# FRACTIONAL DIRICHLET PROBLEMS WITH SINGULAR AND NON-LOCALLY CONVECTIVE REACTION

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ABSTRACT. In this paper, the existence of positive weak solutions to a Dirichlet problem driven by the fractional $(p,q)$-Laplacian and with reaction both weakly singular and non-locally convective (i.e., depending on the distributional Riesz gradient of solutions) is established. Due to the nature of the right-hand side, we address the problem via sub-super solution methods, combined with variational techniques, truncation arguments, as well as fixed point results.

## 1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$, let $0 < s_2 \leq s \leq s_1 \leq 1$, and let $2 < q < p < \frac{N}{s_1}$ with $s_1 p > 1$. Consider the problem

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x,u) + g(x, D^s u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

where, given $r > 1$ and $0 < t < 1$, $(-\Delta)_r^t$ denotes the (negative) fractional $r$-Laplacian, formally defined by

$$(-\Delta)_r^t u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{r-2}(u(x) - u(y))}{|x - y|^{N+tr}} \, dy, \quad x \in \mathbb{R}^N.$$

The symbol $D^s u$ indicates the *distributional Riesz fractional gradient* of $u$ according to [23, 24]. If $u$ is sufficiently smooth and appropriately decays at infinity then

$$D^s u(x) := c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+s}} \frac{x - y}{|x - y|} \, dy, \quad x \in \mathbb{R}^N,$$

with $c_{N,s} > 0$; cf. [24, p. 289]. Moreover, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}_0^+$ and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}_0^+$ stand for Carathéodory's functions such that

$$\begin{cases} \liminf\limits_{t \to 0^+} f(x,t) =: L > 0 & \text{uniformly in } x \in \Omega, \\ f(x,t) \leq c_1 t^{-\gamma} + c_2 t^r & \forall\, (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{$\mathrm{H_{f_1}}$}$$

$$t \mapsto \frac{f(\cdot,t)}{t^{q-1}} \text{ is strictly decreasing on } \mathbb{R}^+, \tag{$\mathrm{H_{f_2}}$}$$

$$g(x,\xi) = c_3(1 + |\xi|^\zeta), \quad (x,\xi) \in \Omega \times \mathbb{R}^N, \tag{$\mathrm{H_g}$}$$

for suitable $\gamma \in (0,1)$, $c_i > 0$, $i = 1,2,3$, $r, \zeta \in (1, p-1)$.

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Since $s_2 \le s_1$, we are naturally led to solve problem (P) in the fractional Sobolev space $W_0^{s_1,p}(\Omega)$. Precisely,

**Definition 1.1.** A function $u \in W_0^{s_1,p}(\Omega)$ is called a *weak solution* of (P) when $u > 0$ a.e. in $\Omega$ and

$$\begin{aligned}\int_{\mathbb{R}^{2N}} & \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y \\ & + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}\,\mathrm{d}x\,\mathrm{d}y \\ = & \int_\Omega f(\cdot,u)\varphi\,\mathrm{d}x + \int_\Omega g(\cdot,D^s u)\varphi\,\mathrm{d}x \quad \forall\,\varphi \in W_0^{s_1,p}(\Omega).\end{aligned}$$

Let us next point out some hopefully newsworthy aspects, namely

- the driving differential operator is neither local nor homogeneous and no parameters appear on the right-hand side,
- $f(x,\cdot)$ can be singular at zero, which means $\lim_{t\to 0^+} f(x,t) = +\infty$, and
- the reaction is also non-locally convective, because $g$ depends on the distributional fractional gradient of solutions.

In latest years, the study of non-local differential equations has seen significant growth, mainly due to their many applications in real-world problems, such as game theory, finance, image processing, and materials science. A wealth of existence and uniqueness or multiplicity theorems are already available; see, e.g., [10, 17, 19]. Several regularity results have also been published; let us mention [15, 16] for the fractional $p$-Laplacian and [12] as regards the fractional $(p,q)$-Laplacian. Finally, the survey [8] provides an exhaustive account on the corresponding functional framework, i.e., fractional Sobolev spaces.

Dirichlet problems driven by non-local operators and with singular reactions were well investigated in [5, 12]. The work [5] treats existence and uniqueness of solutions to the equation

$$(-\Delta)_p^s u = \frac{a(x)}{u^\sigma} \ \text{ in } \ \Omega,$$

where $\sigma > 0$ and $a : \Omega \to \mathbb{R}^+$ fulfills appropriate conditions, while [12] studies the more general situation

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \frac{a_\delta(x)}{u^\sigma} \ \text{ in } \ \Omega,$$

being $a_\delta \in L^\infty_{\mathrm{loc}}(\Omega)$ a function that behaves like $d(x)^{-\delta}$, with $d(x) := \operatorname{dist}(x,\partial\Omega)$ and $\delta \in [0, s_1p)$. It should be noted that, contrary to [5, 12], here, the reaction term $f(\cdot,u)$ is not necessarily a perturbation of $u^{-\sigma}$.

Finally, distributional fractional gradients were first introduced by Horváth [14], but gained traction and saw wider application especially after the two seminal papers of Shieh - Spector [23, 24]. Among recent contributions on this subject, we mention [22, 6, 4, 7, 1], as well as the references therein. The main reasons that contributed to spreading the use of Riesz gradients probably are: 1) Non-local versions of many classical results on Sobolev spaces can be obtained through them. 2) $D^s u$ formally tends to $\nabla u$ as

$s \to 1^-$. 3) Significant geometrical and physical properties (invariance under translations or rotations, homogeneity of order $s$, etc.) remain true in this new context; cf. [25].

Although weakly singular (namely $\gamma < 1$) problems are normally investigated via variational methods, here, the presence of $D^s u$ inside the reaction prevents this approach. That's way the existence of at least one positive solution to (P) is established by means of sub-super solution arguments, variational and truncation techniques, besides Schauder's fixed point theorem.

The paper is organized as follows. Preliminary facts are collected in Section 2. The next one shows that a suitable auxiliary problem, obtained by freezing the convection term, admits a unique solution, while (P) is solved in Section 4.

