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Variational Methods on Finite Dimensional Banach Spaces and Discrete Problems*

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Abstract

In this paper, existence and multiplicity results for a class of second-order difference equations are established. In particular, the existence of at least one positive solution without requiring any asymptotic condition at infinity on the nonlinear term is presented and the existence of two positive solutions under a superlinear growth at infinity of the nonlinear term is pointed out. The approach is based on variational methods and, in particular, on a local minimum theorem and its variants. It is worth noticing that, in this paper, some classical results of variational methods are opportunely rewritten by exploiting fully the finite dimensional framework in order to obtain novel results for discrete problems.

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1 Introduction

The aim of this paper is to study the existence of non-zero solutions for the following second-order discrete boundary value problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f_k(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \qquad (D_\lambda^f)$$

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where N is a positive integer, $[1, N]$ denotes the discrete interval $\{1, ..., N\}$, and, for every $k \in [1, N]$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\Delta^2 u(k-1) := u(k+1) - 2u(k) + u(k-1)$ is the second order difference operator, $f_k : \mathbb{R} \to \mathbb{R}$ are the components of a continuous function $\underline{f} : \mathbb{R} \to \mathbb{R}^N$ and λ is a positive real parameter.

For general references on difference equations and their applications, we refer the reader to monograph [22] and for an overview on applied methods, as upper and lower solutions and fixed point theory, we also cite [2, 20] and the references therein. Recently, existence results for difference equations have been obtained by using variational methods (see, for instance, [7, 9, 12, 13, 14, 15, 23]). To be precise, the main tools used in order to obtain the existence of one non-zero solution have been the direct methods theorem and the mountain pass theorem.

In this paper, again in the framework of variational methods, we use a local minimum theorem established in [4] (see [4, Theorem 3.1]) to obtain the existence of one positive solution for (D^f_λ) (see, as an example, Theorem 1.1 below). Moreover, by applying a two-critical points theorem established in [5] (see [5, Theorem 3.2]), which has been obtained by a suitable combination of such a local minimum theorem with the classical mountain pass theorem, the existence of two positive solutions to (D^f_λ) is guaranteed (see, as an example, Theorem 1.2 below).

It is also worth noticing that the local minimum theorem obtained in [4] is given in infinite dimensional spaces and it is based on the Ekeland variational principle, applied to the non-smooth analysis. Here, a version for functionals defined in finite dimensional spaces is obtained directly by using basic results instead of the Ekeland variational principle (see Theorem 3.3 and its variant Theorem 3.4). Moreover, on the mountain pass theorem some remarks are made. It is shown that one of its fundamental assumptions, the mountain pass geometry, is equivalent to the existence of a local minimum which is not strictly global (see Proposition 3.1). As a consequence, the mountain pass theorem can be directly stated by substituting the mountain pass geometry with the existence of a local minimum (see Theorem 3.2). Furthermore, the mountain pass theorem is stated distinguishing two cases according to whether the functional is bounded from below or not (see Corollaries 3.1 and 3.2). From the local minimum theorem (Theorem 3.3) and the mountain pass theorem, as given in Corollaries 3.1 and 3.2, multiple critical points theorems are obtained (see Section 4). In particular, a three-critical points theorem (Theorem 4.1) and a two-critical points theorem (Theorem 4.2) are pointed out. The proofs of these theorems are simpler than the corresponding infinite dimensional cases given in [4], [5] and [11], just because the finite dimensional setting is fully exploited.

It is also important to observe that the other main assumption of the mountain pass theorem is the Palais-Smale condition. In the applications in infinite dimensional spaces, it is satisfied by requiring a condition on the nonlinear term stronger than the superlinearity at infinity (see Remark 5.2). Here, it is proved that the superlinearity at infinity of the nonlinear datum is enough to prove the Palais-Smale condition (see Lemma 5.1). As a consequence, we get the existence of one positive solution for (D^f_λ) by requiring the superlinearity at zero and at infinity of the nonlinearity (see Corollary 5.2).

Finally, our main results are in Section 6. Precisely, we wish to highlight the results obtained in Theorems 6.1, 6.2 and 6.4. The first result gives the existence of one positive solution without requiring any asymptotic condition at infinity on the nonlinear term. It improves all that results, as for instance [18, Theorem 3.4] (see, also, Theorem 5.1), where an asymptotic condition at infinity is requested. The second one ensures the existence of one positive solution without requiring conditions neither at zero nor at infinity. As its consequence, the existence of positive solution is obtained when the nonlinear term is sublinear at zero (see Corollary 6.1). To the best of our knowledge, we do not know about results in the literature that provide solutions to discrete problems without conditions at zero or at infinity, so Theorem 6.2 seems the first result in this direction. Theorem 6.4 guarantees the existence of two solutions without requiring the usual assumption of superlinearity at zero (see, for instance, [3, Theorem 4.2]). Moreover, in the same section, multiplicity results are also pointed out. In particular, Theorem 6.3 is a more precise version of [16, Theorem 3.1] (see also [9, Theorem 49]). Theorem 6.5, which uses a better inequality with respect to [8, Theorem 3.3], allows us to obtain, as a consequence, Theorem 6.6 which ensures the existence of one positive solution without requiring that the associated functional satisfies the Palais-Smale condition (see Remark 6.6).

We also emphasize that, in this paper, a general maximum principle has been established (see Proposition 2.2). It allows to obtain positive solutions under a single condition at zero on the non-linear term.

As an example, we present here two special cases of our main results.

Theorem 1.1 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty.$$

Then, for every $\lambda \in \left]0, \bar{\lambda}\right[$*, where* $\bar{\lambda} = \dfrac{2}{N(N+1)} \sup\limits_{c>0} \dfrac{c^2}{\max\limits_{s\in[0,c]} \int_0^s f(t)dt}$*, the problem*

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{A_λ^f}$$

admits at least one positive solution.

Theorem 1.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(0) > 0$. *Assume that*

$$\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty.$$

Then, for every $\lambda \in \left]0, \bar{\lambda}\right[$*, where* $\bar{\lambda}$ *is given as in Theorem 1.1, the problem* (A_λ^f) *admits at least two positive solutions.*

In particular, Theorem 1.1 ensures the existence of at least one positive solution to (A_λ^f) for each $\lambda \in]0, \bar{\lambda}[$ requiring that f is sublinear at zero without assuming asymptotic condition at infinity. While, if f is, in addition, superlinear at $+\infty$, Theorem 1.2 ensures the existence of a second positive solution.

The paper is arranged as follows. In Section 2, we recall some preliminary results, and in Sections 3 and 4 we point out variational methods in finite dimensional spaces. In Section 5 some remarks on classical results for discrete problems are made, while Section 6 is devoted to our main results.

2 Preliminary results

Consider the N-dimensional Banach space

$$S := \{u : [0, N+1] \to \mathbb{R} : u(0) = u(N+1) = 0\},$$

endowed with the norm

$$\|u\|_2 := \Big(\sum_{k=1}^{N} u(k)^2 \Big)^{\frac{1}{2}}, \quad \forall u \in S.$$

It is known that the following norm

$$\|u\| = \left(\sum_{k=1}^{N+1}(\Delta u(k-1))^2\right)^{\frac{1}{2}}, \quad \forall u \in S$$

is an equivalent norm in S. A direct computation shows that,

$$\|u\|^2 = u^t A u, \ \forall u \in S,$$

where u^t denotes the transpose of u, and

$$A := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{N\times N}.$$

The matrix A is symmetric, positive-definite and admits N distinct positive eigenvalues given by

$$\lambda_k := 4\sin^2\frac{k\pi}{2(N+1)}, \quad \forall k \in [1,N]. \tag{2.1}$$

The previous norms are equivalent and, in particular, one has

$$\sqrt{\lambda_1}\|u\|_2 \le \|u\| \le \sqrt{\lambda_N}\|u\|_2, \ \ \forall u \in S. \tag{2.2}$$

In the sequel, we will use also the following equivalent norm,

$$\|u\|_\infty := \max_{k\in[1,N]} |u(k)|, \quad \forall u \in S.$$

One has

$$\frac{1}{\sqrt{N\lambda_N}}\|u\| \le \|u\|_\infty \le \frac{1}{\sqrt{\lambda_1}}\|u\|, \ \ \forall u \in S. \tag{2.3}$$

A more precise estimation is given by the following proposition.

