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Dissipative 2D MHD equations with L^1 vorticity and magnetic current

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Abstract. We consider the two-dimensional (2D) Magneto-HydroDynamics (MHD) equations describing the evolution of a viscous incompressible electrically conducting fluid. In this paper, adopting the vorticity-current formulation and assuming that the initial fluid vorticity and magnetic current are in $L^1(\mathbb{R}^2)$, we prove the local in-time existence and regularity of the solutions.

Keywords: MHD equations; vorticity-current formulation; mild solutions; well posedness.

Mathematics Subject Classification 2020:

1. Introduction

The viscous 2D MHD equations, written in the usual velocity-magnetic field formulation, are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p = \nu \Delta \mathbf{u} + S\mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B}, \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{B} = \mu \Delta \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{u}, \tag{1.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{1.3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{1.4}$$

In the above equations, $\mathbf{u}(\mathbf{x},t) = (u_1, u_2)$ is the incompressible fluid velocity, $p(\mathbf{x},t)$ is the fluid pressure, while $\mathbf{B}(\mathbf{x},t) = (b_1, b_2)$ is the solenoidal magnetic field. The spatial variable $\mathbf{x} = (x_1, x_2)$ and the temporal variable t are in $\mathbb{R}^2$ and $\mathbb{R}^+$, respectively, while $\boldsymbol{\nabla} = (\partial_1, \partial_2)$ and $\Delta = \partial_1^2 + \partial_2^2$, where $\partial_j = \partial/\partial x_j$ for $j = 1, 2$, denote the gradient and the Laplacian, respectively.

The parameters appearing in the equations have the following meaning: ν is the inverse of the fluid Reynolds number (i.e. the ratio between the characteristic length times the characteristic velocity and the kinematic viscosity); μ is the inverse of the magnetic Reynolds number (i.e. characteristic velocity times the characteristic length times the conductivity times the magnetic permeability); $S = \mu\nu(Ha)^2$ where Ha is the Hartmann number, whose square represents the ratio between the Lorentz force to viscous forces. One can see that, keeping the permeability fixed, $Ha \sim (\mu\nu)^{-1/2}$, so that $S = O(1)$ when $\mu, \nu \to 0$. More details can be found in [14]. Equations (1.1)–(1.4) when $\nu = 0$ and $\mu = 0$ are the equations of the ideal Magneto-HydroDynamics (MHD).

The previous equations must be supplemented by the initial data

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0, \quad \mathbf{B}(\mathbf{x}, t = 0) = \mathbf{B}_0. \tag{1.5}$$

The MHD equations (1.1) and (1.3) are the Navier–Stokes equations with an external Lorentz force, while (1.2) and (1.4) derive from the Maxwell's equations for the magnetic field $\mathbf{B}$. The MHD equations (1.1)–(1.4) describe an electrically conducting incompressible viscous flow, and play a fundamental role in astrophysics, geophysics, plasma physics, and industrial fluids like liquid metal and strong electrolytes. More details on the derivation of the MHD equations can be found in [14].

Here, it is only possible to review some of the relevant literature concerning the MHD equations. We limit ourselves to mention some of the most significant results.

In 2D, and when $\nu > 0$ and $\mu > 0$, global well-posedness results for the Cauchy problem (1.1)–(1.5) hold, see, e.g. [18, 46] for bounded domains and [49] for the whole space. In 3D, the same authors proved the local in-time well-posedness. We also mention the result of Masmoudi [39] concerning the global in-time well-posedness for the full 2D MHD system or the Maxwell–Navier–Stokes system, which consists of the incompressible Navier–Stokes equations with a Lorentz forcing term coupled with the Ampere–Maxwell hyperbolic equations.

In 3D, global in-time well-posedness with small initial data was proved in [8]. The large-time behavior was addressed in [1, 44].

Concerning the inviscid MHD equations, i.e. when $\nu = 0$ and $\mu > 0$, global well-posedness was established in [35] for the 2D case, while the local well-posedness was proved in [19] for the 3D case. In [52], the authors, working in critical Besov spaces, showed the global existence of a solution and the local in-time well-posedness.

When the fluid has zero resistivity, $\mu = 0$ $\nu > 0$, the paper [30], later improved in [20, 55], contains the proof of the local existence and uniqueness of the solution in 2D spatial domains. The local existence of solutions in Besov spaces, both for the

2D and 3D cases, can be found in [11, 21]. Global well-posedness has been achieved in [48] for a special geometry (an infinite slab) and particular initial data (a uniform non-parallel magnetic field).

For the ideal case, $\mu = \nu = 0$, the local existence of strong solutions was established in [43, 45]. In [6], the authors proved a conditional regularity result analogous to the Beale–Kato–Majda result for the ideal MHD equations. The papers [10, 22] contain a stronger version of this criterion. Some existence results are proved in [2, 12, 13] and in [37] for current-vortex-sheet type data.

Concerning the vanishing viscosity limit problem, we just mention the papers [49, 54] and, for a bounded domain, [50, 51]. The existence of solutions for the MHD equations with fractional Laplacian is investigated in [53].

In this work, we are interested in the well-posedness problem of the MHD equations in the vorticity-current formulation. Define the vorticity ω and the current density j as

$$\omega = \boldsymbol{\nabla} \times \mathbf{u} = \partial_1 u_2 - \partial_2 u_1, \tag{1.6}$$

$$j = \boldsymbol{\nabla} \times \mathbf{B} = \partial_1 b_2 - \partial_2 b_1. \tag{1.7}$$

The MHD equations for the vorticity ω and current j follow by taking the curl of Eqs. (1.1) and (1.2) (with our assumption that $S = 1$) and use (1.3) and (1.4). They are

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \omega = \nu \Delta \omega + S\mathbf{B} \cdot \boldsymbol{\nabla} j, \tag{1.8}$$

$$\frac{\partial j}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} j = \mu \Delta j + \mathbf{B} \cdot \boldsymbol{\nabla} \omega + 2\partial_1 \mathbf{B} \cdot \boldsymbol{\nabla} u_2 - 2\partial_2 \mathbf{B} \cdot \boldsymbol{\nabla} u_1. \tag{1.9}$$

The velocity field $\mathbf{u}$ and the magnetic field $\mathbf{B}$ are determined by the well-known Biot–Savart law (see [38]):

$$\mathbf{u}(\mathbf{x}, t) = (\mathbf{K} * \omega)(\mathbf{x}, t) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y}) \omega(\mathbf{y}, t) d\mathbf{y}, \tag{1.10}$$

$$\mathbf{B}(\mathbf{x}, t) = (\mathbf{K} * j)(\mathbf{x}, t) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y}) j(\mathbf{y}, t) d\mathbf{y}, \tag{1.11}$$

with

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi} \left(-\frac{x_2}{|\mathbf{x}|^2}, \frac{x_1}{|\mathbf{x}|^2} \right). \tag{1.12}$$

The initial condition for (1.8) and (1.9) is

$$(\omega(\mathbf{x}, 0), j(\mathbf{x}, 0)) = (\omega_0(\mathbf{x}), j_0(\mathbf{x})). \tag{1.13}$$

It is well known in fluid dynamics that some interesting physical initial data can be formulated with the vorticity, as point-vortex, vortex-sheet, vortex-layer or vortex patch initial data type [5, 7, 29, 36]. For current-vortex sheet data, some results are established in [37, 41, 42].

For the well-posedness for the Cauchy problem of the MHD equations in the vorticity-current formulation there are very few results knowing in literature. In [28], the author considered the ideal MHD equation in the vorticity-current formulation in the plane $\mathbb{R}^2$ and in particular they proved in time existence and uniqueness of a weak solution of Youdovich type for a smooth vortex patch initial datum whose boundary is a current-vortex sheet.

In our paper, we consider the viscous two-dimensional MHD equations in the vorticity-current formulation, and we investigate the existence of solutions on the critical space $L^1(\mathbb{R}^2)$.