## 2. Preliminaries

Let $X$ be a real Banach space with topological dual $X^*$ and duality brackets $\langle \cdot, \cdot \rangle$. A function $A : X \to X^*$ is called:

- *monotone* when $\langle A(x) - A(z), x - z \rangle \geq 0$ for all $x, z \in X$.
- *of type* $(\mathrm{S})_+$ provided

$$x_n \rightharpoonup x \text{ in } X, \; \limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0 \implies x_n \to x \text{ in } X.$$

The next elementary result will ensure that condition $(\mathrm{S})_+$ holds true for the fractional $(p, q)$-Laplacian.

**Proposition 2.1.** *Let $A : X \to X^*$ be of type $(\mathrm{S})_+$ and let $B : X \to X^*$ be monotone. Then $A + B$ satisfies condition $(\mathrm{S})_+$.*

*Proof.* Suppose $x_n \rightharpoonup x$ in $X$ and

$$\limsup_{n \to +\infty} \langle A(x_n) + B(x_n), x_n - x \rangle \leq 0. \tag{2.1}$$

The monotonicity of $B$ entails

$$\begin{aligned} \langle A(x_n), x_n - x \rangle &= \langle A(x_n) + B(x_n), x_n - x \rangle - \langle B(x_n), x_n - x \rangle \\ &= \langle A(x_n) + B(x_n), x_n - x \rangle - \langle B(x_n) - B(x), x_n - x \rangle - \langle B(x), x_n - x \rangle \\ &\leq \langle A(x_n) + B(x_n), x_n - x \rangle - \langle B(x), x_n - x \rangle \quad \forall \, n \in \mathbb{N}. \end{aligned}$$

Using (2.1) we thus get

$$\limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

whence $x_n \to x$ because $A$ is of type $(\mathrm{S})_+$. □

Finally, if $X$ and $Y$ are two topological spaces then $X \hookrightarrow Y$ means that $X$ continuously embeds in $Y$.

Hereafter, $\Omega$ is a bounded domain of the real Euclidean $N$-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 2$, with a $C^2$-boundary $\partial\Omega$, $|E|$ indicates the $N$-dimensional Lebesgue measure of $E \subseteq \mathbb{R}^N$,

$$t_\pm := \max\{\pm t, 0\}, \quad t \in \mathbb{R},$$

while $C$, $C_1$, etc. are positive constants, which may change value from line to line, whose dependencies will be specified when necessary. Denote by $d : \overline{\Omega} \to \mathbb{R}^+_0$ the distance function of $\Omega$, i.e.,

$$d(x) := \mathrm{dist}(x, \partial\Omega) \quad \forall \, x \in \overline{\Omega}.$$

It enjoys a useful summability property (see [13, Proposition 2.1]), namely

**Proposition 2.2.** *If* $0 < \sigma < 1 < q < \frac{1}{\sigma}$ *then* $d^{-\sigma} \in L^q(\Omega)$.

Let $X(\Omega)$ be a real-valued function space on $\Omega$ and let $u, v \in X(\Omega)$. We simply write $u \le v$ when $u(x) \le v(x)$ a.e. in $\Omega$. Analogously for $u < v$, etc. To shorten notation, define

$$\Omega(u \le v) := \{x \in \Omega : u(x) \le v(x)\}, \quad X(\Omega)_+ := \{w \in X(\Omega) : w > 0\}.$$

Henceforth, $p'$ indicates the conjugate exponent of $p \ge 1$, the Sobolev space $W_0^{1,p}(\Omega)$ is equipped with Poincaré's norm

$$\|u\|_{1,p} := \||\nabla u|\|_p, \quad u \in W_0^{1,p}(\Omega),$$

where, as usual,

$$\|v\|_q := \begin{cases} \left(\int_\Omega |v(x)|^q \, \mathrm{d}x\right)^{1/q} & \text{if } 1 \le q < +\infty, \\ \operatorname*{ess\,sup}_{x \in \Omega} |v(x)| & \text{when } q = +\infty, \end{cases}$$

and, given any $u \in W_0^{1,p}(\Omega)$, we set $u := 0$ a.e. in $\mathbb{R}^N \setminus \Omega$; cf. [8, Section 5]. Moreover, $W^{-1,p'}(\Omega) := (W_0^{1,p}(\Omega))^*$ while $p^*$ is the Sobolev critical exponent for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. It is known that $p^* = \frac{Np}{N-p}$ once $p < N$.

Fix $s \in (0,1)$. The Gagliardo semi-norm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is

$$[u]_{s,p} := \left(\int_{R^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y\right)^{1/p}.$$

$W^{s,p}(\mathbb{R}^N)$ denotes the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : \ [u]_{s,p} < +\infty\right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := (\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p)^{1/p}.$$

On the space

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \ \text{ a.e. in } \ \mathbb{R}^N \setminus \Omega\}$$

we will consider the equivalent norm

$$\|u\|_{s,p} := [u]_{s,p}, \quad u \in W_0^{s,p}(\Omega).$$

As before, $W^{-s,p'}(\Omega) := (W_0^{s,p}(\Omega))^*$ and $p_s^*$ indicates the fractional Sobolev critical exponent, i.e., $p_s^* = \frac{Np}{N-sp}$ when $sp < N$, $p_s^* = +\infty$ otherwise. Thanks to Propositions 2.1–2.2, Theorem 6.7, and Corollary 7.2 of [8] one has

**Proposition 2.3.** *If* $1 \le p < +\infty$ *then:*

(a) $0 < s' \le s'' \le 1 \implies W_0^{s'',p}(\Omega) \hookrightarrow W_0^{s',p}(\Omega)$.
(b) $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1, p_s^*]$.
(c) *The embedding in* (b) *is compact once* $q < p_s^* < +\infty$.

However, contrary to the non-fractional case,

$$1 \le q < p \le +\infty \quad \not\Longrightarrow \; W_0^{s,p}(\Omega) \subseteq W_0^{s,q}(\Omega);$$

cf. [20]. Define, for every $u, v \in W_0^{s,p}(\Omega)$,

$$\langle (-\Delta)_p^s u, v\rangle := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\, \mathrm{d}x\, \mathrm{d}y\,.$$

The operator $(-\Delta)_p^s$ is called (negative) $s$-fractional $p$-Laplacian. It possesses the following properties.