Proposition 2.1 *One has*

$$\|u\|_\infty \le K_2\|u\|, \ \ \forall u \in S, \tag{2.4}$$

where

$$K_2 = \begin{cases} \dfrac{\sqrt{N+1}}{2} & \text{if } N \text{ is odd,} \\[2ex] \dfrac{1}{\left(\dfrac{2}{N}+\dfrac{2}{N+2}\right)^{\frac{1}{2}}} & \text{if } N \text{ is even.} \end{cases} \tag{2.5}$$

Proof. Fix $j \in [1,N]$. On one hand, one has

$$u(j) = (u(1)-u(0)) + (u(2)-u(1)) + \dots + (u(j)-u(j-1)) = \sum_{k=1}^{j}\Delta u(k-1),$$

$$|u(j)| \le \sum_{k=1}^{j} |\Delta u(k-1)| \le j^{\frac{1}{2}} \left(\sum_{k=1}^{j} |\Delta u(k-1)|^2 \right)^{\frac{1}{2}},$$

$$\frac{1}{j}|u(j)|^2 \le \sum_{k=1}^{j} |\Delta u(k-1)|^2. \tag{2.6}$$

On the other hand, one has

$$u(j) = -[(u(j+1)-u(j)) + (u(j+2)-u(j+1)) + ... + (u(N+1)-u(N))] = -\sum_{k=j+1}^{N+1} \Delta u(k-1),$$

$$|u(j)| \le \sum_{k=j+1}^{N+1} |\Delta u(k-1)| \le (N+1-j)^{\frac{1}{2}} \left(\sum_{k=j+1}^{N+1} |\Delta u(k-1)|^2 \right)^{\frac{1}{2}},$$

$$\frac{1}{N+1-j}|u(j)|^2 \le \sum_{k=j+1}^{N+1} |\Delta u(k-1)|^2. \tag{2.7}$$

From (2.6) and (2.7), it follows

$$\left(\frac{1}{j} + \frac{1}{N+1-j} \right) |u(j)|^2 \le \sum_{k=1}^{N+1} |\Delta u(k-1)|^2. \tag{2.8}$$

Now, we claim that $\min_{1\le j\le N} \left(\frac{1}{j} + \frac{1}{N+1-j} \right) = \frac{1}{K_2^2}$. In fact, the real function $\varphi(x) = \left(\frac{1}{x} + \frac{1}{N+1-x} \right)$ is decreasing in $\left]0, \frac{N+1}{2}\right[$, is increasing in $\left]\frac{N+1}{2}, N+1\right[$ and $\frac{N+1}{2}$ is its global minimum point. So, if N is odd, one has $\min_{1\le j\le N} \left(\frac{1}{j} + \frac{1}{N+1-j} \right) = \varphi\left(\frac{N+1}{2}\right) = \frac{4}{N+1} = \frac{1}{K_2^2}$, while, if N is even, one has $\min_{1\le j\le N} \left(\frac{1}{j} + \frac{1}{N+1-j} \right) = \varphi\left(\frac{N}{2}\right) = \varphi\left(\frac{N+2}{2}\right) = \left(\frac{2}{N} + \frac{2}{N+2}\right) = \frac{1}{K_2^2}$. Hence, our claim is proved. Therefore, from (2.8) one has $\frac{1}{K_2^2}|u(j)|^2 \le \sum_{k=1}^{N+1} |\Delta u(k-1)|^2$ and the proof is complete.

Remark 2.1 Clearly, $K_2 \le \frac{\sqrt{N+1}}{2}$ for all $N \in \mathbb{N}$. So, from previous proposition we obtain

$$\|u\|_\infty \le \frac{\sqrt{N+1}}{2}\|u\|, \quad \forall u \in S,$$

which is the inequality obtained in [16]. By a different proof, it is proved in [12] that K_2 is the best constant.

The following result is fundamental to our aims.

Proposition 2.2 *Fix $u \in S$ and assume*

$$\text{either} \quad u(k) > 0 \quad \text{or} \quad -\Delta^2 u(k-1) \ge 0 \quad \text{for each} \quad k \in [1, N]. \tag{2.9}$$

Then, either $u \equiv 0$ or $u > 0$.

Proof. Fix $u \in S$ satisfying (2.9) and such that $u \not\equiv 0$. From (2.9), if $u(j) \leq 0$, $j \in [1, N]$, one has

$$u(j+1) - u(j) \leq u(j) - u(j-1). \tag{2.10}$$

Now, we claim that $u(1) > 0$. Arguing by contradiction, we assume $u(1) \leq 0$. Applying (2.10) with $j = 1$, we obtain $u(2) - u(1) \leq u(1) - 0 \leq 0$, $u(2) \leq u(1)$. So, we have $u(2) \leq 0$ and, applying again (2.10) with $j = 2$, we obtain $u(3) \leq u(2)$. Repeating the reasoning we get $u(N+1) \leq u(N) \leq ... \leq u(1) \leq 0$ and, since $u(N+1) = 0$, we have $u \equiv 0$ and this is absurd. So our claim is proved.
Moreover, we claim that $u(2) > 0$. Arguing again by contradiction, we assume $u(2) \leq 0$ and, applying (2.10) with $j = 2$, we obtain $u(3) - u(2) \leq u(2) - u(1) < u(2) \leq 0$, $u(3) < u(2)$. So, repeating the reasoning, we get $u(N+1) < u(N) < ... < u(2) \leq 0$, that is $u(N+1) < 0$, and this is absurd, for which our claim is proved.
Repeating the same computation, we obtain $u(3) > 0$, $u(4) > 0$,..., $u(N-1) > 0$ and, finally, in the same way, we obtain $u(N) > 0$. In fact, assuming $u(N) \leq 0$, from (2.10) one has $u(N+1) - u(N) \leq u(N) - u(N-1) < u(N) \leq 0$, $u(N+1) < u(N) \leq 0$, $u(N+1) < 0$, which is absurd. Hence, the proof is complete.

Now, let $I_\lambda : S \to \mathbb{R}$ be the functional defined as follows

$$I_\lambda(u) := \frac{1}{2}\sum_{k=1}^{N+1}(\Delta u(k-1))^2 - \lambda \sum_{k=1}^{N} \int_0^{u(k)} f_k(t)dt,$$

for all $u \in S$.

We have the following result, see for instance [21].

Proposition 2.3 *The critical points of I_λ are exactly the solutions of the problem (D^f_λ).*

3 Variational methods on finite dimensional spaces

Let $(X, \|\cdot\|)$ be a finite dimensional Banach space and let $I : X \to \mathbb{R}$ be a functional on X. We recall the following classical definitions.

We say that I is coercive on X if $\lim_{\|u\|\to+\infty} I(u) = +\infty$. Moreover, if I is a continuously Gâteaux differentiable functional, we say that I satisfies the Palais-Smale condition, (in short (PS)), if any sequence $\{u_n\} \subseteq X$ such that

1. $\{I(u_n)\}$ is bounded,

2. $\{I'(u_n)\}$ is convergent to 0 in X^*,

admits a subsequence which is convergent in X.

Since here X is a finite dimensional space, it is enough to require that any sequence $\{u_n\}$ satisfying 1. and 2. admits a bounded subsequence.

Now, we recall the classical direct methods theorem, see for instance [27, Theorem 2.1].

Theorem 3.1 *Assume that I is continuous and coercive on X. Then, I admits a global minimum u_0. Moreover, if I is continuously Gâteaux differentiable on X, then u_0 is a critical point of I.*

Another fundamental tool in variational methods is the mountain pass theorem. It has been established by A. Ambrosetti and P.H. Rabinowitz (see, for instance, [1, Theorem 8.2]) and, subsequently, some important remarks on zero altitude and nature of critical points were given by P. Pucci and J. Serrin (see [24] and [25]) and N. Ghoussoub and D. Preiss (see, for instance, [19]). Its assumptions are the Palais-Smale condition and the mountain pass geometry. In [5], it was proved that the mountain pass geometry is equivalent to the existence of one or two local minima according to whether I is unbounded from below or not. Indeed, if I satisfies the Palais-Smale condition and it is bounded from below, the mountain pass geometry is equivalent to the existence of two local minima (see also [6, Theorem 2.1]), while if I satisfies the Palais-Smale condition and it is unbounded from below, the mountain pass geometry is equivalent to the existence of one local minimum. To be precise, from [5, Theorem 2.1], taking into account that here X is finite dimensional space, we obtain the following immediate consequence.

Proposition 3.1 *Let X be a real finite dimensional Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional which satisfies (PS). Then, the following assertions are equivalent:*

(a) *There are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(b) *I admits at least one local minimum which is not strictly global.*

Proof. It is enough to verify that I is bounded from below on every bounded set of X and from [5, Theorem 2.1] the conclusion follows. In fact, fixed $M > 0$, since X is a finite dimensional space, one has $\inf_{\|x\|\leq M} I(x) > -\infty$.