Critical spaces arise naturally in the study of the Cauchy problem of a PDE. In fact, these spaces have the same scaling invariance as the equations [34, 26, 9].

Moreover, we recall that for the MHD equations in $\mathbb{R}^2$ in the velocity-magnetic field formulation, the critical space is $L^2(\mathbb{R}^2)$.

But in $\mathbb{R}^2$, if the vorticity (or the current) is in $L^1(\mathbb{R}^2)$, then it is not true that its corresponding velocity (magnetic) field is in $L^2(\mathbb{R}^2)$. This implies that, in the plane, the Cauchy problem, in the critical space $L^1(\mathbb{R}^2)$, of the MHD equations in the vorticity-current formulation can be considered independently from the corresponding Cauchy problem for its velocity-magnetic field formulation in the critical space $L^2(\mathbb{R}^2)$.

The motivations of our research are derived from the classical Navier–Stokes theory in $\mathbb{R}^2$ for initial vorticity in $L^1(\mathbb{R}^2)$ or more generally for initial vorticity as a finite measure (see [4, 3, 23, 27, 31–33, 40]).

In particular, we follow the ideas developed in [31]: We write Eqs. (1.8)–(1.13) in a mild integral formulation and we prove the existence of the solution applying the fixed point theorem. The Banach spaces used are the space of continuous in time functions which are in Lebesgue spaces, with respect to the spatial variable, and with norms which control the smoothing properties of the heat semigroup. More details are given in the following section.

The organization of the paper is the following: In Sec. 2, we give the mathematical setting and we give some useful a priori lemmas used in the sequel. In Sec. 3, we give our main result concerning the local in time existence of the viscous MHD equations in the vorticity-current formulation with initial vorticity and current in $L^1(\mathbb{R}^2)$. In Sec. 4, we prove that these solutions are regular. Section 5 concludes.

2. Mathematical Settings and Preliminary Lemmas

Lebesgue spaces are denoted as usual by $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, with norms

$$\|g\|_p = \left(\int_{\mathbb{R}^2} |g(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad \|g\|_\infty = \operatorname*{ess\,sup}_{\mathbb{R}^2} |g(\mathbf{x})|.$$

In the rest of this paper, to simplify the notation, we shall denote $L^p(\mathbb{R}^2)$ by L^p. We shall indicate the Fourier transform by $\mathcal{F}$, the Schwartz class by $\mathcal{S}$, and the space of tempered distribution by $\mathcal{S}'$, while $<, >$ is the pairing between $\mathcal{S}$ and $\mathcal{S}'$.

Given an interval $I \subset \mathbb{R}$ and X a Banach space, $C(I,X)$ and $BC(I,X)$ denote the set of continuous and bounded continuous function $f : I \to X$, respectively.

We shall use the Kato spaces, see [25, 31], denoted by $K_\alpha((t_0,T),L^p)$ with $\alpha \geq 0$ and $t_0 < T \leq \infty$; they are the Banach spaces of (t_0,T)-continuous functions on L^p with finite weighted norm:

$$|||f|||_{p,\alpha} = \sup_{t\in(t_0,T)} (t-t_0)^\alpha \|f(\cdot,t)\|_p.$$

Defining the seminorm

$$[[[f]]]_{p,\alpha} = \limsup_{t\to t_0^+} (t-t_0)^\alpha \|f(\cdot,t)\|_p,$$

one can introduce the space $\dot{K}_\alpha((t_0,T),L^p)$ as follows:

$$\dot{K}_\alpha((t_0,T),L^p) = \{f : \ f \in K_\alpha((t_0,T),L^p) \text{ such that } [[[f]]]_{p,\alpha} = 0\}.$$

As usual, when there is no ambiguity on T, we shall denote the space $K_\alpha((0,T),L^p)$ as $K_{p,\alpha}$ and $\dot{K}_\alpha((0,T),L^p)$ as $\dot{K}_{p,\alpha}$.

From the above definitions it is clear that

$$\text{if } f \in K_{p,\alpha}, \quad \text{then} \quad \|f(\cdot,t)\|_p = \mathcal{O}(t^{-\alpha}), \quad t \to 0^+$$

and

$$\text{if } f \in \dot{K}_{p,\alpha}, \quad \text{then } \|f(\cdot,t)\|_p = o(t^{-\alpha}), \quad t \to 0^+.$$

Note that when $\alpha < \beta$ and $T < \infty$, then $K_{p,\alpha} \subset \dot{K}_{p,\beta}$.

Moreover, $\dot{K}_{p,0} \subsetneq BC([0,T),L^p) \subsetneq K_{p,0}$.

We now introduce some inequalities that will be useful in the rest of the paper. In the Kato spaces K and $\dot{K}$ the following Hölder inequalities hold

$$|||fg|||_{r,\gamma} \leq |||f|||_{p,\alpha}|||g|||_{q,\beta} \quad \text{and} \quad [[[fg]]]_{r,\gamma} \leq [[[f]]]_{p,\alpha}[[[g]]]_{q,\beta}, \tag{2.1}$$

where $1/r = 1/p + 1/q$ and $\gamma = \alpha + \beta$, with $1 \leq p,q \leq \infty$ and $\alpha,\beta \geq 0$.

Using the Hardy–Littlewood–Sobolev inequality, see [47], one obtains the following estimates for the Biot–Savart operator

$$|||\mathbf{K} * f|||_{p,\alpha} \leq \sigma_q |||f|||_{q,\alpha} \quad \text{and} \quad [[[\mathbf{K} * f]]]_{p,\alpha} \leq \sigma_q [[[f]]]_{q,\alpha}, \tag{2.2}$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{2}$, with $1 < q < 2$ and $\alpha \geq 0$, and σ_q is a constant which depends on q.

The heat semigroup $e^{\lambda t\Delta}$, where $\lambda > 0$, is defined using the convolution with the heat kernel: for $\phi : \mathbb{R}^2 \to \mathbb{R}$ one writes

$$(e^{\lambda t\Delta}\phi)(\mathbf{x},t) = \frac{1}{4\pi\lambda t}\int_{\mathbb{R}^2} e^{-|\mathbf{x}-\mathbf{y}|^2/(4\lambda t)}\phi(\mathbf{y})d\mathbf{y},$$

where $t \geq 0$.

The solution of the heat equation with zero initial data and with source $\psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, that we shall denote by $H_\lambda\psi$, can be written in terms of time-convolution with $e^{\lambda t\Delta}$:

$$H_\lambda\psi = \int_0^t e^{\lambda(t-s)\Delta}\psi(\cdot, s)ds.$$

What follows are classical estimates for the heat semigroup, see, e.g. [25, 27, 31]. The first part of Lemma 2.1 expresses the regularizing properties of the heat semigroup, here in the sense of gain of integrability, with the initial transient regularized by the use of the weighted (in time) Kato spaces. The same properties hold for the gradient of the heat semigroup, which, however, needs an extra $t^{1/2}$ regularization to tame the initial transient, as stated in the second part of Lemma 2.1.

Lemma 2.1 ($L^q - L^p$ estimates). *Let $1 \le p \le q \le \infty$, $\gamma \equiv 1/p - 1/q$.*

(1) *$e^{\lambda t\Delta}$ is bounded on L^p to $\dot{K}_{q,\gamma}$ on $(0, \infty)$, with*

$$|||e^{\lambda t\Delta}\phi|||_{q,\gamma} \le \delta_\gamma\|\phi\|_p. \tag{2.3}$$

where $\delta_\gamma \equiv (4\pi\lambda)^{-\gamma}(1-\gamma)^{1-\gamma}$. If $q = p = 1$ then $e^{\lambda t\Delta}$ is bounded on L^1 to $BC([0, T), L^1)$.

