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Weakly Arf property for amalgamation of semigroups, amalgamation of rings and quadratic quotients of the Rees algebra

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Abstract

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Weakly Arf property for amalgamation of semigroups, amalgamation of rings and quadratic quotients of the Rees algebra

by Damiano SACCONE

This thesis aims to characterize the weakly Arf property for some classes of rings obtained with several different constructions: amalgamation algebras, quadratic quotients of Rees algebra, double covering of rings, rings of algebraic integers.

The weakly Arf property, introduced and studied in [6], generalizes the Arf property, which was introduced in order to study one-dimensional rings connected to singularity of curves; in particular, the authors of [6] study when this property is satisfied for the idealization, the amalgamated duplication and some fiber products.

Because the idealization and the amalgamated duplication can be obtained both from the amalgamation algebra and the quadratic quotients of Rees algebra, we are interested in understanding when these rings are weakly Arf.

We also study the amalgamation of two numerical semigroups, a construction related to the the amalgamation of algebroid branches, where the Arf and weakly Arf properties coincide.

In the case of semigroups we get a complete characterization of the weakly Arf property that leads us to understand better what happens at the ring level. However, in all the constructions we study we will have to assume some extra hypotheses to obtain necessary and/or sufficient conditions for the weakly Arf property.

In particular, if we have two integral domains, a ring homomorphism $f : R \to S$ and an ideal J of S, assuming that $f(R) + J$ has integral closure which is a DVR, we are able to give a complete characterization of the weakly Arf property of the amalgamation algebra $R \bowtie^f J$.

As for quadratic quotients of the Rees algebra with respect to an ideal I, denoted by $R(I)_{a,b}$, the integral closure depends on the polynomial $t^2 + at + b$ that we use to quotient. When it is reducible in $R[t]$ and the roots satisfy some extra conditions, we can prove the following result: if I is also an ideal of $\overline{R}$, then $R(I)_{a,b}$ is weakly Arf if and only if R is weakly Arf.

We also investigate the double covering of an integral domain R, i.e. rings of the form $R + It$, with $t^2 = b$, where I is a fractional ideal of R and $b \in R$ is such that $bI^2 \subseteq R$. In this case, we are able to prove that: if R is integrally closed, then $R + It$ is weakly Arf. If R is not integrally closed, we can only prove the $R + It$ weakly Arf implies that also R is weakly Arf.

We conclude this thesis by showing some examples of weakly Arf rings arising from number rings: we show that $\mathbb{Z}[\sqrt{m}]$ and $\mathbb{Z}[\sqrt[3]{d}]$ are weakly Arf rings.

Contents

Introduction

The notion of Arf ring was formally introduced by J. Lipman in 1971 (see [14]), that studied one-dimensional rings satisfying particular properties, connected, in the geometric case of singularity of curves, to the desingularization process. More precisely, in 1945 (see [1]) C. Arf proved that, given a complete local ring R associated to an algebraic curve singularity, one can compute its multiplicity sequences by constructing a specific overring of R (that Lipman called Arf closure) that shares with R the same multiplicity sequence. Lipman generalized Arf's results, defining and studying the notion of Arf ring and connecting it to the concept of saturated rings and to the strict closure.

Since then, the study of Arf rings played a relevant role in the curve singularity theory, from the algebraic point of view. Moreover, since to a curve singularity it is possible to associate a value semigroup $S \subseteq \mathbb{N}^h$ (where h is the number of branches of the singularity), it was natural to define the notion of Arf semigroup (see [7]), which also plays a relevant role in the theory of numerical semigroups and of good semigroups (see [2]).

Recently, in a paper by E. Celikbas, O. Celikbas, C. Ciupercă, N. Endo, S. Goto, R. Isobe and N. Matsuoka, the notion of Arf ring has been generalized to that of weakly Arf ring (see [6]). This new notion almost coincides with (and includes) the concept of Arf ring in the one-dimensional case, but can be given for any Krull dimension.

The definition of Arf ring requires, in particular, that the ring is Noetherian semilocal and that every localization with respect to a maximal ideal is one-dimensional and Cohen-Macaulay. In the above mentioned context, one possible definition of an Arf ring R is the following:

1. Every integrally closed ideal I in R that contains a non-zerodivisor has a principal reduction, i.e., $I^{n+1} = aI^n$ for some $n \geq 0$ and $a \in I$;

2. if $x, y, z \in R$ are such that x is a non-zerodivisor on R and $y/x, z/x \in Q(R)$ are integral over R, then $yz/x \in R$, where $Q(R)$ denotes the total ring of fractions of R.

Since Condition 1 becomes superfluous when R is also local with infinite residue field (see [12, Proposition 8.3.7]), it was natural for the authors of [6] to explore deeply Condition 2. So, they proposed the following more general definition, without any assumption on R: a commutative and unitary ring R is said to be *weakly Arf*, if it satisfies the above Condition 2.

In the same paper, the authors develop a general theory of weakly Arf rings, (briefly WA, from now on), studying many interesting properties of the rings in this class. In particular, they study when some ring constructions as the idealization, the amalgamated duplication and some particular fiber products produce a WA ring (cf. [6, Theorem 5.2, Theorem 6.2, Proposition 6.11]). A related problem was considered also in [5] where the author studies when a quadratic quotient of the Rees algebra of an algebroid branch, with respect to an ideal I, produces an Arf algebroid branch.

More precisely, let us recall that Nagata's idealization is a classical construction (cite [16]) that, given a commutative ring R and an R-module M produces a new ring $R \ltimes M$, whose supporting set is $R \times M$, with sum defined componentwise and product given by $(r,m)(t,n) = (rt, r \cdot n + t \cdot m)$ (with $r,t \in R$ and $m,n \in M$). More recently, in [11], was defined the amalgamated duplication of a ring R with respect to an ideal I as the following subring of $R \times R$:

$$R \bowtie I = \{(r, r+i) \mid r \in R,\ i \in I\}.$$

Successively, in [8] it was introduced a construction that generalizes the amalgamated duplication and it is also connected to Nagata's idealization, when M is an ideal of R. This construction, starting by two rings R, S, a ring homomorphism $f\colon R \to S$ and an ideal J of S, produces the following R-algebra called *amalgamation of R with J with respect to f*

$$R \bowtie^f J = \{(r, f(r) + j \mid r \in R,\ j \in J\} \subseteq R \times S.$$

The amalgamation can be also viewed as a particular kind of fiber product (see [8, Proposition 4.2]). Therefore, taking into account the cited results of [6] it is natural to ask under which conditions the amalgamation produces a WA ring.

A possible approach to this problem was suggested by the paper [7], where it was defined the amalgamation in the context of numerical semigroups (see Chapter 2 for the definition) and where it has been proved that, starting from two algebroid branches R, S (i.e. one dimensional rings of the form $K[[x_1, \ldots, x_n]]/P$, with P prime ideal) the amalgamation (at ring level) produces an algebroid curve with two branches (i.e. one dimensional rings of the form $K[[x_1, \ldots, x_n]]/(P \cap Q)$, with P, Q prime ideals), whose value semigroup is the amalgamation of the two value semigroups of R and S. Hence, to study when the amalgamation at semigroup level produces an Arf semigroup could give a hint to prove more general results at the ring level.

Another unifying context for idealization and amalgamated duplication is given by a different construction, introduced and studied in [3]: the quadratic quotients of the Rees algebra. More precisely, let us consider a commutative and unitary ring R and an ideal I. The Rees algebra associated to R with respect to I is the ring $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$. Given a monic polynomial $f(t) = t^2 + at + b \in R[t]$ the ring

$$R(I)_{a,b} = \frac{\mathcal{R}_+}{(t^2 + at + b)R[t] \cap \mathcal{R}_+}$$

is said to be a *quadratic quotient of the Rees algebra*. Varying the coefficients of the polynomial we get a family of rings whose members can be different with respect to some properties (e.g. they can be integral domains, reduced rings or not reduced rings), but thay share many other properties like Cohen-Macaulay property or Gorenstein property. Moreover, Nagata's idealization w.r.t. I and amalgamated duplication are particular members of this family: $R(I)_{0,0} \cong R \ltimes I$ and $R(I)_{-1,0} \cong R \bowtie I$. Hence, in light of the results of [6] cited above, also in this case it is natural to ask under which conditions it is produced a WA ring and if this property depends only on R and I and is independent of the polynomial $f(t)$.

A variation of the last construction is obtained considering a fractional ideal I of R and an element $b \in R$ such that $bI^2 \subseteq R$; with these choices, defining on the R-module $R \oplus I$ the following product

$$(r,i)(t,j) = (rt + ijb, rj + ti)$$

we get a subring of $Q(R)[t]/(t^2-b)$ (where, as usual, $Q(R)$ is the total ring of fractions of R). These kind of rings were introduced and studied in [13], where in particular, it is described their integral closure in the domain case. This construction can be used to construct double coverings of irreducible affine algebraic curves, but also rings of the form $\mathbb{Z}[\sqrt{m}]$ can be viewed as particular case of this construction. Once again we are naturally interested to understand when the weakly Arf property is fulfilled by these rings. Finally, generalizing the study of the WA property for $\mathbb{Z}[\sqrt{m}]$, it naturally arises the question whether rings of integers in algebraic extensions of $\mathbb{Q}$ are WA.

Hence, the aim of this thesis is to study the WA property in the following contexts:

- amalgamation of two numerical semigroups;
- amalgamated algebras and, in particular, algebroid curves obtained as amalgamation of two algebroid branches;
- quadratic quotients of Rees algebra and double coverings of rings;
- rings of integers of algebraic extensions of $\mathbb{Q}$.

The structure of this thesis is the following:

- In Chapter 1 we introduce Arf and weakly Arf properties and we recall some basic results that we will need in the sequel of the thesis. Moreover, we produce some examples, mainly taken by [6, Example 2.3], that show that Arf and WA rings appear in many different contexts.
- In Chapter 2 we study the amalgamation of numerical semigroups; after recalling some basic facts in numerical semigroups we give the definition of good subsemigroup of $\mathbb{N}^2$. Good semigroups are a class of monoids that contains the value semigroups of curve singularities and this fact motivates the interest in this theory. Then, we give the definition of amalgamation of two numerical semigroups, which produces a good subsemigroup of $\mathbb{N}^2$. For these semigroups Arf and WA properties coincide and we prove the characterization of the Arf property of the amalgamation of two numerical semigroups (see Theorem 2.19).
- In Chapter 3 we turn our attention to amalgamated algebras; in order to study WA property it is necessary to know the total ring of fractions and the integral closure of the amalgamation $R \bowtie^f J$. This consideration leads to assume that both J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Moreover, to control the regular elements (i.e. non-zerodivisors) of the amalgamation it is useful to assume that the image of a regular element of R is always regular in S. Under these assumptions, the first result we prove is that if $R \bowtie^f J$ is WA, then also R is WA (see Proposition 3.5). To obtain a sufficient condition we have to assume that J is also an ideal of the integral closure of $f(R)+J$ (see Proposition 3.6). This result is not very satisfying, since the ring $f(R)+J$ is as difficult to study as the amalgamation itself.

  We can obtain more efficient results assuming that both R and S are integral domains. In particular, if $\overline{f(R)+J}$ is a DVR we can obtain a complete characterization of the WA property of the amalgamation (see Theorems 3.13 and 3.14). The content of Chapters 2 and 3 is mainly contained in the paper [18].

- In Chapter 4 we focus on a quadratic quotient of the Rees algebra; also in this case, in order to study WA property it is necessary to know the total ring of fractions and the integral closure of the quadratic quotients of the Rees algebra $R(I)_{a,b}$. The calculation of the total ring of fractions was done in [3], for this reason it is necessary to calculate its integral closure. After some considerations made in the particular cases in which the polynomial is the form $t^2 + b$ or $t^2 + at$, we will give a conjecture of its value. This result, interesting, but not sufficient, leads us to add further hypotheses in order to calculate the integral closure of $R(I)_{a,b}$; more precisely, we assume that $t^2 + at + b = (t - \xi)(t - \eta)$ with $\xi, \eta \in R$ such that $\xi - \eta$ is an invertible element of R; under these hypotheses we have $\overline{R(I)_{a,b}} \cong \overline{R} + \overline{R}t$. In this setting, to verify the WA property we need to add additional assumptions in order to compare the regular elements of $R(I)_{a,b}$ with those of R. For this purpose, we assume that R is a local Cohen-Macaulay ring and that I is a Cohen-Macaulay R-module of the same dimension as R. With these additional assumptions, we are able to associate a regular element of R with each regular element of $R(I)_{a,b}$. Furthermore, assuming that I is an ideal of $\overline{R}$, we can prove that $R(I)_{a,b}$ is WA if and only if R is WA (see Theorem 4.15).

  We conclude this chapter by studying the WA property for double coverings of an integral domain R, with respect to b, i.e. rings of the form $R + It$, where I is a fractional ideal of R and $b \in R$ is an element of R, such that $bI^2 \subseteq \overline{R}$. When R is integrally closed, the integral closure of $R + It$ has been computed in [13]; and in this case we prove that $R + It$ is WA; as a consequence, we have that $R(I)_{0,-b}$ with R integrally closed and b squarefree is WA (see Corollary 4.21). Following the same ideas, we study the WA property for $R + It$, with R not necessarily integrally closed, by proving Theorem 4.25 which states: if $R + It$ is WA, then R is WA. The contents of this chapter will appear in the work in progress [17].

- In Chapter 5 we show some examples of WA rings rising from number rings. More precisely, we prove that the rings $\mathbb{Z}[\sqrt{m}]$ and $\mathbb{Z}[\sqrt[3]{d}]$ are WA (see Propositions 5.10, 5.11 and Theorem 5.13). It remains open the more general situation of rings of the form $\mathbb{Z}[\alpha]$, where α is an algebraic complex number over $\mathbb{Q}$.

Chapter 1

Arf and Weakly Arf rings

In this chapter we formally introduce the notions of Arf and WA rings, recalling also those properties that we will use in the remaining part of the thesis.

All the rings considered in this thesis are commutative with unity element, so we will omit this hypothesis from now on.

1.1 Arf rings

The notion of Arf ring was formally introduced by J. Lipman in [14] generalizing a class of rings defined by C. Arf in order to study curve singularities (see [1]). The definition of Arf ring requires that the ring is Noetherian semi-local and that every localization with respect to a maximal ideal is one-dimensional and Cohen-Macaulay. Hence, for this section we assume these hypotheses.

We denote with $\overline{R}$ the *integral closure of R in its total ring of fractions* $Q(R)$. Before giving the definition of Arf ring we need some preliminary notions and properties.

An ideal I in R is *regular* if I contains a regular element of R; we denote with $\mathcal{F}_R$ the set of the regular ideals in R. An element x of the ideal $I \in \mathcal{F}_R$ is called a *principal reduction of* I if $xI^n = I^{n+1}$ for some integer $n > 0$.

An element $x \in R$ is said to be *integral over the ideal* I if x satisfies a relation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \quad (n > 0)$$

with $a_i \in I^i$ for $i = 1, 2, \ldots, n$; the set of all the elements in R that are integral over I is called the *integral closure of* I *in* R and it is denoted with $\overline{I}$, we say that I is *integrally closed ideal* if $\overline{I} = I$.

Remark 1.1. We recall some properties highlighted by Lipman in [14]:

1. If $x \in R$ is regular, then $z \in \overline{xR}$ if and only if $z/x \in \overline{R}$.
2. If $z \in R$, J is an ideal of R, and $zI \subseteq JI$, where I is a regular ideal of R, then z is integral over J.
3. Let $J \subseteq I$ be two ideals of R, where I is a regular ideal of R, then $I \subseteq \overline{J}$ if and only if $I^{n+1} = JI^n$ for some integer $n \geq 0$.
4. An element x of an ideal $I \in \mathcal{F}_R$ is a principal reduction of I if and only if every element of I is integral on xR, i.e. x is regular and $z \in I$ implies $z/x \in \overline{R}$.

Definition 1.2. Let R be a Noetherian semi-local ring such that the localization $R_\mathfrak{m}$ is a one-dimensional Cohen-Macaulay local ring for every maximal ideal $\mathfrak{m}$ of R. The ring R is called *Arf ring* if the following conditions are satisfied:

1. if $I \in \mathcal{F}_R$ is an integrally closed ideal in R, then I has a principal reduction;

2. whenever $x, y, z \in R$ are such that x is regular and both y and z are integral over xR (i.e. $y/x, z/x \in \overline{R}$), then $yz \in xR$.

For completeness, we give the following definitions.

- For any $I \in \mathcal{F}_R$, we define the *blow-up* of R at I as $R^I := \bigcup_{n \geq 0}(I^n : I^n)$.
- A regular ideal I of R is *stable* (in R) if $R^I = I : I$, or, equivalently, $IR^I = I$.
- Define the sequence of semilocal rings

$$R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq \overline{R}$$

where, for each $i \geq 0$, R_{i+1} is the ring obtained from R by blowing-up the Jacobson radical of R_i. A local ring of the form $(R_i)_P$, where P is a maximal ideal in R_i, is said to be *infinitely near to* R.