Taking Proposition 3.1 into account, the mountain pass theorem as given by N. Ghoussoub and D. Preiss (see, for instance, [19, Corollary 5.11]), in finite dimensional spaces, becomes as follows.

Theorem 3.2 *Let X be a real finite dimensional Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional which satisfies (PS). Further assume that I admits a local minimum u_1 which is not strictly global. Then, there exist $u_2 \in X$ and a critical point of I u_3, distinct from u_1, such that*

$$I(u_3) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$.

Proof. Since u_1 is not strict global minimum, there is $u_2 \in X$ such that condition (a) of Proposition 3.1 is verified. Hence, [19, Corollary 5.11] ensures the conclusion.

As a special case of the previous mountain pass theorem, we present the following two versions. The first deals with functionals which are bounded from below and it is the classical Courant principle (see, for instance, [27, Theorem 1.1, Chapter II]). The second one is given for functionals which are unbounded from below.

Corollary 3.1 *Let X be a real finite dimensional Banach space and $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional. Assume that I is coercive on X. Further assume that I admits two local minima u_1 and u_2. Then, I admits a distinct third critical point u_3.*

Corollary 3.2 *Let X be a real finite dimensional Banach space and $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional. Assume that I satisfies (PS) and it is unbounded from below. Further assume that I admits a local minimum u_1. Then, I admits a distinct second critical point.*

Owing to previous remarks, it is clear that in order to apply the mountain pass theorem is fundamental the existence of a local minimum. Usually, in the applications, the choice of a local minimum is 0 assuming that the datum is superlinear at 0 so, in particular, the nonlinear term is zero at zero (see Lemma 5.2 and Theorem 5.2). In [4], a local minimum theorem for continuously Gâteaux differentiable functionals has been established (see [4, Theorem 3.1]). So, owing to a such result or its consequences (that is, [4, Theorems 5.1, 5.2 and 5.3]) we can obtain a local minimum that can be different from zero. For completeness, here we give a direct proof of such results exploiting the finite dimensional setting. To this end, we assume that the following structural hypothesis is fulfilled.

(H) *Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals with Φ coercive and such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Clearly, if $\Phi(\bar{x}) = 0$, then $\bar{x} = 0$.

Now, we present our main result of this section.

Theorem 3.3 *Assume that (H) holds and let $r > 0$.*

Then, for each $\lambda \in \Lambda := \left]0, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[$, the function $I_\lambda = \Phi - \lambda\Psi$ admits at least a local minimum $\overline{u} \in X$ such that $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([0, r])$ and $I'_\lambda(\overline{u}) = 0$.

Proof. If $\sup_{\Phi^{-1}([0,r])} \Psi = 0$ we read the interval Λ as $]0, +\infty[$. In this case, fixed $\lambda > 0$, one has $0 \leq \Phi(u) \leq \Phi(u) - \lambda \sup_{\Phi^{-1}([0,r])} \Psi \leq \Phi(u) - \lambda\Psi(u)$ for all $u \in \Phi^{-1}([0, r])$. Hence, our conclusion follows by choosing $\bar{u} = 0$. Now, assume $\sup_{\Phi^{-1}([0,r])} \Psi \neq 0$ and fix $\lambda \in \Lambda$. The set $\Phi^{-1}([0, r])$ is closed since Φ is continuous. Moreover, it is bounded thanks to the coercivity of Φ. Therefore, there exists $u^* \in \Phi^{-1}([0, r])$ such that

$$I_\lambda(u^*) = \min_{\Phi^{-1}([0,r])} I_\lambda. \tag{3.11}$$

Obviously, if we prove that $u^* \in \Phi^{-1}([0, r[)$, we achieve our conclusion choosing $\overline{u} = u^*$. To this end, we observe that if $\Phi(u^*) = r$ then u^* is not a global minimum of I_λ on $\Phi^{-1}([0, r])$. Indeed, since one has that $\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{1}{\lambda}$, we get $\frac{\Psi(u^*)}{r} < \frac{1}{\lambda}$, that is $\frac{\Psi(u^*)}{\Phi(u^*)} < \frac{1}{\lambda}$, and so $0 < I_\lambda(u^*)$. Hence, the proof is complete.

Now, we present the following variant of the local minimum theorem which ensures the existence of a non-zero local minimum.

Theorem 3.4 *Assume that (H) holds. In addition, suppose that there exist $r \in \mathbb{R}$ and $w \in X$, with $0 < \Phi(w) < r$, such that*

$$\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{\Psi(w)}{\Phi(w)}. \tag{3.12}$$

Then, for each $\lambda \in \Lambda_w := \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *admits at least a local minimum* $\overline{u} \in X$ *such that* $\overline{u} \neq 0$, $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}([0, r])$ *and* $I'_\lambda(\overline{u}) = 0$.

Proof. Fix $\lambda \in \Lambda_w$, arguing as in Theorem 3.3 we have that (3.11) holds. Clearly, if we prove that $u^* \in \Phi^{-1}(]0, r[)$ we choose $\overline{u} = u^*$ and we obtain our conclusion. To this end, we reason by contradiction. If $\Phi(u^*) = 0$, from condition (H), one has $u^* = 0$. Therefore, from $\frac{1}{\lambda} < \frac{\Psi(w)}{\Phi(w)}$ one has $I_\lambda(w) < 0 = I_\lambda(u^*)$, and this is in contradiction with (3.11). If $\Phi(u^*) = r$, from $\frac{\sup_{\Phi^{-1}([0,r])} \Psi(u)}{r} < \frac{1}{\lambda}$, one has $\frac{\Psi(u^*)}{r} < \frac{1}{\lambda}$, that is $\frac{\Psi(u^*)}{\Phi(u^*)} < \frac{1}{\lambda}$, and hence $0 < I_\lambda(u^*)$ for which we have again a contradiction.

Remark 3.1 Theorems 3.3 and 3.4 are special cases of [4, Theorem 5.1] (see also [5, Theorems 2.4 and 2.3]), which is a consequence of the local minimum theorem established in the same paper (see [4, Theorem 3.1]). We also observe that, since the weak and strong topology in this case coincide, Theorem 3.3 is the finite dimensional version of [26, Theorem 2.3 (a)]. Finally, it is worth noticing that the proofs presented in Theorems 3.3 and 3.4 are simpler than those developed in the infinite dimensional case.

4 Some consequences of the local minimum theorem

We point out some consequences of Theorem 3.3. The first deals with the existence of three distinct critical points assuming that the functional I_λ is coercive; in the second the existence of two distinct critical points is obtained under the assumptions that the functional satisfies (PS) and it is unbounded from below. Finally, the third consequence is a result of infinitely many critical points.
The first result is the following theorem.

Theorem 4.1 *Assume that* (H) *holds and there exist* $r \in \mathbb{R}$ *and* $w \in X$, *with* $0 < r < \Phi(w)$, *such that*

$$\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{\Psi(w)}{\Phi(w)}. \tag{4.13}$$

Further, assume that, for each $\lambda \in \Lambda := \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *is coercive. Then, for each* $\lambda \in \Lambda$, *the function* I_λ *admits at least three distinct critical points.*

In the proof we need the following lemma.

Lemma 4.1 *Under the assumptions of Theorem 4.1, for each* $\lambda \in \Lambda$ *the function* I_λ *admits a local minimum* u_2 *such that* $\Phi(u_2) > r$.

Proof. Fix $\lambda \in \Lambda$ and put

$$\Phi^r(u) = \begin{cases} r & \text{if } \Phi(u) \leq r \\ \Phi(u) & \text{if } \Phi(u) > r. \end{cases}$$

Clearly, the functional $\Phi^r : X \to \mathbb{R}$ is continuous and the functional $I^r_\lambda := \Phi^r - \lambda\Psi$ is continuous and, taking into account that $\Phi^r(u) - \lambda\Psi(u) \geq \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, it is also coercive.