($\mathrm{p}_1$) $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is monotone, continuous, and of type $(\mathrm{S})_+$; vide [10, Lemma 2.1].

($\mathrm{p}_2$) $(-\Delta)_p^s$ maps bounded sets into bounded sets. In fact,

$$\|(-\Delta)_p^s u\|_{W^{-s,p'}(\Omega)} \le \|u\|_{s,p}^{p-1} \quad \forall\, u \in W_0^{s,p}(\Omega).$$

To deal with distributional fractional gradients, we first introduce the Bessel potential spaces $L^{\alpha,p}(\mathbb{R}^N)$, where $\alpha > 0$. Set, for every $x \in \mathbb{R}^N$,

$$g_\alpha(x) := \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{\delta}} e^{\frac{-\delta}{4\pi}} \delta^{\frac{\alpha-N}{2}} \frac{\mathrm{d}\delta}{\delta}\,.$$

On account of [21, Section 7.1] one can assert that:

1) $g_\alpha \in L^1(\mathbb{R}^N)$ and $\|g_\alpha\|_{L^1(\mathbb{R}^N)} = 1$.
2) $g_\alpha$ enjoys the semigroup property, i.e., $g_\alpha * g_\beta = g_{\alpha+\beta}$ for any $\alpha, \beta > 0$.

Now, put

$$L^{\alpha,p}(\mathbb{R}^N) := \{u :\ u = g_\alpha * \tilde{u} \text{ for some } \tilde{u} \in L^p(\mathbb{R}^N)\}$$

as well as

$$\|u\|_{L^{\alpha,p}(\mathbb{R}^N)} = \|\tilde{u}\|_{L^p(\mathbb{R}^N)} \ \text{ whenever } \ u = g_\alpha * \tilde{u}.$$

Using 1)–2) easily yields

$$0 < \alpha < \beta \implies L^{\beta,p}(\mathbb{R}^N) \subseteq L^{\alpha,p}(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N).$$

Moreover (see [23, Theorem 2.2]),

**Theorem 2.4.** *If $1 < p < +\infty$ and $0 < \varepsilon < \alpha$ then*

$$L^{\alpha+\varepsilon,p}(\mathbb{R}^N) \hookrightarrow W^{\alpha,p}(\mathbb{R}^N) \hookrightarrow L^{\alpha-\varepsilon,p}(\mathbb{R}^N).$$

Finally, define

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Thanks to Theorem 2.4 we clearly have

$$L_0^{s+\varepsilon,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow L_0^{s-\varepsilon,p}(\Omega) \quad \forall\, \varepsilon \in (0,s). \tag{2.2}$$

The next basic notion is taken from [23]. For $0 < \alpha < N$, let

$$\gamma(N,\alpha) := \frac{\Gamma((N-\alpha)/2)}{\pi^{N/2}2^\alpha\Gamma(\alpha/2)}, \quad I_\alpha(x) := \frac{\gamma(N,\alpha)}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

If $u \in L^p(\mathbb{R}^N)$ and $I_{1-s} * u$ makes sense then the vector

$$D^s u := \left( \frac{\partial}{\partial x_1}(I_{1-s} * u), \dots, \frac{\partial}{\partial x_N}(I_{1-s} * u) \right),$$

where partial derivatives are understood in a distributional sense, is called distributional Riesz $s$-fractional gradient of $u$. Theorem 1.2 in [23] ensures that

$$D^s u = I_{1-s} * Du \quad \forall\, u \in C_c^\infty(\mathbb{R}^N).$$

Further, $D^s u$ looks like the natural extension of $\nabla u$ to the fractional framework; cf., e.g., [11] for details. According to [23, Definition 1.5], $X^{s,p}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X^{s,p}(\mathbb{R}^N)} := (\|u\|^p_{L^p(\mathbb{R}^N)} + \|D^s u\|^p_{L^p(\mathbb{R}^N)})^{1/p}.$$

Since, by [23, Theorem 1.7], $X^{s,p}(\mathbb{R}^N) = L^{s,p}(\mathbb{R}^N)$ we can deduce many facts about $X^{s,p}(\mathbb{R}^N)$ from the existing literature on $L^{s,p}(\mathbb{R}^N)$. In particular, if

$$X_0^{s,p}(\Omega) := \{u \in X^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

then $X_0^{s,p}(\Omega) = L_0^{s,p}(\Omega)$.

## 3. FREEZING THE CONVECTION TERM

To address the two troubles (singularity and convection) separately, here, we will study an auxiliary equation patterned after that of (P), but with $D^s u$ replaced by $D^s v$ for fixed $v \in W_0^{s_1,p}(\Omega)$.

**Lemma 3.1.** *Under hypothesis* ($\mathrm{H_{f_1}}$)*, the problem*

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathrm{P}_f$}$$

*possesses a positive sub-solution* $\underline{u} \in W_0^{s_1,p}(\Omega) \cap C^{0,\tau}(\overline{\Omega})$*, where* $\tau \in (0, s_1)$*.*

*Proof.* Thanks to ($\mathrm{H_{f_1}}$), for every $\varepsilon \in (0, L)$ there exists $\delta \in (0,1)$ such that

$$f(x,t) > \varepsilon \quad \forall\, (x,t) \in \Omega \times (0,\delta).$$

Let $\sigma > 0$. Theorem 3.15 of [12] provides a positive solution $u_\sigma \in W_0^{s_1,p}(\Omega) \cap C^{0,\tau}(\overline{\Omega})$, $\tau \in (0, s_1)$, of the torsion problem

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \sigma & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover, $u_\sigma \to 0$ in $C^{0,\tau}(\overline{\Omega})$ as $\sigma \to 0^+$. Thus, for any $\sigma$ sufficiently small one has both $\sigma < \varepsilon$ and $\|u_\sigma\|_\infty < \delta$. This evidently implies

$$(-\Delta)_p^{s_1} u_\sigma + (-\Delta)_q^{s_2} u_\sigma = \sigma < \varepsilon < f(\cdot, u_\sigma),$$

i.e., $\underline{u} := u_\sigma$ is a positive sub-solution of ($\mathrm{P}_f$). □

**Remark 3.2.** If $s_1 \neq q' s_2$ then Hopf's theorem [12, Proposition 2.12] ensures that

$$\eta d(x)^{s_1} \le \underline{u}(x) \quad \forall\, x \in \Omega, \tag{3.1}$$

with suitable $\eta > 0$. Otherwise, $\eta d^\alpha \le \underline{u}$, being $\alpha > s_1$, $\alpha \neq q' s_2$, and $\alpha \neq p' s_1$; cf. [12, Remark 2.14].