Two more interesting characterizations of Arf rings is given in [14, Theorem 2.2]:

Theorem 1.3. *The following conditions on R are equivalent:*

1. *R is an Arf ring.*
2. *Every integrally closed open ideal in R is stable.*
3. *If S is any local ring infinitely near to R, then the embedding dimension of S is equal to the multiplicity of S.*

We note that: if X is an affine variety and $\boldsymbol{p} \in X$ is a nonsingular point, then the ring $\mathcal{O}_{\boldsymbol{p},X}$ is a regular local ring (i.e. $\mathcal{O}_{\boldsymbol{p},X}$ has the same embedding dimension and Krull dimension), which implies that the ring is Arf, being a UFD and hence integrally closed. On the other hand, if $\boldsymbol{p}$ is a singular point, a desingularization process of $\boldsymbol{p}$ starts by blowing-up $\mathcal{O}_{\boldsymbol{p},X}$ with respect to its maximal ideal. This blowing-up is not necessarily local and the desingularization process proceed by blowing-up the Jacobson radical. The previous Theorem 1.3, in particular, implies that if the ring $\mathcal{O}_{\boldsymbol{p},X}$ is an Arf ring, then all its infinitely near rings are again Arf and, using Arf's results, gives a method to compute the multiplicity sequences at each branch of the singularity.

1.2 Weakly Arf rings

Weakly Arf rings were introduced by E. Celikbas, O. Celikbas, C. Ciupercă, N. Endo, S. Goto, R. Isobe and N. Matsuoka in [6], generalizing the notion of Arf rings.

As written in the introduction, the authors of this paper consider only Condition 2 of the definition of Arf rings, without assuming any hypothesis on the ring except the fact it is commutative with unity:

Definition 1.4. A ring R is said to be *weakly Arf* (WA from now on) if for all $x, y, z \in R$ such that x is a non-zerodivisor element and $y/x, z/x \in \overline{R}$, it follows that $yz/x \in R$.

The authors in [6] gave a list of some example of weakly Arf rings that show the wide typology of rings that fullfill this property:

Example 1.5. Let k be a field.

1. Let $k[s,t]$ be the polynomial ring over k. Then, for each $\ell \geq 1$, $k[s, t^2, t^{2\ell+1}]$ is a WA ring.

2. Let $S = k[t]$ be the polynomial ring over k. Then, for each $\ell \geq 2$, $k[t^\ell + t^{\ell+1}] + t^{\ell+2}S$ is a WA ring.

3. Let A be an integral domain and let G be a finite subgroup of Aut$A$. Suppose that the order of G is invertible in A. If A is a WA ring, then so is the invariant subring A^G.

4. Let A be a Noetherian integral domain with Serre's (S_2) condition, containing an infinite field. Then A is a WA ring if and only if so is the polynomial ring $A[X_1, X_2, \ldots, X_n]$ for every $n \geq 1$.

5. Let $S = k[t]$ be the polynomial ring over k and set $R = k[t^2, t^{2\ell+1}]$ $(\ell \geq 1)$. Then the idealization $R \ltimes S^{\oplus n}$ is a WA ring for every $n \geq 0$.

6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then R is a WA ring if and only if so is the fiber product $R \times_{R/\mathfrak{m}} R$.

7. Let $k[X, Y, Z]$ be the polynomial ring over k. Then $k[X, Y, Z]/I_2\left(\begin{smallmatrix} X & Y^2 & Z \\ 0 & Z & Y \end{smallmatrix}\right)$ is a WA ring, where $I_2(M)$ denotes the ideal of $k[X, Y, Z]$ generated by 2×2-minors of a matrix M.

8. The ring $\mathbb{Z}_2[[X, Y]]/(XY(X+Y))$ is a WA ring, but not an Arf ring.

In the same paper the authors study when some ring constructions as the idealization, the amalgamated duplication and some particular fiber products produce a WA ring.

We recall the definition of idealization: let R be a ring and let M be an R-module, the *idealization of M over R*, it is denoted by $R \ltimes M$, is the R-module $R \oplus M$ with multiplication given by $(r, m) \cdot (s, n) = (rs, rn + sm)$.

Theorem 1.6 ([6, Theorem 5.2])**.** *Let $A = R \ltimes M$ be the idealizations of a torsion-free module M over a Noetherian ring R; assume that M is finitely generated. Then, the following conditions are equivalent:*

1. *A is WA.*

2. *R is WA and $\overline{(a)} \cdot M = aM$ for every non-zerodivisor $a \in R$.*

3. *R is WA and M is an $\overline{R}$-module.*

As for fiber products, let us recall the definition and its basic properties. Let R, S, and T be arbitrary commutative rings, and let $f\colon R \to T$ and $g\colon S \to T$ be ring homomorphisms; the *fiber product* $R \times_T S$ of R and S over T with respect to f and g is the set

$$R \times_T S := \{(a, b) \in R \times S \mid f(a) = g(b)\},$$

which forms a subring of $R \times S$. We then have a commutative diagram

$$\begin{tikzcd} & R \times_T S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S \arrow[dr, "g"'] & & R \arrow[dl, "f"] \\ & T & \end{tikzcd}$$

of rings, where $p_1\colon R \times_T S \to R$ and $p_2\colon R \times_T S \to S$ stand for the projections. Hence, we get an exact sequence

$$0 \longrightarrow R \times_T S \xrightarrow{i} R \times S \xrightarrow{\varphi} T$$

of $R \times_T S$-modules, where $\varphi = \begin{bmatrix} f \\ -g \end{bmatrix}$. The map φ is surjective if either f or g is surjective. Moreover, if both f and g are surjective, then T is cyclic as an $R \times_T S$-module, so that $R \times S$ is a module-finite extension over $R \times_T S$. Using this fact it is possible to prove the following result.

Theorem 1.7 ([6, Theorem 6.2]). *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be Noetherian local rings with a common residue class field $k = R/\mathfrak{m} = S/\mathfrak{n}$. Suppose that depth$(R) > 0$ and depth$(S) > 0$. Then the following conditions are equivalent:*

1. *$R \times_k S$ is WA.*
2. *R and S are WA.*

Finally, let R be a commutative ring and let I be an ideal of R, the *amalgamated duplication of R with respect to I* is the folowing subring of $R \times R$:

$$R \bowtie I := \{(r, r+i) \mid r \in R, i \in I\}$$

with product given by $(r, r+i) \cdot (s, s+j) = (rs, rs + rj + si + ij)$. We recall when the amalgamated duplication $R \bowtie I$ produces a WA ring (whit I a regular ideal of R). In this situation $Q(R \bowtie I) = Q(R) \bowtie Q(R)$ and $\overline{R \bowtie I} = \overline{R} \bowtie \overline{R}$ (see [11, Corollary 3.3]).

Proposition 1.8 ([6, Proposition 6.11]). *The amalgamated duplication $R \bowtie I$ is WA if and only if so is R and I is an ideal of $\overline{R}$.*

Chapter 2

Amalgamation of numerical semigroups

In this chapter we will present the amalgamation of two numerical semigroups, a construction that produces a good subsemigroup of $\mathbb{N}^2$, and we will study when the obtained semigroup is an Arf good semigroup. This construction is motivated by the fact that a particular amalgamation of two algebroid branches gives rise to a good semigroup of $\mathbb{N}^2$ that is the amalgamation of the value semigroups of the given branches.

2.1 Basic properties of the numerical semigroups

In this section we recall the basic definition and properties of numerical semigroups that we will need in the sequel.

Definition 2.1. Let $S \subseteq (\mathbb{N}, +)$ be a submonoid; we say that S is a *numerical semigroup* if its complement in $\mathbb{N}$ is finite. That is, if $|\mathbb{N} \setminus S| < \infty$.

For example, the set $S = \{0, 2, 4, 6, 8, 10, \rightarrow\}$ (where with "$\rightarrow$" we mean that all natural numbers greater than 10 belong to S) is a numerical semigroup. Instead, the set $\{0, 2, 4, 6, \dots\}$ of the even numbers is a submonoid of $(\mathbb{N}, +)$, but it is not a numerical semigroup.

Definition 2.2. Let E be a subset of a numerical semigroup S; we say that E is a *semigroup ideal* (or simply an ideal) of S if $E + S \subseteq E$.

Remark 2.3. If $E \subseteq S$ is a semigroup ideal, we can consider $\tilde{e} = \min(E)$; set

$$\overline{E} = \{s \in S \colon s \geq \tilde{e}\}.$$

Clearly, for any $e_1, e_2 \in \overline{E}$ we have $e_1 + e_2 \in S$ and $e_1 + e_2 \geq \tilde{e}$ and so $e_1 + e_2 \in \overline{E}$. Hence, $\overline{E}$ is a semigroup ideal of S called *the integral closure* of E in S.

Definition 2.4. Let S be a numerical semigroup; a semigroup ideal E of S is said to be *integrally closed* if $\overline{E} = E$.

Definition 2.5. Let S and T be two numerical semigroups, a function $g \colon S \to T$ is called *semigroup homomorphism* if $g(a + b) = g(a) + g(b)$ for all $a, b \in S$.

It is not difficult to check that, if $g \colon S \to T$ is a semigroup homomorphism, then it is the multiplication by a non-negative integer s such that $sa \in T$ for all $a \in S$.

Lemma 2.6. *Let S and T be two numerical semigroups and $g \colon S \to T$ a semigroup homomorphism. Let E be a semigroup ideal of T. If E is integrally closed, then $g^{-1}(E)$ is an integrally closed ideal of S.*

Proof. Let $a \in g^{-1}(E)$ and let $b \in S$; then $g(a) \in E$ and $g(b) \in T$. Since E is an ideal of T, we get $g(a+b) = g(a) + g(b) \in E + T \subseteq E$. It follows that $a + b \in g^{-1}(E)$, i.e. $g^{-1}(E)$ is a semigroup ideal of S.

Let $e = \min(g^{-1}(E))$ and let $a \in S$ be such that $a \geq e$. Since $g(a) \geq g(e)$, with $g(e) \in E$ and E is integrally closed in T, we get $g(a) \in E$. Thus $a \in g^{-1}(E)$, hence $g^{-1}(E)$ is integrally closed. □

2.2 Amalgamation of numerical semigroups

In this section we first define the notion of good subsemigroup of $\mathbb{N}^2$, and then we will give the definition of amalgamation of numerical semigroups.

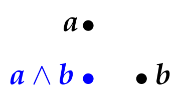
Let us start by fixing some notations for submonoids of $(\mathbb{N}^2, +)$ (where the sum is defined componentwise). For an element $\boldsymbol{a} \in \mathbb{N}^2$ we denote by a_1 and a_2 its first and second coordinate, respectively. Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^2$, we say that $\boldsymbol{a} \leq \boldsymbol{b}$ if $a_1 \leq b_1$ and $a_2 \leq b_2$ with respect to the usual ordering of $\mathbb{N}$. We observe that $\mathbb{N}^2$ with this order is a partially ordered set. The infimum of the set $\{\boldsymbol{a}, \boldsymbol{b}\}$ will be denoted by

$$\boldsymbol{a} \wedge \boldsymbol{b} = (\min\{a_1, b_1\}, \min\{a_2, b_2\}).$$

Definition 2.7. Following [2], we say that a submonoid S of $\mathbb{N}^2$ is a *good semigroup* if satisfies the following conditions:

1. for all $\boldsymbol{a}, \boldsymbol{b} \in S$, it holds $\boldsymbol{a} \wedge \boldsymbol{b} \in S$;
2. if $\boldsymbol{a} \neq \boldsymbol{b} \in S$ and $a_i = b_i$ for some $i \in \{1, 2\}$, then there exists $\boldsymbol{c} \in S$ such that $c_i > a_i = b_i$ and $c_j = \min\{a_j, b_j\}$, with $j \in \{1, 2\} \setminus \{i\}$;
3. there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^2 \subseteq S$.

Remark 2.8. Graphically, Condition 1. of the previous definition says that for every pair of points $\boldsymbol{a}, \boldsymbol{b} \in S$, we have $\boldsymbol{a} \wedge \boldsymbol{b} \in S$



On the other hand, Condition 2. of the definition says that, for every pair of points $\boldsymbol{a}, \boldsymbol{b} \in S$ that are aligned horizontally (or vertically), we have an element $\boldsymbol{c} \in S$ as in the figure below:

Definition 2.9. Let $S \subseteq \mathbb{N}^2$ be a good semigroup, a subset $E \subseteq \mathbb{Z}^2$ is called *relative ideal* of S if $E + S \subseteq S$ and $\boldsymbol{a} + E \subseteq S$ for some $\boldsymbol{a} \in S$. If E satisfies Conditions 1 and 2 of Definition 2.7, then E is said *good relative ideal*.

Notice that Condition 3. of Definition 2.7 is always verified by a good relative ideal.

Now we are ready to define the amalgamation of two numerical semigroups (introduced in [7]).

Definition 2.10. Let S and T be two numerical semigroups and let $g\colon S \to T$ a semigroup homomorphism (i.e. the multiplication by a positive integer s); let E be an ideal of T. Then the *amalgamation of S with T along E with respect to g* is the following subset of $\mathbb{N}^2$:

$$S \bowtie^g E = D \cup (g^{-1}(E) \times E) \cup \{\boldsymbol{a} \wedge \boldsymbol{b} : \boldsymbol{a} \in D,\ \boldsymbol{b} \in g^{-1}(E) \times E\}$$

where $D = \{(a, sa)\colon a \in S\}$.

It is not difficult to check that $S \bowtie^g E$ is a good semigroup (see [7]).

Let us see some useful properties of the amalgamation.

Remark 2.11. We observe that $g(S) \cup E$ is closed with respect to the sum and $\mathbb{N} \setminus (g(S) \cup E)$ is finite, thus $g(S) \cup E$ is a numerical semigroup. Now, we denote by $\pi_1\colon S \bowtie^g E \to S$ and $\pi_2\colon S \bowtie^g E \to T$ the canonical projections on the first and on the second component, respectively. We get $\pi_1(S \bowtie^g E) = S$ and $\pi_2(S \bowtie^g E) = g(S) \cup E$. Indeed, for the first equality it is enough to observe that $\pi_1(D) = S$; as for the second equality, if we take an element of the form $(a, sa) \wedge \boldsymbol{b}$, with $a \in S$ and $\boldsymbol{b} \in g^{-1}(E) \times E$, then either $((a, sa) \wedge \boldsymbol{b})_2 = sa \in g(S)$ or $((a, sa) \wedge \boldsymbol{b})_2 = b_2 \in E$. It follows immediately that $\pi_2(S \bowtie^g E) = g(S) \cup E$.

Notice also that the set $g(S) \cup E$ is a numerical semigroup contained in T.

Lemma 2.12. *Keeping the notations introduced above, if E is an integrally closed ideal of $g(S) \cup E$, then $S \bowtie^g E = D \cup (g^{-1}(E) \times E)$.*

Proof. Let $\boldsymbol{a} = \boldsymbol{b} \wedge \boldsymbol{c}$, with $\boldsymbol{b} \in D$ and $\boldsymbol{c} \in g^{-1}(E) \times E$. If we exclude the trivial cases $\boldsymbol{a} = \boldsymbol{b}$ and $\boldsymbol{a} = \boldsymbol{c}$, then $\boldsymbol{a} = (b_1, c_2)$ or $\boldsymbol{a} = (c_1, b_2)$.

In the first case we have $c_2 \leq b_2 = sb_1$, so $b_2 \in E$ and therefore $b_1 \in g^{-1}(E)$. In the second case we have $c_1 \leq b_1$, so $sc_1 \leq sb_1 = b_2$, with $sc_1 \in E$; therefore $b_2 \in E$.

In conclusion, we have $\boldsymbol{a} = \boldsymbol{b} \in D$ or $\boldsymbol{a} \in g^{-1}(E) \times E$ and so

$$S \bowtie^g E = D \cup (g^{-1}(E) \times E). \qquad \square$$

This construction of semigroups is related to amalgamation of algebroid branches as stated in [7]. In particular, in that paper it is proved the following result:

Proposition 2.13 ([7, Proposition 3.6])**.** *Let $R = k[[t^s : s \in S]]$, $U = k[[u^t : t \in T]]$ and $J = (u^e : e \in E)$; assume that $cs \in T$, for all $s \in S$ and let $f\colon R \to U$ be the homomorphism defined by $f(F(t)) = F(u^c)$. Let v_1 snd v_2 be the usual discrete valuations on R and S, respectively. Let $R \bowtie^f J$ the ring $\{(r, f(r) + j) : r \in R,\ j \in J\}$ with the discrete valutation $v = (v_1, v_2)$. Then $v(R \bowtie^f J) = S \bowtie^g E$, where $g\colon S \to T$ is the multiplication by c.*

2.3 Arf property

In this section we want to study when the Arf property is fullfilled by a semigroup amalgamation. Let us notice that in the one dimensional local case with residue field infinite, any regular ideal has a principal reduction (see [12, Proposition 8.3.7]).

