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SOME REMARKS ON HILBERT'S (WEAK) NULLSTELLENSATZ

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Abstract

Certain remarks are provided related to weak nullstellensatz exploiting some problems proposed in Fulton's book entitled "An Introduction to Algebraic Geometry" and elementary notions of Functional Analysis.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

I. *If* $f \in \mathbb{K}[X_1, ..., X_n]$, $a_1, ..., a_n \in \mathbb{K}$, *then*

$$f = \sum_{\substack{i_l \geq 0 \\ l=1, \dots, n}} \lambda_{i_1 \cdots i_n} \prod_{l=1}^{n} (X_l - a_l)^{i_l} \tag{1}$$

for certain $\lambda_{i_1 \cdots i_n} \in \mathbb{K}$ (see [1, Problem 1-7, (a)]).

Example. If $\mathbb{K} = \mathbb{C}$, $f(X) = X^2 + 1 \in \mathbb{C}[X]$ and $a \in \mathbb{C}$, then $f(X) = \lambda(X - a)^2 + \mu(X - a) + \sigma$ with $\lambda = 1$, $\mu = 2a$, $\sigma = 1 + a^2$.

If $n = 1$, then $\mathbb{K}[X]$ is a Euclidean domain (ED). By Division Algorithm, we

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obtain[1] $f(X) = (X - a)q(X) + r(X)$ with $r = 0$ or $0 \leq \partial eg\, r < 1 = \partial eg(X - a)$, that is, r is a constant. Repeating this procedure on $q(X)$, we have $q(X) = (X - a)t(X) + s(X)$ with $s = 0$ or $0 \leq \partial eg\, s < 1 = \partial eg(X - a)$, and hence $f(X) = (X - a)^2 t(X) + (X - a)s(X) + r(X)$. Thus, iterating[2] as far as $t(X)$ is a constant, we get (1) for $n = 1$.

We suppose that the hypothesis is true for $n - 1$ $(n > 1)$.

In $\mathbb{K}[X_1, ..., X_n] = \mathbb{K}[X_1, ..., X_{n-1}][X_n] = D[X_n]$ (with $D = \mathbb{K}[X_1, ..., X_{n-1}]$ only UFD), the polynomial division is possible[3] if the (polynomial) quotient has invertible leading coefficient, as in $X_n - a_n$, so that

$$f(X_1, ..., X_n) = q(X_1, ..., X_n)(X_n - a_n) + r(X_1, ..., X_n)$$

with r, however, constant $(\text{because } 0 \leq \partial eg\, r < 1 = \partial eg(X_n - a_n))$, and thus $\partial eg_{X_n} f = \partial eg_{X_n} q + 1 > \partial eg_{X_n} q$, with $q \in D[X_n]$. In the same manner, we proceed by successive division of $q(X_1, ..., X_n)$ by $X_n - a_n$, obtaining

$$q(X_1, ..., X_n) = t(X_1, ..., X_n)(X_n - a_n) + s(X_1, ..., X_n)$$

with s, as usual, constant, $t(X_1, ..., X_n) \in D[X_n]$ and

$$\partial eg_{X_n} t(X_1, ..., X_n) < \partial eg_{X_n} q(X_1, ..., X_n).$$

Therefore,

$$f(X_1, ..., X_n) = t(X_1, ..., X_n)(X_n - a_n)^2 + r(X_1, ..., X_n)(X_n - a_n) + s(X_1, ..., X_n),$$

and we proceed in similar manner until t is devoid of degree in X_n, with the final result (generalizing the last expression for f, bearing in mind that r and s are constants) that

$$f(X_1, ..., X_n) = l(X_1, ..., X_{n-1})(X_n - a_n)^{i_n} + \sum_{j=0}^{i_n} \lambda_{i_n - j}(X_n - a_n)^j \qquad (2)$$

[1]$\partial eg\, p$ is the degree of the polynomial p.

[2]The iterative procedure is finite because $\partial eg\, t$ is finite.

[3]By means of the algorithm of the Gröbner's basis, see [2].

with $i_n \in \mathbb{N}$, $\lambda_{i_n - j} \in \mathbb{K}$, $\forall j$. But, by the inductive hypothesis (that holds true for $n - 1 \geq 1$), we have

$$l(X_1, \ldots, X_{n-1}) = \sum_{\substack{j_l \geq 0 \\ l=1, \ldots, n-1}} \lambda_{j_1 \cdots j_{n-1}} \prod_{l=1}^{n-1} (X_{j_l} - a_l)^{j_l},$$

and substituting in (2), it becomes

$$f(X_1, \ldots, X_n) = \sum_{\substack{j_l \geq 0 \\ l=1,\ldots,n-1}} \lambda_{j_1 \cdots j_{n-1}} \prod_{l=1}^{n-1} (X_{j_l} - a_l)^{j_l} (X_n - a_n)^{i_n} + \sum_{j=0}^{i_n} \lambda_{i_n - j} (X_n - a_n)^j$$

with $\lambda_{j_1 \cdots j_{n-1}}, \lambda_{i_n - j} \in \mathbb{K}$, $\forall j$. Except changes in indices, the last expression is of the type (1), and, hence the result is true for n.

