_g'(t) + \tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u_g'(t) + \tau u_g''(t)\mu_g(\{t\})|)} \cdot |f(t,u(t),u_g'(t))|\mathrm{d}\tau \right) \mathrm{d}g(t) \\ &\leq \int_{[t_1,\bar{t}_2)} \left(\int_0^1 |u_g'(t) + \tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}} \cdot h(t)\mathrm{d}\tau \right) \mathrm{d}g(t) \end{aligned}$$

by $(H3)$ since

$$|u_g'(t) + \tau u_g''(t)\mu_g(\{t\}) - u_g'(t)| \leq |f(t,u(t),u_g'(t))| \cdot \mu_g(\{t\}).$$

It follows that

$$\begin{aligned}
\int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u &\le \int_{[t_1,\bar{t}_2)} \left(\int_0^1 |u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}}\mathrm{d}\tau\right)\cdot h(t)\mathrm{d}g(t)\\
&\le \int_{[t_1,\bar{t}_2)} \left(\int_0^1 |u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|\mathrm{d}\tau\right)^{\frac{q-1}{q}}\cdot h(t)\mathrm{d}g(t)\\
&\le \left[\int_{[t_1,\bar{t}_2)} \left(\int_0^1 |u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|\mathrm{d}\tau\right)\mathrm{d}g(t)\right]^{\frac{q-1}{q}}\cdot\left(\int_{[t_1,\bar{t}_2)} h^q(t)\mathrm{d}g(t)\right)^{\frac{1}{q}}\\
&\le \left[\int_{[t_1,\bar{t}_2)} \left(\int_0^1 u_g'(t)+\tau |u_g''(t)|\mu_g(\{t\})\mathrm{d}\tau\right)\mathrm{d}g(t)\right]^{\frac{q-1}{q}}\cdot\|h\|_q\\
&\le \left[\int_{[t_1,\bar{t}_2)} \left(u_g'(t)+\frac{1}{2}h(t)\cdot\sup_{v\in[-N,N]}\chi(|v|)\mu_g([0,T))\right)\mathrm{d}g(t)\right]^{\frac{q-1}{q}}\cdot\|h\|_q\\
&\le \left(\sup_{t\in[0,T]}\beta(t)-\inf_{t\in[0,T]}\alpha(t)+\frac{\mu_g([0,T))\eta_N}{2}\right)^{\frac{q-1}{q}}\cdot\|h\|_q.
\end{aligned}$$

If $q=1$, similarly one has

$$\begin{aligned}
\int_{\eta_N}^{N} \frac{1}{\chi(|u|)}\mathrm{d}u &\le \int_{u_g'(t_1)}^{u_g'(\bar{t}_2)} \frac{1}{\chi(|u|)}\mathrm{d}u\\
&= \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{1}{\chi(|u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|)}\mathrm{d}\tau\cdot u_g''(t)\right)\mathrm{d}g(t)\\
&\le \int_{[t_1,\bar{t}_2)} h(t)\mathrm{d}g(t)\le\|h\|_1.
\end{aligned}$$

In both cases, we contradict the choice of N.

Case II. If $u_g'(\bar{t}_2)<N$ then in the case where $q>1$ (and similarly when $q=1$)

$$\begin{aligned}
\int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u &< \int_{u_g'(t_1)}^{u_g'(t_2)} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u\\
&= \int_{u_g'(t_1)}^{u_g'(\bar{t}_2)} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u + \int_{u_g'(\bar{t}_2)}^{u_g'(t_2)} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u.
\end{aligned}$$

It follows by Lemma 2.4 that

$$\begin{aligned}
\int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u &< \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{|u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u_g''(t)\mathrm{d}g(t)\\
&+ \int_{[\bar{t}_2,t_2)} \left(\int_0^1 \frac{|u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u_g'(t)+\tau u_g''(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u_g''(t)\mathrm{d}g(t)
\end{aligned}$$

whence

$$\int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u \leq \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u''_g(t)\mathrm{d}g(t)$$

$$+\lim_{t_2\to\bar{t}_2,\ t_2>\bar{t}_2} \int_{[\bar{t}_2,t_2)} \left(\int_0^1 \frac{|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u''_g(t)\mathrm{d}g(t)$$

$$= \int_{[t_1,\bar{t}_2)} \left(\int_0^1 \frac{|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u''_g(t)\mathrm{d}g(t)$$

$$+\int_{\{\bar{t}_2\}} \left(\int_0^1 \frac{|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|^{\frac{q-1}{q}}}{\chi(|u'_g(t)+\tau u''_g(t)\mu_g(\{t\})|)}\mathrm{d}\tau\right)\cdot u''_g(t)\mathrm{d}g(t).$$

Consequently, as before,

$$\int_{\eta_N}^{N} \frac{|u|^{\frac{q-1}{q}}}{\chi(|u|)}\mathrm{d}u$$

$$\leq 2\left(\sup_{t\in[0,T]}\beta(t) - \inf_{t\in[0,T]}\alpha(t) + \frac{\mu_g([0,T))\eta_N}{2}\right)^{\frac{q-1}{q}} \cdot \|h\|_q$$

and again we get to a contradiction.