Hence for algebroid curves Arf and WA properties coincide. At semigroup level it is well known that any good semigroup ideal I has a principal reduction, i.e. there exists an integer r such that for any $n \geq r$, $(n+1)I = \tilde{i} + nI$, where $\tilde{i}$ is the minimum element in I (that does exist, since I is good).

To see this fact it is enough to observe that we have the chain of good relative ideals:

$$S \subseteq I - \tilde{i} \subseteq 2I - 2\tilde{i} \subseteq \cdots \subseteq nI - n\tilde{i} \subseteq \cdots \subseteq \mathbb{N}^2;$$

where $I - \tilde{i} = \{z \in \mathbb{Z} \mid z + \tilde{i} \in I\}$. Since the distance $d(\mathbb{N}^2 \setminus S)$ (for a formal definition of the distance see [2]) is finite, it follows that the chain eventually stabilizes, that is exactly our claim.

Because of these observations, it is clear that, at semigroup level, the translation of the WA property has to consider only Property 2 of Definition 1.2. Therefore it will coincide with the numerical analogous of the Arf property, that was introduced in [4] for numerical semigroups and was extended to good semigroups in [2] and it is an additive translation of the same property given for rings, taking into account that the role of the integral closure is played by $\mathbb{N}$ for numerical semigroups and by $\mathbb{N}^2$ in the case of good subsemigroups of $\mathbb{N}^2$. This means that the condition $y/x \in \overline{R}$ is translated saying that for two elements $a, b \in S$, $b - a \in \mathbb{N}$ ($b - a \in \mathbb{N}^2$, respectively) or equivalently that $a \leq b$, where in the case of $\mathbb{N}^2$ we are considering the componentwise partial ordering defined in the previous section.

Definition 2.14. Let S be a numerical semigroup, we say that S is an *Arf semigroup* if, for all $a, b, c \in S$ with $a \leq b$ and $a \leq c$, we have $b + c - a \in S$. Similary, if S is a good semigroup, we say that S is an *Arf semigroup* if for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$ with $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a} \leq \boldsymbol{c}$ we have $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \in S$.

In what follows we want to give a characterization of when $S \bowtie^g E$ is an Arf semigroup. The next lemma should be well known, but we did not find a direct reference for it, so we include its proof for the sake of completeness.

Lemma 2.15. *Let $S \subseteq \mathbb{N}^2$ be a good semigroup and let $\pi_1 \colon S \to \mathbb{N}$ and $\pi_2 \colon S \to \mathbb{N}$ be the canonical projections on the first and on the second component, respectively. Set $S_1 = \pi_1(S)$ and $S_2 = \pi_2(S)$; if S is an Arf semigroup, then S_1 and S_2 are Arf semigroups.*

In particular, if $S \bowtie^g E$ is an Arf semigroup, then both S and $g(S) \cup E$ are Arf semigroups.

Proof. It is enough to prove the lemma for one projection, since it is independent of the choice of the component. Let $a_1, b_1, c_1 \in S_1$ be such that $a_1 \leq b_1 \leq c_1$. By definition, there are $a_2, b_2, c_2 \in S_2$ such that $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in S$. Moreover, we may assume $(a_1, a_2) \leq (b_1, b_2)$ and $(a_1, a_2) \leq (c_1, c_2)$. Indeed, if $(a_1, a_2) \not\leq (b_1, b_2)$ since $a_1 \leq b_1$ it follows that $b_2 < a_2$; therefore $(a_1, a_2) \wedge (b_1, b_2) = (a_1, b_2) \in \pi^{-1}(a_1)$ and $(a_1, b_2) \leq (b_1, b_2)$. Similarly, we have $(a_1, c_2) \leq (c_1, c_2)$ with $(a_1, c_2) \in \pi^{-1}(a_1)$. Hence, it is enough to replace (a_1, a_2) with

$$\left[(a_1, a_2) \wedge (b_1, b_2)\right] \wedge \left[(a_1, a_2) \wedge (c_1, c_2)\right]$$

to get $(a_1, a_2) \leq (b_1, b_2)$ and $(a_1, a_2) \leq (c_1, c_2)$.

Now, if S is an Arf semigroup, we have $(b_1 + c_1 - a_1, b_2 + c_2 - a_2) \in S$; therefore $b_1 + c_1 - a_1 \in S_1$, and thus S_1 is an Arf semigroup. $\square$

Remark 2.16. Let E be an integrally closed ideal in $g(S) \cup E$ and let $\boldsymbol{a}, \boldsymbol{b} \in S \bowtie^g E$ such that $\boldsymbol{a} \leq \boldsymbol{b}$. If $\boldsymbol{a} \in g^{-1}(E) \times E$, then $\boldsymbol{b} \in g^{-1}(E) \times E$.

In fact, keeping in mind Lemma 2.12, the only other possibility for $\boldsymbol{b}$ is that it belongs to the set D; now, if $\boldsymbol{b} \in D$, since $\boldsymbol{b} \geq \boldsymbol{a} \in g^{-1}(E) \times E$, we have $a_2 \leq b_2 = sb_1$ so that $b_2 \in E$. Therefore $b_1 \in g^{-1}(E)$, hence $\boldsymbol{b} \in g^{-1}(E) \times E$.

Proposition 2.17. *Assume that S and $g(S) \cup E$ are both Arf numerical semigroups and let E be an integrally closed ideal of $g(S) \cup E$. Then $S \bowtie^g E$ is an Arf semigroup.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S \bowtie^g E$ be such that $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a} \leq \boldsymbol{c}$; since S and $g(S) \cup E$ are Arf semigroups we have $b_1 + c_1 - a_1 \in S$ and $b_2 + c_2 - a_2 \in g(S) \cup E$. By Lemma 2.12 and the above remark $S \bowtie^g E = D \cup (g^{-1}(E) \times E)$ and, if $\boldsymbol{a} \in g^{-1}(E) \times E$ then $\boldsymbol{b}, \boldsymbol{c} \in g^{-1}(E) \times E$ too. So we need to consider the following cases:

- if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in D$, then $b_2 + c_2 - a_2 = s(b_1 + c_1 - a_1) \in g(S)$, hence $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \in D \subseteq S \bowtie^g E$;

- if $\boldsymbol{a} \in D$ and at least one of $\boldsymbol{b}$ and $\boldsymbol{c}$ is in $g^{-1}(E) \times E$, for instance $\boldsymbol{c} \in g^{-1}(E) \times E$, then we have $s(b_1 + c_1 - a_1) \in g(S) \cup E$ and $s(b_1 + c_1 - a_1) \geq sc_1$, with $sc_1 \in E$. Since E is integrally closed in $g(S) \cup E$, it follows that $s(b_1 + c_1 - a_1) \in E$, so $b_1 + c_1 - a_1 \in g^{-1}(E)$. Hence $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \in g^{-1}(E) \times E \subseteq S \bowtie^g E$;

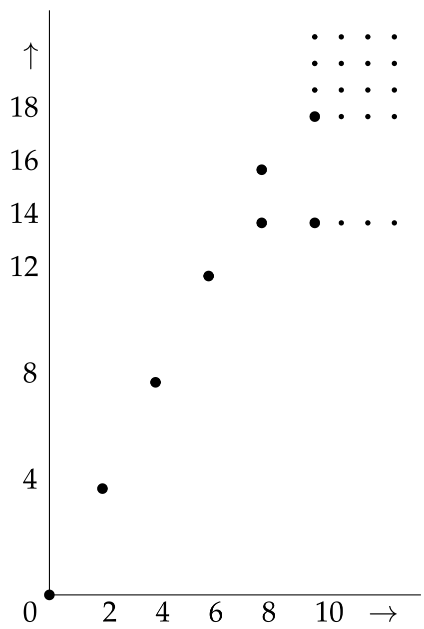
- if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in g^{-1}(E) \times E$ we have $s(b_1 + c_1 - a_1) \geq sc_1$ and $b_2 + c_2 - a_2 \geq c_2$ with $sc_1, c_2 \in E$, therefore $b_1 + c_1 - a_1 \in g^{-1}(E)$ and $b_2 + c_2 - a_2 \in E$. Hence $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \in g^{-1}(E) \times E \subseteq S \bowtie^g E$ (remembering that, by Lemma 2.6, if E is integrally closed, also $g^{-1}(E)$ is integrally closed). □

The assuption that E is integrally closed in $g(S) \cup E$ cannot be removed in order to obtain that $S \bowtie^g E$ is an Arf semigroup, as the following example shows.

Example 2.18. Let $S = \{0, 2, 4, 6, 8, 10, \rightarrow\}$ and $T = \{0, 4, \rightarrow\}$; both S and T are Arf semigroups. Let $E = \{14, 18, \rightarrow\} \subset T$ and let $g \colon S \to T$ the multiplication by 2, that is $g(a) = 2a$ for all $a \in S$. We have $g(S) = \{0, 4, 8, 12, 16, 20, 22, \dots\}$ and therefore

$$g(S) \cup E = \{0, 4, 8, 12, 14, 16, 18, \rightarrow\}.$$

Notice that $16 \notin E$, so E is not integrally closed in $g(S) \cup E$. Computing $g^{-1}(E)$ we get $\{10, \rightarrow\}$; thus $S \bowtie^g E$ is the good semigroup depicted in the following picture.



The obtained semigroup $S \bowtie^g E$ is not an Arf semigroup: if we consider $\boldsymbol{a} = (8,14), \boldsymbol{b} = (8,16)$ and $\boldsymbol{c} = (10,14) \in S \bowtie^g E$, we have $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a} \leq \boldsymbol{c}$, but $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} = (10,16) \notin S \bowtie^g E$.

We conclude the section by showing that the sufficient condition of Proposition 2.17 is in fact a characterization.

Theorem 2.19. *Given two numerical semigroups $S, T \subseteq \mathbb{N}$ let $g\colon S \to T$ a semigroup homomorphism and E a semigroup ideal of T. The following conditions are equivalent.*

1. *$S \bowtie^g E$ is an Arf semigroup.*

2. *S and $g(S) \cup E$ are Arf semigroups with E integrally closed in $g(S) \cup E$.*

Proof. We need to show only the implication $1 \Rightarrow 2$. By Lemma 2.15 we know that both S and $g(S) \cup E$, are Arf numerical semigroups. So it remains to prove that E in integrally closed in $g(S) \cup E$.

Suppose, by contradiction, that E is not integrally closed in $g(S) \cup E$ and set $\tilde{e} = \min E$; then there is an element $b_2 \in g(S) \cup E$ such that $b_2 > \tilde{e}$ and $b_2 \notin E$; hence $b_2 \in g(S)$, and so $b_2 = g(b_1)$, for some $b_1 \in S$. Consider $\boldsymbol{b} = (b_1, b_2) \in S \bowtie^g E$ and let c_2 be an element in E such that $c_2 < b_2$; let $\boldsymbol{c} = (c_1, c_2) \in g^{-1}(E) \times E$ be such that $c_1 > b_1$ (such an element exists, since $g^{-1}(E)$ is an ideal of S, so it has finite complement in $\mathbb{N}$). Let $\boldsymbol{a} = \boldsymbol{b} \wedge \boldsymbol{c} \in S \bowtie^g E$; by construction we have $\boldsymbol{a} \leq \boldsymbol{b}$, $\boldsymbol{a} \leq \boldsymbol{c}$ and $\boldsymbol{a} = (b_1, c_2)$.

We claim that $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} = (c_1, b_2) \notin S \bowtie^g E$. Since $c_1 \neq b_1$ and $b_2 = g(b_1) \notin E$, the unique possibility to have $(c_1, b_2) \in S \bowtie^g E$ is that $(c_1, b_2) = \boldsymbol{d} \wedge \boldsymbol{e}$ for some $\boldsymbol{d} \in D$ and $\boldsymbol{e} \in g^{-1}(E) \times E$. Hence $(c_1, b_2) = (d_1, e_2)$ or $(c_1, b_2) = (e_1, d_2)$.

- If $(c_1, b_2) = (d_1, e_2)$, then $b_2 = e_2 \in E$, that is a contradiction.

- If $(c_1, b_2) = (e_1, d_2)$, then $g(b_1) = sb_1 = b_2 = d_2 = sd_1$, and so $b_1 = d_1$; then $c_1 = e_1 \leq d_1 = b_1$. That is a contradiction, because we chose $c_1 > b_1$.

This shows that $\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \notin S \bowtie^g E$, a contradiction against the assumption that $S \bowtie^g E$ is Arf. $\square$

Chapter 3

Amalgamated algebras

In this chapter we talk about the weakly Arf property for amalgamated algebras. To do this we will try to follow a proof similar to the one done for the amalgamation of semigroups in Chapter 2.

3.1 Basic properties of the amalgamated algebra

In this section we recall properties of the amalgamation, that we will need in the sequel. We start recalling the definition (cf. [8]). As we declared in the first chapter, all the rings considered are commutative with unity element.

Definition 3.1. Let R and S be two rings, let J be an ideal of S and $f\colon R \to S$ be a ring homomorphism. We define *the the amalgamation of R with S along J with respect to f* as the following subring of $R \times S$

$$R \bowtie^f J = \{(r, f(r)+j) \mid r \in R, j \in J\}.$$

The next proposition shows the connection between the amalgamation and its projections.

Remark 3.2. By [8, Proposition 5.1] we have:

1. Let $\iota := \iota_{R,f,J}\colon R \to R \bowtie^f J$ be the natural ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in R$. Then ι is an embedding, making $R \bowtie^f J$ a ring extension of R (with $\iota(R) = \Gamma(f) := \{(a, f(a)) \mid a \in R\}$ a subring of $R \bowtie^f J$).

2. Let $p_R\colon R \bowtie^f J \to R$ and $p_S\colon R \bowtie^f J \to S$ be the natural projections of $R \bowtie^f J \subseteq R \times S$ into R and S, respectively. Then p_R is surjective and $\ker(p_R) = \{0\} \times J$. Moreover, $p_S(R \bowtie^f J) = f(R) + J$ and $\ker(p_S) = f^{-1}(J) \times \{0\}$. Hence, the following canonical isomorphisms hold:

$$\frac{R \bowtie^f J}{\{0\} \times J} \cong R \text{ and } \frac{R \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(R) + J.$$

3. Let $\gamma\colon R \bowtie^f J \to (f(R)+J)/J$ be the natural ring homomorphism, defined by $(a, f(a)+j) \mapsto f(a)+J$. Then γ is surjective and $\ker(\gamma) = f^{-1}(J) \times J$. Thus, there exists a natural isomorphism

$$\frac{R \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{f(R)+J}{J}.$$

For greater clarity and to shorten we set $S_\diamond = f(R) + J$.

The next two results show sufficient conditions to determine the total ring of fractions and the integral closure of the amalgamation, whose knowledge is necessary to work with the WA property.

Proposition 3.3 ([9, Proposition 3.1]). *Assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Then $Q(R \bowtie^f J)$ is canonically isomorphic to $Q(R) \times Q(S)$.*

Proposition 3.4 ([9, Proposition 3.4]). *Assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Then $\overline{R \bowtie^f J}$ (i.e., the integral closure of $R \bowtie^f J$ in its total ring of fractions) coincides with $\overline{R} \times \overline{S_\diamond}$. In particular, if f is an integral homomorphism (i.e. $f(R) \subseteq S$ is an integral extension), then $\overline{R \bowtie^f J} = \overline{R} \times \overline{S}$.*

3.2 The weakly Arf property

In light of the last two results, in order to study the weakly Arf property for $R \bowtie^f J$ we will need to assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively. Moreover, to control the non-zerodivisors of the amalgamation we will assume some extra hypothesis. For a ring R, let $W(R)$ be the set of non-zerodivisors of R.

Proposition 3.5. *Assume J and $f^{-1}(J)$ are regular ideals of S and R, respectively, and that $f(a) \in W(S)$, for all $a \in W(R)$. Then, if $R \bowtie^f J$ is a WA ring, also R is WA.*

Proof. Let $a, b, c \in R$ be such that $c \in W(R)$ and $a/c, b/c \in \overline{R}$. By hypothesis $f(c) \in W(S)$, then $\alpha = (a/c, f(a)/f(c))$ and $\beta = (b/c, f(b)/f(c))$ are elements of $Q(R) \times Q(S)$. Moreover $(c, f(c)) \in W(R \bowtie^f J)$, therefore

$$\frac{(a, f(a))}{(c, f(c))}, \frac{(b, f(b))}{(c, f(c))} \in Q(R \bowtie^f J)$$

and we can write

$$\alpha = \frac{(a, f(a))}{(c, f(c))} \quad \text{and} \quad \beta = \frac{(b, f(b))}{(c, f(c))}.$$

Since $a/c \in \overline{R}$, there exists a monic polynomial $h(X) \in R[X]$ such that a/c is a root of $h(X)$, say $h(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$; define the polynomial

$$f(h)(X) = X^n + \sum_{i=0}^{n-1} f(a_i) X^i \in S[X].$$

Then, it is straightforward to see that $\alpha = (a/c, f(a)/f(c))$ is a root of the polynomial $X^n + \sum_{i=0}^{n-1} (a_i, f(a_i)) X^i$, that is $\alpha \in \overline{R \bowtie^f J}$. Similarly we get that $\beta \in \overline{R \bowtie^f J}$.