II. *If* $f \in \mathbb{K}[X_1, \ldots, X_n]$, $\partial eg\ f > 0$ *and* $f(a_1, \ldots, a_n) = 0$ *for* $a_1, \ldots, a_n \in \mathbb{K}$, *then*

$$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} G_i(X_1, \ldots, X_n)(X_i - a_i) \tag{3}$$

for certain (*not unique*) $G_i \in \mathbb{K}[X_1, \ldots, X_n]$, $\forall i$ (see [1, Problem 1-7, (b)]).

We deduce (3) from (1), proceeding by finite induction on n. If $n = 1$, then $f(a) = 0$ with $f \in \mathbb{K}[X]$, and by the well-known Ruffini's theorem, we have $f(X) = (X - a)g(X)$. We suppose that the statement is true for $n - 1$ ($n > 1$). Then

$$f(X_1, \ldots, X_n) = \sum_{\substack{i_l \geq 0 \\ l=1, \ldots, n}} \lambda_{i_1 \cdots i_n} \prod_{l=1}^{n} (X_l - a_l)^{i_l}$$

$$\overset{4}{=} \sum_{i_n} \left(\sum_{\substack{i_l \geq 0 \\ l=1, \ldots, n-1}} \lambda_{i_1 \cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l} \right) (X_n - a_n)^{i_n},$$

[4] We may suppose $i_l \geq 1$ for some l, because, if $i_l = 0$, $\forall l$, by (1) follows that $f = \text{const.}$, against $\partial eg\ f > 0$. Hence, without loss of generality, let $l = n$.

and, isolating the term $\lambda_{0\cdots 0i_n}$, we have

$$\sum_{\substack{i_l \geq 0 \\ l=1,\ldots,n-1}} \lambda_{i_1\cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l} = \sum_{\substack{i_l \geq 0 \\ (i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)}} \lambda_{i_1\cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l} + \lambda_{0\cdots 0i_n},$$

so that

$$f = \sum_{i_n} \left(\sum_{\substack{i_l \geq 0 \\ l=1,\ldots,n-1}} \lambda_{i_1\cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l} \right) (X_n - a_n)^{i_n}$$

$$= \sum_{i_n} \left(\sum_{\substack{i_l \geq 0 \\ (i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)}} \lambda_{i_1\cdots i_n} \prod_{i=1}^{n-1} (X_l - a_l)^{i_l} + \lambda_{0\cdots 0i_n} \right) (X_n - a_n)^{i_n}$$

$$= \sum_{i_n} \left(\sum_{\substack{i_l \geq 0 \\ (i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)}} \lambda_{i_1\cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l} \right) (X_n - a_n)^{i_n}$$

$$+ \sum_{i_n} \lambda_{0\cdots 0i_n} (X_n - a_n)^{i_n}. \qquad (\clubsuit)$$

Putting

$$g(X_1, \ldots, X_{n-1}) = \sum_{\substack{i_l \geq 0 \\ (i_1,\ldots,i_{n-1}) \neq (0,\ldots,0)}} \lambda_{i_1\cdots i_n} \prod_{l=1}^{n-1} (X_l - a_l)^{i_l},$$

since $(i_n, \ldots, i_{n-1}) \neq (0, \ldots, 0)$, we have $g \in \mathbb{K}[X_1, \ldots, X_{n-1}]$ to be nonconstant such that $g(a_1, \ldots, a_{n-1}) (= g(a_1, \ldots, a_n)) = 0$; and thus, by the inductive hypothesis for $n-1$ (and being verified the conditions of the theorem for such g), we have

$$g(X_1, \ldots, X_{n-1}) = \sum_{l=1}^{n-1} H_l(X_1, \ldots, X_{n-1})(X_l - a_l)$$

for certain $H_l \in \mathbb{K}[X_1, ..., X_{n-1}](\subset \mathbb{K}[X_1, ..., X_n])$, so that, substituting in the last formula (♣) for f, we have

$$f(X_1, ..., X_n)$$

$$= \sum_{i_n} g(X_1, ..., X_{n-1})(X_n - a_n)^{i_n} + \sum_{i_n} \lambda_{0\cdots 0 i_n}(X_n - a_n)^{i_n}$$