From Theorem 3.1, there exists $u_2 \in X$ such that $I_\lambda^r(u_2) = \min_{u\in X} I_\lambda^r(u)$, that is,

$$\Phi^r(u_2) - \lambda\Psi(u_2) \le \Phi^r(u) - \lambda\Psi(u) \quad \forall u \in X. \tag{4.14}$$

Now, we claim that $\Phi(u_2) > r$. Arguing by contradiction, we assume $\Phi(u_2) \le r$. Owing to (4.13), one has $\dfrac{\Psi(w) - \sup_{u\in\Phi^{-1}([0,r])}\Psi(u)}{\Phi(w) - r} > \dfrac{\Psi(w) - r\dfrac{\Psi(w)}{\Phi(w)}}{\Phi(w) - r} = \dfrac{\Psi(w)}{\Phi(w)} > \dfrac{1}{\lambda}$, that is

$$\frac{\Psi(w) - \sup_{u\in\Phi^{-1}([0,r])}\Psi(u)}{\Phi(w) - r} > \frac{1}{\lambda}.$$

It follows $r - \lambda\sup_{u\in\Phi^{-1}([0,r])}\Psi(u) > \Phi(w) - \lambda\Psi(w)$. Therefore, being $\Phi(w) > r$ and $\Phi(u_2) \le r$, one has $\Phi^r(u_2) - \lambda\Psi(u_2) > \Phi^r(w) - \lambda\Psi(w)$ and this is contrary to (4.14). So our claim is proved.

Hence, from (4.14), in particular, we have $\Phi(u_2) - \lambda\Psi(u_2) \le \Phi(u) - \lambda\Psi(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$. Taking into account that $\Phi^{-1}(]r, +\infty[)$ is an open set, then u_2 is a local minimum of I_λ and the conclusion is achieved.

Proof of Theorem 4.1. Fix $\lambda \in \Lambda$. Since $\lambda < \dfrac{r}{\sup_{u\in\Phi^{-1}([0,r])}\Psi(u)}$, from Theorem 3.3, I_λ admits a local minimum u_1 such that $\Phi(u_1) < r$. Moreover, from Lemma 4.1, I_λ admits a local minimum u_2 such that $\Phi(u_2) > r$. Hence the mountain pass theorem as given in Corollary 3.1 ensures the conclusion.

The second consequence of the local minimum theorem is the following.

Theorem 4.2 *Assume that* (H) *holds and fix* $r > 0$. *Assume that for each* $\lambda \in \Lambda := \left]0, \dfrac{r}{\sup\limits_{\Phi^{-1}([0,r])}\Psi}\right[$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *satisfies* (PS) *and it is unbounded from below. Then, for each* $\lambda \in \Lambda$, *the function* I_λ *admits at least two distinct critical points.*

Proof. It follows from Theorem 3.3 and from mountain pass theorem as given in Corollary 3.2.

The third consequence is the following infinitely many critical points theorem.
Assume that (H) holds and put

$$\varphi(r) = \frac{\sup\limits_{\Phi^{-1}([0,r])}\Psi}{r}$$

for all $r > 0$,

$$\varphi_\infty = \liminf_{r\to+\infty}\varphi(r), \qquad \varphi_0 = \liminf_{r\to 0^+}\varphi(r).$$

Theorem 4.3 *The following propositions hold:*

(*i*) *Assume that* $\varphi_\infty < +\infty$ *and for each* $\lambda \in \left]0, \dfrac{1}{\varphi_\infty}\right[$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is unbounded from below. Then, there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim\limits_{n\to+\infty}\Phi(u_n) = +\infty$.

(*ii*) *Assume that* $\varphi_0 < +\infty$ *and for each* $\lambda \in \left]0, \frac{1}{\varphi_0}\right[$ 0 *is not a local minimum for the function* $I_\lambda = \Phi - \lambda\Psi$. *Then, there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *such that* $\lim_{n\to+\infty} u_n = 0$.

Proof. It follows from Theorem 3.3 by arguing exactly as in the infinite dimensional case (see, for instance, [4, Theorem 7.4]).

Remark 4.1 Theorems 4.1, 4.2 and 4.3 are the finite dimensional versions of [11, Theorem 3.6], [5, Theoren 3.2] and [4, Theorem 7.4] (see also [26, Theorem 2.3] and observations in Remark 3.1) respectively. For an infinite dimensional version of Lemma 4.1 we refer to [4, Theorem 5.3] and [10, Theorem 2.1].

5 Some remarks on classical results for discrete problems

Let $\underline{f} : \mathbb{R} \to \mathbb{R}^N$ be a continuous function and $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, ..., N$, its components. Here and in the sequel we assume

$$f_k(0) \geq 0$$

for all $k \in [1, N]$. Moreover, taking Proposition 2.2 into account, we can assume, without loss of generality for our purposes since we get positive solutions, that $f_k(x) = f_k(0)$ for all $x < 0$ and for all $k \in [1, N]$. Put

$$F_k(s) := \int_0^s f_k(t)dt, \quad \forall s \in \mathbb{R}, \quad \forall k \in [1, N],$$

$$L^\infty(k) := \limsup_{s\to+\infty} \frac{F_k(s)}{s^2} \qquad L^\infty := \max_{1\leq k\leq N} L^\infty(k),$$

$$L_\infty(k) := \liminf_{s\to+\infty} \frac{F_k(s)}{s^2} \qquad L_\infty := \min_{1\leq k\leq N} L_\infty(k).$$

Moreover, the energy functional $I_\lambda : S \to \mathbb{R}$ defined in Section 2, can be written as follows

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \frac{1}{2}\sum_{k=1}^{N+1}(\Delta u(k-1))^2, \qquad \Psi(u) := \sum_{k=1}^{N} F_k(u_k), \quad \forall u \in S.$$

We have the following result.

Lemma 5.1 *The following propositions hold:*

(*i*) *If* $0 \leq L^\infty < +\infty$ *then* I_λ *is coercive for all* $\lambda \in \left]0, \frac{\lambda_1}{2L^\infty}\right[$;

(*ii*) *If* $L_\infty > 0$ *then* I_λ *satisfies* (PS) *and it is unbounded from below for all* $\lambda \in \left]\frac{\lambda_N}{2L_\infty}, +\infty\right[$.

Proof. (i) Let $0 < L^\infty < +\infty$ and λ a positive parameter such that $\lambda < \dfrac{\lambda_1}{2L^\infty}$. So, we can fix $l > 0$ such that $L^\infty < l < \dfrac{\lambda_1}{2\lambda}$. Taking into account that $F_k(s) = f_k(0)s$ for all $s < 0$, one has $\limsup_{|s|\to+\infty} \dfrac{F_k(s)}{s^2} \le L^\infty < l$. If $L^\infty = 0$ and $\lambda \in]0, +\infty[$ we can fix again $l > 0$ such that $l < \dfrac{\lambda_1}{2\lambda}$ and $\limsup_{|s|\to+\infty} \dfrac{F_k(s)}{s^2} < l$. Therefore, in both cases, there is a nonnegative constant Q such that

$$F_k(s) \le ls^2 + Q,$$

for all $s \in \mathbb{R}$ and for all $k \in [1, N]$. It follows that $\sum_{k=1}^{N} F_k(u_k) \le \sum_{k=1}^{N} \left(lu_k^2 + Q\right) = l \sum_{k=1}^{N} |u_k|^2 + NQ$, that is

$$\sum_{k=1}^{N} F_k(u_k) \le l\|u\|_2^2 + NQ$$

for all $u \in S$. Hence, one has
$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) = \dfrac{1}{2}\|u\|^2 - \lambda \sum_{k=1}^{N} F_k(u_k) \ge \dfrac{\lambda_1}{2}\|u\|_2^2 - \lambda \sum_{k=1}^{N} F_k(u_k) \ge \dfrac{\lambda_1}{2}\|u\|_2^2 - \lambda l\|u\|_2^2 - \lambda NQ = \left(\dfrac{\lambda_1}{2} - \lambda l\right)\|u\|_2^2 - \lambda NQ$ for all $u \in S$. So, since $\dfrac{\lambda_1}{2} - \lambda l > 0$, one has $\lim_{\|u\|_2\to+\infty} I_\lambda(u) = +\infty$.
(ii) Taking into account that $L_\infty > 0$ we fix $\lambda > \dfrac{\lambda_N}{2L_\infty}$ and, fix l such that $L_\infty > l > \dfrac{\lambda_N}{2\lambda}$.