By hypothesis $R \bowtie^f J$ is a WA ring, therefore

$$\frac{(a, f(a))(b, f(b))}{(c, f(c))} = \left(\frac{ab}{c}, \frac{f(ab)}{f(c)}\right) \in R \bowtie^f J$$

and so $ab/c \in R$, which means that R is a WA ring. □

Proposition 3.6. *Assume that J and $f^{-1}(J)$ are regular ideals of S and R, respectively, and that $f(a) \in W(S)$ for all $a \in W(R)$; furthermore, assume that J is also an ideal of $\overline{S_\diamond}$. Then, if R is a WA ring, also $R \bowtie^f J$ is WA.*

Proof. Let $\alpha, \beta, \gamma \in R \bowtie^f J$ such that $\gamma \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$. By Proposition 3.4, we have $\overline{R \bowtie^f J} \cong \overline{R} \times \overline{S_\diamond}$, so we can write $\alpha/\gamma = (a, y)$ and

$\beta/\gamma = (b,z)$ with $a,b \in \overline{R}$ and $y,z \in \overline{S_\diamond}$. Let $\gamma = (c, f(c)+x)$ with $c \in R$ and $x \in J$, Then

$$\alpha = \gamma\frac{\alpha}{\gamma} = (c, f(c)+x)(a,y) = (ac, f(c)y + xy)$$
$$\beta = \gamma\frac{\beta}{\gamma} = (c, f(c)+x)(b,z) = (bc, f(c)z + xz).$$

It follows that

$$\frac{\alpha\beta}{\gamma} = \gamma\frac{\alpha}{\gamma}\frac{\beta}{\gamma} = (ac, f(c)y+xy)(b,z) = (abc, f(c)yz + xyz).$$

Since $f(W(R)) \subseteq W(S)$, we can define $f^*\colon Q(R) \to Q(S)$ as follows: for any $r/u \in Q(R)$ we set $f^*(r/u) = f(r)/f(u) \in Q(S)$ (with a little abuse of notation we will continue to identify f^* with f). Since $\alpha, \beta \in R \bowtie^f J$ we can write $\alpha = (ac, f(ac)+x')$ and $\beta = (bc, f(bc)+x'')$ for some $x', x'' \in J$. Now, $\alpha = \gamma(\alpha/\gamma)$ implies $f(ac)+x' = f(c)y + xy$, then it follows

$$\begin{aligned} f(c)yz + xyz = f(ac)z + x'z = f(a)\big(f(c)z + xz\big) + x'z - f(a)xz = \\ = f(a)\big(f(bc) + x''\big) + x'z - f(a)xz = \\ = f(abc) + f(a)x'' + x'z - f(a)xz \end{aligned}$$

with $f(a)x'' \in J$ and $x'z, f(a)xz \in J\overline{S_\diamond} = J$.

Moreover, by Proposition 3.3, we have $Q(R \bowtie^f J) \cong Q(R) \times Q(S)$ and by $\gamma \in W(R \bowtie^f J)$, we get that γ is unit in $Q(R \bowtie^f J)$, and so c is unit in $Q(R)$, hence $c \in W(R)$. On the other hand, since $a, b \in \overline{R}$, we have $ca, cb \in c\overline{R}$.

The ring R is WA by hypothesis; since we also have $ac, bc \in R$, we obtain

$$\frac{(ca)(cb)}{c} = abc \in R$$

it follows that $\alpha\beta/\gamma \in R \bowtie^f J$, that is $R \bowtie^f J$ is a WA ring. □

With the previous proposition we obtained a sufficient condition for the amalgamation to be WA. It would be interesting to understand if it is also necessary.

In the remaining part of this section we will give a characterization for the WA property of the amalgamation, adding some additional hypotheses.

First of all, assume that R and S are integral domains; in this case we have the following.

Proposition 3.7 ([8, Proposition 5.2]). *Under the notation of the previous section, if we assume that $J \neq \{0\}$, then the following conditions are equivalent:*

1. *$R \bowtie^f J$ is an integral domain.*
2. *$S_\diamond$ is an integral domain and $f^{-1}(J) = \{0\}$.*

We note that, if $J = \{0\}$ we have $R \bowtie^f J \cong R$. On the other hand, if $f^{-1}(J) = \{0\}$ we have $R \bowtie^f J \cong S_\diamond$. Hence, the case that is interesting to study is when $J \neq \{0\}$ and $f^{-1}(J) \neq \{0\}$. In particular, by Proposition 3.3, we have that $\overline{R \bowtie^f J} \cong \overline{R} \times \overline{S_\diamond}$ and, although R and $S_\diamond$ are integral domain, $R \bowtie^f J$ is not an integral domain.

The following result will be very useful in the sequel.

Lemma 3.8. *Let I be an ideal of a ring R, let $x, i \in R$ be such that $x/i \in \overline{R}$ and $i \in I$. Then $x \in \overline{I}$.*

Proof. We have $x \in i\overline{R} \subseteq I\overline{R}$; by [12, Proposition 1.6.1.] $\overline{I} = \overline{I\overline{R}} \cap R$, therefore $x \in \overline{I}$. ◻

Lemma 3.9. *Assume that $\overline{S_\diamond}$ is a valuation domain and J is integrally closed in $S_\diamond$. Let $x, y \in S_\diamond$ be such that $x \notin J$ and $y = f(a) + j$, with $a \in R$ and $j \in J$; then $y/x \in \overline{S_\diamond}$ if and only if $f(a)/x \in \overline{S_\diamond}$.*

Proof. Since $j \in J$, we have $x/j \in \overline{S_\diamond}$ if and only if $x \in j\overline{S_\diamond} \cap S_\diamond \subseteq \overline{J} = J$. Thus, $x/j \notin \overline{S_\diamond}$, since by hypothesis $x \notin J$. Moreover, since $\overline{S_\diamond}$ is a valuation domain, $x/j \notin \overline{S_\diamond}$ implies $j/x \in \overline{S_\diamond}$. Now, by

$$\frac{y}{x} = \frac{f(a)}{x} + \frac{j}{x}$$

the thesis follows immediately. ◻

Notice that the assumption in Proposition 3.5, that J is an ideal of $\overline{S}_\diamond$, is different from the assumption in the next proposition, that J is integrally closed in $S_\diamond$. Moreover, we could not find any implication between the two assumptions.

Proposition 3.10. *Let R and S be two integral domains, let $J \neq (0)$ be an ideal of S such that $f^{-1}(J) \neq (0)$. We also assume that $\overline{S_\diamond}$ is a valuation domain. Assume that one of the following two conditions holds:*

(1) $f(W(R)) \subseteq J$;

(2) $f(R)$ *is a field, namely* J *is a maximal ideal of* $S_\diamond$.

Then, if R and $S_\diamond$ are WA and J is integrally closed in $S_\diamond$, $R \bowtie^f J$ is WA.

Proof. Let $\alpha, \beta, \gamma \in R \bowtie^f J$ such that $\gamma \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$, we write $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ and $\gamma = (c_1, c_2)$. So $a_1/c_1, b_1/c_1 \in \overline{R}$ and $a_2/c_2, b_2/c_2 \in \overline{S_\diamond}$. Since R and $S_\diamond$ are WA, we have $a_1b_1/c_1 \in R$ and $a_2b_2/c_2 \in S_\diamond$, hence $\alpha\beta/\gamma \in R \times S_\diamond$. Since J is integrally closed in $S_\diamond$, also $f^{-1}(J)$ is integrally closed in R; thus, if $c_2 \in J$ (that implies also $c_1 \in f^{-1}(J)$), by Lemma 3.8 we get $\alpha, \beta, \gamma \in f^{-1}(J) \times J$. Furthermore, we note that

$$\frac{a_1b_1}{c_1} = c_1\frac{a_1}{c_1}\frac{b_1}{c_1} \in c_1\overline{R} \cap R \subseteq \overline{(f^{-1}(J))\overline{R}} \cap R = \overline{f^{-1}(J)}$$
$$\frac{a_2b_2}{c_2} = c_2\frac{a_2}{c_2}\frac{b_2}{c_2} \in c_2\overline{S_\diamond} \cap S_\diamond \subseteq \overline{J\overline{S_\diamond}} \cap S_\diamond = \overline{J}$$

hence

$$\frac{\alpha\beta}{\gamma} \in f^{-1}(J) \times J \subseteq R \bowtie^f J.$$

In particular, if condition *(1)* holds, this is the only possible case.

On the other hand, by definition of $R \bowtie^f J$, it follows that a pair $(x, y) \in R \bowtie^f J$ if and only if $f(x) - y \in J$. If $c_2 \notin J$ and condition *(2)* holds, we want show that

$$f\left(\frac{a_1b_1}{c_1}\right) - \frac{a_2b_2}{c_2} \in J.$$

We observe that

$$f\left(\frac{a_1b_1}{c_1}\right) - \frac{f(a_1)f(b_1)}{f(c_1)} = \frac{f(c_1)f(\frac{a_1b_1}{c_1}) - f(a_1)f(b_1)}{f(c_1)} =$$
$$= \frac{f(c_1\frac{a_1b_1}{c_1}) - f(a_1b_1)}{f(c_1)} = 0$$

Therefore, $f(a_1b_1/c_1) = f(a_1)f(b_1)/f(c_1)$, and so, we can write

$$f\left(\frac{a_1b_1}{c_1}\right) - \frac{a_2b_2}{c_2} = \frac{f(a_1)f(b_1)}{f(c_1)} - \frac{a_2b_2}{c_2} =$$
$$= \frac{c_2f(a_1)f(b_1) - f(c_1)a_2b_2}{f(c_1)c_2}.$$

Since $c_2 = f(c_1) + h$ for same $h \in J$ and $f(a_1)f(b_1) - a_2b_2 \in J$, we have

$$c_2f(a_1)f(b_1) - f(c_1)a_2b_2 = f(c_1)\big(f(a_1)f(b_1) - a_2b_2\big) + hf(a_1)f(b_1) \in J.$$

Then, there is $j \in J$ such that

$$f\left(\frac{a_1b_1}{c_1}\right) - \frac{a_2b_2}{c_2} = \frac{j}{f(c_1)c_2}.$$

Now, by condition *(2)*, $f(c_1)$ and c_2 are invertible in $S_\diamond$, end so $f(c_1)c_2$ is invertible in $S_\diamond$; it follows that $j/f(c_1)c_2 \in J$. Hence, also in this case we have $\alpha\beta/\gamma \in R \bowtie^f J$. $\square$

It remains to be determined whether the proposition holds even when conditions *(1)* and *(2)* are removed. Further studies on this matter are desirable, and it is our intention to pursue them.

Now we will study when the converse of the above proposition holds; to this aim we will see that we do not need to assume the two technical conditions *(1)* and *(2)*. On the other hand, we will have to assume that $\overline{S_\diamond}$ is a DVR. First of all we recall that, if R is a local, Noetherian, one-dimensional domain, whose integral closure $\overline{R}$ is a DVR, then, every ideal I has a principal reduction (x), with x any element of minimal value in I; moreover $\overline{I} = x\overline{R} \cap R$. Indeed, since $x\overline{R}$ is integrally closed in $\overline{R}$, then $\overline{(x)} = x\overline{R} \cap R$ by [12, Proposition 1.6.1]. On the other hand, since $\overline{R}$ is a DVR, then $I\overline{R}$ is a principal ideal generated by any element x of minimal value, and thus, $I\overline{R} = x\overline{R}$. Therefore, $(x) \subseteq I \subseteq \overline{(x)}$, and so, being I finitely generated, (x) is a principal reduction of I by [12, Corollary 1.2.5]. As a consequence of this fact, we have that: for any $i \in I$, then $i/x \in \overline{R}$.

Lemma 3.11. *Let R be a local Noetherian domain, such that $\overline{R}$ is a DVR and $(R\colon \overline{R}) \neq (0)$; denote the associated valuation by v. Let I be an ideal of R and let $e_0 \in I$ be a minimal reduction of I. Let $c \in (R\colon \overline{R}) \setminus \{0\}$.*

Then, for any $a \in \overline{R}$ such that $v(a) > v(ce_0)$, we have $a \in I$.

Proof. By $c \in (R\colon \overline{R})$ it follows that $c\overline{R} \subseteq R$; therefore, if $v(a) > v(ce_0)$, we have $a \in e_0(c\overline{R}) \subseteq I$. Thus $a \in I$. $\square$

Lemma 3.12. *Let R be a local Noetherian domain such that $\overline{R}$ is a DVR and $(R\colon \overline{R}) \neq (0)$; denote the associated valuation by v. Denote by $\mathfrak{n}$ the maximal ideal of $\overline{R}$ and by $\mathfrak{m} = \mathfrak{n} \cap R$ the maximal ideal of R. Let I be an ideal of R not integrally closed and let $e_0 \in I$ be a minimal reduction of I.*

If $\overline{R}/\mathfrak{n} \cong R/\mathfrak{m}$, then there exists $a \in R \setminus I$ such that $a/e_0 \in \overline{R}$ and $v(a) \notin v(I) = \{v(i) | i \in I \setminus \{0\}\}$.

Proof. Since I is not integrally closed, there exists $a \in \overline{I} \setminus I \subseteq R$; the choice of e_0 implies that $v(a) - v(e_0) > 0$, and so $a/e_0 \in \overline{R}$.

If $v(a) \in v(I)$, that is $v(a) = v(i_1)$ for some $i_1 \in I$, $a/i_1 \in \overline{R} \setminus \mathfrak{n}$; since $\overline{R}/\mathfrak{n} \cong R/\mathfrak{m}$ there exists $b_1 \in R \setminus \mathfrak{m}$ such that $b_1 - a/i_1 \in \mathfrak{n}$, that implies $v(b_1 i_1 - a) = v(i_1(b_1 - a/i_1)) > v(i_1) = v(a)$; furthermore $a - b_1 i_1 \in R \setminus I$ and $(a - b_1 i_1)/e_0 \in \overline{R}$.

If $v(a - b_1 i_1) \notin v(I)$ we get the thesis replacing a with $a - b_1 i_1$. Otherwise, as above, we can find $i_2 \in I$ and $b_2 \in R \setminus \mathfrak{m}$ such that $v(a - b_1 i_1 - b_2 i_2) > v(a - b_1 i_1)$; furthermore $a - b_1 i_1 - b_2 i_2 \in R \setminus I$ and $(a - b_1 i_1 - b_2 i_2)/e_0 \in \overline{R}$.

Inductively, for any $r \in \mathbb{N}_+$ we find the element $a_{r-1} = a - b_1 i_1 - \ldots - b_{r-1} i_{r-1} \in R \setminus I$ such that $a_{r-1}/e_0 \in \overline{R}$, if $v(a_{r-1}) \in v(I)$, we can find $i_r \in I$ and $b_r \in R \setminus I$ such that the element $a_r = a - b_1 i_1 - \ldots - b_r i_r \in R \setminus I$, $a_r/e_0 \in \overline{R}$ and $v(a_r) > v(a_{r-1})$.

We claim that, after a finite number of steps we get $v(a_r) \notin v(I)$. Indeed, if $c \in (R \colon \overline{R}) \setminus \{0\}$, since the sequence of the $v(a_i)$ is strictly increasing, there exists $r \in \mathbb{N}_+$ such that $v(a_r) > v(c e_0)$ and, by Lemma 3.11, we get $a_r \in I$ that is a contradiction. $\square$

Now, we assume that both R and S are integral domains and $\overline{S_\diamond}$ is a DVR with valuation v. We assume that J is a finitely generated ideal of S and that $f^{-1}(J) \neq \{0\}$. Then, by Proposition 3.4, we have $\overline{R \bowtie^f J} = \overline{R} \times \overline{S_\diamond}$. Assume also that $(S_\diamond \colon \overline{S_\diamond}) \neq (0)$ and that the residue fields of $\overline{S_\diamond}$ and $S_\diamond$ are isomorphic (so we can apply the previous lemma).

With this setting, we have the following.

Theorem 3.13. *If J is not integrally closed in $S_\diamond$, then $R \bowtie^f J$ is not WA.*

Proof. Let $e_0 \in J$ such that $J\overline{S_\diamond} = e_0 \overline{S_\diamond}$, since J is not integrally closed in $S_\diamond$ we can take $y \in S_\diamond$ such that $y/e_0 \in \overline{S_\diamond}$; by Lemma 3.12 we can assume that $v(y) \notin v(J)$; we write $y = f(x) + j$ with $x \in R$ and $j \in J$, we set $a = (x, y) \in R \bowtie^f J$. We can choose $i \in J$ such that $v(y) > v(i)$, and we set $b = (0, i) \in R \bowtie^f J$. Let $c = a + b \in R \bowtie^f J$, we observe that $a/c, b/c \in \overline{R \bowtie^f J}$; indeed we have

$$\frac{a}{c} = \frac{(x,y)}{(x, y+i)} = \left(1, \frac{y}{y+i}\right) \qquad \text{and} \qquad \frac{b}{c} = \frac{(0,i)}{(x, y+i)} = \left(0, \frac{i}{y+i}\right)$$

clearly $0, 1 \in \overline{R}$; moreover $v(y) - v(y+i) = v(y) - v(i) > 0$ and $v(i) - v(y+i) = 0$, thus

$$\frac{y}{y+i}, \frac{i}{y+i} \in \overline{S_\diamond}.$$

Now, we want to show that $ab/c \notin R \bowtie^f J$, which is equivalent to prove that $yi/(y+i) \notin J$.