$$= \sum_{i_n}\left(\sum_{l=1}^{n-1} H_l(X_1, ..., X_{n-1})(X_l - a_l)\right)(X_n - a_n)^{i_n} + \sum_{i_n} \lambda_{0\cdots 0 i_n}(X_n - a_n)^{i_n}$$

$$= \sum_{l=1}^{n-1}(X_l - a_l)H_l(X_1, ..., X_{n-1})\sum_{i_n}(X_n - a_n)^{i_n}$$

$$+ \left(\sum_{i_n} \lambda_{0\cdots 0 i_n}(X_n - a_n)^{i_n - 1}\right)(X_n - a_n)$$

$$\stackrel{5}{=} \sum_{l=1}^{n-1} G_l(X_1, ..., X_n)(X_l - a_l) + G_n(X_1, ..., X_n)(X_n - a_n)$$

$$= \sum_{l=1}^{n} G_l(X_1, ..., X_n)(X_l - a_l)$$

with

$$G_l(X_1, ..., X_n) = H_l(X_1, ..., X_{n-1})\sum_{i_n}(X_n - a_n)^{i_n}, \quad l = 1, ..., n-1,$$

$$G_n(X_1, ..., X_n) = \sum_{i_n}(X_n - a_n)^{i_n - 1}.$$

Thus the statement holds true also for n.

III. *For each* $(a_1, ..., a_n) \in \mathbb{K}^n$, $I = \langle X_1 - a_1, ..., X_n - a_n\rangle$ *is a maximal ideal of* $\mathbb{K}[X_1, ..., X_n]$ (see [1, Problem 1-21]).

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[5]Let us remember that we have supposed $i_n \geq 1$, that is, $i_n - 1 \geq 0$.

First proof. If J is another proper ideal such that $\langle X_1 - a_1, ..., X_n - a_n \rangle \subseteq J \subset \mathbb{K}[X_1, ..., X_n]$, then by **I**, we have, for each $f \in J$, that

$$f(X_1, ..., X_n) = \sum_{\substack{i_l \geq 0 \\ l = 1, ..., n}} \lambda_{i_1 \cdots i_n} \prod_{l=1}^{n} (X_l - a_l)^{i_l} \in \langle X_1 - a_1, ..., X_n - a_n \rangle$$

with $\partial eg\ f > 0$ (because, if $\partial eg\ f = 0$, $f(\in J)$ would be constant, that is, invertible, against to the fact that J is a proper ideal[6]), and then $i_l \geq 1$ for some l. So $J \subseteq \langle X_1 - a_1, ..., X_n - a_n \rangle$, whence $J = \langle X_1 - a_1, ..., X_n - a_n \rangle$.

Second proof. If $P = (a_1, ..., a_n) \in \mathbb{K}^n$, let $\phi_P : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}$ be the (polynomial) evaluation map $f(X_1, ..., X_n) \rightsquigarrow f(a_1, ..., a_n)$. It is a surjective map, so that $\mathrm{Im}\,\phi_P = \mathbb{K}[a_1, ..., a_n]$ is a finitely generated integral $\mathbb{K}$-algebra. If $f \in Ker\,\phi_P$, then $f(a_1, ..., a_n) = 0$, and therefore, by **II**, it follows that $f \in \langle X_1 - a_1, ..., X_n - a_n \rangle$, that is, $Ker\,\phi_P \subseteq \langle X_1 - a_1, ..., X_n - a_n \rangle$. Obviously, $f(a_1, ..., a_n) = 0$, $\forall f \in \langle X_1 - a_1, ..., X_n - a_n \rangle$, so $\langle X_1 - a_1, ..., X_n - a_n \rangle \subseteq Ker\,\phi_P$, and we conclude that $Ker\,\phi_P = \langle X_1 - a_1, ..., X_n - a_n \rangle$. But ϕ_P is an epimorphism, so that $\mathbb{K} \cong \mathbb{K}[X_1, ..., X_n]/Ker\,\phi_P$, and since $\mathbb{K}$ is a field, it follows that $Ker\,\phi_P$ is a maximal ideal; hence $\langle X_1 - a_1, ..., X_n - a_n \rangle (= Ker\,\phi_P)$ is a maximal ideal.

IV (Pre-weak Nullstellensatz). $\mathbb{K} \cong \mathbb{K}[X_1, ..., X_n]/\langle X_1 - a_1, ..., X_n - a_n \rangle$ (see [1, Problem 1-21]).