(2) *$\boldsymbol{\nabla} e^{\lambda t\Delta}$ is bounded on $(L^p)^2$ to $\dot{K}_{q,1/2+\gamma}$, with*

$$|||\boldsymbol{\nabla} e^{\lambda t\Delta} \cdot \phi|||_{q,1/2+\gamma} \le \chi_\gamma\|\phi\|_p, \tag{2.4}$$

where $\chi_\gamma \equiv \frac{1}{2}(2\pi\lambda)^{-\gamma}(1-\gamma)^{1/2}\Gamma(1/(2-2\gamma))^{1-\gamma}$ and Γ is the gamma function.

For a proof, see Lemma 1.1, part (i) and (ii), in [34].

We give analogous estimates for the operator H_λ where, besides the regularizing properties, one also has the possibility of trading between spatial integrability and initial time regularization. We also explicitly state estimates for the operator $H_\lambda\boldsymbol{\nabla}\cdot$, which will help treat the convective-type terms present in the MHD equations.

Lemma 2.2. *Let $0 < T \le \infty$, $1 \le p \le q \le \infty$, $\gamma \equiv 1/p - 1/q$ and $0 < \alpha < \beta < 1$. Denote with B the beta function.*

(1) *If $\beta + 1/p = 1 + \alpha + 1/q$ then H_λ is a bounded operator from $K_{p,\beta}$ to $K_{q,\alpha}$ and from $\dot{K}_{p,\beta}$ to $\dot{K}_{q,\alpha}$, with*

$$|||H_\lambda\psi|||_{q,\alpha} \le \delta_\gamma B(\beta-\alpha, 1-\beta)|||\psi|||_{p,\beta}. \tag{2.5}$$

(2) *If $\beta + 1/p = 1/2 + \alpha + 1/q$ then $H_\lambda\boldsymbol{\nabla}\cdot$ is a bounded operator from $(K_{p,\beta})^2$ to $K_{q,\alpha}$ and from $(\dot{K}_{p,\beta})^2$ to $\dot{K}_{q,\alpha}$, with*

$$|||H_\lambda\boldsymbol{\nabla}\cdot\psi|||_{q,\alpha} \le \chi_\gamma B(\beta-\alpha, 1-\beta)|||\psi|||_{p,\beta}. \tag{2.6}$$

For a proof, see Lemma 2.1, part (i), in [34].

The following lemmas are classical extensions of the $L^q - L^p$ estimate for the kth derivative of the heat semigroup and the H_λ operator.

Lemma 2.3. *Let $1 \le p \le q \le \infty$, $\gamma = \frac{1}{r} - \frac{1}{q}$ and $k > 0$, then $D^k e^{\lambda t \Delta}$ is a bounded operator on L^p to $\dot{K}_{q,k/2+\gamma}$, with bound depending on γ and k.*

For a proof, see Lemma 1.1, part (iii), in [34].

Lemma 2.4. *Let $0 < T \le \infty$, $1 \le p \le q \le \infty$, $0 \le \alpha < \beta < 1$ and $0 \le h < 1$, then:*

- *if $\beta + 1/p = 1 - h/2 + \alpha + 1/q$, then $D^h H_\lambda$ maps $K_{p,\beta}$ into $K_{q,\alpha}$ ($\dot{K}_{p,\beta}$ into $\dot{K}_{q,\alpha}$).*
- *if $\beta + 1/p = 1/2 - h/2 + \alpha + 1/q$, then $D^h H_\lambda \boldsymbol{\nabla}\cdot$ maps $(K_{p,\beta})^2$ into $K_{q,\alpha}$ ($(\dot{K}_{p,\beta})^2$ into $\dot{K}_{q,\alpha}$).*

For a proof, see Lemma 2.1, part (ii), in [34].

3. Local Existence

In this section, we shall establish, for L^1 data, the local in-time existence of mild solutions of the vorticity-current MHD equations. We shall also prove a uniqueness result. First, we rewrite Eqs. (1.8) and (1.9) as

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + \boldsymbol{\nabla} \cdot (\omega \mathbf{K} * \omega) - S \boldsymbol{\nabla} \cdot (j \mathbf{K} * j) = 0, \tag{3.1}$$

$$\frac{\partial j}{\partial t} - \mu \Delta j + \boldsymbol{\nabla} \cdot (j \mathbf{K} * \omega) - \boldsymbol{\nabla} \cdot (\omega \mathbf{K} * j) - \mathcal{H}(\omega, j) = 0, \tag{3.2}$$

where

$$\mathcal{H}(\omega, j) = 2\partial_1 \mathbf{B} \cdot \boldsymbol{\nabla} u_2 - 2\partial_2 \mathbf{B} \cdot \boldsymbol{\nabla} u_1.$$

If one introduces the Riesz operators $\mathcal{R}_h$, defined through the Fourier multiplier by $\mathcal{F}(\mathcal{R}_h f)(\xi) \equiv i\xi_h |\xi|^{-1} \mathcal{F}(f)(\xi)$, then $\mathcal{H}$ can be written as

$$\mathcal{H}(\omega, j) = 2\mathcal{R}_{12}\omega(\mathcal{R}_{11} j - \mathcal{R}_{22} j) - 2\mathcal{R}_{12} j(\mathcal{R}_{11}\omega - \mathcal{R}_{22}\omega), \tag{3.3}$$

with $\mathcal{R}_{hk} = \mathcal{R}_h \mathcal{R}_k = \partial_{hk} \Delta^{-1}$. If $f \in L^q(\mathbb{R}^2)$, we have the following well-known estimate, see, e.g. [47]

$$\|\mathcal{R}_{hk} f\|_q \le c_q \|f\|_q, \quad 1 < q < \infty, \tag{3.4}$$

where c_q is a constant depending on q.

If one defines the map $\Phi(\omega, j) = (\Phi_1(\omega, j), \Phi_2(\omega, j))$ whose components are

$$\Phi_1(\omega, j) = e^{\nu t \Delta}\omega_0 - H_\nu \boldsymbol{\nabla} \cdot (\omega \mathbf{K} * \omega - S j \mathbf{K} * j), \tag{3.5}$$

$$\Phi_2(\omega, j) = e^{\mu t \Delta} j_0 - H_\mu \boldsymbol{\nabla} \cdot (j \mathbf{K} * \omega - \omega \mathbf{K} * j) + H_\mu \mathcal{H}(\omega, j), \tag{3.6}$$

one immediately sees that the MHD equations written in the vorticity-current formulation, Eqs. (3.1) and (3.2), can be written as

$$(\omega, j) = \Phi(\omega, j). \tag{3.7}$$

The solutions of (3.7) will be called mild solutions of the MHD equations.

We can now state the main result of this section.

Theorem 3.1. *Suppose* $(\omega_0, j_0) \in (L^1)^2$. *Then there exists a time* $T > 0$ *such that the MHD equations in the mild formulation* (3.7)

(1) (*Existence*) *admit a solution* (ω, j) *which, for all* $q \in (1, \infty)$, *belongs to* $(\dot{K}_{q,1-1/q})^2$;
(2) (*Uniqueness*) *the solution* (ω, j) *is the only solution with the property of belonging to* $(\dot{K}_{q,1-1/q})^2$ *for all* $q \in [4/3, 2)$;
(3) (*Time regularity in* L^1) *moreover,* $(\omega, j) \in BC([0, T), (L^1)^2)$.

One of the main tools in the proof of the above theorem is a fixed point argument. Therefore, we define, in $(K_{q,1-1/q})^2$, the ball of radius R_q

$$E_{R_q,q} = \{(\omega, j) \in K_{q,1-1/q} \times K_{q,1-1/q} : |||\omega|||_{q,1-1/q} \leq R_q, |||j|||_{q,1-1/q} \leq R_q\}, \tag{3.8}$$

where R_q is defined as

$$R_q = (1 - \sqrt{1 - 4D_q\bar{\rho}})/(2D_q) \quad \text{where } \bar{\rho} < 1/(4D_q), \tag{3.9}$$

being the constant D_q given by

$$D_q = 2B(1 - 1/q, 2/q - 1)(\sigma_q\chi_{1/q-1/2}(1 + S) + 4c_q^2\delta_{1/q}). \tag{3.10}$$

At this stage, $\bar{\rho}$ is any number smaller than $1/(4D_q)$ so that R_q is real. Later, see (3.13), we shall fix $\bar{\rho}$ to be the norm in $K_{q,1-1/q}$ of the heat semiflow generated by the initial data so that the condition $\bar{\rho} < 1/(4D_q)$ will correspond to a short-time existence condition.