Now, let $\{u_n\}$ be a sequence such that $\lim_{n\to+\infty} I_\lambda(u_n) = c$ and $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$. Put $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in \mathbb{N}$. We claim that $\{u_n^-\}$ is bounded. In fact, one has $\Delta u_n^-(k-1)\Delta u_n^-(k-1) \le -\Delta u_n(k-1)\Delta u_n^-(k-1)$ for all $k \in [1, N+1]$ as a direct computation shows. It follows that

$$\sum_{k=1}^{N+1} \Delta u_n^-(k-1)\Delta u_n^-(k-1) \le -\sum_{k=1}^{N+1} \Delta u_n(k-1)\Delta u_n^-(k-1).$$

So,

$$\|u_n^-\|^2 = \sum_{k=1}^{N+1} \Delta u_n^-(k-1)\Delta u_n^-(k-1) \le -\sum_{k=1}^{N+1} \Delta u_n(k-1)\Delta u_n^-(k-1) = -\Phi'(u_n)(u_n^-).$$

Moreover,

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{N} f_k(u_n(k))u_n^-(k) \ge 0.$$

Therefore, one has $\|u_n^-\|^2 \le -\Phi'(u_n)(u_n^-) \le -\Phi'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-)$, that is,

$$\|u_n^-\|^2 \le -I'_\lambda(u_n)(u_n^-) \tag{5.15}$$

for all $n \in \mathbb{N}$. Now, from $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$ one has $\lim_{n\to+\infty} \sup_{\|v\|\le 1} I'_\lambda(u_n)(v) = 0$, $\lim_{n\to+\infty} \dfrac{I'_\lambda(u_n)(u_n^-)}{\|u_n^-\|} = 0$, for which, taking (5.15) into account, one has $\lim_{n\to+\infty} \|u_n^-\| = 0$. Hence, our claim is proved. So, there is $M > 0$ such that $\|u_n^-\| \le M$, $\|u_n^-\|_2 \le \dfrac{M}{\sqrt{\lambda_1}} = L$, $0 \le u_n^-(k) \le L$ for all $k \in [1, N]$ for all $n \in \mathbb{N}$.

Now, arguing by a contradiction, assume that $\{u_n\}$ is unbounded (that is, $\{u_n^+\}$ is unbounded). From $\liminf_{s\to+\infty} \frac{F_k(s)}{s^2} = L_\infty(k) \geq L_\infty > l$ there is $\delta_k > 0$ such that $F_k(s) > ls^2$ for all $s > \delta_k$. Moreover,

$$F_k(s) \geq \min_{s\in[-L,\delta_k]} F_k(s) \geq ls^2 - l(\max\{\delta_k, L\})^2 + \min_{s\in[-L,\delta_k]} F_k(s) \geq$$

$$ls^2 - \max\{l(\max\{\delta_k, L\})^2 - \min_{s\in[-L,\delta_k]} F_k(s), 0\} = ls^2 - Q_k$$

for all $s \in [-L, \delta_k]$. Hence, $F_k(s) \geq ls^2 - Q_k$ for all $s > -L$. It follows that $F_k(u_n(k)) \geq l(u_n(k))^2 - Q_k$ for all $n \in \mathbb{N}$, $\sum_{k=1}^{N} F_k(u_n(k)) \geq \sum_{k=1}^{N}\left(l|u_n(k)|^2 - Q_k\right) = l\|u_n\|_2^2 - \sum_{k=1}^{N} Q_k = l\|u_n\|_2^2 - Q$, that is,

$$\Psi(u_n) \geq l\|u_n\|_2^2 - Q$$

for all $n \in \mathbb{N}$. Therefore, one has $I_\lambda(u_n) = \Phi(u_n) - \lambda\Psi(u_n) = \frac{1}{2}\|u_n\|^2 - \lambda\Psi(u_n) \leq \frac{\lambda_N}{2}\|u\|_2^2 - \lambda\Psi(u_n) \leq \frac{\lambda_N}{2}\|u_n\|_2^2 - \lambda l\|u_n\|_2^2 + \lambda Q$, that is

$$I_\lambda(u_n) \leq \left(\frac{\lambda_N}{2} - \lambda l\right)\|u_n\|_2^2 + \lambda Q$$

for all $n \in \mathbb{N}$. Since $\|u_n\|_2 \to +\infty$ and $\frac{\lambda_N}{2} - \lambda l < 0$, one has $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$ and this is absurd. Hence, I_λ satisfies (PS).

Finally, we prove that I_λ is unbounded from below. Let $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded. Arguing as before one has $\Psi(u_n) \geq l\|u_n\|_2^2 - Q$ for all $n \in \mathbb{N}$ and, consequently, $I_\lambda(u_n) \leq \left(\frac{\lambda_N}{2} - \lambda l\right)\|u_n\|_2^2 + \lambda Q$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$ and the proof is complete.

From the direct methods theorem (see Theorem 3.1) we obtain the following existence result of one positive solution. Some versions of it have been already presented in other papers (see, for instance, [7], [18]).

Theorem 5.1 *Assume $L^\infty < +\infty$. Then, for each $\lambda \in \left]0, \frac{\lambda_1}{2L^\infty}\right[$, the problem (D_λ^f) admits at least one solution. Moreover, the solution is positive provided that $f_k(0) \neq 0$ for some $k \in [1, N]$.*

Proof. We recall that the space S is equipped with the norm $\|\cdot\|_2$. Owing to Lemma 5.1 (i), the function I_λ is coercive. Hence, from Theorem 3.1, I_λ admits at least one critical point, which is a solution for the problem (D_λ^f). Finally, if $f_k(0) \neq 0$ for some $k \in [1, N]$, the maximum principle (see Proposition 2.2) ensures the conclusion.

We point out the following immediate consequence of Theorem 5.1.

Corollary 5.1 *Assume that*

$$\lim_{t\to+\infty} \frac{f_k(t)}{t} = 0$$

for all $k \in [1, N]$ and $f_k(0) \neq 0$ for some $k \in [1, N]$. Then, the problem

$$\begin{cases} -\Delta^2 u(k-1) = f_k(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \qquad (D^f)$$

admits at least one positive solution.

Now, we apply the mountain pass theorem in order to obtain again an existence result of one positive solution for problem (D^f_λ). We point out that the existence of a nontrivial solution for a discrete problem through mountain pass theorem has been already considered in [3, 18]. First, we give a lemma and the following notations. Put

$$L^0(k) := \limsup_{s\to 0^+} \frac{F_k(s)}{s^2} \qquad L^0 := \max_{1\le k\le N} L^0(k).$$

Lemma 5.2 *If* $0 \le L^0 < +\infty$ *then* 0 *is a local minimum for* I_λ *for each* $\lambda \in \left]0, \frac{\lambda_1}{2L^0}\right[$.

Proof. Since $L^0 < +\infty$ we can fix a positive parameter $\lambda < \frac{\lambda_1}{2L^0}$. Moreover, fix l such that $L^0 < l < \frac{\lambda_1}{2\lambda}$. Taking into account that $F_k(s) = f_k(0)s$, $s \le 0$, from $\limsup_{s\to 0^+} \frac{F_k(s)}{s^2} = L^0(k) \le L^0 < l$, there is $\delta_k > 0$ such that $F_k(s) \le ls^2$ for all $s \in]-\delta_k, \delta_k[$. Therefore, by setting $\delta = \sqrt{\sum_{k=1}^N \delta_k^2}$, one has

$$\sum_{k=1}^N F_k(u_k) \le l \sum_{k=1}^N |u_k|^2$$

for all $u \in S$ such that $\|u\|_2 < \delta$. Hence, one has $I_\lambda(u) = \Phi(u) - \lambda\Psi(u) = \frac{1}{2}\|u\|^2 - \lambda \sum_{k=1}^N F_k(u_k) \ge \frac{\lambda_1}{2}\|u\|_2^2 - \lambda \sum_{k=1}^N F_k(u_k) \ge \frac{\lambda_1}{2}\|u\|_2^2 - \lambda l\|u\|_2^2 = \left(\frac{\lambda_1}{2} - \lambda l\right)\|u\|_2^2 \ge 0$, that is $I_\lambda(u) \ge I_\lambda(0)$ for all $u \in S$ such that $\|u\|_2 < \delta$.

From the mountain pass theorem we obtain the following existence result.

Theorem 5.2 *Assume that*

(i) $L^0 = 0$, *that is* $\limsup_{s\to 0^+} \frac{F_k(s)}{s^2} = 0$ *for all* $k \in [1, N]$;

(ii) $L_\infty > 0$, *that is* $\liminf_{s\to +\infty} \frac{F_k(s)}{s^2} > 0$ *for all* $k \in [1, N]$.

Then, for each $\lambda \in \left]\frac{\lambda_N}{2L_\infty}, +\infty\right[$, *the problem* (D^f_λ) *admits at least one positive solution.*

Proof. Owing to Lemma 5.2, I_λ admits 0 as local minimum and by Lemma 5.1 part (ii) I_λ satisfies (PS) and it is unbounded from below. Hence, the mountain pass theorem as given in Corollary 3.2 ensures a critical point distinct from 0, which is a non-zero solution for the problem (D^f_λ). So, the maximum principle (see Proposition 2.2) ensures the conclusion.