$\mathbb{K}[X_1, ..., X_n]$ is an UFD, so that $\mathbb{K}[X_1, ..., X_n]/\langle X_1 - a_1, ..., X_n - a_n \rangle$ is a field. If

$$i : \mathbb{K} \hookrightarrow \mathbb{K}[X_1, ..., X_n]$$

is the natural monomorphism of immersion, and

$$\pi : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}[X_1, ..., X_n]/\langle X_1 - a_1, ..., X_n - a_n \rangle$$

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[6]We recall that the group of units of $\mathbb{K}[X_1, ..., X_n]$ is $\mathbb{K}$.

is the canonical epimorphism of projection (quotient map), then it follows that

$$\sigma = \pi \circ i : \mathbb{K} \to \mathbb{K}[X_1, \ldots, X_n]/\langle X_1 - a_1, \ldots, X_n - a_n\rangle$$

is a ring homomorphism with functional law

$$\sigma(k) = \pi(i(k)) = i(k) + \langle X_1 - a_1, \ldots, X_n - a_n\rangle, \quad \forall k \in \mathbb{K}.$$

We prove that σ is bijective. If $\sigma(k) = \sigma(k')$, then we have $\sigma(k - k') = 0$, that is, $k - k' \in Ker\, \sigma$, where

$$Ker\, \sigma = \{k;\ k \in \mathbb{K},\ i(k) \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle\}.$$

But, in any case, $i(k)$ is a scalar constant (in $\mathbb{K}[X_1, \ldots, X_n]$) and, if $k - k' = \xi \neq 0$, then it would be $0 \neq i(\xi) \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle$, that is, the maximal ideal $\langle X_1 - a_1, \ldots, X_n - a_n\rangle$ would contain an invertible element, which is not possible. Hence $i(\xi) = 0$, or $\xi = 0$, that is, $k = k'$, and thus σ is 1-1. Therefore, $\dim_{\mathbb{K}} Ker\, \sigma = 0$. Thus $\mathbb{K}$ is isomorphic to $\mathrm{Im}\, \sigma = \sigma(\mathbb{K})$. Since $\dim_{\mathbb{K}} \mathbb{K} = 1$, it follows that

$$1 = \dim_{\mathbb{K}} \mathbb{K} = \dim_{\mathbb{K}} Ker\, \sigma(\mathbb{K}) + \dim_{\mathbb{K}} \sigma(\mathbb{K}) = \dim_{\mathbb{K}} \sigma(\mathbb{K}). \tag{$\spadesuit$}$$

On the other hand, if

$$f(X_1, \ldots, X_n) + \langle X_1 - a_1, \ldots, X_n - a_n\rangle \in \mathbb{K}[X_1, \ldots, X_n]/\langle X_1 - a_1, \ldots, X_n - a_n\rangle,$$

then it is $f + \langle X_1 - a_1, \ldots, X_n - a_n\rangle = \sigma(k)$ for some $k \in \mathbb{K}$ if and only if $f - i(k) \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle$. Then, for the maximality of $\langle X_1 - a_1, \ldots, X_n - a_n\rangle$ (by **III**), first of all $f - i(k)$ must be not invertible. We obtain this if, for example, $i(k)$ is the zero-degree term of the polynomial *f*, provided it exists and is nonzero. Thus, if *f* is constant, for $i(k) = f$, then we have $f - i(k) = 0 \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle$, whereas, if *f* is not constant, then it is $\partial eg\, f > 0$ and $f - i(k)$ is not a constant polynomial, for any $i(k) \in \mathbb{K}$; therefore, the last is not a unit of $\mathbb{K}[X_1, \ldots, X_n]$, namely, not invertible. Thus we can consider the equation (in $i(k)$) $f - i(k) \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle$, for any choice of $i(k)$. In the same hypothesis, if *I* is an arbitrary maximal ideal of $\mathbb{K}[X_1, \ldots, X_n]$ and $f \in \mathbb{K}[X_1, \ldots, X_n]$ is given, a priori, then it is not easy to do in such a way that $f - i(k) \in I$ for some[7] $k \in \mathbb{K}$. But, for our purpose, if $I = \langle X_1 - a_1, ..., X_n - a_n \rangle$, supposing that $f - i(k)$ is not a constant (as above), it is enough, simply, to observe that, if $(f - i(k))(a_1, ..., a_n) = 0$, then, by **II**, it follows that $(f - i(k)) \in \langle X_1 - a_1, ..., X_n - a_n \rangle$ as required, whereas, if $(f - i(k))(a_1, ..., a_n) \neq 0$, then we have this last zero condition for $i(k) = f(a_1, ..., a_n)$. Therefore, σ is also surjective.