We shall achieve the proof of Theorem 3.1 in two steps. First, we shall show that for $4/3 \leq q < 2$, in $E_{R_q,q}$ and locally in time, there exists a unique solution; see Proposition 3.2. We shall denote this solution as (ω_q, j_q). The second step will consist of showing that the above-constructed solution (ω_q, j_q) belongs to $(\dot{K}_{q',1-1/q'})^2$ for all $1 < q' < \infty$, see Proposition 3.7. This will lead us to conclude that (ω_q, j_q) belongs to $E_{R_{q'},q'}$ for all $4/3 \leq q' < 2$, from which uniqueness will easily follow.

Proposition 3.2 (Uniqueness and local existence in $E_{R_q,q}$, for $4/3 \leq q < 2$). *Suppose* $4/3 \leq q < 2$ *and* $(\omega_0, j_0) \in (L^1)^2$. *Then there exists* $T_q > 0$ *such that Eq.* (3.7) *has a unique solution* (ω_q, j_q) *in* $E_{R_q,q}$.

The preparatory work for the proof of the above proposition is in the following three lemmas.

The first of these lemmas gives the estimate of the nonlinearities present in the definition of $\Phi = (\Phi_1, \Phi_2)$, see (3.5) and (3.6).

Lemma 3.3. *Let* $4/3 \leq q < 2$ *and* $(\phi, \psi) \in K_{q,1-1/q} \times K_{q,1-1/q}$. *Then the following estimates hold*:

(1)

$$\begin{aligned} &|||H_\nu \boldsymbol{\nabla} \cdot (\phi \mathbf{K} * \psi)|||_{q,1-1/q} \\ &\quad \leq \sigma_q\chi_{1/q-1/2}B(1 - 1/q, 2/q - 1)\,|||\phi|||_{q,1-1/q}\,|||\psi|||_{q,1-1/q}. \end{aligned} \tag{3.11}$$

(2)

$$|||H_\mu(\mathcal{H}(\phi,\psi))|||_{q,1-1/q} \leq 8c_q^2\delta_{1/q}B(1-1/q,2/q-1)|||\phi|||_{q,1-1/q}|||\psi|||_{q,1-1/q}. \tag{3.12}$$

The proof is reported in Appendix A.

Next lemma shows that if the heat semiflow originated by (ω_0, j_0) is small enough in $K_{q,1-1/q}$, then $\Phi(E_{R_q,q}) \subseteq E_{R_q,q}$.

Lemma 3.4. *Let $4/3 \leq q < 2$, $(\omega_0, j_0) \in (L^1(\mathbb{R}^2))^2$ and $0 < T < \infty$ small enough so that*

$$\bar\rho = \max(|||e^{\nu t\Delta}\omega_0|||_{q,1-1/q}, |||e^{\mu t\Delta}j_0|||_{q,1-1/q}) < 1/(4D_q), \tag{3.13}$$

with D_q given in (3.10). *Then Φ maps $E_{R_q,q}$ into $E_{R_q,q}$.*

Proof. Suppose that $(\omega, j) \in E_{R_q,q}$. By (3.11), we have

$$\begin{aligned}|||H_\nu\boldsymbol{\nabla}\cdot(\omega\mathbf{K}*\omega - Sj\mathbf{K}*j)|||_{q,1-1/q}\\ \leq \sigma_q\chi_{1/q-1/2}B(1-1/q,2/q-1)(|||\omega|||^2_{q,1-1/q} + S|||j|||^2_{q,1-1/q})\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}|||H_\mu\boldsymbol{\nabla}\cdot(j\mathbf{K}*\omega - \omega\mathbf{K}*j)|||_{q,1-1/q}\\ \leq 2\sigma_q\chi_{1/q-1/2}B(1-1/q,2/q-1)|||\omega|||_{q,1-1/q}|||j|||_{q,1-1/q}.\end{aligned} \tag{3.15}$$

By (3.12), we have

$$|||H_\mu(\mathcal{H}(\omega,j))|||_{q,1-1/q} \leq 8c_q^2\delta_{1/q}B(1-1/q,2/q-1)|||\omega|||_{q,1-1/q}|||j|||_{q,1-1/q}. \tag{3.16}$$

Combining (3.14)–(3.16), we obtain

$$\begin{aligned}|||\Phi_1(\omega,j)|||_{q,1-1/q} &\leq |||e^{\nu t\Delta}\omega_0|||_{q,1-1/q}\\ &\quad+\sigma_q\chi_{1/q-1/2}B(1-1/q,2/q-1)(|||\omega|||^2_{q,1-1/q} + S|||j|||^2_{q,1-1/q}),\\ |||\Phi_2(\omega,j)|||_{q,1-1/q} &\leq |||e^{\nu t\Delta}j_0|||_{q,1-1/q}\\ &\quad+2\sigma_q\chi_{1/q-1/2}B(1-1/q,2/q-1)||||\omega|||_{q,1-1/q}||j|||_{q,1-1/q}\\ &\quad+8c_q^2\delta_{1/q}B(1-1/q,2/q-1)|||\omega|||_{q,1-1/q}||j|||_{q,1-1/q}.\end{aligned}$$

Given that $(\omega, j) \in E_{R_q,q}$, and recalling the definition of D_q and (3.13), we have that

$$\begin{aligned}|||\Phi_1(\omega,j)|||_{q,1-1/q} &\leq |||e^{\nu t\Delta}\omega_0|||_{q,1-1/q} + D_qR_q^2 \leq \bar\rho + D_qR_q^2 = R_q,\\ |||\Phi_2(\omega,j)|||_{q,1-1/q} &\leq |||e^{\mu t\Delta}j_0|||_{q,1-1/q} + D_qR_q^2 \leq \bar\rho + D_qR_q^2 = R_q,\end{aligned}$$

where the last equality is a straightforward consequence of the definition of R_q, see (3.9). Hence $\Phi(\omega, j) \in E_{R_q,q}$. □

Next lemma shows that the map Φ is a contraction.

Lemma 3.5. *Suppose the hypotheses of Lemma* 3.4 *hold true. Then the map* $\Phi : E_{R_q,q} \to E_{R_q,q}$ *is a contraction.*

Proof. Let $(\omega', j') = (\omega_1 - \omega_2, j_1 - j_2)$ with $(\omega_i, j_i) \in E_{R_q,q}$ for $i = 1, 2$. Therefore

$$\begin{aligned}
&\Phi_1(\omega_1, j_1) - \Phi_1(\omega_2, j_2)\\
&\quad = -H_\nu \boldsymbol{\nabla} \cdot (\omega' \mathbf{K} * \omega_1 + \omega_2 \mathbf{K} * \omega') + S H_\nu \boldsymbol{\nabla} \cdot (j' \mathbf{K} * j_1 + j_2 \mathbf{K} * j'),\\
&\Phi_2(\omega_1, j_1) - \Phi_2(\omega_2, j_2)\\
&\quad = -H_\mu \boldsymbol{\nabla} \cdot (j' \mathbf{K} * \omega_1 + j_2 \mathbf{K} * \omega')\\
&\qquad + H_\mu \boldsymbol{\nabla} \cdot (\omega' \mathbf{K} * j_1 + \omega_2 \mathbf{K} * j') + H_\mu(\mathcal{H}(\omega', j_1)) + H_\mu(\mathcal{H}(\omega_2, j')).
\end{aligned}$$

Using (3.11) and (3.12) and with estimates similar to those used in the proof of Lemma 3.4, we have that

$$\begin{aligned}
&|||\Phi_1(\omega_1, j_1) - \Phi_1(\omega_2, j_2)|||_{q,1-1/q} \le D_q R_q (|||\omega'|||_{q,1-1/q} + |||j'|||_{q,1-1/q}),\\
&|||\Phi_2(\omega_1, j_1) - \Phi_2(\omega_2, j_2)|||_{q,1-1/q} \le D_q R_q (|||\omega'|||_{q,1-1/q} + |||j'|||_{q,1-1/q}).
\end{aligned}$$

Given that, from the definition of R_q, one has $2D_q R_q < 1$, the fact that Φ is a contraction follows. □

We can now give the proof of Proposition 3.2.