Remark 5.1 The conclusion of Theorem 5.2 holds true for each $\lambda \in \left]0, \frac{\lambda_1}{2L^0}\right[$ if we assume

(i) $L^0 < +\infty$, that is $\limsup_{s\to 0^+} \frac{F_k(s)}{s^2} < +\infty$ for all $k \in [1, N]$;

(ii) $L_\infty = +\infty$, that is $\liminf_{s\to +\infty} \frac{F_k(s)}{s^2} = +\infty$ for all $k \in [1, N]$.

More generally, the conclusion of Theorem 5.2 holds true for each $\lambda \in \left]\frac{\lambda_N}{2L_\infty}, \frac{\lambda_1}{2L^0}\right[$ if we assume

$$0 \leq L^0 < \frac{\lambda_1}{\lambda_N} L_\infty.$$

In particular, in the autonomous case and assuming $f(t) \geq 0$ for all $t > 0$, we get one positive solution to problem (A_λ^f) if

$$\limsup_{s\to 0^+} \frac{F(s)}{s^2} < \frac{\lambda_1}{\lambda_N} \liminf_{s\to+\infty} \frac{F(s)}{s^2}.$$

Now, we point out the following particular case of Theorem 5.2 (see also the previous remark).

Corollary 5.2 *Assume that*

(i) $\displaystyle\lim_{t\to 0^+} \frac{f_k(t)}{t} = 0$ *for all* $k \in [1, N]$*;*

(ii) $\displaystyle\lim_{t\to+\infty} \frac{f_k(t)}{t} = +\infty$ *for all* $k \in [1, N]$.

Then, the problem

$$\begin{cases} -\Delta^2 u(k-1) = f_k(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0 \end{cases} \qquad (D^{\underline{f}})$$

admits at least one positive solution.

Remark 5.2 In nonlinear differential problems, the corresponding energy functional is defined in a infinite dimensional space and, in order to verify the Palais-Smale condition, an assumption on nonlinear term, called Ambrosetti-Rabinowitz condition, is requested (see [1, 27]). We know that, in the infinite dimensional case, the Ambrosetti-Rabinowitz condition implies the superlinearity of the nonlinear term at infinity and the superlinearity of the nonlinear term at infinity may also not imply the Palais-Smale condition of the associated functional.
Several authors have used a type of the Ambrosetti-Rabinowitz condition also in finite dimensional spaces in order to verify the Palais-Smale condition of the corresponding functional (see for instance [3]). In this setting, the Ambrosetti-Rabinowitz condition is the following:

(AR) *There exist* $\mu > 2$ *and* $R > 0$ *such that for* $s \geq R$

$$0 < \mu F_k(s) \leq s f_k(s) \quad \forall k \in [1, N].$$

A standard computation shows that it implies the following condition

(S) *For every* $k \in [1, N]$*, there exist two positive constants,* M_k *and* Q_k*, such that*

$$F_k(s) \geq M_k s^\mu - Q_k, \quad \forall s \geq 0,$$

for which $L_\infty = +\infty$. In fact, fix $k \in [1, N]$ and let $s > R$. Integrating the following inequality, between R and s,

$$\frac{\mu}{s} \leq \frac{f_k(s)}{F_k(s)},$$

being k arbitrary and $\mu > 2$, an easy computation shows that (S) is verified.

Hence, in the framework of discrete problems, the superlinearity of the nonlinear term at infinity is enough to obtain that the corresponding functional satisfies (PS), as Lemma 5.1 part (ii) shows, without requiring the Ambrosetti-Rabinowitz condition as seen, for instance, in Corollary 5.2. We present the following example.

Example 5.1 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by putting

$$f(s) = \begin{cases} s(2\ln(s) + 1), & \text{if } s > e^{-1/2}; \\ 0, & \text{if } s \le e^{-1/2}. \end{cases}$$

Simple computations show that f verifies conditions (i) and (ii) of Corollary 5.2. On the other hand, since

$$sf(s) - \mu F(s) = s^2[(2-\mu)\ln(s) + 1] - \mu\frac{1}{2e} \to -\infty,$$

for $s \to +\infty$ and for all $\mu > 2$, it is clear that condition (AR) is not satisfied. Hence, owing to Corollary 5.2 the problem

$$\begin{cases} -\Delta^2 u(k-1) = f(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0 \end{cases}$$

admits at least one positive solution. Finally, since condition (AR) is not satisfied, we observe that Theorem 4.2 of [3] cannot be applied to obtain the same conclusion.

In this last part of the section, we get the existence of solutions with no data on the sign of the nonlinearity. Let $\underline{g} : \mathbb{R} \to \mathbb{R}^N$ be a continuous function and $g_k : \mathbb{R} \to \mathbb{R}$, $k \in [1, N]$ its components. Put $G_k(s) := \int_0^s g_k(t)dt$ for all $s \in \mathbb{R}$, for all $k \in [1, N]$, and

$$M_\infty(k) := \liminf_{|s|\to+\infty} \frac{G_k(s)}{s^2} \qquad M_\infty := \min_{1\le k\le N} M_\infty(k).$$

Moreover, in this case, denote $\tilde{I}_\lambda(u) = \Phi(u) - \lambda\tilde{\Psi}(u) = \frac{1}{2}\|u\|^2 - \lambda\sum_{k=1}^{N} G_k(u_k)$ for all $u \in S$. Clearly, Proposition 2.3 holds again true for $\tilde{I}_\lambda$ and $(D_\lambda^{\underline{g}})$.

We point out the following result.

Lemma 5.3 *If $M_\infty > 0$ then $\tilde{I}_\lambda$ is anti-coercive for each $\lambda \in \left]\frac{\lambda_N}{2M_\infty}, +\infty\right[$.*

Proof. Fix $l > 0$ such that $M_\infty > l > \frac{\lambda_N}{2\lambda}$. From $\liminf_{|s|\to+\infty} \frac{G_k(s)}{s^2} > l$, arguing as in the proof of Lemma 5.1 part (ii), one has $\tilde{\Psi}(u) \ge l\|u\|_2^2 - Q$ for all $u \in S$ and for some $Q \ge 0$. It follows $\tilde{I}_\lambda(u) \le \left(\frac{\lambda_N}{2} - \lambda l\right)\|u\|_2^2 + \lambda Q$ for all $u \in S$ and, hence, the conclusion is achieved.

Theorem 5.3 *Assume that*

(i) $\limsup_{s\to 0} \frac{G_k(s)}{s^2} = 0$ *for all* $k \in [1, N]$;

(ii) $M_\infty > 0$, *that is* $\liminf_{|s|\to+\infty} \frac{G_k(s)}{s^2} > 0$ *for all* $k \in [1, N]$.

Then, for each $\lambda \in \left]\frac{\lambda_N}{2M_\infty}, +\infty\right[$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda g_k(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \qquad (D_\lambda^{\underline{g}})$$

admits at least two non-zero solutions.

Proof. Arguing as in the proof of Lemma 5.2, condition (i) implies that 0 is a local minimum of $\tilde{I}_\lambda$. Owing to Lemma 5.3 the functional $\tilde{I}_\lambda$ is anti-coercive. Therefore, Theorem 3.1 ensures that $\tilde{I}_\lambda$ admits a global maximum. So, from [17], arguing as in the proof of [7, Theorem 2.3], we obtain three distinct critical points, that are two non-zero critical points which are two non-zero solutions of (D_λ^{g}).

Remark 5.3 Theorem 5.3 does not provide information on the sign of the two solutions. On the contrary, taking into account that condition (i) implies $g_k(0) = 0$ for all $k \in [1, N]$, by applying Theorem 5.2 to $\underline{g}$ and $-\underline{g}$, the existence of one positive solution and of one negative solution is obtained.

6 Main results

In this section we present our main results which are Theorems 6.1, 6.2 and 6.4. The first result gives the existence of one positive solution without requiring asymptotic condition at infinity on the nonlinear term. It improves Theorem 5.1. The second one ensures the existence of one positive solution without requiring conditions neither at zero nor at infinity. As its consequence the existence of positive solution is obtained when the nonlinear term is sublinear at zero (see Corollary 6.1). Theorem 6.4 guarantees the existence of two solutions and it improves Theorem 5.2 since no condition at zero is requested.