Hence, by (♠), it follows that

$$\dim_{\mathbb{K}} \frac{\mathbb{K}[X_1, ..., X_n]}{\langle X_1 - a_1, ..., X_n - a_n \rangle} = \dim_{\mathbb{K}} \sigma(\mathbb{K}) = 1$$

and so $[\mathbb{K}[X_1, ..., X_n]/\langle X_1 - a_1, ..., X_n - a_n \rangle : \mathbb{K}] = 1$.

Remark 1. However, among the results established in **IV**, the only useful is the injectivity of σ (see final proof of **VIII**), whose proof, besides, remains unchanged also for any maximal ideal I of $\mathbb{K}[X_1, ..., X_n]$, so that we have the immersion (restriction of the scalars):

$$\sigma : \mathbb{K} \hookrightarrow \mathbb{K}[X_1, ..., X_n]/I,$$

where, for the maximality of I, $A = \mathbb{K}[X_1, ..., X_n]/I$ is a field $(= \mathbb{A})$, hence a finitely generated field extension[8] of $\mathbb{K}$.

On the other hand, in view of the next result **VIII**, we might call the results of **IV** to be the *pre-Weak Nullstellensatz*.

Remark 2. The Weak Nullstellensatz that we prove further in **VIII**, concerns with an arbitrary proper ideal I of $\mathbb{K}[X_1, ..., X_n]$, in the proof of which we use an auxiliary maximal ideal J containing it. Only, a posteriori (see final proof of VIII,

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[7]We shall deal with that question in VIII.

[8]The usual proof of $\mathbb{K} \cong A$ (Weak Nullstellensatz) follows from the subsequently proof (conducted through the E. Nöether's Normalization Lemma, and others field theory tools) that $tr\, A/\mathbb{K} = 0$, namely, that the transcendence degree of the field extension $A/\mathbb{K}$ is zero, so that $A/\mathbb{K}$ is a finitely generated algebraic extension (hence finite) of $\mathbb{K}$. Then, the thesis descend by the algebraic closure of $\mathbb{K}$.

and Remark 6), we deduce that every maximal ideal of $\mathbb{K}[X_1, \ldots, X_n]$ is of the form $\langle X_1 - a_1, \ldots, X_n - a_n\rangle$, a statement that is an equivalent form of the Weak Nullstellensatz (see [2]).

Let $\mathbb{K}$ be an arbitrary field of characteristic zero, not necessarily algebraically closed.

Let $A_{\mathbb{K}}$ be an arbitrary linear unitary commutative $\mathbb{K}$-algebra. For each $a \in A_{\mathbb{K}}$, let[9]

$$\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \{\lambda; \lambda \in \mathbb{K} \text{ such that } \nexists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}\}$$

be the $(A_{\mathbb{K}}, \mathbb{K})$-*spectrum* of a in $A_{\mathbb{K}}$; $r_{A_{\mathbb{K}}, \mathbb{K}}(a) = \mathbb{K} \setminus \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ is said the $(A_{\mathbb{K}}, \mathbb{K})$-*resolvent* of a in $A_{\mathbb{K}}$.

There exists linear unitary commutative $\mathbb{K}$-algebras in which such spectrum may be empty for some of its elements: for instance, if $A_{\mathbb{K}}$ is a linear unitary commutative integral $\mathbb{K}$-algebra of finite degree (> 1) over $\mathbb{K}$, then it follows that it is a field, and so $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \emptyset$ for each $a \in A_{\mathbb{K}} \setminus \mathbb{K}\mathbb{A}_{\mathbb{K}} (\neq \emptyset)$ because, since $a - \lambda 1_{A_{\mathbb{K}}} \in A_{\mathbb{K}} \setminus \{0\}$, $\forall \lambda \in \mathbb{K}$, there always exists $(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$.

Likewise, if $A_{\mathbb{K}} = \mathbb{K}(X)$, then $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(X) = \emptyset$.

It follows that the question relative to the emptiness, or not, of the spectrum of the generic element of a given linear unitary commutative $\mathbb{K}$-algebra, is not trivial (see **IX**).

If $G(A_{\mathbb{K}})$ denotes the group of units of $A_{\mathbb{K}}$ and $\mathcal{M}(A_{\mathbb{K}})$ denotes the set of all maximal ideals of $A_{\mathbb{K}}$, then we have $A_{\mathbb{K}} \setminus G(A_{\mathbb{K}}) \subseteq \bigcup_{M \in \mathcal{M}(A_{\mathbb{K}})} M$.

For each $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset$, the correspondence

$$\Omega(a, \cdot) : \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \to A_{\mathbb{K}} \setminus G(A_{\mathbb{K}}), \quad \lambda \rightsquigarrow \Omega(a, \lambda) = (a - \lambda 1_{\mathbb{K}}), \; \lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$$

is called *spectral map* on a; it is injective.