In conclusion u is a solution of (3.5). □

A remark about the assumptions (i) and (ii) of the previous theorem: their aim is to ensure the control of the jumps of (the first derivative of) solutions at the discontinuity points in connection with the jumps of (the first derivative of) the lower and upper solutions; of course, they are superfluous when g is continuous (in particular, when the usual derivative is involved).

Similar hypotheses were used for first order measure differential equations in [20] (to obtain extremal solutions), in [26, Theorem 2.5] and also for Stieltjes differential equations or systems in [12,13] and they are necessary (as it can be seen from [20, Example 4.2]). In the set-valued framework, for first order Stieltjes differential problems such hypotheses can be found in [18].

As far as the authors know, here is the first time where this kind of assumptions are imposed for second order differential equations with Stieltjes derivative.

Now we are going to present sufficient conditions in order that (i), (ii) are satisfied.

Proposition 3.11. *The hypotheses* (i) *and* (ii) *in the previous theorem are satisfied if we assume that for every* $t\in D_g$*, the following conditions hold:*

1. *for every solution* $\overline{u}$ *of* (3.7) *and any* $u\in[\alpha(t),\beta(t)]$*,*

(i') *if* $Au-u_1(t)\leq-\beta'_g(t)$ *then*

$$Au-u_1(t)+\mu_g(\{t\})\cdot\tilde{f}(t,u,-u+u_1(t))\leq-\beta'_g(t)+\mu_g(\{t\})\cdot f(t,\beta(t),\beta'_g(t));$$

(ii') *if* $Au-u_1(t)\geq-\alpha'_g(t)$ *then*

$$Au-u_1(t)+\mu_g(\{t\})\cdot\tilde{f}(t,u,-u+u_1(t))\geq-\alpha'_g(t)+\mu_g(\{t\})\cdot f(t,\alpha(t),\alpha'_g(t));$$

here u_1 *is the solution of*

$$\begin{cases} u'_g(t) - Au(t) = \tilde{f}(t, p(t, \overline{u}), p'_g(t, \overline{u})) + A^2 p(t, \overline{u}), \quad \mu_g \text{ -a.e. in } [0,T] \\ u(0) = u(T) \end{cases}$$

given by Theorem 2.5.

2. *For every* $u < \alpha(t)$*, if* $Au - u_1^\alpha(t) \leq -\beta'_g(t)$ *then*

$$Au - u_1^\alpha(t) + \mu_g(\{t\}) \cdot \tilde{f}(t, u, -u + u_1^\alpha(t)) \leq -\beta'_g(t) + \mu_g(\{t\}) \cdot f(t, \beta(t), \beta'_g(t));$$

here u_1^α *is the solution of*

$$\begin{cases} u'_g(t) - Au(t) = \tilde{f}(t, \alpha(t), \alpha'_g(t)) + A^2 \alpha(t), \quad \mu_g \text{ -a.e. in } [0,T] \\ u(0) = u(T) \end{cases}$$

given by Theorem 2.5.

3. *For every* $u > \beta(t)$*, if* $Au - u_1^\beta(t) \leq -\alpha'_g(t)$ *then*

$$Au - u_1^\beta(t) + \mu_g(\{t\}) \cdot \tilde{f}(t, u, -u + u_1^\beta(t)) \geq -\alpha'_g(t) + \mu_g(\{t\}) \cdot f(t, \alpha(t), \alpha'_g(t));$$

here u_1^β *is the solution of*

$$\begin{cases} u'_g(t) - Au(t) = \tilde{f}(t, \beta(t), \beta'_g(t)) + A^2 \beta(t), \quad \mu_g \text{ -a.e. in } [0,T] \\ u(0) = u(T) \end{cases}$$

given by Theorem 2.5.

Proof. Indeed, we only need to prove (3.12). We know that $u(t^*) \leq \beta(t^*)$; we could encounter two situations.