Moreover, in this section, multiplicity results are also pointed out. They are Theorems 6.3 and 6.5. The first is a more precise version of [16, Theorem 3.1] (see also [9, Theorem 49]), the second, which uses a better inequality with respect to [8, Theorem 3.3], allows us to obtain, as a consequence, Theorem 6.6 which ensures the existence of one positive solution without requiring that the associated functional satisfies (PS) (see Remark 6.6).
The energy functional $I_\lambda : S \to \mathbb{R}$ defined in Section 2, can be written as follows

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \frac{1}{2}\sum_{k=1}^{N+1}(\Delta u(k-1))^2, \qquad \Psi(u) := \sum_{k=1}^{N} F_k(u_k), \qquad \forall u \in S.$$

We recall that, as in Section 5, the condition

$$f_k(0) \geq 0$$

for all $k \in [1, N]$, is assumed, while on the function $\underline{g}$ (see the last part of Section 5) no sign assumption is a priori imposed. Finally, in this section, I_λ and $\widetilde{I_\lambda}$ are as in Section 5.

Our first result is the following.

Theorem 6.1 *Fix $c > 0$ and assume that $f_k(0) \neq 0$ for some $k \in [1, N]$. Then, for each* $\lambda \in \left]0, \dfrac{1}{2K_2^2}\dfrac{c^2}{\sum_{k=1}^{N}\max_{s\in[0,c]} F_k(s)}\right[$, *the problem (D_λ^{f}) admits at least one positive solution $\bar{u} \in S$ such that* $\|\bar{u}\|_\infty < c$.

Proof. Put $r = \dfrac{c^2}{2K_2^2}$. Taking (2.4) into account, for all $u \in S$ such that $\Phi(u) \leq r$ one has $\|u\|_\infty \leq K_2\|u\| \leq K_2\sqrt{2r} = c$, that is,

$$\|u\|_\infty \leq c$$

for all $u \in S$ such that $\Phi(u) \leq r$. Therefore, $\Psi(u) = \sum_{k=1}^{N} F_k(u_k) \leq \sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)$ for all $u \in S$ such that $\Phi(u) \leq r$. It follows that

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq 2K_2^2 \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2}.$$

Hence, owing to Theorem 3.3 for each $\lambda < \dfrac{1}{2K_2^2} \dfrac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \leq \dfrac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}$, the functional I_λ admits a non-zero critical point $\bar{u} \in S$ such that $\Phi(\bar{u}) < r$, that is $\bar{u}$ is a non-zero solution of (D^{f}_{λ}) such that $\|\bar{u}\|_\infty < c$. Hence, the maximum principle (see Proposition 2.2) ensures the conclusion.

Theorem 6.2 *Assume that there are two positive constants c, d, with $d < c$, such that*

(*a*) $$\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{1}{2K_2^2} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}.$$

Then, for each $\lambda \in \left] \dfrac{d^2}{\sum_{k=1}^{N} F_k(d)}, \dfrac{1}{2K_2^2} \dfrac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \right[$, *the problem (D^{f}_{λ}) admits at least one positive solution $\bar{u}$ such that $\|\bar{u}\|_\infty < c$.*

Proof. Put $r = \frac{1}{2K_2^2}c^2$. Arguing as in the proof of Theorem 6.1 we obtain

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq 2K_2^2 \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2}. \tag{6.16}$$

Now, let $w \in \mathbb{R}^{N+2}$ be such that $w(k) = d$ for all $k \in [1, N]$ and $w(0) = w(N + 1) = 0$. Clearly, $w \in S$. Moreover, one has $\Phi(w) = \dfrac{1}{2} \sum_{k=1}^{N} (\Delta w(k-1))^2 = d^2$ and $\Psi(w) = \sum_{k=1}^{N} F_k(d)$. It follows that

$$\frac{\Psi(w)}{\Phi(w)} = \frac{\sum_{k=1}^{N} F_k(d)}{d^2}. \tag{6.17}$$

Hence, from (6.16), (6.17) and assumption (a) one has

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Moreover, from $0 < d < c$, taking (a) into account, one has $0 < d < \frac{\sqrt{2}}{2K_2}c$ (by definition of K_2, one has $\frac{\sqrt{2}}{2K_2} < 1$ if $N \ge 2$). In fact, arguing by contradiction, if we assume $\frac{\sqrt{2}}{2K_2}c \le d < c$ we obtain $\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} \ge \frac{\sum_{k=1}^{N} F_k(d)}{c^2} \ge \frac{\sum_{k=1}^{N} F_k(d)}{2K_2^2 d^2}$ against to (a). Therefore, one has $0 < \Phi(w) < r$. Hence, Theorem 3.4 ensures the conclusion.

Now, we point out the following consequences of Theorem 6.2.

Corollary 6.1 *Assume that $f_k(t) \ge 0$ for all $t \ge 0$ and for all $k \in [1, N]$, and*

(b) $\limsup_{s\to 0^+} \frac{F_k(s)}{s^2} = +\infty$ *for some* $k \in [1, N]$.

Then, for each $\lambda \in \left]0, \frac{1}{2K_2^2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^{N} F_k(c)}\right[$, *the problem* (D_λ^f) *admits at least one positive solution.*

Proof. Fix λ as in the conclusion and $\bar{c} > 0$ such that $\lambda < \frac{1}{2K_2^2} \frac{\bar{c}^2}{\sum_{k=1}^{N} F_k(\bar{c})}$. From (b) there is $d < \bar{c}$ such that $\frac{F_k(d)}{d^2} > \frac{1}{\lambda}$. Hence, $\frac{\sum_{k=1}^{N} F_k(d)}{d^2} > \frac{1}{\lambda} > 2K_2^2 \frac{\sum_{k=1}^{N} F_k(\bar{c})}{\bar{c}^2}$, for which $\frac{\sum_{k=1}^{N} F_k(\bar{c})}{\bar{c}^2} < \frac{1}{2K_2^2} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}$ and $\lambda \in \left]\frac{d^2}{\sum_{k=1}^{N} F_k(d)}, \frac{1}{2K_2^2} \frac{\bar{c}^2}{\sum_{k=1}^{N} F_k(\bar{c})}\right[$. So, the conclusion follows from Theorem 6.2.

Corollary 6.2 *Assume that $f(t) \ge 0$ for all $t \ge 0$ and there are two positive constants c, d, with $d < c$, such that*

$$\frac{F(c)}{c^2} < \frac{2}{N+1} \frac{F(d)}{d^2}.$$

Then, for each $\lambda \in \left]\frac{1}{N} \frac{d^2}{F(d)}, \frac{2}{N+1} \frac{1}{N} \frac{c^2}{F(c)}\right[$, *the problem*

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0 \end{cases} \qquad (A_\lambda^f)$$

admits at least one positive solution $\bar{u}$ such that $\|\bar{u}\|_\infty < c$.

Proof. It is enough to apply Theorem 6.2 by choosing $f = f_k$ for all $k \in [1, N]$.

Remark 6.1 Theorem 1.1 in Introduction is a consequence of Corollary 6.1.

Now, we present some examples that bring out the usefulness of these results.

Example 6.1 Owing to Theorem 1.1, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \sqrt{u(k)}, & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases}$$

admits at least one positive solution.

Example 6.2 For each $\lambda \in \left]0, \frac{1}{N(N+1)} \frac{\pi^2}{e^{\pi/2}-1}\right[$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda e^{u(k)} \cos(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{6.18}$$

admits at least one positive solution. It is enough to apply Corollary 6.1 to the function $f(s) = e^s \cos(s)$, $s \in \mathbb{R}$.
It is a simple matter to see that the same conclusion cannot be obtained by classical variational methods.

Now, we give a result of three solutions.

Theorem 6.3 *Assume that*

$$\limsup_{s \to +\infty} \frac{F_k(s)}{s^2} = 0 \qquad \forall k \in [1, N].$$

Moreover, assume that there are two positive constants c, d, with $c < d$, such that

$$\frac{\sum_{k=1}^{N} \max_{s \in [0,c]} F_k(s)}{c^2} < \frac{1}{2K_2^2} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}. \tag{c}$$

Then, for each $\lambda \in \left] \frac{d^2}{\sum_{k=1}^{N} F_k(d)}, \frac{1}{2K_2^2} \frac{c^2}{\sum_{k=1}^{N} \max_{s \in [0,c]} F_k(s)} \right[$, the problem (D_λ^f) admits at least three nonnegative solutions.