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[9] $1_{A_{\mathbb{K}}}$ denotes the unit of such $\mathbb{K}$-algebra.

For each $a \in A_{\mathbb{K}}$ such that $r_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset,$ the correspondence

$$R(a, \cdot) : r_{A_{\mathbb{K}}, \mathbb{K}}(a) \to G(A_{\mathbb{K}}), \quad \lambda \rightsquigarrow R(a, \lambda) = (a - \lambda 1_{A_{\mathbb{K}}})^{-1}, \ \lambda \in r_{A_{\mathbb{K}}, \mathbb{K}}(a)$$

is called *resolvent map* on a; it is immediate to prove that $R(a, \cdot)$ is injective, and also verify the following relations:

$$aR(a, \lambda) = 1_{A_{\mathbb{K}}} + \lambda R(a, \lambda), \quad \lambda \in r_{A_{\mathbb{K}}, \mathbb{K}}(a),$$

said to be the *Hilbert's identity*, and

$$R(a, \lambda_1) - R(a, \lambda_2) = (\lambda_1 - \lambda_2) R(a, \lambda_1) R(a, \lambda_2), \ \ \lambda_1, \lambda_2 \in r_{A_{\mathbb{K}}, \mathbb{K}}(a),$$

said to be the *resolvent equation*.

If $A_{\mathbb{K}}$ is a linear unitary commutative $\mathbb{K}$-algebra, then $A_{\mathbb{K}}^*$ denotes the (algebraic) dual $\mathbb{K}$-linear space of $A_{\mathbb{K}}$, namely, the set of all linear functionals from the $\mathbb{K}$-linear space $A_{\mathbb{K}}$ to $\mathbb{K}$ endowed with the usual structure of $\mathbb{K}$-linear space.

Among the elements of $A_{\mathbb{K}}^*$, there exists particular linear functionals, namely, the *multiplicative* linear functionals, that are also (nonzero) ring homomorphisms between the ring structure of the $\mathbb{K}$-algebra $A_{\mathbb{K}}$ and the ring structure of $\mathbb{K}$: we denote their set by $\Delta(A_{\mathbb{K}})(\subseteq A_{\mathbb{K}}^* \setminus \{0\})$, and, a priori, it has no particular algebraic structure. More specifically, the elements of $\Delta(A_{\mathbb{K}})$ are said to be the *characters* of $A_{\mathbb{K}}$.

If $A_{\mathbb{K}}$ is an arbitrary linear unitary commutative $\mathbb{K}$-algebra, with support A, then an $A_{\mathbb{K}}$*-ideal* I of it, is a subset $I \subseteq A$ such that $\lambda a + \mu b \in I$, $\lambda, \mu \in \mathbb{K}$, $a, b \in I$, and $ab \in I$, $a \in A$, $b \in I$. If $(A, +, \cdot)$ is the ring structure of $A_{\mathbb{K}}$, then the class of all ideals of $(A, +, \cdot)$ is equal to the class of all ideals of $A_{\mathbb{K}}$: in fact, it is evident that an $A_{\mathbb{K}}$-ideal is also an ideal of $(A, +, \cdot)$, whereas, if I is an ideal of $(A, +, \cdot)$, from $ab \in I$, $a \in A$, $b \in I$, since $a = \lambda 1_{A_{\mathbb{K}}} \in A$, $\lambda \in \mathbb{K}$, it follows that $\lambda b \in I$ for each $\lambda \in \mathbb{K}$.

Remark 3. Let $A_{\mathbb{K}}$ be a linear unitary commutative $\mathbb{K}$-algebra, with $\mathbb{K}$ algebraically closed. Let $\mathcal{L}(A_{\mathbb{K}})$ be the $\mathbb{K}$-linear space of all linear endomorphisms of the $\mathbb{K}$-linear space $A_{\mathbb{K}}$.

Then we have the following representation (of $\mathbb{K}$ -algebras):

$$T : A_{\mathbb{K}} \to \mathcal{L}(A_{\mathbb{K}}), \quad a \rightsquigarrow \varphi_a, \quad \forall a \in A_{\mathbb{K}},$$

where $\varphi_a : x \to ax, \ x \in A_{\mathbb{K}}$. We say T to be the *regular representation* of $A_{\mathbb{K}}$.