Proof. As $(\omega_0, j_0) \in (L^1(\mathbb{R}^2))^2$, from Lemma 2.1 one has that, for each $1 < q < \infty$, $[[[e^{\nu t \Delta}\omega_0]]]_{q,1-1/q} = 0$ and $[[[e^{\mu t \Delta} j_0]]]_{q,1-1/q} = 0$.

Consider $q \in [4/3, 2)$; for the continuity of the heat semi-flow, there exists $T_q > 0$ small enough such that

$$\bar{\rho} = \max(|||e^{\nu t \Delta}\omega_0|||_{q,1-1/q}, |||e^{\mu t \Delta} j_0|||_{q,1-1/q}) < 1/(4D_q),$$

where D_q is given by (3.10) and, computing the $||| \cdot |||$ norm, the sup is taken on $t \in [0, T_q]$. By Lemmas 3.4 and 3.5, the map Φ, in $E_{R_q,q}$, has a unique fixed point (ω_q, j_q). □

It is important to remark that the uniqueness holds only in $E_{R_q,q}$ so that the solution constructed in $E_{R_{\bar{q}},\bar{q}}$, with $q \neq \bar{q}$, could be a different one. However, the following corollary, which will help establish uniqueness, is a straightforward consequence of Proposition 3.2.

Corollary 3.6. *Suppose there exists a solution of* (3.7) *with the property* $(\omega, j) \in (\dot{K}_{q,1-1/q})^2$ *for all* $q \in [4/3, 2)$ *and* $t \in [0, T]$. *Then, for all* $q \in [4/3, 2)$ *one has that* $(\omega, j) = (\omega_q, j_q)$, $t \in [0, \min(T, T_q)]$.

Proof. Let $(\omega, j) \in (\dot{K}_{q,1-1/q})^2$ be a solution of (3.7). Therefore, by definition

$$\limsup_{t \to 0^+} t^{1-1/q} \|\omega\|_q = 0 \qquad \limsup_{t \to 0^+} t^{1-1/q} \|j\|_q = 0.$$

It follows that, for $T' > 0$ small enough

$$\sup_{t\in(0,T']} t^{1-1/q}\|\omega\|_q < R_q \qquad \sup_{t\in(0,T']} t^{1-1/q}\|j\|_q < R_q.$$

This means that, up to time T', one has that $(\omega, j) \in E_{R_q,q}$. However, by Proposition 3.2, (ω_q, j_q) is the only solution that belongs to $E_{R_q,q}$. Therefore, up to time T', $(\omega, j) = (\omega_q, j_q)$. One can reinitialize the solution at time T', and continue up to time $\min(T, T_q)$. The statement of the corollary follows. □

In the following proposition, we shall improve on Proposition 3.2, showing that the solution constructed in Proposition 3.2 belongs also to $\dot{K}_{q',1-1/q'}$ for all $1 < q' < \infty$, the case $q' = 1$ being however excluded.

Proposition 3.7. *Suppose $4/3 \le q < 2$ and $(\omega_0, j_0) \in (L^1)^2$. Let $(\omega_q, j_q) \in E_{R_q,q} \subset (K_{q,1-1/q})^2$ be the solution of* (3.7) *constructed in Proposition* 3.2. *Then $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$ for all $q' \in (1,\infty)$.*

Proof. Let $4/3 \le q < 2$ and let $(\omega_q, j_q) \in E_{R_q,q} \subset (K_{q,1-1/q})^2$ be the solution of (3.7) constructed in Proposition 3.2. We split the proof into two steps: The first will consist in proving that $(\omega_q, j_q) \in (\dot{K}_{q,1-1/q})^2$; the second, which we shall accomplish using a bootstrap argument, in showing that $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$ for all $q' \in (1,\infty)$.

By the definition of R_q given in (3.9), it is obvious that $R_q < 2\bar\rho$, where $\bar\rho$ is given in (3.13). Recalling the definition of the norm $|||\cdot|||$, one can therefore write:

$$\begin{aligned}\sup_{0<t<T_q} & t^{1-1/q}\|\omega_q\|_q \le 2\bar\rho \\ &\equiv 2\max\left\{\sup_{0<t<T_q} t^{1-1/q}\|e^{\nu t\Delta}\omega_0\|_q,\ \sup_{0<t<T_q} t^{1-1/q}\|e^{\mu t\Delta}j_0\|_q\right\},\end{aligned} \tag{3.17}$$

with an analogous bound holding for the current j_q. The time T_q was chosen small enough so that one could carry out the fixed point argument, see Lemma 3.4 and the proof of Proposition 3.2, where we chose T_q so that $\bar\rho < 1/(4D_q)$. It is however obvious that the fixed point argument works for any T' such that $0 < T' < T_q$, and that the same bound (3.17) holds with T' instead of T_q

$$\begin{aligned}\sup_{0<t<T'} & t^{1-1/q}\|\omega_q\|_q \le 2\bar\rho \\ &\equiv 2\max\left\{\sup_{0<t<T'} t^{1-1/q}\|e^{\nu t\Delta}\omega_0\|_q,\ \sup_{0<t<T'} t^{1-1/q}\|e^{\mu t\Delta}j_0\|_q\right\}.\end{aligned}$$

Taking the limit $T' \to 0$ in the above expression, one gets

$$[[[\omega_q]]]_{q,1-1/q} \le 2\max([[[e^{\nu t\Delta}\omega_0]]]_{q,1-1/q},\ [[[e^{\mu t\Delta}j_0]]]_{q,1-1/q}).$$

The right-hand side of the previous inequality is zero by Lemma 2.1. Therefore, $[[[\omega_q]]]_{q,1-1/q} = 0$. Analogously one has that $[[[j_q]]]_{q,1-1/q} = 0$. This proves that (ω_q, j_q) is in $(\dot{K}_{q,1-1/q})^2$.

Now we show that $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$ for all $q' \in (1,\infty)$. The proof of this statement is based on a bootstrap argument. We begin with the estimate of the nonlinear term appearing in (3.7) given in the following remark.

Remark 3.8. Given that $(\omega_q, j_q) \in (\dot{K}_{q,1-1/q})^2$, the quantities

$$H_\nu \boldsymbol{\nabla} \cdot (\omega_q \mathbf{K} * \omega_q), \quad H_\nu \boldsymbol{\nabla} \cdot (\omega_q \mathbf{K} * j_q), \quad H_\mu \boldsymbol{\nabla} \cdot (j_q \mathbf{K} * \omega_q),$$
$$H_\mu \boldsymbol{\nabla} \cdot (j_q \mathbf{K} * j_q), \quad H_\mu \mathcal{H}(\omega_q, j_q),$$

all belong to $\dot{K}_{q',1-1/q'}$ where the range of q' is given by $2/q - 1 < 1/q' \leq 2/q - 1/2$.

In fact, using HLS inequality (2.2) and then Hölder inequality (2.1) one has that $\omega_q \mathbf{K} * \omega_q \in \dot{K}_{p,2-2/q}$ with $p = 2/q - 1/2$. We can use Lemma 2.2 to say that $H_\nu \boldsymbol{\nabla} \cdot (\omega_q \mathbf{K} * \omega_q) \in \dot{K}_{q',1-1/q'}$. The condition $p \leq q'$ ($p \leq q$ in the notation of Lemma 2.2) leads to $1/q' \leq 2/q - 1/2$; while the condition $\alpha < \beta$ of Lemma 2.2 leads to $2/q - 1 < 1/q'$.