Now, fixed any $v \in W_0^{s_1,p}(\Omega)$, consider the following problem, where the convective term has been frozen:

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x,u) + g(x, D^s v) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P$_v$}$$

**Theorem 3.3.** *Let* ($H_{f_1}$) *and* ($H_g$) *be satisfied. If* $v \in W_0^{s_1,p}(\Omega)$ *then* ($P_v$) *admits a weak solution* $u_v \in W_0^{s_1,p}(\Omega) \cap C^{0,\tau}(\overline{\Omega})$, *where* $\tau \in (0, s_1)$. *Moreover,* $u_v \geq \underline{u}$.

*Proof.* Recalling Lemma 3.1, define

$$\tilde{f}(x,t) := f(x, \max\{\underline{u}(x), t\}), \quad (x,t) \in \Omega \times \mathbb{R}. \tag{3.2}$$

Without loss of generality we can suppose $s_1 \neq q's_2$. In fact, by Remark 3.2, the case $s_1 = q's_2$ is entirely analogous. Thanks to ($H_{f_1}$) and (3.1) one has

$$\begin{aligned} \tilde{f}(\cdot,t) &\leq c_1(\max\{\underline{u},t\})^{-\gamma} + c_2(\max\{\underline{u},t\})^r \leq c_1\underline{u}^{-\gamma} + c_2(\underline{u}^r + t^r) \\ &\leq c_1(\eta d^{s_1})^{-\gamma} + c_2(\max_{\overline{\Omega}} \underline{u})^r + c_2 t^r \leq C_1(d^{-\gamma s_1} + t^r + 1), \quad t \in \mathbb{R}^+. \end{aligned} \tag{3.3}$$

The energy functional $\tilde{J} : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ associated with the problem

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \tilde{f}(x,u) + g(x, D^s v) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.4}$$

is written as

$$\begin{aligned} \tilde{J}(u) := \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}} \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{q} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2 q}} \, \mathrm{d}x \, \mathrm{d}y \\ - \int_\Omega \tilde{F}(\cdot, u) \, \mathrm{d}x - \int_\Omega G(\cdot, D^s v) \, \mathrm{d}x, \quad u \in W_0^{s_1,p}(\Omega), \end{aligned}$$

where

$$\tilde{F}(x,\tau) := \int_0^\tau \tilde{f}(x,t) \, \mathrm{d}t, \qquad G(x,\xi) := \int_0^\tau g(x,\xi) \, \mathrm{d}t = \tau g(x,\xi).$$

Obviously, $\tilde{J}$ turns out well defined and of class $C^1$. Moreover, (3.3) easily entails

$$\tilde{F}(x,\tau) \leq \int_0^{|\tau|} \tilde{f}(x,t) \, \mathrm{d}t \leq C_1 \left[ (d^{-\gamma s_1} + 1)|\tau| + \frac{|\tau|^r}{r+1} \right] \quad \forall \tau \in \mathbb{R}, \tag{3.5}$$

because $\tilde{f}(x,t) \geq 0$. From (3.5), ($\mathrm{H_g}$), Hölder's inequality, and fractional Hardy's inequality [9, Theorem 1.1] (recall that $s_1 p > 1$) it follows

$$\begin{aligned}\tilde{J}(u) &\geq \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y - C_1\int_\Omega d^{-\gamma s_1}|u|\,\mathrm{d}x - \frac{C_1}{r+1}\int_\Omega |u|^{r+1}\,\mathrm{d}x\\ &\quad - c_3\int_\Omega |D^s v|^\zeta |u|\,\mathrm{d}x - (C_1+c_3)\int_\Omega |u|\,\mathrm{d}x\\ &\geq \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y - C_1\int_\Omega d^{(p-\gamma)s_1}\frac{|u|}{d^{s_1p}}\,\mathrm{d}x - \frac{C_1}{r+1}\int_\Omega |u|^{r+1}\,\mathrm{d}x\\ &\quad - c_3\int_\Omega |D^s v|^\zeta |u|\,\mathrm{d}x - C_2\|u\|_p\\ &\geq \frac{1}{p}\|u\|_{s_1,p}^p - C_3(\|u\|_{s_1,p} + \|u\|_p^{r+1} + \|D^s v\|_p^\zeta \|u\|_{\frac{p}{p-\zeta}} + \|u\|_p), \quad u\in W_0^{s_1,p}(\Omega).\end{aligned}$$

Since $r,\zeta\in(1,p-1)$, through Proposition 2.3 (b) we see that $\tilde{J}$ is coercive. Thus, by Weierstrass-Tonelli's theorem, there exists $u_v\in W_0^{s_1,p}(\Omega)$ fulfilling

$$\tilde{J}(u_v) = \inf_{u\in W_0^{s_1,p}(\Omega)} \tilde{J}(u),$$

whence $u_v$ turns out a weak solution to (3.4). As in the proof of [15, Proposition 2.10] one has $(\underline{u}-u_v)_+\in W_0^{s_1,p}(\Omega)$. Testing (3.4) with $\varphi := (\underline{u}-u_v)_+$ yields

$$\langle(-\Delta)_p^{s_1}u_v + (-\Delta)_q^{s_2}u_v,\varphi\rangle = \int_\Omega \tilde{f}(\cdot,u_v)\varphi\,\mathrm{d}x + \int_\Omega g(\cdot,D^s v)\varphi\,\mathrm{d}x.$$

Lemma 3.1 and the inequality $g(x,\xi)\geq 0$ produce

$$\langle(-\Delta)_p^{s_1}\underline{u} + (-\Delta)_q^{s_2}\underline{u},\varphi\rangle \leq \int_\Omega f(\cdot,\underline{u})\varphi\,\mathrm{d}x \leq \int_\Omega f(\cdot,\underline{u})\varphi\,\mathrm{d}x + \int_\Omega g(\cdot,D^s v)\varphi\,\mathrm{d}x.$$