Let (with abuse of notation) $[A_{\mathbb{K}} : \mathbb{K}] = \dim_{\mathbb{K}} A_{\mathbb{K}} = n < \infty$; if $\mathcal{B} = \{v_1, ..., v_n\}$ is a base of the $\mathbb{K}$-linear space $A_{\mathbb{K}}$, let $M_a^{\mathcal{B}}$ be the (n, n)-matrix associated to φ_a with respect to $\mathcal{B}$, so that let[10] $\mathfrak{p}_a^{\mathcal{B}}(\lambda) = \det(M_a^{\mathcal{B}} - \lambda I)$ be the characteristic polynomial of φ_a with respect to $\mathcal{B}$, where I is the (n, n)-identity matrix, and $\lambda \in \mathbb{K}$. It is $\mathfrak{p}_a^{\mathcal{B}}(\lambda) \in \mathbb{K}[\lambda]$ with $\partial eg\, \mathfrak{p}_a^{\mathcal{B}}(\lambda) = n$, and, if $Eig_{\mathbb{K}}(\varphi_a) = \{\lambda;\ \lambda \in \mathbb{K}, \mathfrak{p}_a^{\mathcal{B}}(\lambda) = 0\}$ is the set of eigenvalues of φ_a for every $a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$, we have $Eig_{\mathbb{K}}(\varphi_a) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ because, if $\lambda \in Eig_{\mathbb{K}}(\varphi_a)$, then $\varphi_a(x) = \lambda x$ for, at least, one eigenvector $x \neq 0_{A_{\mathbb{K}}}$ of λ, so that $ax = \lambda x$, hence $(a - \lambda 1_{A_{\mathbb{K}}})x = 0_{A_{\mathbb{K}}}$, that is, $(a - \lambda 1_{A_{\mathbb{K}}}) = 0_{A_{\mathbb{K}}}$, whence $\nexists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}$. Therefore, since $Eig_{\mathbb{K}}(\varphi_a) \neq \varnothing$, $a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$ because of the algebraic closure of $\mathbb{K}$, it follows that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \varnothing$ for each $a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$, and for $0_{\mathbb{K}} \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(0_{A_{\mathbb{K}}})$, we conclude that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \varnothing$, $a \in A_{\mathbb{K}}$.

Nevertheless, in infinite dimension, may be $Eig_{\mathbb{K}}(\varphi_a) = \varnothing$ for some $a \in A_{\mathbb{K}}$, and therefore, in such a case, we proceed in another manner.

V. *We have the following results*:

(1) *If* $A_{\mathbb{K}}$ *and* $A'_{\mathbb{K}}$ *are linear unitary commutative* $\mathbb{K}$-*algebras and* $\varphi : A_{\mathbb{K}} \to A'_{\mathbb{K}}$ *is a homomorphism of linear unitary commutative* $\mathbb{K}$-*algebras*, *then* $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(\varphi(a)) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for each* $a \in A_{\mathbb{K}}$. *If* φ *is an isomorphism*, *then* $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(\varphi(a)) = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for each* $a \in A_{\mathbb{K}}$.

(2) *If* $A_{\mathbb{K}}$ *and* $A'_{\mathbb{K}}$ *are linear unitary commutative* $\mathbb{K}$-*algebras such that* $A_{\mathbb{K}} \subseteq A'_{\mathbb{K}}$, *then* $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for each* $a \in A_{\mathbb{K}}$.

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[10]The determinant that follows, is computed in the abstract field $\mathbb{K}$.

(3) *If* $\mathbb{K}'$ *is a* (*over*) *field such that* $\mathbb{K} \subseteq \mathbb{K}'$ *and* $A_{\mathbb{K}'}$ *is a linear unitary commutative* $\mathbb{K}'$*-algebra*, *then*, *by restriction of the scalars*, *said* $A_{\mathbb{K}}$ *the corresponding linear unitary commutative* $\mathbb{K}$*-algebra*, *we have* (i) $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$ *for each* $a \in A_{\mathbb{K}}$; (ii) $\sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for each* $a \in A_{\mathbb{K}'}$; (iii) $\sigma_{A_{\mathbb{K}'}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$ *for each* $a \in A_{\mathbb{K}}$.

(4) *If* a *is invertible in* $A_{\mathbb{K}}$, *then* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a^{-1}) = \{\lambda^{-1}; \lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)\}$. *Furthermore*, *for each* $a \in A_{\mathbb{K}}$, *we have* $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(\mu a)$ *if and only if* $\lambda/\mu \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for every* $\mu \in \mathbb{K}\backslash\{0\}$.

(5) *If* $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, *then* $\lambda^n \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(A^n)$, $\forall n \in \mathbb{N}$, $n \geq 2$.

(6) **(Polynomial spectral mapping theorem)** *If* $\mathbb{K}$ *is algebraically closed and* $p(X) \in \mathbb{K}[X]$, *then* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(p(a)) = p(\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a))$ *for each* $a \in A_{\mathbb{K}}$.