The analysis of the terms $H_\nu \boldsymbol{\nabla} \cdot (\omega_q \mathbf{K} * j_q)$, $H_\mu \boldsymbol{\nabla} \cdot (j_q \mathbf{K} * \omega_q)$, $H_\mu \boldsymbol{\nabla} \cdot (j_q \mathbf{K} * j_q)$ is analogous.

Passing to the last term, using (2.1) and (3.4), we have that $\mathcal{H}(\omega_q, j_q)$ is in $\dot{K}_{q/2,2-2/q}$. From Lemma 2.2, we have that

$$H_\mu \mathcal{H}(\omega_q, j_q) \in \dot{K}_{q',1-1/q'}, \quad \text{for } 2/q - 1 < 1/q' \leq 2/q.$$

This concludes the proof of the remark.

Moreover, $(\omega_0, j_0) \in (L^1)^2$ and Lemma 2.1 gives

$$e^{\nu t \Delta}\omega_0, \quad e^{\mu t \Delta} j_0 \in \dot{K}_{q',1-1/q'}, \quad \text{for all } 1 < q' < \infty.$$

Note however, that $q' = 1$ cannot be included given that, in this case, no initial time regularization is present, and the heat semiflow of the initial datum is only bounded continuous; see Lemma 2.1 in the case $q = p = 1$.

We can now use the fact that (ω_q, j_q) is a solution of (3.7) to say that it belongs to $(\dot{K}_{q',1-1/q'})^2$, with $2/q - 1 < 1/q' \leq 2/q - 1/2$ and $q' \neq 1$.

Repeating this process a finite number of times, we have that ω_q and j_q are in $\dot{K}_{q',1-1/q'}$, with $0 < 1/q' < 1$. Equivalently $1 < q' < \infty$. $\square$

We now give the proof of the main result of this section using Theorem 3.1.

Proof. For a fixed $q \in [4/3, 2)$ Proposition 3.2 gives the existence, up to time $T_q > 0$, of a solution $(\omega_q, j_q) \in (K_{q,1-1/q})^2$. Proposition 3.7 states that, for all $1 < q' < \infty$, $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$. Therefore, the existence part of Theorem 3.1 is proved.

Given that $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$ for all $1 < q' < \infty$, in particular $(\omega_q, j_q) \in (\dot{K}_{q',1-1/q'})^2$ for all $4/3 \leq q' < 2$. Suppose there exists a solution (ω, j) with the same property. Applying Corollary 3.6 one has that $(\omega, j) = (\omega_q, j_q)$. The uniqueness part of Theorem 3.1 is thus proved.

From now on, the solution constructed in part (i) of Theorem 3.1, and with the uniqueness property of part (ii), shall be denoted by (ω, j)

We now prove that $(\omega, j) \in BC([0,T), L^1)$. This is easily shown using a standard bootstrapping argument.

In fact, the solution (ω, j) we have constructed belongs, in particular, to $(\dot{K}_{4/3,1/4})^2$. Hence, by the HSL inequality (2.2), we have that $\mathbf{K} * \omega$ and $\mathbf{K} * j$ belong to $(\dot{K}_{4,1/4})^2$. By Hölder inequality (2.1) one has that $\omega\mathbf{K} * \omega$, $j\mathbf{K} * j$, $j\mathbf{K} * \omega$ and $\omega\mathbf{K} * j$ are in $\dot{K}_{1,1/2}$. By Lemma 2.2

$$H_\nu \boldsymbol{\nabla} \cdot (\omega\mathbf{K} * \omega - Sj\mathbf{K} * j) \quad \text{and}$$

$$H_\mu \boldsymbol{\nabla} \cdot (j\mathbf{K} * \omega - \omega\mathbf{K} * j) \in \dot{K}_{1,0} \subset BC([0,T), L^1).$$

An analogous argument shows that

$$H_\mu(\mathcal{H}(\omega, j)) \in \dot{K}_{1,0} \subset BC([0,T), L^1).$$

Finally, from Lemma 2.1, one gets

$$e^{\nu t\Delta}\omega_0, \quad e^{\mu t\Delta} j_0 \quad \text{belong to } BC([0,T), L^1).$$

Using Eq. (3.7) one gets that $(\omega, j) \in BC([0,T), L^1)$. The proof of Theorem (3.1) is therefore complete. □

4. Regularity

In this section, we shall prove that the solution given in Theorem 3.1 is regular, so the mild solution constructed in the previous section is a strong solution. The main result of this section is the following theorem.

Theorem 4.1. *Let $(\omega_0, j_0) \in (L^1(\mathbb{R}^2))^2$, and let (ω, j) be the solution of* (3.7) *constructed in the previous section. Then, for all $k \geq 0$,*

(1) *$(D^k\omega, D^k j)$ is in $(K_{q,\alpha})^2$, where $\frac{2}{k+2} < q \leq \infty$ and $\alpha = 1 + k/2 - 1/q$; if $k = 0$ then $1 \leq q \leq \infty$;*
(2) *$(D^k\mathbf{u}, D^k\mathbf{B})$ is in $(K_{q,\alpha})^2$, where $\frac{2}{k+1} < q \leq \infty$ and $\alpha = 1/2 + k/2 - 1/q$.*

All the above statements hold for $t \in (0,T)$, where T is the time of existence of the solution given in Theorem 3.1.

In the proof of Theorem 4.1, the following lemma will be useful.

Lemma 4.2. *Let $f \in K_{p_0,\alpha_0}$, $D^k f \in K_{p,\alpha}$, $g \in K_{r_0,\beta_0}$, $D^k g \in K_{r,\beta}$, with p, p_0, r and r_0 in $(1,\infty)$ and $1/p + 1/r_0 = 1/p_0 + 1/r$ and α, α_0, β and β_0 positive, with $\alpha + \beta_0 = \beta + \alpha_0$. Then $D^k(fg) \in K_{q,\gamma}$ with $1/q = 1/p + 1/r_0 = 1/p_0 + 1/r$ and $\gamma = \alpha + \beta_0 = \beta + \alpha_0$.*

The proof of the above lemma is in [31]. We now give the proof of Theorem 4.1. We shall follow the procedure of [31] and refer to that paper for more details.

Proof of Theorem 4.1. First, we observe that, from the Biot–Savart law (1.10) and (1.11), statement (2) of the theorem is a consequence of statement (1), see [31, 38]. Moreover, once we show the statements (1) and (2) for $q \neq \infty$, the case $q = \infty$ follows from the Gagliardo–Nirenberg inequality. Therefore, we prove statement (1) when $q \neq \infty$. We proceed by induction on k.

For $k = 0$ and $1 < q < \infty$, statement (1) is given by the local existence Theorem 3.1 proven in the previous section.

Assume that (1), and therefore (2), hold for a given $k > 0$ and $\frac{2}{k+2} < q < \infty$; we have to show that (1) holds for $k + h$, with $0 < h < 1$.

In order to do so, we consider the equations satisfied by $t^\delta D^{k+h}\omega$ and $t^\delta D^{k+h}j$, with $t \in (0,T)$ and $\delta > 0$; one can derive these equations, formally, from (3.1) and (3.2), and rigorously justify them adapting to the MHD case, the procedure outlined in [31]:

$$D^{k+h}\omega = \delta t^{-\delta} D^h H_\nu(t^{\delta-1}D^k\omega) - t^{-\delta} D^h H_\nu \boldsymbol{\nabla}\cdot(t^\delta D^k(\mathbf{u}\omega - S\mathbf{B}j)), \tag{4.1}$$

$$\begin{aligned} D^{k+h}j = {} & \delta t^{-\delta} D^h H_\mu(t^{\delta-1}D^k j) - t^{-\delta} D^h H_\mu \boldsymbol{\nabla}\cdot(t^\delta D^k(\mathbf{u}j - \mathbf{B}\omega)) \\ & + t^{-\delta} D^h H_\mu(t^\delta D^k \mathcal{H}(\omega, j)). \end{aligned} \tag{4.2}$$

We choose $\delta \equiv (k+h)/2+1-1/q > 0$ and show that all the terms on the right-hand sides of the previous equations belong to $K_{q,\delta}$.