Therefore, by ($\mathrm{p_1}$) and (3.2),

$$\begin{aligned}&\langle(-\Delta)_p^{s_1}\underline{u} - (-\Delta)_p^{s_1}u_v,\varphi\rangle\\ &\qquad\leq \langle(-\Delta)_p^{s_1}\underline{u} - (-\Delta)_p^{s_1}u_v,\varphi\rangle + \langle(-\Delta)_q^{s_2}\underline{u} - (-\Delta)_q^{s_2}u_v,\varphi\rangle\\ &\qquad\leq \int_{\Omega(u_v<\underline{u})}(f(\cdot,\underline{u}) - \tilde{f}(\cdot,u_v))(\underline{u}-u_v)\,\mathrm{d}x\\ &\qquad= \int_{\Omega(u_v<\underline{u})}(f(\cdot,\underline{u}) - f(\cdot,\underline{u}))(\underline{u}-u_v)\,\mathrm{d}x = 0.\end{aligned}$$

Now, Lemma 9 in [18] forces

$$0<\underline{u}\leq u_v.$$

Consequently, $u_v\in W_0^{s_1,p}(\Omega)_+$ weakly solves ($\mathrm{P_v}$). Corollary 2.10 of [12] then ensures that $u_v\in C^{0,\tau}(\overline{\Omega})$ for all $\tau\in(0,s_1)$. □

**Lemma 3.4.** *If* $0<s<1$ *while* $\Phi: W_0^{s,p}(\Omega)\to\mathbb{R}_0^+$ *is defined by*

$$\Phi(u) := \frac{1}{p}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \quad \forall\, u\in W_0^{s,p}(\Omega)$$

*then the operator*

$$\hat{\Phi}(w) := \begin{cases} \Phi\left(w^{1/q}\right) & \text{when } w \geq 0 \text{ and } w^{1/q} \in W_0^{s,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

*has a nonempty domain and is convex.*

*Proof.* Pick $l > q$ and a non-negative $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. One has $u^{l/q} \in W_0^{1,p}(\Omega)$, because

$$\int_\Omega |\nabla(u^{l/q})|^p \, \mathrm{d}x = \int_\Omega \left(\frac{l}{q} u^{l/q-1} |\nabla u|\right)^p \mathrm{d}x \leq C \int_\Omega |\nabla u|^p \, \mathrm{d}x < +\infty.$$

So, $u^{l/q} \in W_0^{s,p}(\Omega)$ by Proposition 2.3. Here, as usual, $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. We claim that $\hat{\Phi}(u^l) < +\infty$. In fact,

$$\begin{aligned} \hat{\Phi}(u^l) = \Phi(u^{l/q}) &= \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x)^{l/q} - u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{p} \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_1(x)} \frac{|u(x)^{l/q} - u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}y + \frac{1}{p} \int_{\mathbb{R}^N} \mathrm{d}x \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u(x)^{l/q} - u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}y. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathrm{d}x \int_{B_1(x)} \frac{|u(x)^{l/q} - u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}y = \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_1(0)} \frac{|u(x)^{l/q} - u(x+y)^{l/q}|^p}{|y|^{N+sp}} \, \mathrm{d}y \\ &= \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_1(0)} \frac{|u(x)^{l/q} - u(x+y)^{l/q}|^p}{|y|^p} \frac{1}{|y|^{N+(s-1)p}} \, \mathrm{d}y \\ &\leq \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_1(0)} \left(\int_0^1 |\nabla u(x+\tau y)^{l/q}| \, \mathrm{d}\tau\right)^p \frac{1}{|y|^{N+(s-1)p}} \, \mathrm{d}y \\ &\leq C \int_{\mathbb{R}^N} \int_{B_1(0)} \int_0^1 \frac{|\nabla u(x+\tau y)^{l/q}|^p}{|y|^{N+(s-1)p}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau \\ &\leq C \int_{B_1(0)} \int_0^1 \frac{\|\nabla u^{l/q}\|^p_{L^p(\mathbb{R}^N)}}{|z|^{N+(s-1)p}} \, \mathrm{d}y \, \mathrm{d}\tau \leq C_1 \|\nabla u^{l/q}\|^p_{L^p(\mathbb{R}^N)} < +\infty. \end{aligned}$$

Likewise,

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathrm{d}x \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u(x)^{l/q} - u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}y \\ &\qquad \leq 2^{p-1} \int_{\mathbb{R}^N} \mathrm{d}x \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u(x)^{l/q}|^p + |u(y)^{l/q}|^p}{|x-y|^{N+sp}} \, \mathrm{d}y \\ &\qquad = 2^{p-1} \int_{\mathbb{R}^N} \mathrm{d}x \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|u(x)^{l/q}|^p + |u(x+y)^{l/q}|^p}{|y|^{N+sp}} \, \mathrm{d}y \\ &\qquad \leq C_3 \int_{\mathbb{R}^N} |u(x)^{l/q}|^p \, \mathrm{d}x = C_3 \|u^{l/q}\|_p^p < +\infty \end{aligned}$$

because $u^{l/q} \in W_0^{1,p}(\Omega)$. Hence, $u^l \in \operatorname{dom}\hat{\Phi}$, and the first conclusion follows. Next, let $u_1, u_2 \in \operatorname{dom}\hat{\Phi}$ and let $t \in (0,1)$. If $v_i := u_i^{1/q}$, $i = 1, 2$, and

$$v_3 := ((1-t)u_1 + tu_2)^{1/q}$$

then, thanks to discrete hidden convexity [2, Proposition 4.1],

$$|v_3(x) - v_3(y)|^p \le (1-t)|v_1(x) - v_1(y)|^p + t|v_2(x) - v_2(y)|^p \quad \forall\, x, y \in \mathbb{R}^N.$$