If $\frac{yi}{y+i} \in J$ we would have $yi \in (y+i)J$, so $yi = (y+i)h$ for some $h \in J$; then $v(y) + v(i) = v(y+i) + v(h)$ with $v(y+i) = v(i)$, therefore $v(y) = v(h) \in v(J)$ that is a contradiction. Thus $ab/c \notin R \bowtie^f J$, that is $R \bowtie^f J$ is not WA. $\square$

Now, we are ready for the proof of the main result. First, we will make a recap of the hypotheses.

Let R and S be two local Noetherian domains, let $f \colon R \to S$ be a ring homomorphism. Let $J \neq (0)$ be a finitely generated ideal of S which is integrally closed in $S_\diamond$

and such that $f^{-1}(J) \neq (0)$. We assume that $\overline{S_\diamond}$ is a DVR, that $(S_\diamond : \overline{S_\diamond}) \neq (0)$ and that the residue fields of $S_\diamond$ and $\overline{S_\diamond}$ are isomorphic.

Theorem 3.14. *With our assumptions, consider the following conditions:*

(i) $R \bowtie^f J$ *is WA.*

(ii) R *and* $S_\diamond$ *are WA and* J *is integrally closed in* $S_\diamond$*.*

Then, the implication (i) $\Rightarrow$ (ii) holds and the converse holds if one of the two conditions of Proposition 3.10 is satisfied.

Proof. By Proposition 3.10 we only need to prove that *(i)* implies *(ii)*. Furthermore, if *(i)* holds then J is integrally closed in $S_\diamond$ by Theorem 3.13. So it remains to prove only that R and $S_\diamond$ are WA.

Let $a, b, c \in R$ be such that $c \neq 0$ and $a/c, b/c \in \overline{R}$, we set $\alpha = (a, f(a))$ and $\beta = (b, f(b)) \in R \bowtie^f J$. We can have two cases:

1. if $f(c) \neq 0$, then we have $\gamma = (c, f(c)) \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$. Since $R \bowtie^f J$ is WA, we get $\alpha\beta/\gamma \in R \bowtie^f J$. Thus $ab/c \in R$;

2. if $f(c) = 0$, then we have $c \in f^{-1}(J)$, and therefore $a \in c\overline{R} \cap R \subseteq \overline{f^{-1}(J)} = f^{-1}(J)$. Since J is finitely generated there is $e \in J$ such that $v(e) = \min v(J)$. We note that $v(f(a)) \geq v(e)$ and $v(f(b)) \geq v(e)$, hence $f(a)/e, f(b)/e \in \overline{S_\diamond}$. Let $\gamma = (c, e) \in W(R \bowtie^f J)$, then $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$; by hypothesis we get $\alpha\beta/\gamma \in R \bowtie^f J$, thus $ab/c \in R$.

This shows that R is WA.

Now, we check that $S_\diamond$ is WA; let $x, y, z \in S_\diamond$ such that $x \neq 0$ and $y/x, z/x \in \overline{S_\diamond}$, we say $x = f(c) + h$, $y = f(a) + j$ and $z = f(b) + i$ with $a, b, c \in R$ and $j, i, h \in J$. We can have two cases:

1. if $x \in J$, since $y, z \in x\overline{S_\diamond}$ we have $y, z \in J$ by Lemma 3.8. We take $d \in f^{-1}(J)$ such that $d \neq 0$ and we set $\alpha = (d, y)$, $\beta = (d, z)$ and $\gamma = (d, x)$; then $\alpha, \beta \in R \bowtie^f J$ and $\gamma \in W(R \bowtie^f J)$. We compute
$$\frac{\alpha}{\gamma} = \left(\frac{d}{d}, \frac{y}{x}\right) = \left(1, \frac{y}{x}\right) \in \overline{R \bowtie^f J}$$
$$\frac{\beta}{\gamma} = \left(\frac{d}{d}, \frac{z}{x}\right) = \left(1, \frac{z}{x}\right) \in \overline{R \bowtie^f J}$$
since $R \bowtie^f J$ is WA we get $\alpha\beta/\gamma \in R \bowtie^f J$, thus $yz/x \in S_\diamond$;

2. if $x \notin J$, by Lemma 3.9 we have $f(a)/x \in \overline{S_\diamond}$, therefore
$$f(a) \in x\overline{S_\diamond} \subseteq f(c)(\overline{S_\diamond}) + J.$$
Then there is $k \in J$ such that $(f(a) + k)/f(c) \in \overline{S_\diamond}$, again by Lemma 3.9 we get $f(a)/f(c) \in \overline{S_\diamond}$, similarly $f(b)/f(c) \in \overline{S_\diamond}$. Therefore there is a monic polynomial $P(t) \in S_\diamond[t]$ such that $P(f(a)/f(c)) = 0$. We write
$$P(t) = t^n + \sum_{i=0}^{n-1} w_i t^i$$

where $w_i = f(m_i) + k_i$ with $m_i \in R$ and $k_i \in J$; we set

$$Q(t) = t^n + \sum_{i=0}^{n-1} m_i t^i \in R[t].$$

By $f(Q(t)) = P(t) - \sum_{i=0}^{n-1} k_i t^i$ we get $f(Q(\frac{a}{c})) \in J\overline{S_\diamond} \subseteq J$, therefore $Q(t) - Q(\frac{a}{c}) \in R[t]$ and $(Q(t) - Q(\frac{a}{c}))(\frac{a}{c}) = 0$; thus $a/c \in \overline{R}$, similarly $b/c \in \overline{R}$. We set $\alpha = (a, y), \beta = (b, z), \gamma = (c, x) \in R \bowtie^f J$. Then, Since $x \notin J$, we have $c \neq 0$ so that $\gamma \in W(R \bowtie^f J)$ and $\alpha/\gamma, \beta/\gamma \in \overline{R \bowtie^f J}$, so that $\alpha\beta/\gamma \in R \bowtie^f J$ and thus $yz/x \in S_\diamond$.

This completes the proof of *(ii)*. $\square$

Example 3.15. Let $R = k[[X_1, \dots, X_n]]$ and $S = k[[Y_1, \dots, Y_m]]$, let $f \colon R \to S$ be the ring homomorphism defined by $f(X_i) = Y_1$ for any $i = 1, \dots, n$ and $f(1) = 1$. Let J be the ideal of S generated by Y_1. It's easy to check that all the hypotheses of Theorem 3.14 are satisfied observing that $f(R) + J = k[[Y_1]]$ is a DVR. Since R and $k[[Y_1]]$ are UFDs, by [12, Proposition 2.1.5] they are integrally closed, and hence they are WA rings. Moreover (Y_1) is an ideal integrally closed in $k[[Y_1]]$, therefore J is integrally closed in $f(R) + J$. Hence, by Theorem 3.14 the ring $R \bowtie^f J$ is a WA ring.

Chapter 4

A quadratic quotient of the Rees algebra

In this chapter we study the weakly Arf property of the quadratic quotients of the Rees algebra. The first problem that arises is the calculation of the integral closure, because it depends on the integral closure of R, on the polynomial over which we make the quotient and on the ideal I. In Section 4.3 we will calculate the integral closure in the case where the polynomial is reducible and in Section 4.4 we will describe when the obtained ring is WA. In Section 4.5 we will study the weakly Arf property for a construction similar to amalgamated duplication, which includes some cases of the quadratic quotient of the Rees algebra, when the polynomial is irreducible.

4.1 Basic properties

As stated at the beginning of this thesis, the ring R will be commutative and unitary. Let I be a proper ideal of R and let t be an indeterminate. We denote the *Rees algebra associated to R and I* by

$$\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n \subseteq R[t].$$

Given a monic polynomial of the type $t^2 + at + b \in R[t]$, the ring

$$R(I)_{a,b} := \frac{\mathcal{R}_+}{(t^2 + at + b)R[t] \cap \mathcal{R}_+}$$

is called a quadratic quotient of the Rees algebra. These rings were considered and studied in [3], where, in particular, it is observed that $R(I)_{a,b} = R \oplus It$ as abelian groups, with the multiplication

$$(r + it) \cdot (s + jt) = rs - bij + (rj + si - aij)t.$$

Proposition 4.1 ([3, Proposition 1.3])**.** *The following ring extensions are both integral*

$$R \subseteq R(I)_{a,b} \subseteq \frac{R[t]}{(t^2 + at + b)}.$$

In particular, the three rings have the same Krull dimension.

We recall that the Nagata's idealization of R with respect to I, denoted by $R \ltimes I$, is defined as the R-module $R \oplus I$ endowed with the multiplication

$$(r, i) \cdot (s, j) = (rs, rj + si).$$

We also recall that the amalgamated duplication of R with respect to I, denoted by $R \bowtie I$, is defined as the R-module $R \oplus I$ endowed with the multiplication

$$(r,i) \cdot (s,j) = (rs, rj + si + ij).$$

Remark 4.2. We have the following ring isomorphisms (see [3, Proposition 1.4]):

1. $R(I)_{0,0} \cong R \ltimes I$.
2. $R(I)_{-1,0} \cong R \bowtie I$.

4.2 The integral closure

Let Q be the total ring of fractions of $R(I)_{a,b}$ and let $\overline{R(I)_{a,b}}$ be the integral closure of $R(I)_{a,b}$ in Q.

In order to understand the WA property for $R(I)_{a,b}$, it is necessary to determine explicitly $\overline{R(I)_{a,b}}$. The following two proposition are our starting point.

Proposition 4.3 ([3, Proposition 1.7])**.** *Each element of Q is of the form $\frac{r+it}{u}$, where u is a regular element of R.*

Proposition 4.4 ([3, Corollary 1.8])**.** *Assume that I is a regular ideal; then the rings $R(I)_{a,b}$ and $R[t]/(t^2+at+b)$ have the same total ring of fractions and the same integral closure.*

In particular, we have $Q = Q(R) + Q(R)t$, where $Q(R)$ is the total ring of fractions of R. For this reason, throughout the rest of the chapter we will assume that I is a regular ideal.

Remark 4.5. Given $\alpha \in Q$, we can write $\alpha = \frac{r}{u} + \frac{s}{u}t$ with $r, s \in R$ and $u \in W(R)$. Then the following quadratic relation holds for α:

$$\alpha^2 - \alpha\left(2\frac{r}{u} - a\frac{s}{u}\right) + \frac{r^2}{u^2} + b\frac{s^2}{u^2} - a\frac{rs}{u^2} = 0. \tag{4.1}$$

It follows that $\alpha \in \overline{R(I)_{a,b}}$ if and only if the coefficients

$$2\frac{r}{u} - a\frac{s}{u} \in \overline{R} \qquad\qquad \frac{r^2}{u^2} + b\frac{s^2}{u^2} - a\frac{rs}{u^2} \in \overline{R},$$

where $\overline{R}$ is the integral closure of R in $Q(R)$.

As special cases we have the following:

- If $a = 0$ and R contains a field of characteristic different by two, we have $\alpha \in \overline{R(I)_{0,b}}$ if and only if the coefficients of (4.1) are in $\overline{R}$, which in this case means

  $$2\frac{r}{u} \in \overline{R} \qquad\qquad \frac{r^2}{u^2} + b\frac{s^2}{u^2} \in \overline{R}.$$

  By $2\frac{r}{u} \in \overline{R}$ it follows that $\frac{r}{u} \in \overline{R}$, and so $b\frac{s^2}{u^2} \in \overline{R}$.

  We observe that, if $\gamma^2 \in \overline{R}$, there is a monic polynomial $f(T) \in R[T]$ such that $f(\gamma^2) = 0$; let's consider $g(T) = f(T^2) \in R[T]$ which is a monic polynomial such that $g(\gamma) = f(\gamma^2) = 0$, thus $\gamma \in \overline{R}$. In particular, since $b^2\frac{s^2}{u^2} \in \overline{R}$, we get $b\frac{s}{u} \in \overline{R}$ and therefore $\frac{s}{u} \in (\overline{R} \colon b)$. Hence $\alpha \in \overline{R} + (\overline{R} \colon b)t$.

- if $b = 0$ we have $\alpha \in \overline{R(I)_{a,0}}$ if and only if

$$2\frac{r}{u} - a\frac{s}{u} \in \overline{R} \qquad\qquad \frac{r^2}{u^2} - a\frac{rs}{u^2} \in \overline{R}$$

If $\frac{r}{u} \in \overline{R}$, we get $a\frac{s}{u} \in \overline{R}$; therefore $\frac{s}{u} \in (\overline{R} : a)$.

If $\frac{r}{u} \notin \overline{R}$, we note that

$$\frac{r^2}{u^2} - \left(2\frac{r}{u} - a\frac{s}{u}\right)\frac{r}{u} + \left(\frac{r^2}{u^2} - a\frac{rs}{u^2}\right) = 0.$$

We consider the polynomial $p(X) = X^2 - (2\frac{r}{u} - a\frac{s}{u})X + \frac{r^2}{u^2} - a\frac{rs}{u^2} \in Q(R)[X]$, then $p(\alpha) = p(\frac{r}{u}) = 0$.

Moreover, if $\alpha \in \overline{R(I)_{a,0}}$ there exists a monic polynomial $q(X) \in (R(I)_{a,0})[X]$ such that $q(\alpha) = 0$. If we take $q(X)$ of minimal degree, then $q(X) = h(X)p(X)$ for some $h(X) \in Q[X]$. Hence $q(\frac{r}{u}) = 0$, that is $\frac{r}{u} \in \overline{R(I)_{a,0}}$. Now, because the extension $R \subseteq R(I)_{a,0}$ is integral, $\frac{r}{u} \in \overline{R(I)_{a,0}}$ implies that $\frac{r}{u} \in \overline{R}$, that is a contradiction.

Hence, if $\alpha \in \overline{R(I)_{a,0}}$ we must have $\frac{r}{u} \in \overline{R}$, and so $\alpha \in \overline{R} + (\overline{R}\colon a)t$.

Remark 4.6. Set $L = (\overline{R} : a) \cap (\overline{R} : b) \subseteq Q$ and assume that $bL^2 \subseteq \overline{R}$. Let $\alpha = \frac{r}{u} + \frac{s}{u}t \in \overline{R} + Lt$; it follows that $\frac{r}{u}, \frac{as}{u}, \frac{bs}{u}, \frac{bs^2}{u^2} \in \overline{R}$. Thus

$$2\frac{r}{u} - a\frac{s}{u} \in \overline{R}$$

and

$$\frac{r}{u}\left(\frac{r}{u} - a\frac{s}{u}\right) + b\frac{s^2}{u^2} = \frac{r^2}{u^2} + b\frac{s^2}{u^2} - a\frac{rs}{u^2} \in \overline{R}.$$

Hence $\alpha \in \overline{R(I)_{a,b}}$, that is $\overline{R} + Lt \subseteq \overline{R(I)_{a,b}}$.

Moreover, for the Nagata's idealization $R \ltimes I = R(I)_{0,0}$ and for the amalgamated duplication $R \bowtie I = R(I)_{-1,0}$ of R with respect to I, it is known that their integral closures are, respectively, $\overline{R \ltimes I} = \overline{R} + Q(R)t$ and $\overline{R \bowtie I} = \overline{R} + \overline{R}t$. On the other hand, it results $(\overline{R} : 0) = Q(R)$ and $(\overline{R} : -1) = \overline{R}$; in both cases we have $\overline{R(I)_{a,b}} = \overline{R} + Lt$.

Therefore, it is reasonable to believe that $\overline{R(I)_{a,b}} = \overline{R} + Lt$ for any $a, b \in R$ with $bL^2 \subseteq \overline{R}$; however, in general, we were not able to prove the equality.

4.3 The case of a reducible polynomial

In this section we consider the particular case when the polynomial $t^2 + at + b$ is reducible in $R[X]$. So we assume that there exist $\xi, \eta \in R$ such that $t^2 + at + b = (t - \xi)(t - \eta)$.

Then, we have the following diagram:

$$\begin{tikzcd}
Q \arrow[r, "\sim"] & \frac{Q(R)[t]}{(t-\xi)(t-\eta)} \arrow[r, "\psi"] & Q(R)\times Q(R) \\
& \frac{R[t]}{(t-\xi)(t-\eta)} \arrow[u, hook] \arrow[r] & R\times R \arrow[u, hook] \\
& R(I)_{a,b} \arrow[u, hook] \arrow[ur, dashed, "\vartheta"'] &
\end{tikzcd}$$

where ψ is defined by $\psi(f(t)) = (f(\xi), f(\eta))$ for any $f(t) \in Q(R)[t]$.