Analysis of the terms $\boldsymbol{t^{-\delta}D^hH_\nu(t^{\delta-1}D^k\omega)}$ and $\boldsymbol{t^{-\delta}D^hH_\mu(t^{\delta-1}D^kj)}$. We give the procedure for the analysis of $t^{-\delta}D^hH_\nu(t^{\delta-1}D^k\omega))$; the analysis of $t^{-\delta}D^hH_\mu(t^{\delta-1}D^kj)$ is analogous.

By the induction hypothesis $D^k\omega$ is in $K_{q,\alpha}$, where $\frac{2}{k+2} < q < \infty$ and $\alpha + 1/q = 1 + k/2$. Therefore, $t^{\delta-1}D^k\omega$ is in $K_{q,1-h/2}$. Moreover, using Lemma 2.4, one obtains that $D^hH_\nu(t^{\delta-1}D^k\omega) \in K_{q,0}$.

Hence $t^{-\delta}D^hH_\nu(t^{\delta-1}D^k\omega)$ is in $K_{q,\delta}$.

Analysis of the term $\boldsymbol{t^{-\delta}D^hH_\nu\nabla\cdot t^\delta D^k(\mathbf{u}\omega)}$, and of the analogous terms involving $\boldsymbol{D^k(\mathbf{B}j)}$, $\boldsymbol{D^k(\mathbf{B}\omega)}$ and $\boldsymbol{D^k(\mathbf{u}j)}$). First, we make the following remark.

Remark 4.3. $D^k(\mathbf{u}\omega)$ is in $K_{q,\gamma}$, with $1/q + \gamma = k/2 + 3/2$.

Proof of Remark 4.3. To prove the above remark, first, we observe that $\omega \in K_{r_0,\beta_0}$, with $1 \leq r_0 < \infty$ and $\beta_0 = 1 - 1/r_0$, because ω has been constructed in Theorem 3.1. As a consequence, $\mathbf{u} \in K_{p_0,\beta_0}$, with $2 < p_0 < \infty$ and$1/p_0 + \beta_0 = 1/2$, as $1/r_0 - 1/p_0 = 1/2$, because of the Biot–Savart law and of the Hardy–Littlewood–Sobolev estimate.

By the induction hypothesis, $D^k\omega \in K_{r,\beta}$, with $\frac{2}{k+2} < r < \infty$ and $\beta = 1 + k/2 - 1/r$, and $D^k\mathbf{u} \in K_{p,\beta}$, with $\frac{2}{k+1} < p < \infty$ and moreover $\beta = 1/2 + k/2 - 1/p$ as $1/r - 1/p = 1/2$ because of the Biot–Savart law and of the Hardy–Littlewood–Sobolev estimate.

Using Lemma 4.2, we have that $D^k(\mathbf{u}\omega)$ is in $K_{q,\gamma}$ with $1/q = 1/p_0 + 1/r = 1/p + 1/r_0$ and $\gamma = \beta_0 + \beta$. Given that $\beta = 1 + k/2 - 1/r$ and $1/p_0 + \beta_0 =$

$1/2$, it follows immediately that $1/q + \gamma = k/2 + 3/2$. Therefore, Remark 4.3 is proved. $\square$

Now, we return to analyze the term $t^{-\delta} D^h H_\nu \boldsymbol{\nabla} \cdot t^\delta D^k(\mathbf{u}\omega)$. From the Remark 4.3, we have that $t^\delta D^k(\mathbf{u}\omega)$ is in $K_{q,1/2-h/2}$ and, consequently, $D^h H_\nu \boldsymbol{\nabla} \cdot t^\delta D^k(\mathbf{u}\omega)$ is in $K_{q,0}$ by Lemma 2.4. Finally, $t^{-\delta} D^h H_\nu \boldsymbol{\nabla} \cdot t^\delta D^k(\mathbf{u}\omega)$ is in $K_{q,\delta}$.

Analysis of the term $\boldsymbol{t^{-\delta} D^h H_\mu (t^\delta D^k \mathcal{H}(\omega, j))}$. First we prove the following remark.

Remark 4.4. $D^k \mathcal{H}(\omega, j) \in K_{q,\gamma}$, with $1/q + \gamma = k/2 + 2$.

Proof of Remark 4.4. To prove the above remark, first, we observe that ω and $j \in K_{r_0,\beta_0}$, with $1 \le r_0 < \infty$ and $\beta_0 = 1 - 1/r_0$, because ω and j have been constructed in Theorem 3.1. From the continuity of the Riesz operator $\mathcal{R}_{hk}$, see (3.4), we have that $\mathcal{R}_{hk}\omega$ and $\mathcal{R}_{hk} j \in K_{r_0,\beta_0}$, with $1 < r_0 < \infty$ and $\beta_0 = 1 - 1/r_0$.

By the induction hypothesis, $D^k \omega$ and $D^k j \in K_{r,\beta}$, with $\frac{2}{k+2} < r < \infty$ and $\beta = 1 + k/2 - 1/r$. Therefore, once again by the continuity of the Riesz operator, $D^k \mathcal{R}_{hk}\omega$ and $D^k \mathcal{R}_{hk} j \in K_{r,\beta}$.

The term $\mathcal{H}$ is expressed by terms of the type $\mathcal{R}_{hk}\omega \mathcal{R}_{lm} j$, see (3.3); therefore, we can use Lemma 4.2 to get that $D^k(\mathcal{H}(\omega, j)) \in K_{q,\gamma}$ with $1/q = 1/r_0 + 1/r$ and $\gamma = \beta_0 + \beta$. Given that $\beta = 1 + k/2 - 1/r$ and $\beta_0 = 1 - 1/r_0$, one immediately gets that $\gamma = k/2 + 2 - 1/q$. Remark 4.4 is thus proved. $\square$

We can now analyze the term $t^{-\delta} D^h H_\mu (t^\delta D^k \mathcal{H}(\omega, j))$. Using the above Remark 4.4, Lemma 2.4, and recalling the definition of δ as $\delta \equiv (k+h)/2 + 1 - 1/q > 0$, one easily gets that $t^{-\delta} H_\mu (t^\delta D^k \mathcal{H}(\omega, j))$ is in $K_{q,\delta}$.

This concludes the proof that that $D^{k+h}\omega$ and $D^{k+h} j$, as given in (4.1) and (4.2), are in $K_{q,(k+h)/2+1-1/q}$ and Theorem 4.1 is, therefore, proved.

Using the above regularity Theorem 4.1, one can easily prove that the solution (ω, j) constructed in Theorem 3.1 is a classical solution of the MHD vorticity-current Eqs. (1.8) and (1.9). Moreover, the corresponding vector fields $\mathbf{u}$ and $\mathbf{B}$ are classical solutions of (1.1)–(1.4).

Corollary 4.5. *Suppose* (ω_0, j_0) *in* $(L^1(\mathbb{R}^2))^2$, *and let* (ω, j) *be the solution given by Theorem* 3.1. *Then* (ω, j) *is a classical solutions of* (1.8) *and* (1.9) *with* $\omega(t) \to \omega_0$ *and* $j(t) \to j_0$ *in* $\mathcal{S}'$, *as* $t \to 0$. *Moreover,* $(\mathbf{u}, \mathbf{B})$ *with* $\mathbf{u} = \mathbf{K} * \omega$ *and* $\mathbf{B} = \mathbf{K} * j$, *is the classical solution of* (1.1)–(1.4) *with* $\mathbf{u}(t) \to \mathbf{K} * \omega_0$ *and* $\mathbf{B}(t) \to \mathbf{K} * j_0$ *in* $\mathcal{S}'$, *as* $t \to 0$.