This entails

$$\begin{aligned}\hat{\Phi}((1-t)u_1 + tu_2) &= \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|v_3(x) - v_3(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{1-t}{p}\int_{\mathbb{R}^{2N}} \frac{|v_1(x) - v_1(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \frac{t}{p}\int_{\mathbb{R}^{2N}} \frac{|v_2(x) - v_2(y)|^r}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= (1-t)\hat{\Phi}(u_1) + t\hat{\Phi}(u_2),\end{aligned}$$

thus completing the proof. □

**Remark 3.5.** The above result holds true even when $q = p$, with the same proof.

**Theorem 3.6.** *Under* ($\mathrm{H_{f_1}}$)–($\mathrm{H_{f_2}}$) *and* ($\mathrm{H_g}$), *for every fixed* $v \in W_0^{s_1,p}(\Omega)$, *the solution* $u_v \in W_0^{s_1,p}(\Omega) \cap C^{0,\tau}(\overline{\Omega})$ *to problem* ($\mathrm{P_v}$) *given by Theorem 3.3 is unique.*

*Proof.* Suppose $u_v, w_v \in W_0^{s_1,p}(\Omega) \cap C^{0,\tau}(\overline{\Omega})$ solve ($\mathrm{P_v}$), namely

$$\langle(-\Delta)_p^{s_1} u_v + (-\Delta)_q^{s_2} u_v, \varphi\rangle = \int_\Omega f(\cdot, u_v)\varphi\,\mathrm{d}x + \int_\Omega g(\cdot, D^s v)\varphi\,\mathrm{d}x, \tag{3.6}$$

$$\langle(-\Delta)_p^{s_1} w_v + (-\Delta)_q^{s_2} w_v, \psi\rangle = \int_\Omega f(\cdot, w_v)\psi\,\mathrm{d}x + \int_\Omega g(\cdot, D^s v)\psi\,\mathrm{d}x \tag{3.7}$$

for all $\varphi, \psi \in W_0^{s_1,p}(\Omega)$. The functions

$$\varphi := \frac{{u_v}^q - {w_v}^q}{{u_v}^{q-1}} \quad \text{and} \quad \psi := \frac{{u_v}^q - {w_v}^q}{w_v^{q-1}}$$

lie in $W_0^{s_1,p}(\Omega)$, because $u_v, w_v \in C^{0,\tau}(\overline{\Omega})_+$. Hence, via (3.6)–(3.7) we achieve

$$\begin{aligned}&\langle(-\Delta)_p^{s_1} u_v, \varphi\rangle - \langle(-\Delta)_p^{s_1} w_v, \psi\rangle + \langle(-\Delta)_q^{s_2} u_v, \varphi\rangle - \langle(-\Delta)_q^{s_2} w_v, \psi\rangle \\ &\quad = \int_\Omega \left(\frac{f(\cdot, u_v)}{u_v^{q-1}} - \frac{f(\cdot, w_v)}{w_v^{q-1}}\right)(u_v^q - w_v^q)\,\mathrm{d}x + \int_\Omega g(\cdot, D^s v)(\varphi - \psi)\,\mathrm{d}x.\end{aligned} \tag{3.8}$$

Lemma 3.4 ensures that the functional $\hat{J}$ associated with

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+s_1 p}}\,\mathrm{d}x\,\mathrm{d}y, \quad u \in W_0^{s_1,p}(\Omega),$$

turns out convex. Therefore, after a standard computation,

$$0 \le q\langle\hat{J}'(u_v^q) - \hat{J}'(w_v^q), u_v^q - w_v^q\rangle = \langle(-\Delta)_p^{s_1} u_v, \varphi\rangle - \langle(-\Delta)_p^{s_1} w_v, \psi\rangle. \tag{3.9}$$

An analogous argument produces

$$\langle(-\Delta)_q^{s_2} u_v, \varphi\rangle - \langle(-\Delta)_q^{s_2} w_v, \psi\rangle \ge 0. \tag{3.10}$$

Now, gathering (3.8)–(3.10) together yields

$$\int_\Omega \left( \frac{f(\cdot,u_v)}{u_v^{q-1}} - \frac{f(\cdot,w_v)}{w_v^{q-1}} \right) (u_v^q - w_v^q) \,\mathrm{d}x + \int_\Omega g(\cdot, D^s v)(\varphi - \psi) \,\mathrm{d}x \geq 0. \tag{3.11}$$

By ($\mathrm{H_{f_2}}$) the function $t \mapsto \frac{f(\cdot,t)}{t^{q-1}}$ is decreasing on $\mathbb{R}^+$. This implies

$$\int_\Omega \left( \frac{f(\cdot,u_v)}{u_v^{q-1}} - \frac{f(\cdot,w_v)}{w_v^{q-1}} \right) (u_v^q - w_v^q) \,\mathrm{d}x \leq 0. \tag{3.12}$$

Moreover,

$$\begin{aligned} \int_\Omega g(\cdot, D^s v)(\varphi - \psi) \,\mathrm{d}x &\leq \int_{\Omega(u_v \geq w_v)} g(\cdot, D^s v) \left( \frac{u_v^q - w_v^q}{w_v^{q-1}} - \frac{u_v^q - w_v^q}{w_v^{q-1}} \right) \mathrm{d}x \\ &\quad + \int_{\Omega(u_v < w_v)} g(\cdot, D^s v) \left( \frac{u_v^q - w_v^q}{u_v^{q-1}} - \frac{u_v^q - w_v^q}{w_v^{q-1}} \right) \mathrm{d}x \\ &\leq \int_{\Omega(u_v < w_v)} g(\cdot, D^s v) \left( -\frac{w_v^q - u_v^q}{u_v^{q-1}} + \frac{w_v^q - u_v^q}{u_v^{q-1}} \right) \mathrm{d}x = 0. \end{aligned} \tag{3.13}$$

From (3.11)–(3.13) it finally follows

$$\int_\Omega \left( \frac{f(\cdot,u_v)}{u_v^{q-1}} - \frac{f(\cdot,w_v)}{w_v^{q-1}} \right) (u_v^q - w_v^q) \,\mathrm{d}x = 0,$$

whence, due to ($\mathrm{H_{f_2}}$) again, $u_v \equiv w_v$, as desired. □

## 4. Main result

Define, for every $v \in W_0^{s_1,p}(\Omega)$,

$$T(v) := u_v, \tag{4.1}$$

$u_v \in W_0^{s_1,p}(\Omega)_+$ being the unique solution of ($\mathrm{P_v}$) found in Theorem 3.3.