In the first one, $u(t^*) \in [\alpha(t^*), \beta(t^*)]$. As in the proof of Theorem 3.1, the solution u verifies

$$u'_g(t^*) + Au(t^*) = u_1(t^*)$$

and so, the condition

$$Au(t^*) - u_1(t^*) = -u'_g(t^*) \leq -\beta'_g(t^*)$$

is satisfied and then by hypothesis $1.i')$

$$\begin{aligned} Au(t^*) - u_1(t^*) + \mu_g(\{t^*\}) \cdot \tilde{f}(t^*, u(t^*), -u(t^*) + u_1(t^*)) \\ \leq -\beta'_g(t^*) + \mu_g(\{t^*\}) \cdot f(t^*, \beta(t^*), \beta'_g(t^*)), \end{aligned}$$

whence

$$\begin{aligned} -u'_g(t^*) + \mu_g(\{t^*\}) \cdot \tilde{f}(t^*, u(t^*), u'_g(t^*)) &\leq -\beta'_g(t^*) + \mu_g(\{t^*\}) \cdot f(t^*, \beta(t^*), \beta'_g(t^*)) \\ &\leq -\beta'_g(t^*) - \mu_g(\{t^*\}) \cdot \beta''_g(t^*) \end{aligned}$$

i.e., (by the definition of the g-derivative at a point in D_g),

$$u'_g(t^*+) \geq \beta'_g(t^*+).$$

In the second situation, $u(t^*) < \alpha(t^*)$. Then

$$-u''_g(t^*) + A^2 u(t^*) = f(t^*, \alpha(t^*), \alpha'_g(t^*)) + A^2 \alpha(t^*)$$

so, as in the proof of Theorem 3.1,

$$u'_g(t^*) + Au(t^*) = u_1^\alpha(t^*).$$

We can see that the condition

$$Au(t^*) - u_1^\alpha(t^*) = -u_g'(t^*) \leq -\beta_g'(t^*)$$

is satisfied and then by hypothesis 2.

$$\begin{aligned} &Au(t^*) - u_1^\alpha(t^*) + \mu_g(\{t^*\}) \cdot \tilde{f}(t^*, u(t^*), -u(t^*) + u_1^\alpha(t^*)) \\ &\quad \leq -\beta_g'(t^*) + \mu_g(\{t^*\}) \cdot f(t^*, \beta(t^*), \beta_g'(t^*)), \end{aligned}$$

whence

$$\begin{aligned} -u_g'(t^*) + \mu_g(\{t^*\}) \cdot \tilde{f}(t^*, u(t^*), u_g'(t^*)) &\leq -\beta_g'(t^*) + \mu_g(\{t^*\}) \cdot f(t^*, \beta(t^*), \beta_g'(t^*)) \\ &\leq -\beta_g'(t^*) - \mu_g(\{t^*\}) \cdot \beta_g''(t^*) \end{aligned}$$

i.e., (since $t^* \in D_g$),

$$u_g'(t^*+) \geq \beta_g'(t^*+).$$

□

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References

[1] Cabada, A.: An overview of the lower and upper solutions method with nonlinear boundary value conditions. Bound. Value Probl. **2011**, Article ID 893753, 18 pages (2011)

[2] Di Piazza, L., Marraffa, V., Satco, B.: Measure differential inclusions: existence results and minimum problems. Set-Valued Var. Anal. **29**(2), 361–382 (2021)

[3] Di Piazza, L., Marraffa, V., Satco, B.: Approximating the solutions of differential inclusions driven by measures. Ann. Mat. Pura Appl. **198**, 2123–2140 (2019)

[4] Federson, M., Mesquita, J.G., Slavík, A.: Measure functional differential equations and functional dynamic equations on time scales. J. Differ. Equ. **252**, 3816–3847 (2012)

[5] Fernández, F.J., Marquéz Albés, I., Tojo, F.A.F.: On first and second order linear Stieltjes differential equations. J. Math. Anal. Appl. **511**, 126010 (2022)

[6] Fernández, F.J., Marquéz Albés, I., Tojo, F.A.F.: On second order linear Stieltjes differential equations with non-constant coefficients. Open Math. **22**, 1 , Paper No. 20240018 (2024)

[7] Folland, G.B.: Real Analysis -Modern Techniques and Their Applications, 2nd edn. John Wiley & Sons, Inc., Canada (1999)

[8] Fraňková, D.: Regulated functions. Math. Bohem. **116**, 20–59 (1991)

[9] Frigon, M., Pouso, R.L.: Theory and applications of first-order systems of Stieltjes differential equations. Adv. Nonlinear Anal. **6**, 13–36 (2017)

[10] Gao, W., Wang, J.: On a nonlinear second order periodic boundary value problem with Carathéodory functions. Ann. Polon. Math. **LXII**(3), 283–291 (1995)

[11] Kurzweil, J.: Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. **7**, 418–449 (1957)

[12] Maia, L., El Khattabi, N., Frigon, M.: Existence and multiplicity results for first-order Stieltjes differential equations. Adv. Nonlinear Stud. **22**, 648–710 (2022)