Proof. From Theorem 4.1, ω and j are smooth solutions of (3.5) and (3.6), then it is obvious that they are classical solutions of (1.8) and (1.9).

The fact that $\omega(t) \to \omega_0$ and $j(t) \to j_0$ in $\mathcal{S}'$, as $t \to 0$, is a consequence of Theorem 3.1.

It remains to prove that $\mathbf{u} = \mathbf{K} * \omega$ and $\mathbf{B} = \mathbf{K} * j$ are classical solutions of (1.1)–(1.4).

It is obvious, see [38], that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\boldsymbol{\nabla} \times \mathbf{u} = \omega$. Analogous relations hold for j and $\mathbf{B}$ which are denoted by

$$\mathbf{F} = \partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p - \nu \Delta \mathbf{u} - S\mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B}$$

and by

$$\mathbf{G} = \partial_t \mathbf{B} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{B} - \mu \Delta \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{u}.$$

Using the equations satisfied by ω and j, one has that $\boldsymbol{\nabla} \times \mathbf{F} = 0$ and $\boldsymbol{\nabla} \times \mathbf{G} = 0$. Moreover, $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$ and, as usual, it is possible to solve an elliptic problem for the pressure p so that $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. Then $\mathbf{F}$ and $\mathbf{G}$ are harmonic vectors in $L^s(\mathbb{R}^2)$, for some index s.

This implies that $\mathbf{F} \equiv 0$ and $\mathbf{G} \equiv 0$, which means that $\mathbf{u}$ and $\mathbf{B}$ are solutions of (1.1) and (1.4).

We finally prove that $\mathbf{u}(t) \to \mathbf{K} * \omega_0$ in $\mathcal{S}'$ as $t \to 0$.

Using the Fourier transform, we have that $\widehat{\mathbf{K}}(\xi)(\widehat{\omega}(t) - \widehat{\omega}_0) \to 0$ in $\mathcal{S}'$ as $t \to 0$, because $\mathcal{F}(\omega(t) - \omega_0) \to 0$ in $\mathcal{S}'$, as $t \to 0$, and $\widehat{\mathbf{K}}(\xi) = \mathrm{const}\, \xi^T / |\xi|^2$.

An analogous proof holds for $\mathbf{B}$, and the corollary is proved. □

In the next corollary, we prove that the solution (ω, j) of the MHD vorticity-current equations with initial datum in $(L^1)^2$ is in $(L^1 \cap L^\infty)^2$.

Corollary 4.6. *Let ω and j be the solutions given by Theorem* 3.1, *with initial datum (ω_0, j_0) in $(L^1)^2$. Then, for each $t \in (0, T)$, $\|\omega(t)\|_q$ and $\|j(t)\|_q$ are bounded, for each $q \in [1, \infty]$.*

Proof. It is an easy consequence of Theorems 3.1 and 4.1. □

5. Conclusions

In this paper, we have proved the short-time well-posedness of the MHD equations in the vorticity-current formulation. MHD is an important model of mathematical physics, and it will probably play a crucial role in developing future technologies. Starting with the pioneering work of DiPerna and Majda [16, 17] and the result of Delort, [15] the advancements of the mathematical theory of the fluid dynamics equations with vorticity measure initial data have attracted a great deal of interest, both for the intrinsic interest of the problem and for the practical relevance of these flows. For the same reasons, we believe that the corresponding problem for the MHD will play a significant role in the future development of mathematical fluid dynamics.

In this paper, we have proved that the MHD equations with L^1 data admit a regular solution for a short time. To prove the global well-posedness for the corresponding NS problem, Kato relied on the global a priori bound for the L^p norm of the vorticity. The lack of a similar estimate for the MHD equations is the main obstacle to making our result global in time. A further problem that would be interesting to address is considering, as initial data, atomic measures or vortex-sheets and currents-sheets, which are data of particular significance in astrophysics and plasma physics. These would be challenging configurations to deal with, particularly in the ideal case or in the limit of small viscosity and resistivity, where singularity formation or concentration phenomena are likely to occur, see [7, 5] for the Euler and NS equations case.

Finally, if the solutions we have built in this paper will be shown to be global in time, it would be important to study their long-time behavior. In the NS case, it is well known that "any solution of the two-dimensional Navier–Stokes equation whose initial vorticity distribution is integrable converges to an explicit self-similar solution called Oseen's vortex" [24]. One can easily see that an analogous self-similar solution exists for the MHD equations and a natural question is whether this solution is stable and is an attractor.

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Appendix A. Proof of Lemma 3.3

We give the proof of Lemma 3.3.

Proof. Since $4/3 \le q < 2$, by (2.2), we have

$$|||\mathbf{K} * \psi|||_{p,1-1/q} \le \sigma_q |||\psi|||_{q,1-1/q}, \quad 1/p = 1/q - 1/2.$$

By Hölder's inequality (2.1),we have that

$$|||\phi\mathbf{K} * \psi|||_{r,2-2/q} \le |||\phi|||_{q,1-1/q} |||\mathbf{K} * \psi|||_{p,1-1/q} \le \sigma_q |||\phi|||_{q,1-1/q} |||\psi|||_{q,1-1/q},$$

with $1/r = 2/q - 1/2$.

Using (2.6), since $1/r = 2/q - 1/2$ and $4/3 \le q < 2$ ($q \ge 4/3$ is necessary to have $r \ge 1$), we obtain

$$\begin{aligned} &|||H_\nu \boldsymbol{\nabla} \cdot (\phi\mathbf{K} * \psi)|||_{q,1-1/q} \\ &\quad \le \chi_{1/q-1/2} B(1 - 1/q, 2/q - 1) |||\phi\mathbf{K} * \psi|||_{r,2-2/q} \\ &\quad \le \sigma_q \chi_{1/q-1/2} B(1 - 1/q, 2/q - 1) |||\phi|||_{q,1-1/q} |||\psi|||_{q,1-1/q}, \end{aligned}$$

which is (3.11). We now pass to proving (3.12).

Consider the term $\mathcal{H}(\omega, j)$, by (2.1) and (3.4), since $4/3 \le q < 2$, it follows that

$$\begin{aligned}|||\mathcal{H}(\phi,\psi)|||_{q/2,2-2/q} &\le 2|||\mathcal{R}_{12}\phi\mathcal{R}_{11}\psi|||_{q/2,2-2/q} + 2|||\mathcal{R}_{12}\phi\mathcal{R}_{22}\psi|||_{q/2,2-2/q}\\ &\quad + 2|||\mathcal{R}_{12}j\mathcal{R}_{11}\phi|||_{q/2,2-2/q} + 2|||\mathcal{R}_{12}\psi\mathcal{R}_{22}\phi|||_{q/2,2-2/q}\\ &\le 2|||\mathcal{R}_{12}\phi|||_{q,1-1/q}\left(|||\mathcal{R}_{11}\psi|||_{q,1-1/q} + |||\mathcal{R}_{22}\psi|||_{q,1-1/q}\right)\\ &\quad + 2|||\mathcal{R}_{12}\psi|||_{q,1-1/q}\left(|||\mathcal{R}_{11}\phi|||_{q,1-1/q} + |||\mathcal{R}_{22}\phi|||_{q,1-1/q}\right)\\ &\le 8c_q^2|||\phi|||_{q,1-1/q}|||\psi|||_{q,1-1/q}.\end{aligned}$$

Using (2.5), since $4/3 \le q < 2$, we have

$$\begin{aligned}|||H_\mu(\mathcal{H}(\phi,\psi))|||_{q,1-1/q} &\le \delta_{1/q}B(1-1/q, 2/q-1)|||\mathcal{H}(\phi,\psi)|||_{q/2,2-2/q}\\ &\le 8c_q^2\delta_{1/q}B(1-1/q,2/q-1)|||\phi|||_{q,1-1/q}|||\psi|||_{q,1-1/q}\end{aligned}$$

and the proof is complete. □

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