**Lemma 4.1.** *Let ($\mathrm{H_{f_1}}$), ($\mathrm{H_{f_2}}$), ($\mathrm{H_g}$) be satisfied and let $q's_2 \neq s_1 < \frac{1}{p'\gamma}$. Then $T$ possesses a fixed point $u \in W_0^{s_1,p}(\Omega)$.*

*Proof.* Given any $v \in W_0^{s_1,p}(\Omega)$, test problem ($\mathrm{P_v}$) with its solution $u_v$. Through ($\mathrm{H_{f_1}}$) and ($\mathrm{H_g}$) we thus arrive at

$$\begin{aligned} \|u_v\|_{s_1,p}^p &\leq \int_{\mathbb{R}^{2N}} \frac{|u_v(x)-u_v(y)|^p}{|x-y|^{N+s_1 p}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^{2N}} \frac{|u_v(x)-u_v(y)|^q}{|x-y|^{N+s_2 q}} \,\mathrm{d}x\,\mathrm{d}y \\ &\leq c_1 \int_\Omega u_v^{1-\gamma} \,\mathrm{d}x + c_2 \int_\Omega u_v^{r+1} \,\mathrm{d}x + c_3 \int_\Omega (u_v + |D^s v|^\zeta u_v) \,\mathrm{d}x. \end{aligned}$$

Thanks to Young's inequality with $\varepsilon > 0$, each term of the right-hand side is estimated as follows (recall that $\gamma < 1$ while $r, \zeta < p-1$):

$$\int_\Omega u_v^\alpha \,\mathrm{d}x \leq \frac{\alpha}{p}\varepsilon \|u_v\|_p^p + \frac{p-\alpha}{p} C_\varepsilon(\alpha)|\Omega|, \quad \alpha \in \{1-\gamma, r+1, 1\};$$

$$\int_\Omega |D^s v|^\zeta u_v \,\mathrm{d}x \leq \frac{1}{p}\varepsilon \|u_v\|_p^p + \frac{1}{p'} C_\varepsilon \int_\Omega |D^s v|^{\zeta p'} \,\mathrm{d}x.$$

Consequently, by Proposition 2.3 (b) and (2.2),

$$\begin{aligned}\|u_v\|_{s_1,p}^p &\le c\varepsilon\|u_v\|_p^p + C_\varepsilon(1+\|D^s v\|_{\zeta p'}^{\zeta p'}) \le c'\varepsilon\|u_v\|_{s_1,p}^p + C'_\varepsilon(1+\|D^s v\|_p^{\zeta p'})\\ &\le c'\varepsilon\|u_v\|_{s_1,p}^p + C'_\varepsilon\left(1+\|v\|_{X_0^{s,p}(\mathbb{R}^N)}^{\zeta p'}\right) \le c'\varepsilon\|u_v\|_{s_1,p}^p + C^*_\varepsilon(1+\|v\|_{s_1,p}^{\zeta p'}).\end{aligned}$$

This entails

$$(1-c'\varepsilon)\|u_v\|_{s_1,p}^p \le C^*_\varepsilon(1+\|v\|_{s_1,p}^{\zeta p'}),$$

i.e., after choosing $\varepsilon < \frac{1}{c'}$,

$$\|T(v)\|_{s_1,p}^p = \|u_v\|_{s_1,p}^p \le \hat{C}(1+\|v\|_{s_1,p}^{\zeta p'}), \tag{4.2}$$

with $\hat{C} := \frac{C^*_\varepsilon}{1-c'\varepsilon}$. Since $\zeta p' < p$, there exists $\rho > 0$ such that $\hat{C}(1+\rho^{\zeta p'}) \le \rho^p$. Thus, due to (4.2), $\|v\|_{s_1,p} \le \rho$ implies $\|T(v)\|_{s_1,p} \le \rho$, which clearly means $T(K) \subseteq K$, provided

$$K := \{u \in W_0^{s_1,p}(\Omega) : \|u\|_{s_1,p} \le \rho\}.$$

*Claim 1:* The operator $T\lfloor_K$ is compact.
Let $\{v_n\} \subseteq K$ and let $u_n := T(v_n)$, $n \in \mathbb{N}$. The reflexivity of $W_0^{s_1,p}(\Omega)$ yields $v_n \rightharpoonup v$ in $W_0^{s_1,p}(\Omega)$ while Proposition 2.3 (c) ensures that

$$\forall\, r \in [1, p_{s_1*}) \ \text{ one has } \ v_n \to v \ \text{ in } \ L^r(\Omega),$$

where a sub-sequence is considered when necessary. Likewise, from $\{u_n\} \subseteq K$ it follows $u_n \rightharpoonup u$ in $W_0^{s_1,p}(\Omega)$ and, as before,

$$\forall\, r \in [1, p_{s_1*}) \ \text{ one has } \ u_n \to u \ \text{ in } \ L^r(\Omega). \tag{4.3}$$

Now, testing ($\mathrm{P_v}$) with $\varphi_n := u_n - u$ and using ($\mathrm{H_{f_1}}$), Theorem 3.3, and (3.1), we obtain

$$\begin{aligned}&\langle(-\Delta)_p^{s_1}u_n + (-\Delta)_q^{s_2}u_n, \varphi_n\rangle \le \int_\Omega f(\cdot,u_n)|\varphi_n|\,\mathrm{d}x + \int_\Omega g(\cdot, D^s v_n)|\varphi_n|\,\mathrm{d}x\\ &\le c_1\int_\Omega u_n^{-\gamma}|\varphi_n|\,\mathrm{d}x + c_2\int_\Omega u_n^r|\varphi_n|\,\mathrm{d}x + c_3\int_\Omega (1+|D^s v_n|^\zeta)|\varphi_n|\,\mathrm{d}x\\ &\le \int_\Omega [c_1(\eta d^{s_1})^{-\gamma} + c_3]|\varphi_n|\,\mathrm{d}x + c_2\int_\Omega u_n^r|\varphi_n|\,\mathrm{d}x + c_3\int_\Omega |D^s v_n|^\zeta|\varphi_n|\,\mathrm{d}x.\end{aligned}$$