[13] Maia, L., El Khattabi, N., Frigon, M.: Systems of Stieltjes differential equations and applications to a predator-prey model of an exploited fishery. Discrete Cont. Dyn. Syst. **43**, 4244–4271 (2023)

[14] Marquéz Albés, I.: Differential problems with Stieltjes derivatives and applications. Ph.D thesis, Universidade de Santiago de Compostela (2021) https://minerva.usc.es/xmlui/handle/10347/24663

[15] Marquéz Albés, I., Slavík, A.: Generalized chain rules and applications to Stieltjes differential and integral equations. Results Math. **79**, 78 (2024)

[16] Marraffa, V., Satco, B.: Relaxation results for differential inclusions with Stieltjes derivative. J. Math. Anal. Appl. **528**(127533), 22 (2023)

[17] Marraffa, V., Satco, B.: Relaxation theorem for Stieltjes differential inclusions on infinite intervals. Mediterr. J. Math. **20**(331), 23 (2023)

[18] Marraffa, V., Satco, B.: Stieltjes differential problems with general boundary value conditions. Existence and bounds of solutions. Czech. Math. J. (2024). https://doi.org/10.21136/CMJ.2024.0125-23 https://doi.org/10.21136/CMJ.2024.0125-23

[19] Monteiro, G.A., Satco, B.: Distributional, differential and integral problems: equivalence and existence results. Electron. J. Qual. Theory Differ. Equ. **7**, 1–26 (2017)

[20] Monteiro, G.A., Slavík, A.: Extremal solutions of measure differential equations. J. Math. Anal. Appl. **444**, 568–597 (2016)

[21] Monteiro, G.A., Slavík, A., Tvrdy, M.: Kurzweil–Stieltjes Integral, Theory and Its Applications in Series in Real Analysis, vol. 15. World Scientific, Singapore (2018)

[22] Nagumo, M.: Uber die differentialgleichung $y'' = f(x, y, y')$. Proc. Phys. Math. Soc. Japan **19**, 861–866 (1937)

[23] Picard, E.: Mémoire sur la théorie des équations aux derivés partielles et la méthode des approximations succesives. J. Math. Pures Appl. **6**, 145–210 (1890)

[24] Picard, E.: Sur l'application des méthodes d'approximations succesives à l'étude de certains équations différentielles ordinaires. J. Math. Pures Appl. **9**, 217–271 (1893)

[25] Pouso, R.L., Rodriguez, A.: A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives. Real Anal. Exchange **40**, 319–353 (2015)

[26] Pouso, R.L., Marquéz Albés, I.: General existence principles for Stieltjes differential equations with applications to mathematical biology. J. Differ. Equ. **264**, 5388–5407 (2018)

[27] Pouso, R.L., Marquéz Albés, I.: Resolution methods for mathematical models based on differential equations with Stieltjes derivatives. Electron. J. Qual. Theory Differ. Equ. **2019**, 1–15 (2019)

[28] Satco, B.: Nonlinear Volterra integral equations in Henstock integrability setting. Electron. J. Differ. Equ. **39**, 1–9 (2008)

[29] Satco, B.: Continuous dependence results for set-valued measure differential problems. Electron. J. Qual. Theory Differ. Equ. **79**, 1–15 (2015)

[30] Satco, B.: Ulam-type stability for differential equations driven by measures. Math. Nachr. **293**(1), 147–157 (2020)

[31] Satco, B., Smyrlis, G.: Periodic boundary value problems involving Stieltjes derivatives. J. Fixed Point Theory Appl. **22**(4), Art. No. 24, 23 pp (2020)

[32] Satco, B., Smyrlis, G.: Applications of Stieltjes derivatives to periodic boundary value inclusions. Mathematics **8**, 2142 (2020)

[33] Satco, B., Smyrlis, G.: Viability and Filippov-type lemma for Stieltjes differential inclusions. Set-Valued Var. Anal. **31**(3), 23 (2023)

[34] Slavík, A.: Dynamic equations on time scales and generalized ordinary differential equations. J. Math. Anal. Appl. **385**, 534–550 (2012)

[35] Slavík, A.: Well-posedness results for abstract generalized differential equations and measure functional differential equations. J. Differ. Equ. **259**, 666–707 (2015)

[36] Young, W.H.: On integrals and derivatives with respect to a function. Proc. Lond. Math. Soc. **1**(s2–15), 35–63 (1917)

[37] Wang, M.-X., Cabada, A., Nieto, J.J.: Monotone method for nonlinear second order periodic boundary value problem with Carathéodory functions. Ann. Polon. Math. **LVII**(3), 221–235 (1993)

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