Assume that $\xi - \eta$ is a nonzero-divisor in R. This implies that $\xi - \eta$ is invertible in $Q(R)$ and, since it belongs to ideal $(t-\xi) + (t-\eta) \subseteq Q(R)[t]$, we obtain that $(t-\xi)$ and $(t-\eta)$ are comaximal in $Q(R)[t]$. Therefore, by Chinese Remainder Theorem, ψ is a ring isomorphism. Clearly, we can see ϑ as the restriction of ψ to $R(I)_{a,b}$. In particular, it is injective. Hence, we need to compute $\mathrm{Im}(\vartheta) = \psi(R(I)_{a,b})$.

Lemma 4.7. *Let $K = \{(r + i\xi, r + i\eta) \mid r \in R,\ i \in I\}$. Then, $Im(\vartheta) = K$.*

Proof. Let $f(t) \in R(I)_{a,b}$. We can write $f(t) = r + it$ with $r \in R$ and $i \in I$; therefore $\psi(f(t)) = \psi(r + it) = (r + i\xi, r + i\eta)$ and hence $\mathrm{Im}(\vartheta) \subseteq K$.

Conversely, let $(x, y) \in K$. Then, there are $r \in R$ and $i \in I$ such that $(x, y) = (r + i\xi, r + i\eta)$. We take $r + it \in R(I)_{a,b}$; then $\psi(r + it) = (x, y)$, that is $K \subseteq \mathrm{Im}(\vartheta)$. Hence $\mathrm{Im}(\vartheta) = K$. □

Now, we want to determine the integral closure of K in $Q(R) \times Q(R)$; let's consider the canonical projections $\pi_1 \colon Q(R) \times Q(R) \to Q(R)$ on the first factor and $\pi_2 \colon Q(R) \times Q(R) \to Q(R)$ on the second factor.

Lemma 4.8. *With these hypotheses, $\overline{K} = \overline{R} \times \overline{R}$.*

Proof. Let $(\frac{r}{u}, \frac{s}{v}) \in Q(R) \times Q(R)$ be integral over K, and let $f(t) \in K[t]$ be a monic polynomial such that $f(\frac{r}{u}, \frac{s}{v}) = 0$; write

$$f(t) = t^n + \sum_{h=0}^{n-1} a_h t^h \text{ with } a_h \in K;$$

for any $h = 0, \dots, n-1$, we can write $a_h = (r_h + i_h\xi, r_h + i_h\eta)$ with $r_h \in R$ and $i_h \in I$, therefore

$$\begin{aligned} f(t) &= t^n + \sum_{h=0}^{n-1} (r_h + i_h\xi, r_h + i_h\eta) t^h \\ &= \left(t^n + \sum_{h=0}^{n-1} (r_h + i_h\xi) t^h, t^n + \sum_{h=0}^{n-1} (r_h + i_h\eta) t^h \right) \end{aligned}$$

This shows that if $(\frac{r}{u}, \frac{s}{v})$ is integral over K then $\frac{r}{u}$ and $\frac{s}{v}$ are integral over $\pi_1(K)$ and $\pi_2(K)$, respectively.

Conversely, let $(\frac{r}{u}, \frac{s}{v}) \in \overline{\pi_1(K)} \times \overline{\pi_2(K)} \subseteq Q(R) \times Q(R)$; then, there are two monic polynomials $f_1(t) \in \pi_1(K)[t]$ and $f_2(t) \in \pi_2(K)[t]$ such that $f_1(\frac{r}{u}) = 0$ and $f_2(\frac{s}{v}) = 0$. We observe that, if $(x, y) \in \pi_1(t) \times \pi_2(t)$, there are $x', y' \in R$ such that $(x, x'), (y', y) \in K$. Therefore, if $f_1(t) = t^n + a_{n-1}t^{n-1} + \ \dots \ + a_1 t + a_0$ there are $a'_n, a'_{n-1}, \dots, a'_0 \in R$ such that $(a_h, a'_h) \in K$ for any $h = 0, \dots, n$ where $a_n = 1$. Hence, we can get a polynomial $f_1'(t) = (1, a_n')t^n + (a_{n-1}, a_{n-1}')t^{n-1} + \ldots + (a_0, a_0') \in K[t]$. Moreover, since $\vartheta(1) = (1,1)$ we have $(1,1) \in K$ and so we can choose $a_n' = 1$. Hence, $f'(t) \in K[t]$ is monic; similarly we can get $f_2'(t) \in K[t]$. Now, we can define $f(t) = f_1'(t)f_2'(t) \in K[t]$; by construction, we have that $f(t)$ is monic and $f(\frac{r}{u}, \frac{s}{v}) = 0$. Hence, $(\frac{r}{u}, \frac{s}{v}) \in \overline{K}$. This shows that $\overline{K} = \overline{\pi_1(K)} \times \overline{\pi_2(K)}$. Now, since $\pi_1(K) = \pi_2(K) = R$, we have $\overline{\pi_1(K)} = \overline{\pi_2(K)} = \overline{R}$. Hence $\overline{K} = \overline{R} \times \overline{R}$. □

Corollary 4.9. *With our hypotheses, $\overline{R(I)_{a,b}} = \psi^{-1}(\overline{R} \times \overline{R})$.*

Proof. Since ψ is injective, it follows that $\overline{R(I)_{a,b}} = \psi^{-1}(\overline{K}) = \psi^{-1}(\overline{R} \times \overline{R})$. □

Remark 4.10. For any element $(\frac{r}{u}, \frac{s}{v}) \in Q(R) \times Q(R)$ we can write

$$\begin{aligned} \frac{r}{u} &= \frac{r}{u} - \alpha\xi + \alpha\xi = \frac{r - u\alpha\xi}{u} + \alpha\xi \\ \frac{s}{v} &= \frac{s}{v} - \alpha\eta + \alpha\eta = \frac{s - v\alpha\eta}{v} + \alpha\eta \end{aligned}$$

where α is any element in $Q(R)$; if we take $\alpha = \frac{rv-su}{uv(\xi-\eta)}$ we get $\frac{r}{u} - \alpha\xi = \frac{s}{v} - \alpha\eta$, indeed

$$\begin{aligned} \frac{r}{u} - \alpha\xi &= \frac{r}{u} - \frac{rv - su}{uv(\xi - \eta)}\xi \\ &= \frac{r}{u} - \left(\frac{r}{u} - \frac{s}{v}\right)\frac{\xi}{\xi - \eta} \\ &= \frac{r}{u}\left(\frac{(\xi - \eta) - \xi}{\xi - \eta}\right) + \frac{s}{v} \cdot \frac{\xi}{\xi - \eta} \\ &= \frac{s}{v} \cdot \frac{\xi}{\xi - \eta} - \frac{r}{u} \cdot \frac{\eta}{\xi - \eta} \\ &= \frac{s}{v} + \frac{s}{v} \cdot \frac{\eta}{\xi - \eta} - \frac{r}{u} \cdot \frac{\eta}{\xi - \eta} \\ &= \frac{s}{v} - \frac{rv - su}{uv(\xi - \eta)}\eta = \frac{s}{v} - \alpha\eta \end{aligned}$$

therefore $(\frac{r}{u}, \frac{s}{v}) = \psi(\frac{r}{u} - \alpha\xi + \alpha t)$.

Remark 4.11. Since $(t - \xi)$ and $(t - \eta)$ are comaximal in $Q(R)[t]$, we have $1 = g(t)(t - \xi) + h(t)(t - \eta)$ with $g(t), h(t) \in Q(R)[t]$. It follows that $\psi(1) = (h(\xi)(\xi - \eta), g(\eta)(\eta - \xi))$ and therefore

$$h(\xi) = -g(\eta) = (\xi - \eta)^{-1} \in Q(R)$$

in particular, we can write $\alpha = (\frac{r}{u} - \frac{s}{v})h(\xi)$.

Theorem 4.12. *Let R be a commutative ring and let I be a regular ideal of R. Let $t^2 + at + b = (t - \xi)(t - \eta) \in R[t]$ be such that $\xi - \eta$ is a nonzero-divisor in R and invertible in $\overline{R}$. Then, $\overline{R(I)_{a,b}} = \overline{R} + \overline{R}t$.*

Proof. Let $r + st \in \overline{R} + \overline{R}t$, then $\psi(r + st) = (r + s\xi, r + s\eta) \in \overline{R} \times \overline{R}$. Hence $\overline{R} + \overline{R}t \subseteq \overline{R(I)_{a,b}}$.

Conversely, let $(\frac{r}{u}, \frac{s}{v}) \in \overline{R} \times \overline{R}$, since

$$\psi\left(\frac{r}{u} - \left(\frac{r}{u} - \frac{s}{v}\right)h(\xi)\xi + \left(\frac{r}{u} - \frac{s}{v}\right)h(\xi)t\right) = \left(\frac{r}{u}, \frac{s}{v}\right),$$

we obtain

$$\frac{r}{u}-\left(\frac{r}{u}-\frac{s}{v}\right)h(\xi)\xi+\left(\frac{r}{u}-\frac{s}{v}\right)h(\xi)t\in\psi^{-1}(\overline{R}\times\overline{R})=\overline{R(I)_{a,b}}.$$

Since $\xi-\eta$ is invertible in $\overline{R}$, it follows that $h(\xi)\in\overline{R}$. Then,

$$\frac{r}{u}-\alpha\xi+\alpha t\in\overline{R}+\overline{R}t.$$

Hence, $\overline{R(I)_{a,b}}=\overline{R}+\overline{R}t$. □

It would be interesting to understand if there are weaker conditions on ξ and η that characterize when $\overline{R(I)_{a,b}}$ coincides with $\overline{R}+\overline{R}t$.

4.4 Weakly Arf property

In this section we assume the setting of the Section 4.3 In particular, $t^2+at+b=(t-\xi)(t-\eta)$ in $R[t]$ and $\xi-\eta$ invertible in $\overline{R}$, so $\overline{R}(I)_{a,b}=\overline{R}+\overline{R}t$.

Moreover, we suppose that R is a Cohen-Macaulay local ring and that I is a Cohen-Macaulay R-module with the same dimension as R (as R-module; this holds e.g. if I is Cohen-Macaulay as R-module that contains a nonzero-divisor). With these hypotheses $R(I)_{a,b}$ is a local, Cohen-Macaulay ring by [3, Proposition 2.7] and the ideal (0) is pure in both R and $R(I)_{a,b}$. Moreover, we assume that I is also an ideal of $\overline{R}$. Before studying the WA property, we observe some facts about the relations between nonzero-divisors of R and $R(I)_{a,b}$.

Remark 4.13. Let $r\in W(R)$. Then $r(s+jt)=rs+rjt=0$ in $R(I)_{a,b}$ if and only if $rs=rj=0$ in R, therefore $s=j=0$. Hence $r\in W(R(I)_{a,b})$.

Remark 4.14. In the Cohen-Macaulay setting it is well known that the set of the zero-divisors is the union of all the minimal prime ideals of the zero ideal. For a ring A, we set $\operatorname{Min}(A)$ to be the set of the minimal prime ideals of the zero ideal. If $r+it\in R(I)_{a,b}$ is a zero-divisor, then $r+it\in\mathfrak{q}$ for some $\mathfrak{q}\in\operatorname{Min}(R(I)_{a,b})$. By [3, Remark 1.10] we have $\mathfrak{p}=\mathfrak{q}\cap R\in\operatorname{Min}(R)$. Because $t^2+at+b=(t-\xi)(t-\eta)$, by [10, Lemma 1.1] we get $\mathfrak{q}=\{s+jt\mid s\in R,\ j\in I \text{ and } s+\xi j\in\mathfrak{p}\}$ or $\mathfrak{q}=\{s+jt\mid s\in R,\ j\in I \text{ and } s+\eta j\in\mathfrak{p}\}$; therefore at least one between $r+\xi i$ and $r+\eta i$ is a zero-divisor of R.

Conversely, let $r\in R$ and $i\in I$ be such that $r+i\xi$ (or $r+i\eta$) is a zerodivisor in R, then $r+i\xi\in\mathfrak{p}$ for some $\mathfrak{p}\in\operatorname{Min}(R)$; we consider $\mathfrak{q}=\{s+jt\mid s\in R,\ j\in I \text{ and } s+j\xi\in\mathfrak{p}\}$. By [10, Lemma 1.1] and [3, Proposition 1.9], there are at most two prime ideals of $R(I)_{a,b}$ lying over $\mathfrak{p}$ and they are $\mathfrak{q}$ and $\mathfrak{q}'=\{s+jt\mid s\in R,\ j\in I \text{ and } s+j\eta\in\mathfrak{p}\}$. Therefore $r+it\in\mathfrak{q}$ and $\mathfrak{q}\in\operatorname{Min}(R(I)_{a,b})$; hence $r+it$ is a zero-divisor in $R(I)_{a,b}$.

Theorem 4.15. *Let R be a Cohen-Macaulay local ring. Let $t^2+at+b=(t-\xi)(t-\eta)$, with $\xi,\eta\in R$ and $\xi-\eta$ invertible in R. Assume that I is also an ideal of $\overline{R}$. Then $R(I)_{a,b}$ is a WA ring if and only if R is a WA ring.*

Proof. Assume that $R(I)_{a,b}$ is a WA ring. Let $r,s,p\in R$ be such that $r\in W(R)$ and $s,p\in r\overline{R}$. Since $\overline{R}\subseteq\overline{R(I)_{a,b}}$, we have $s,p\in r(\overline{R(I)_{a,b}})$, with $r\in W(R(I)_{a,b})$. It follows that $sp/r=q+it$ for some $q\in R$ and $i\in I$, thus $sp=rq+rit\in R$ so that $ri=0$ and $sp/r=q\in R$; hence R is a WA ring.

Conversely, we assume that R is a WA ring. Let $\alpha, \beta, \gamma \in R(I)_{a,b}$ such that $\alpha \in W(R(I)_{a,b})$ and $\beta/\alpha, \gamma/\alpha \in \overline{R} + \overline{R}t$; we write $\alpha = r + it$, $\beta/\alpha = x + yt$ and $\gamma/\alpha = s + pt$ with $r \in R$, $i \in I$ and $x, y, s, p \in \overline{R}$. Then

$$\begin{aligned} \beta &= (r+it)(x+yt) = rx - biy + (ry + ix - aiy)t \\ \gamma &= (r+it)(s+pt) = rs - bip + (rp + is - aip)t. \end{aligned}$$

Since $I \subseteq R$ is an ideal of $\overline{R}$, we get $biy, bip, i(x - ay), i(s - ap) \in I$, it follows that $rx, rs \in R$ and $ry, rp \in I$. Let us compute $(\beta\gamma)/\alpha$:

$$\begin{aligned} \frac{\beta\gamma}{\alpha} &= (r+it)(x+yt)(s+pt) \\ &= (rx - biy + (ry + ix - aiy)t)(s + pt) \\ &= (rx - biy)s - b(ry + ix - aiy)p \\ &\quad + [(rx - biy)p + (ry + ix - aiy)(s - ap)]t. \end{aligned}$$

We have

$$\begin{aligned} &biys + b(ry + ix - aiy)p \in R \\ &(rx - biy)p + (ry + ix - aiy)(s - ap) \in I. \end{aligned} ;$$

Hence, we only need to show that $rxs \in R$.

Since $\alpha \in W(R(I)_{a,b})$, by Remark 4.8 we have $u = r + \xi i \in W(R)$ (and $v = r + \eta i \in W(R)$). Write rxs as follows:

$$rxs = rxs + \xi ixs - \xi ixs = uxs - \xi ixs.$$

Since $\xi ixs \in I$, we need to prove that $uxs \in R$. Clearly $ux, us \in u\overline{R}$, therefore $ux/u \in \overline{R}$ and $us/u \in \overline{R}$. Moreover, $ux = (r + \xi i)x = rx + \xi ix \in R$ (similarly $us \in R$). Now, by the WA property of R we get

$$uxs = \frac{(ux)(us)}{u} \in R.$$

This shows that $\beta\gamma/\alpha \in R(I)_{a,b}$, hence $R(I)_{a,b}$ is a WA ring. $\square$

4.5 Weakly Arf property for double coverings

Let R be an integral domain with unity and let $b \in R$ be an element that is not a square in $Q(R)$. Let I be a fractional ideal of R such that $bI^2 \subseteq R$, where I^2 is the R-module generated by $\{i^2 \mid i \in I\}$; we recall that a fractional ideal of R is an R-submodule of $Q(R)$ such that there exists an element $a \in R$, with $a \neq 0$, for which $aI \subseteq R$.

We consider the ring

$$\tilde{R} = \frac{Q(R)[T]}{(T^2 - b)}.$$

Called t the class of T in $\tilde{R}$, we can define the following subring

$$R + It := \{r + it \mid r \in R,\ i \in I\} \subseteq \tilde{R}.$$

We observe that $R + It \cong R \oplus I$ endowed with the multiplication

$$(r, i) \cdot (s, j) = (rs + ijb, rj + si).$$

We will call these rings *double coverings* of R with respect to b. These rings were studied in [13], and the next remarks and results are proved there. We notice that $R + It$ is again a domain, because of the choice of b.