Proof. Fix λ as in the conclusion and put $r = \frac{1}{2K_2^2} c^2$, $w \in \mathbb{R}^{N+2}$ be such that $w(k) = d$ for all $k \in [1, N]$ and $w(0) = w(N+1) = 0$. Clearly, $w \in S$. Arguing as in the proof of Theorem 6.2, one has $\frac{\sup_{\Phi(u) \leq r} \Psi(u)}{r} \leq 2K_2^2 \frac{\sum_{k=1}^{N} \max_{s \in [0,c]} F_k(s)}{c^2}$, $\frac{\Psi(w)}{\Phi(w)} = \frac{\sum_{k=1}^{N} F_k(d)}{d^2}$, for which

$$\frac{\sup_{\Phi(u) \leq r} \Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Moreover, from $0 < c < d$ one has $0 < c < \sqrt{2}K_2 d$ (in fact, $\sqrt{2}K_2 \geq 1$) and, hence, $0 < r < \Phi(w)$. Finally, taking our assumption into account, Lemma 5.1 (i) ensures that I_λ is coercive. Hence, from Theorem 4.1 the conclusion is achieved.

Remark 6.2 If in Theorem 6.3 we assume $f_k(t) \geq 0$ for all $t \geq 0$ and for all $k \in [1, N]$ the assumption (c) becomes
there are two positive constants c, d, *with* $c < d$, *such that*

$$\frac{\sum_{k=1}^{N} F_k(c)}{c^2} < \frac{1}{2K_2^2} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}. \qquad (c')$$

Hence, to verify (c') it is enough that *there are two positive constants* c, d, *with* $c < d$, *such that*

$$\frac{\sum_{k=1}^{N} F_k(c)}{c^2} < \frac{2}{N+1} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}. \qquad (c'')$$

We point out the following consequence of Theorem 6.3.

Corollary 6.3 *Assume that* f *is a non-zero function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to +\infty} \frac{f(t)}{t} = 0.$$

Then, for each $\lambda \in \left] \frac{1}{N} \inf_{d>0} \frac{d^2}{F(d)}, +\infty \right[$, *the problem*

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0 \end{cases} \qquad (A_\lambda^f)$$

admits at least two positive solutions.

Proof. It is enough to observe that $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$ implies $\lim_{s \to 0^+} \frac{\max_{\xi \in [0,s]} F(\xi)}{s^2} = 0$.

Remark 6.3 Theorem 6.3 is a more precise version of Theorem 3.1 of [16] (see also Theorem 3.2). In fact, Theorem 6.3 ensures a larger interval of parameters λ.

Theorem 6.4 *Assume that*

$$\lim_{s \to +\infty} \frac{F_k(s)}{s^2} = +\infty \qquad \forall k \in [1, N].$$

Then, for each $\lambda \in \left] 0, \frac{1}{2K_2^2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^{N} \max_{s \in [0,c]} F_k(s)} \right[$, *the problem* (D_λ^f) *admits at least two nonnegative solutions.*

Proof. Fix λ as in the conclusion and c such that $\lambda < \frac{1}{2K_2^2}\frac{c^2}{\sum_{k=1}^{N}\max_{s\in[0,c]}F_k(s)}$. Owing to Lemma 5.1(ii), the functional I_λ satisfies (PS) and it is unbounded from below. Hence, taking into account, as seen in the proof of Theorem 6.1, that $\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq 2K_2^2\frac{\sum_{k=1}^{N}\max_{s\in[0,c]}F_k(s)}{c^2}$, from Theorem 4.2 the conclusion is achieved.

Remark 6.4 In Theorem 6.4, if we assume

$$\liminf_{s\to+\infty}\frac{F_k(s)}{s^2} > \lambda_N K_2^2 \inf_{c>0}\frac{\sum_{k=1}^{N}\max_{s\in[0,c]}F_k(s)}{c^2} \qquad \forall k\in[1,N]$$

the conclusion is achieved for each $\lambda \in \left]\frac{\lambda_N}{2L_\infty}, \frac{1}{2K_2^2}\sup_{c>0}\frac{c^2}{\sum_{k=1}^{N}\max_{s\in[0,c]}F_k(s)}\right[$.

Remark 6.5 Theorem 1.2 in Introduction is a consequence of Theorem 6.4.

Example 6.3 For each $\lambda \in \left]0, \frac{2}{N(N+1)}\right[$, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda[2u(k)\ln^2(u(k)) + 2u(k)\ln(u(k)) + 1], & k\in[1,N], \\ u(0) = u(N+1) = 0 \end{cases} \tag{6.19}$$

admits at least two positive solutions. Indeed, it is enough to apply Theorem 1.2 to the function

$$f(s) = \begin{cases} 2s\ln^2(s) + 2s\ln(s) + 1, & \text{if } s > 0 \\ 1, & \text{if } s \leq 0, \end{cases}$$

taking into account that $\sup_{c>0}\frac{c^2}{\max_{s\in[0,c]}F(c)} \geq 1$, where $F(s) = s^2\ln^2(s) + s$, for $s > 0$. In particular, the problem

$$\begin{cases} -\Delta^2 u(k-1) = \frac{1}{2}u(k)\ln^2(u(k)) + \frac{1}{2}u(k)\ln(u(k)) + \frac{1}{4}, & k\in[1,2], \\ u(0) = u(3) = 0, \end{cases} \tag{6.20}$$

admits at least two positive solutions. We also observe that, for the previous problems, the condition (AR) is not satisfied, since

$$sf(s) - \mu F(s) = s^2\ln s[(2-\mu)\ln s + 2] + (1-\mu)s \to -\infty, \qquad \text{for all } \mu > 2.$$

Now, we point out the following result of existence of three solutions when no sign assumption on nonlinear term is requested.

Corollary 6.4 *Assume that*

$$\lim_{|s|\to+\infty}\frac{G_k(s)}{s^2}=+\infty.$$

Then, for each $\lambda\in\left]0,\frac{1}{2K_2^2}\sup_{c>0}\frac{c^2}{\sum_{k=1}^{N}\max_{s\in[0,c]}G_k(s)}\right[$, *the problem* (D_λ^g) *admits at least three solutions.*

Proof. Owing to Lemma 5.3 the functional $\tilde{I}_\lambda$ is anti-coercive, for which from [7, Theorem 2.3] the conclusion is achieved.

Now, we point out a result of infinitely many positive solutions.

Theorem 6.5 *Assume that*

$$\liminf_{s\to+\infty}\frac{\sum_{K=1}^{N}\max_{\xi\in[0,s]}F_k(\xi)}{s^2}<\frac{1}{2K_2^2}\limsup_{s\to+\infty}\frac{\sum_{K=1}^{N}F_k(s)}{s^2}.$$

Then, for each $\lambda\in\left]\frac{1}{\limsup_{s\to+\infty}\frac{\sum_{K=1}^{N}F_k(s)}{s^2}},\frac{1}{2K_2^2}\frac{1}{\liminf_{s\to+\infty}\frac{\sum_{K=1}^{N}\max_{\xi\in[0,s]}F_k(\xi)}{s^2}}\right[$, *the problem* (D_λ^f) *admits infinitely many positive solutions.*

Proof. The conclusion is obtained from Theorem 4.3 (i) arguing as in the proof of [8, Theorem 3.1].

We point out the following consequence of Theorems 6.4 and 6.5. To this end, put

$$\lambda^*=\begin{cases}\frac{1}{2K_2^2}\frac{1}{N}\sup_{c>0}\frac{c^2}{F(c)}, & \text{if } L_\infty=+\infty,\\ \frac{1}{2K_2^2}\frac{1}{L_\infty}, & \text{if } L_\infty<+\infty.\end{cases}$$

Theorem 6.6 *Assume that* $f(t)\geq 0$ *for all* $t\geq 0$ *and*

$$\limsup_{s\to+\infty}\frac{F(s)}{s^2}=+\infty.$$

Then, for each $\lambda\in\,]0,\lambda^*[$, *the problem* (A_λ^f) *admits at least one positive solution.*

Proof. If $\liminf_{s\to+\infty}\frac{F(s)}{s^2}=+\infty$, from Theorem 6.4 we obtain the conclusion. Otherwise, if $\liminf_{s\to+\infty}\frac{F(s)}{s^2}<+\infty$ we can apply Theorem 6.5 and we obtain infinitely many positive solutions.

Remark 6.6 The assumption $\limsup_{s\to+\infty}\frac{F(s)}{s^2}=+\infty$ implies that the associated functional is unbounded from below (see the proof of Lemma 5.1 part (ii)), but it does not imply that a such functional satisfies (PS).

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