Hence, due to Hölder's inequality, Proposition 2.2 (recall that $s_1\gamma < \frac{1}{p'}$), Proposition 2.3 (b), besides (2.2),

$$\begin{aligned}&\langle(-\Delta)_p^{s_1}u_n + (-\Delta)_q^{s_2}u_n, \varphi_n\rangle\\ &\qquad \le C_1\|\varphi_n\|_p + c_2\|u_n\|_p^r\|\varphi_n\|_{\frac{p}{p-r}} + c_3\|D^s v_n\|_p^\zeta\|\varphi_n\|_{\frac{p}{p-\zeta}}\\ &\qquad \le C_1\|\varphi_n\|_p + C_2\rho^r\|\varphi_n\|_{\frac{p}{p-r}} + C_3\rho^\zeta\|\varphi_n\|_{\frac{p}{p-\zeta}}\end{aligned} \tag{4.4}$$

for all $n \in \mathbb{N}$, where

$$C_1 := c_1\eta^{-\gamma}\|d^{-s_1\gamma}\|_{p'} + c_3|\Omega|^{1/p'}.$$

On account of (4.3)–(4.4) one arrives at

$$\limsup_{n\to+\infty}\langle(-\Delta)_p^{s_1}u_n + (-\Delta)_q^{s_2}u_n, u_n - u\rangle \le 0,$$

whence $u_n \to u$ in $W_0^{s_1,p}(\Omega)$ because, by Proposition 2.1 and (p$_4$), the fractional $(p,q)$-Laplacian

$$u \mapsto (-\Delta)_p^{s_1} u_+ (-\Delta)_q^{s_2} u, \quad u \in W_0^{s_1,p}(\Omega),$$

is type $(\mathrm{S})_+$.

*Claim 2:* The operator $T\lfloor_K$ turns out continuous.

Let $\{v_n\} \subseteq K$ satisfy $v_n \to v$ in $W_0^{s_1,p}(\Omega)$ and let $u_n := T(v_n)$, $n \in \mathbb{N}$. Since $T\lfloor_K$ is compact, along a sub-sequence if necessary, we have $u_n \to u$ in $W_0^{s_1,p}(\Omega)$. Moreover, (4.3) holds. Our claim thus becomes $u = T(v)$. Pick any $\varphi \in W_0^{s_1,p}(\Omega)$. From (4.1) it follows

$$\begin{aligned}\int_{\mathbb{R}^{2N}} &\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_\Omega f(\cdot,u_n)\varphi\,\mathrm{d}x+\int_\Omega g(\cdot,D^s v_n)\varphi\,\mathrm{d}x \quad \forall\, n\in\mathbb{N}.\end{aligned} \tag{4.5}$$

Observe that

$$\{u_n\} \subseteq K \implies \left\{ \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1p}{p'}}} \right\} \text{ bounded in } L^{p'}(\mathbb{R}^{2N})$$

and that, by (4.3),

$$\lim_{n\to+\infty} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1p}{p'}}} = \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_1p}{p'}}}$$

for almost every $(x,y) \in \mathbb{R}^{2N}$. So, up to sub-sequences,

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{|x-y|^{\frac{N+s_1p}{p'}}}} \rightharpoonup \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_1p}{p'}}} \text{ in } L^{p'}(\mathbb{R}^{2N}).$$

This implies

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{R}^{2N}} &\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.6}$$

because

$$\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N+s_1p}{p}}} \in L^p(\mathbb{R}^{2N}).$$

An analogous argument, which employs the continuous embedding $W_0^{s_1,p}(\Omega) \hookrightarrow W_0^{s_2,q}(\Omega)$ (cf. Proposition 2.3 (a)), produces

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{R}^{2N}} &\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.7}$$

Let us next focus on the right-hand side of (4.5). Exploiting ($\mathrm{H_{f_1}}$), Theorem 3.3, (3.1), (4.3), and [3, Theorem 4.9], we achieve

$$|f(\cdot,u_n)\varphi| \le [c_1 u_n^{-\gamma} + c_2 u_n^r]|\varphi| \le [c_1 \underline{u}^{-\gamma} + c_2 u_n^r]|\varphi| \le [c_1(\eta d^{s_1})^{-\gamma} + c_2\psi^r]|\varphi|, \quad n \in \mathbb{N},$$

for some $\psi \in L^r(\Omega)$. Now, by (4.3) and [3, Theorem 4.2], one has

$$\lim_{n\to+\infty} \int_\Omega f(\cdot,u_n)\varphi \, \mathrm{d}x = \int_\Omega f(\cdot,u)\varphi \, \mathrm{d}x. \tag{4.8}$$

Finally, observe that, thanks to (2.2),

$$v_n \to v \text{ in } W_0^{s_1,p}(\Omega) \implies v_n \to v \text{ in } L_0^{s,p}(\Omega)$$

as well as

$$v_n \to v \text{ in } L_0^{s,p}(\Omega) \implies D^s v_n \to D^s v \text{ in } L^p(\Omega) \implies (D^s v_n)^\zeta \to (D^s v)^\zeta \text{ in } L^{\frac{p}{\zeta}}(\Omega),$$

where a sub-sequence is considered if necessary. Since ($\mathrm{H_g}$) holds while $\varphi \in L^{\frac{p}{p-\zeta}}(\Omega)$ because $\zeta \in (1,p-1)$, this forces

$$\lim_{n\to+\infty} \int_\Omega g(\cdot,D^s v_n)\varphi \, \mathrm{d}x = \int_\Omega g(\cdot,D^s v)\varphi \, \mathrm{d}x. \tag{4.9}$$

Letting $n \to +\infty$ in (4.5) and using (4.6)–(4.9), we arrive at $u = T(v)$.

Now, Schauder's fixed point theorem can be applied to $T\lfloor_K$, which ends the proof. □

Our main result is the following:

**Theorem 4.2.** *Under hypotheses* ($\mathrm{H_{f_1}}$),($\mathrm{H_{f_2}}$), ($\mathrm{H_g}$) *and the conditions* $q's_2 \neq s_1 < \frac{1}{p'\gamma}$, *problem* (P) *admits a weak solution* $u \in W_0^{s_1,p}(\Omega)$.

*Proof.* Simply use Lemma 4.1 and note that fixed points of $T$ weakly solve (P). □

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