With this setting, assume for now that R is integrally closed in its field of fractions. Moreover, assume that it contains a field k with $\mathrm{char}(k) \neq 2$ and define the following subset of $Q(R)$:

$$\tilde{I} := \{i \in Q(R) \mid bi^2 \in R\}.$$

Remark 4.16. Notice that $bij \in R$, for any $i, j \in \tilde{I}$. In fact, if $i, j \in \tilde{I}$ then $bi^2, bj^2 \in R$, so $(bi^2)(bj^2) = (bij)^2 \in R$ and thus $bij \in \overline{R}$. Since R is integrally closed, it follows that $bij \in R$.

Remark 4.17. The set $\tilde{I}$ is a fractional ideal of R. In fact, if we take $i, j \in \tilde{I}$ then $b(i-j)^2 = bi^2 - 2bij + bj^2 \in R$, i.e. $\tilde{I}$ is an additive group. Moreover, if $r \in R$ and $i \in \tilde{I}$ then $b(ri)^2 = r^2(bi^2) \in R$; hence $\tilde{I}$ is an R-module. Finally, we want to show that $b\tilde{I} \subseteq R$; let $i \in \tilde{I}$ and we consider the element $bi \in \tilde{I}$, then $(bi)^2 = b(bi^2) \in R$; it follows that $bi \in \overline{R} = R$.

The proof of the next proposition can be found in [13, Proposition 4.17].

Proposition 4.18. *In our hypotheses* $Q(R + It) = Q(R) + Q(R)t$ *and* $\overline{R + It} = R + \tilde{I}t$.

Now we are ready to prove that the ring $R + It$ satisfies the WA property.

Theorem 4.19. *Assuming the hypotheses of this section, il R is integrally closed, then the ring $R + It$ is a WA ring.*

Proof. Let $r_h + i_h t \in R + It$ for $h = 1, 2, 3$ such that $\frac{r_h + i_h t}{r_1 + i_1 t} \in \overline{R + It}$ for $h = 2, 3$. By Proposition 4.18, we can write $\frac{r_h + i_h t}{r_1 + i_1 t} = \tilde{r}_h + \tilde{i}_h t$ for any $h = 2, 3$ with $\tilde{r}_h \in R$ and $\tilde{i}_h \in \tilde{I}$. We calculate

$$\frac{(r_2 + i_2 t)(r_3 + i_3 t)}{r_1 + i_1 t} = (r_1 + i_1 t)(\tilde{r}_2 + \tilde{i}_2 t)(\tilde{r}_3 + \tilde{i}_3 t) =$$
$$(r_1\tilde{r}_2\tilde{r}_3 + r_1\tilde{i}_2\tilde{i}_3 b + i_1\tilde{r}_2\tilde{i}_3 b + i_1\tilde{r}_3\tilde{i}_2 b) + (r_1\tilde{r}_2\tilde{i}_3 + r_1\tilde{i}_2\tilde{r}_3 + i_1\tilde{r}_2\tilde{r}_3 + i_1\tilde{i}_2\tilde{i}_3 b)t.$$

Clearly $r_1\tilde{r}_2\tilde{r}_3 \in R$. By hypothesis $bI^2 \subseteq R$, so $I \subseteq \tilde{I}$; furthermore $\tilde{I}$ is an R-module such that $b\tilde{I}^2 \subseteq R$, thus $r_1\tilde{i}_2\tilde{i}_3 b + i_1\tilde{r}_2\tilde{i}_3 b + i_1\tilde{r}_3\tilde{i}_2 b \in R$. Now, $\tilde{r}_2\tilde{r}_3 i_1$ and $i_1\tilde{i}_2\tilde{i}_3 b$ are both in I, but $r_1\tilde{r}_2\tilde{i}_3 + r_1\tilde{i}_2\tilde{r}_3 \in \tilde{I}$.

For $h = 2, 3$ we have $r_h + i_h t \in R + It$ and

$$\begin{aligned} r_h + i_h t &= (r_1 + i_1 t)(\tilde{r}_h + \tilde{i}_h t) \\ &= (r_1\tilde{r}_h + i_1\tilde{i}_h b) + (r_1\tilde{i}_h + \tilde{r}_h i_1)t \end{aligned}$$

therefore $r_1\tilde{i}_h + \tilde{r}_h i_1 \in I$. Since $\tilde{r}_h i_1 \in I$, we obtain

$$r_1\tilde{i}_h = (r_1\tilde{i}_h + \tilde{r}_h i_1) - \tilde{r}_h i_1 \in I.$$

Hence $r_1\tilde{r}_2\tilde{i}_3 + r_1\tilde{i}_2\tilde{r}_3 + i_1\tilde{r}_2\tilde{r}_3 + i_1\tilde{i}_2\tilde{i}_3 b \in I$. This shows that $R + It$ is a WA ring. □

Example 4.20. Let k be an integrally closed field with $\mathrm{char}(k) \neq 2$. Then, the ring $A = k[X, Y, T]/(T^2 - X^3)$ is a WA ring.

Indeed, we take $R = I = k[X, Y]$ and $b = X^3$. Since $A \cong R + It$, by Theorem 4.19 A is a WA ring. (we can find the same result in [6, Example 3.6 (1)]).

Corollary 4.21. *Let R be an integrally closed domain such that it contains a field with characteristic different from 2 and let I be a proper ideal of R. Let $b \in R$ be such that it is not a square in $Q(R)$. Then $R(I)_{0,-b}$ is a WA ring.*

We will now study the case in which R is not integrally closed. Assume again that R contains a field of characteristic different from 2. Let $R + It$ be as above and define the subset of $Q(R)$

$$\tilde{J} := \{j \in Q(R) \mid bj^2 \in \overline{R}\}.$$

Remark 4.22. We observe that $\tilde{J}$ is a fractional ideal of $\overline{R}$. If $i, j \in \tilde{J}$ then $(bij)^2 = (bi^2)(bj^2) \in \overline{R}$, so that $bij \in \overline{R}$. Moreover $b(i-j)^2 = bi^2 + bj^2 - 2bij \in \overline{R}$ and, if $r \in \overline{R}$ we have $b(ri)^2 = r^2(bi^2) \in \overline{R}$; thus $i - j, ri \in \tilde{J}$, and so $\tilde{J}$ is a $\overline{R}$-module. Finally $(bi)^2 = b(bi^2) \in \overline{R}$, therefore $bi \in \overline{R}$; hence $\tilde{J}$ is a fractional ideal of $\overline{R}$.

Remark 4.23. If R is integrally closed we get $\tilde{J} = \tilde{I}$.

Proposition 4.24. *Keeping the above hypotheses, we have $\overline{R + It} = \overline{R} + \tilde{J}t$.*

Proof. For any $r + st \in Q(R) + Q(R)t$ we have

$$(r + st)^2 - 2r(r + st) + r^2 - bs^2 = 0.$$

Then, $r + st$ is a root of the polynomial $p(X) = X^2 - 2rX + r^2 - bs^2 \in Q(R)[X]$. If $r + st \in R + It$, then $p(X) \in R[X]$; therefore the extension $R \subseteq R + It$ is integral, and so $\overline{R} \subseteq \overline{R + It}$.

If $r + st \in \overline{R} + \tilde{J}t$, we have $p(X) \in \overline{R}[X]$; therefore $r + st \in \overline{R} \subseteq \overline{R + It}$, hence $\overline{R} + \tilde{J}t \subseteq \overline{R + It}$.

Conversely, let $r + st \in Q(R) + Q(R)t$ be integral over R. Then there exists a monic polynomial $f(X) \in R[X]$ such that $f(r + st) = 0$; since $r + st$ is a root of $p(X) \in Q(R)[X]$, we have $f(X) = p(X)g(X)$ for some monic polynomial $g(X) \in Q(R)[X]$. Therefore, the coefficients of $p(X)$ are integral over R, that is $2r$ and $r^2 - bs^2$ are both in $\overline{R}$. Since R contains a field of characteristic different from 2, it follows that $r, bs^2 \in \overline{R}$. Hence, $r + st \in \overline{R} + \tilde{J}t$. □

Theorem 4.25. *If $R + It$ is a WA ring, then R is a WA ring.*

Proof. Let $x, y, z \in R$ such that $x \neq 0$ and $y, z \in x\overline{R}$. Since $\overline{R} \subseteq \overline{R + It}$, we have $y/x, z/x \in \overline{R + It}$. By the WA property of $R + It$ we get $yz/x \in R + It$, and so $yz/x = r + it$ with $i = 0$. Hence $yz/x \in R$, and this shows that R is a WA ring. □

We conclude this section observing that, in light of the results of [6], it would be reasonable to expect that $R + It$ is a WA ring if and only if R is WA and I is also an ideal of $\overline{R}$; but we did not find a complete proof yet.

Chapter 5

Some examples of weakly Arf rings

In this chapter we will show some example of weakly Arf rings arising from number rings. In particular we will study the weakly Arf property of the quadratic rings $\mathbb{Z}[\sqrt{m}]$ and of the cubic rings $\mathbb{Z}[\sqrt[3]{d}]$ with $m, d \in \mathbb{Z}$ when they are not integrally closed.

5.1 Preliminary properties

In this section we recall some preliminary results about number rings, useful for understanding the examples given later. All the results shown here can be found in [15].

Definition 5.1. A subfield of $\mathbb{C}$ with finite degree (as a vector space over $\mathbb{Q}$) is called *number field*.

We want to study the number field $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt[3]{d}]$.

Remark 5.2. Because $\sqrt{m}$ is a root of the polynomial $x^2 - m$ and $\sqrt[3]{d}$ is a root of the polynomial $x^3 - d$, whenever these polynomials are irreducible we have

$$\begin{aligned} \mathbb{Q}[\sqrt{m}] &= \{q_0 + q_1\sqrt{m} \mid q_0, q_1 \in \mathbb{Q}\} \\ \mathbb{Q}[\sqrt[3]{d}] &= \{s_0 + s_1\sqrt[3]{d} + s_2(\sqrt[3]{d})^2 \mid s_0, s_1, s_2 \in \mathbb{Q}\} \end{aligned}$$

otherwise we have $\mathbb{Q}[\sqrt{m}] = \mathbb{Q}$ or $\mathbb{Q}[\sqrt[3]{d}] = \mathbb{Q}$.

Definition 5.3. A complex number $\alpha \in \mathbb{C}$ is called *algebraic integer* if α is a root of a monic polynomial $p(x) \in \mathbb{Z}[x]$. We denote with $\mathbb{A}$ the set of algebraic integers.

In particular, $\sqrt{m}$ and $\sqrt[3]{d}$ are algebraic integers.

Proposition 5.4. *Let m be a squarefree integer. The set $\mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$ of the algebraic integers in the quadratic field is*

$$\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} \mid a, b \in \mathbb{Z}\} \quad \textit{if} \quad m \equiv 2 \textit{ or } 3 \pmod 4$$

$$\mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] = \left\{\frac{a+b\sqrt{m}}{2} \mid a, b \in \mathbb{Z},\ a \equiv b \pmod 2\right\} \quad \textit{if} \quad m \equiv 1 \pmod 4$$

Remark 5.5. By the proposition above it follows that the integral closure of $\mathbb{Z}[\sqrt{m}]$ is the ring $\mathbb{Z}[\frac{1+\sqrt{m}}{2}]$ if $m \equiv 1 \pmod 4$, else $\mathbb{Z}[\sqrt{m}]$ it is integrally closed.

Proposition 5.6. *Let K be a number field of degree n over $\mathbb{Q}$, and let R be the ring $\mathbb{A} \cap K$ of algebraic integers in K. Then, R is a free abelian group of rank n.*

Equivalently, there exist $a_1, a_2, \ldots, a_n \in R$ such that every $r \in R$ is uniquely representable in the form $m_1a_1 + m_2a_2 + \cdots + m_na_n$ with $m_i \in \mathbb{Z}$. The set $\{a_1, a_2, \ldots, a_n\}$ is called an ***integral basis*** *for R.*

Theorem 5.7. *Let $\alpha \in \mathbb{C}$ be an algebraic integer over $\mathbb{Q}$ and suppose α has degree n over $\mathbb{Q}$. Then there is an integral basis*

$$1, \frac{f_1(\alpha)}{d_1}, \ldots, \frac{f_{n-1}(\alpha)}{d_{n-1}}$$

where the d_i are in $\mathbb{Z}$ and satisfy $d_1|d_2|\ldots|d_{n-1}$; the f_i are monic polynomials over $\mathbb{Z}$, and f_i has degree i. The d_i are uniquely determined.

Corollary 5.8. *Let $d \in \mathbb{Z}$ cubefree, so that $d = ab^2$ with a and b squarefree. Setting $t = \sqrt[3]{d}$, an integral basis for $\mathbb{A} \cap \mathbb{Q}[t]$ consists of*

$$\begin{aligned} &1, t, \frac{t^2}{b} \quad \textit{if } d \not\equiv \pm 1 \pmod 9 \\ &1, t, \frac{t^2 + b^2t + b^2}{3b} \quad \textit{if } d \equiv 1 \pmod 9 \\ &1, t, \frac{t^2 - b^2t + b^2}{3b} \quad \textit{if } d \equiv -1 \pmod 9. \end{aligned}$$

In particular, if d is squarefree, then an integral basis for $\mathbb{A} \cap \mathbb{Q}[t]$ consists of

$$\begin{aligned} &1, t, t^2 \quad \textit{if } d \not\equiv \pm 1 \pmod 9 \\ &1, t, \frac{t^2 \pm t + 1}{3} \quad \textit{if } d \equiv \pm 1 \pmod 9. \end{aligned}$$

5.2 The weakly Arf property for quadratic rings

In light of Remark 5.5, we are interested in studying $R = \mathbb{Z}[\sqrt{m}]$, with $m \equiv 1 \pmod 4$; in this case $\overline{R} = \mathbb{Z}[\frac{1+\sqrt{m}}{2}] \subseteq \mathbb{Q}(\sqrt{m})$. Let be $\alpha = a + b\sqrt{m} \in R \setminus \{0\}$ and let $\beta, \gamma \in R$ be such that $\beta, \gamma \in \alpha\overline{R}$. We can write

$$\beta = \alpha \cdot \frac{x + y\sqrt{m}}{2} \qquad \gamma = \alpha \cdot \frac{u + w\sqrt{m}}{2},$$

with $x \equiv y$ and $u \equiv w \pmod 2$; hence

$$\frac{\beta\gamma}{\alpha} = \alpha \cdot \frac{x + y\sqrt{m}}{2} \cdot \frac{u + w\sqrt{m}}{2} = \alpha \cdot \frac{xu + ywm + (xw + yu)\sqrt{m}}{4}.$$

We observe that $xu + ywm \equiv xw + yu \equiv 0 \pmod 2$, therefore

$$\frac{\beta\gamma}{\alpha} = \alpha \cdot \frac{s + r\sqrt{m}}{2} = \frac{as + brm + (ar + bs)\sqrt{m}}{2}$$

where $s = (xu + ywm)/2$ and $r = (xw + yu)/2$ are in $\mathbb{Z}$.

Remark 5.9. If $x \equiv y \equiv 0 \pmod 2$ and $u \equiv w \equiv 0 \pmod 2$, then $\beta/\alpha, \gamma/\alpha \in R$, and so $\beta\gamma/\alpha \in R$. Therefore, we assume that $x \equiv y \equiv 1 \pmod 2$, then

$$\beta = \alpha \cdot \frac{x + y\sqrt{m}}{2} = \frac{ax + bym}{2} + \frac{ay + bx}{2}\sqrt{m} \in R$$

hence, we must have $a \equiv b \pmod 2$.

If $r \equiv s \pmod 2$ we have two cases:

- r and s are even, thus $(r + s\sqrt{m})/2 \in R$; hence $\beta\gamma/\alpha \in R$.
- r and s are odd; in this case we have

$$\begin{cases} as + brm \equiv a + b \pmod 2 \\ ar + bs \equiv a + b \pmod 2 \end{cases}$$

now, since $a \equiv b \pmod 2$ it follows that $a + b \equiv 0 \pmod 2$; hence $\beta\gamma/\alpha \in R$.

We still have to consider the case $r + s \equiv 1 \pmod 2$; now we will show that this case is impossible. Indeed

$$r + s = \frac{xu + ywm}{2} + \frac{xw + yu}{2} = x \cdot \frac{u + w}{2} + y \cdot \frac{wm + u}{2};$$

notice that

$$\frac{u + mw}{2} - \frac{u + w}{2} = w \cdot \frac{m - 1}{2}.$$

By hypothesis $m - 1 \equiv 0 \pmod 4$, so $(m - 1)/2 \equiv 0 \pmod 2$; therefore

$$\frac{u + w}{2} \equiv \frac{u + wm}{2} \pmod 2$$

furthermore we know that $x + y \equiv 0 \pmod 2$, it follows that

$$r + s \equiv (x + y) \cdot \frac{u + w}{2} \equiv 0 \pmod 2.$$

Since integrally closed rings are WA, we can summarize the computations of this section in the following statement.

Proposition 5.10. *The ring $\mathbb{Z}[\sqrt{m}]$ is WA for any $m \in \mathbb{Z}$.*

5.3 The weakly Arf property for cubic rings

Keeping in mind the results recalled in Section 5.1, we start studying the WA property for $R = \mathbb{Z}[\sqrt[3]{d}]$, with $d \not\equiv \pm 1 \pmod 9$ and cubefree. Write $d = ab^2$; then there is an integral basis $\{1, t, t^2/b\}$ (where $t = \sqrt[3]{d}$).

Let $\alpha, \beta, \gamma \in R$ be such that $\beta, \gamma \in \alpha\overline{R}$. Then $\beta = \alpha\tilde{\beta}$ and $\gamma = \alpha\tilde{\gamma}$ with $\tilde{\beta}, \tilde{\gamma} \in \overline{R}$. We write

$$\tilde{\beta} = x_0 + x_1 t + \frac{x_2 t^2}{b} \qquad \tilde{\gamma} = y_0 + y_1 t + \frac{y_2 t^2}{b}$$

so, we can write

$$\tilde{\beta}\tilde{\gamma} = z_0 + z_1 t + \frac{z_2 t^2}{b}$$

where $z_0 + z_1 t \in R$ and

$$\frac{z_2}{b} = \frac{x_0 y_2 + x_2 y_0}{b} + x_1 y_1.$$

Therefore $\beta\gamma/\alpha = \alpha(z_0 + z_1 t) + \alpha(z_2 t^2/b) \in R$ if and only if $\alpha \cdot (z_2 t^2/b) \in R$ if and only if $\alpha \cdot (x_0 y_2 + x_2 y_0) \equiv 0 \pmod b$. We write $\alpha = u_0 + u_1 t + u_2 t^2$ with $u_0, u_1, u_2 \in \mathbb{Z}$.

Since $\beta = \alpha\tilde{\beta} \in R$ we must have $\alpha \cdot (x_2 t^2/b) \in R$, that is $u_0 x_2/b \in R$ or equivalently $u_0 x_2 \equiv 0 \pmod{b}$; similarly $u_0 y_2 \equiv 0 \pmod{b}$. This shows that $\alpha \cdot (z_2 t^2/b) \in R$, hence $\beta\gamma/\alpha \in R$.

What is shown above can be summarized with the following proposition.

Proposition 5.11. *Let d be a cubefree integer such that $d \not\equiv \pm 1 \pmod{9}$. Then, the ring $\mathbb{Z}[\sqrt[3]{d}]$ is WA.*

Now we study the case of $d \equiv \pm 1 \pmod{9}$; more precisely, we want to understand when the WA property holds for the ring $R = \mathbb{Z}[t]$, where $t = \sqrt[3]{d}$ with $d = ab^2$ such that a and b are squarefree and $d \equiv \pm 1 \pmod{9}$. With these hypotheses we know that one integral basis of R is $\{1, t, \frac{t^2 \pm b^2 t + b^2}{3b}\}$, where the sign + or - depends on whether $d \equiv 1 \pmod{9}$ or $d \equiv -1 \pmod{9}$, namely any element of $\overline{R}$ can be written as $r_2 \frac{t^2 \pm b^2 t + b^2}{3b} + r_1 t + r_0$ with $r_2, r_1, r_0 \in \mathbb{Z}$.

Remark 5.12. Since $d \equiv \pm 1 \pmod{9}$ and b^2 is a square, b^2 can only assume the values $1, 4$ and $7 \pmod{9}$; consequently, a will assume the values $1, 7, 4$ if $d \equiv 1 \pmod{9}$ and $-1, 2, 5$ if $d \equiv -1 \pmod{9}$. In particular, we have $\gcd(3, b) = 1$.

Now, we investigate the case $d \equiv 1 \pmod{9}$. We take $\alpha, \beta, \gamma \in R$ such that $\alpha \neq 0$ and $\beta, \gamma \in \alpha\overline{R}$, and we write $\alpha = x_2 t^2 + x_1 t + x_0$ and

$$\beta = \alpha \left(y_2 \frac{t^2 + b^2 t + b^2}{3b} + y_1 t + y_0 \right), \qquad \gamma = \alpha \left(z_2 \frac{t^2 + b^2 t + b^2}{3b} + z_1 t + z_0 \right)$$

with $x_i, y_i, z_i \in \mathbb{Z}$. We observe that, because $\beta \in R$ and $y_1 t + y_0$, we must have $\alpha y_2 \frac{t^2 + b^2 t + b^2}{3b} \in R$; similarly we have $\alpha z_2 \frac{t^2 + b^2 t + b^2}{3b} \in R$. It follows that $\beta\gamma/\alpha \in R$ if and only if $\alpha y_2 z_2 (\frac{t^2 + b^2 t + b^2}{3b})^2 \in R$. We compute

$$\begin{aligned}
\alpha y_2 z_2 \left(\frac{t^2 + b^2 t + b^2}{3b} \right)^2 &= \alpha y_2 z_2 \frac{t^4 + b^4 t^2 + b^4 + 2(b^2 t^3 + b^2 t^2 + b^4 t)}{9b^2} \\
&= \alpha y_2 z_2 \frac{b^2(b^2+2)t^2 + (d + 2b^4)t + b^2(b^2 + 2d)}{9b^2} \\
&= \alpha y_2 z_2 \frac{(b^2+2)t^2 + (a + 2b^2)t + (b^2 + 2d)}{9}.
\end{aligned}$$

From this, we deduce that, if

$$\begin{cases} b^2 + 2 \equiv 0 & \pmod{9} \\ a + 2b^2 \equiv 0 & \pmod{9} \end{cases}$$

then $\beta\gamma/\alpha \in R$. Now, if $b^2 \equiv 7$ and $a \equiv 4 \pmod{9}$, we have

$$\begin{cases} b^2 + 2 \equiv 7 + 2 \equiv 0 & \pmod{9} \\ a + 2b^2 \equiv 4 + 5 \equiv 0 & \pmod{9} \end{cases}$$

hence, in this case $\mathbb{Z}[\sqrt[3]{d}]$ is a WA ring.

If $b^2 \equiv a \equiv 1 \pmod{9}$ we have

$$\begin{cases} b^2 + 2 \equiv 1 + 2 \equiv 3 & \pmod{9} \\ a + 2b^2 \equiv 1 + 2 \equiv 3 & \pmod{9} \end{cases}$$

whereas, if $b^2 \equiv 4$ and $a \equiv 7 \pmod 9$ we have

$$\begin{cases} b^2+2 \equiv 4+2 \equiv 6 & \pmod 9 \\ a+2b^2 \equiv 7-1 \equiv 6 & \pmod 9 \end{cases}$$

In both cases, if $y_2 \equiv 3$ or $z_2 \equiv 3 \pmod 9$ we have $\beta\gamma/\alpha \in R$; so we can suppose that $y_2 \not\equiv 3$ and $z_2 \not\equiv 3 \pmod 9$. We observe that $\beta \in \alpha\overline{R} \cap R$, so that $\alpha y_2 \frac{t^2+b^2t+b^2}{3b} \in R$. We compute it

$$\begin{aligned} & y_2(x_2t^2+x_1t+x_0)\cdot\frac{t^2+b^2t+b^2}{3b} \\ &= y_2\frac{x_2(dt+b^2d+b^2t^2)+x_1(d+b^2t^2+b^2t)+x_0(t^2+b^2t+b^2)}{3b} \\ &= y_2\frac{(x_2b^2+x_1b^2+x_0)t^2+(x_2d+x_1b^2+x_0b^2)t+(x_2b^2d+x_1d+x_0b^2)}{3b} \end{aligned}$$

from which it follows that

$$\begin{cases} y_2(x_2b^2+x_1b^2+x_0) \equiv 0 & \text{(mod 3b)} \\ y_2(x_2d+x_1b^2+x_0b^2) \equiv 0 & \text{(mod 3b)} \\ y_2(x_2b^2d+x_1d+x_0b^2) \equiv 0 & \text{(mod 3b)} \end{cases}$$

and, in particular,

$$\begin{cases} x_2b^2+x_1b^2+x_0 \equiv 0 & \pmod 3 \\ x_2d+x_1b^2+x_0b^2 \equiv 0 & \pmod 3 \\ x_2b^2d+x_1d+x_0b^2 \equiv 0 & \pmod 3 \end{cases}$$

Since $d \equiv b^2 \equiv 1 \pmod 3$, we must have $x_2+x_1+x_0 \equiv 0 \pmod 3$.

Going back to the computation of $\alpha y_2 z_2 (\frac{t^2+b^2t+b^2}{3b})^2$, we get that it is equal to

$$\begin{aligned} &= y_2z_2\left[x_2\frac{d(b^2+2)t+d(a+2b^2)+(b^2+2d)t^2}{9}+\right. \\ &\quad + x_1\frac{d(b^2+2)+(a+2b^2)t^2+(b^2+2d)t}{9}+ \\ &\quad \left.+ x_0\frac{(b^2+2)t^2+(a+2b^2)t+(b^2+2d)}{9}\right]= \\ &= y_2z_2\left[\frac{x_2(b^2+2d)+x_1(a+2b^2)+x_0(b^2+2)}{9}t^2+\right. \\ &\quad + \frac{x_2d(b^2+2)+x_1(b^2+2d)+x_0(a+2b^2)}{9}t+ \\ &\quad \left.+ \frac{x_2d(a+2b^2)+x_1d(b^2+2)+x_0(b^2+2d)}{9}\right]. \end{aligned}$$

In each line we have $3(x_2+x_1+x_0) \equiv 0 \pmod 9$; therefore $\beta\gamma/\alpha \in R$. Hence, if $d \equiv 1 \pmod 9$, then $\mathbb{Z}[\sqrt[3]{d}]$ is a WA ring.

Finally, we investigate the case $d \equiv -1 \pmod 9$. We take $\alpha, \beta, \gamma \in R$ such that $\alpha \neq 0$ and $\beta, \gamma \in \alpha\overline{R}$, and we write $\alpha = x_2t^2 + x_1t + x_0$ and

$$\beta = \alpha\left(y_2\frac{t^2 - b^2t + b^2}{3b} + y_1t + y_0\right), \qquad \gamma = \alpha\left(z_2\frac{t^2 - b^2t + b^2}{3b} + z_1t + z_0\right)$$

with $x_i, y_i, z_i \in \mathbb{Z}$. We observe that, because $\beta \in R$ and $y_1t + y_0$, we must have $\alpha y_2\frac{t^2-b^2t+b^2}{3b} \in R$; similarly we have $\alpha z_2\frac{t^2-b^2t+b^2}{3b} \in R$. It follows that $\beta\gamma/\alpha \in R$ if and only if $\alpha y_2z_2(\frac{t^2-b^2t+b^2}{3b})^2 \in R$. In the same way as in the previous case, we compute

$$\begin{aligned}\alpha y_2z_2\left(\frac{t^2 - b^2t + b^2}{3b}\right)^2 &= \alpha y_2z_2\frac{dt + b^4t^2 + b^4 + 2(-b^2d + b^2t^2 - b^4t)}{9b^2}\\ &= \alpha y_2z_2\frac{b^2(b^2+2)t^2 + b^2(a-2b^2)t + b^2(b^2-2d)}{9b^2}\\ &= \alpha y_2z_2\frac{(b^2+2)t^2 + (a-2b^2)t + (b^2-2d)}{9}.\end{aligned}$$

As in the previous case; if $b^2 \equiv 7$ and $a \equiv 5 \pmod 9$, we have

$$\begin{cases} b^2 - 2d \equiv b^2 + 2 \equiv 7 + 2 \equiv 0 & \pmod 9\\ a - 2b^2 \equiv 5 + 4 \equiv 0 & \pmod 9\end{cases}$$

therefore also in this case $\mathbb{Z}[\sqrt[3]{d}]$ is a WA ring.

If $b^2 \equiv 1$ and $a \equiv -1 \pmod 9$, we have

$$\begin{cases} b^2 - 2d \equiv b^2 + 2 \equiv 1 + 2 \equiv 3 & \pmod 9\\ a - 2b^2 \equiv -1 - 2 \equiv -3 & \pmod 9\end{cases}$$

whereas, if $b^2 \equiv 4$ and $a \equiv 2 \pmod 9$ we have

$$\begin{cases} b^2 - 2d \equiv b^2 + 2 \equiv 4 + 2 \equiv 6 & \pmod 9\\ a - 2b^2 \equiv 2 - 8 \equiv -6 & \pmod 9\end{cases}$$

In both cases, if $y_2 \equiv 3$ or $z_2 \equiv 3 \pmod 9$ we have $\beta\gamma/\alpha \in R$; so we can suppose that $y_2 \not\equiv 3$ and $z_2 \not\equiv 3 \pmod 9$. We observe that $\beta \in \alpha\overline{R} \cap R$, so that $\alpha y_2\frac{t^2-b^2t+b^2}{3b} \in R$. We compute it

$$\begin{aligned}&y_2(x_2t^2 + x_1t + x_0)\cdot\frac{t^2 - b^2t + b^2}{3b}\\ &= y_2\frac{x_2(dt - b^2d + b^2t^2) + x_1(d - b^2t^2 + b^2t) + x_0(t^2 - b^2t + b^2)}{3b}\\ &= y_2\frac{(x_2b^2 - x_1b^2 + x_0)t^2 + (x_2d + x_1b^2 - x_0b^2)t + (-x_2b^2d + x_1d + x_0b^2)}{3b}\end{aligned}$$

from which it follows that

$$\begin{cases} y_2(x_2b^2 - x_1b^2 + x_0) \equiv 0 & \pmod{3b}\\ y_2(x_2d + x_1b^2 - x_0b^2) \equiv 0 & \pmod{3b}\\ y_2(-x_2b^2d + x_1d + x_0b^2) \equiv 0 & \pmod{3b}\end{cases}$$

In particular,

$$\begin{cases} x_2b^2 - x_1b^2 + x_0 \equiv 0 & \pmod 3 \\ x_2d + x_1b^2 - x_0b^2 \equiv 0 & \pmod 3 \\ -x_2b^2d + x_1d + x_0b^2 \equiv 0 & \pmod 3 \end{cases}$$

Since $d \equiv -1$ and $b^2 \equiv 1 \pmod 3$, we must have $x_2 - x_1 + x_0 \equiv 0 \pmod 3$.

Returning to the computation of $\alpha y_2 z_2 (\frac{t^2-b^2t+b^2}{3b})^2$, it is equal to

$$\begin{aligned} = y_2z_2 \Bigg[& x_2 \frac{d(b^2+2)t + d(a-2b^2) + (b^2-2d)t^2}{9} + \\ & + x_1 \frac{d(b^2+2) + (a-2b^2)t^2 + (b^2-2d)t}{9} + \\ & + x_0 \frac{(b^2+2)t^2 + (a-2b^2)t + (b^2-2d)}{9} \Bigg] = \end{aligned}$$

$$\begin{aligned} = y_2z_2 \Bigg[& \frac{x_2(b^2-2d) + x_1(a-2b^2) + x_0(b^2+2)}{9} t^2 + \\ & + \frac{x_2d(b^2+2) + x_1(b^2-2d) + x_0(a-2b^2)}{9} t + \\ & + \frac{x_2d(a-2b^2) + x_1d(b^2+2) + x_0(b^2-2d)}{9} \Bigg]. \end{aligned}$$

Now, if $b^2 \equiv 1 \pmod 9$, we have

$$\begin{cases} x_2(b^2-2d) + x_1(a-2b^2) + x_0(b^2+2) \equiv 3x_2 - 3x_1 + 3x_0 & \pmod 9 \\ x_2d(b^2+2) + x_1(b^2-2d) + x_0(a-2b^2) \equiv -3x_2 + 3x_1 - 3x_0 & \pmod 9 \\ x_2d(a-2b^2) + x_1d(b^2+2) + x_0(b^2-2d) \equiv 3x_2 - 3x_1 + 3x_0 & \pmod 9 \end{cases}$$

whereas, if $b^2 \equiv 4 \pmod 9$, we have

$$\begin{cases} x_2(b^2-2d) + x_1(a-2b^2) + x_0(b^2+2) \equiv -3x_2 + 3x_1 - 3x_0 & \pmod 9 \\ x_2d(b^2+2) + x_1(b^2-2d) + x_0(a-2b^2) \equiv 3x_2 - 3x_1 + 3x_0 & \pmod 9 \\ x_2d(a-2b^2) + x_1d(b^2+2) + x_0(b^2-2d) \equiv -3x_2 + 3x_1 - 3x_0 & \pmod 9 \end{cases}$$

in any case we find $3(x_2 - x_1 + x_0) \equiv 0 \pmod 9$; therefore $\beta\gamma/\alpha \in R$.

Therefore, we can conclude the section summarizing all the computations in the following result:

Theorem 5.13. *The ring $\mathbb{Z}[\sqrt[3]{d}]$ is WA for any $d \in \mathbb{Z}$.*

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