(7) **(Rational spectral mapping theorem)** *If* $\mathbb{K}$ *is algebraically closed*, *for each* $a \in A_{\mathbb{K}}$ *and for each rational function* $\chi \in \mathbb{K}(X)$ *regular on* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, *then* $\chi(\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)) = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(\chi(a))$.

Let us prove (1). If $a \in A_{\mathbb{K}}$ and $\lambda \in \sigma_{A'_{\mathbb{K}}, \mathbb{K}}(\varphi(a))$, then $\nexists(\varphi(a) - \lambda 1_{A'_{\mathbb{K}}})^{-1} = (\varphi(a) - \lambda\varphi(1_{A_{\mathbb{K}}}))^{-1} = \varphi((a - \lambda 1_{A_{\mathbb{K}}})^{-1})$ in $A'_{\mathbb{K}}$, so that $(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ cannot exist in $A_{\mathbb{K}}$, otherwise $\varphi((a - \lambda 1_{A_{\mathbb{K}}})^{-1})$ would exist in $A'_{\mathbb{K}}$; thus $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$. The last part of (1) follows immediately from the first part.

(2) follows from (1) with φ the natural immersion.

In the case (3), first of all it is $1_{A_{\mathbb{K}}} = 1_{A_{\mathbb{K}'}}$ because $A_{\mathbb{K}}$ and $A_{\mathbb{K}'}$ have the same (inner) addition and multiplication laws but different (external) scalar multiplication law. If $a \in A_{\mathbb{K}}$ and $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, then $\nexists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ in $A_{\mathbb{K}}$, and since $\lambda \in \mathbb{K} \subseteq \mathbb{K}'$, it is also $\nexists(a - \lambda 1_{A_{\mathbb{K}'}})^{-1}$ in $A_{\mathbb{K}'}$, so that $\lambda \in \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$, whence $\sigma_{A_{\mathbb{K}'}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$, that is (i). Conversely, if $\lambda \in \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$, then $\nexists(a - \lambda 1_{A_{\mathbb{K}'}})^{-1}$ in $A_{\mathbb{K}'}$, with $\lambda \in \mathbb{K}'$ and not necessarily with $\lambda \in \mathbb{K} \subseteq \mathbb{K}'$; then, if it is also $\lambda \in \mathbb{K}$, we have $\lambda 1_{A_{\mathbb{K}'}} = \lambda 1_{A_{\mathbb{K}}} \in A_{\mathbb{K}}$ so that, by $a \in A_{\mathbb{K}}$, it is $\nexists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ in $A_{\mathbb{K}}$, namely, $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, whence $\sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K} \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$; but, by (i), it follows that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$, whereas, by definition of spectrum, it is also $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \mathbb{K}$, so that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K}$; in conclusion $\sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}}$.

For the proof of (iii), we have, for each $a \in A_{\mathbb{K}'}$,

$$\sigma_{A_{\mathbb{K}'}, \mathbb{K}}(a) = \{\lambda; \lambda \in \mathbb{K} \text{ such that } \nexists (a - \lambda 1_{A_{\mathbb{K}'}})^{-1} \text{ in } A_{\mathbb{K}'}\} = \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K};$$

but $\sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}'}$, by (ii), and $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$, by (i), so that $\sigma_{A_{\mathbb{K}'}, \mathbb{K}}(a) = \sigma_{A_{\mathbb{K}'}, \mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$.

For the proof of (4), observing that $0 \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, since a is invertible, it is enough to consider the identities $a^{-1} - \lambda^{-1} 1_{A_{\mathbb{K}}} = -\lambda^{-1} a^{-1} (a - \lambda 1_{A_{\mathbb{K}}})$, $\forall \lambda \neq 0$ for the proof of the first part, and $(\mu a - \lambda 1_{A_{\mathbb{K}}})^{-1} = \mu^{-1} (a - \lambda \mu^{-1} 1_{A_{\mathbb{K}}})^{-1}$, $\forall \mu \neq 0$ for the proof of the second part.

As regard (5), if $\lambda^n \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, then by the identity,

$$\begin{aligned}(a^n - \lambda^n 1_{A_{\mathbb{K}}}) &= (a - \lambda 1_{A_{\mathbb{K}}})(a^{n-1} + \lambda a^{n-2} + \cdots + \lambda^{n-2} a + \lambda^{n-1} 1_{A_{\mathbb{K}}}) \\ &= (a - \lambda 1_{A_{\mathbb{K}}}) b(a, \lambda), \quad \forall n \in \mathbb{N}, \ n \geq 2\end{aligned}$$

with $b(a, \lambda) = a^{n-1} + \lambda a^{n-2} + \cdots + \lambda^{n-1} 1_{A_{\mathbb{K}}} \in A_{\mathbb{K}}$, since $\exists (a^n - \lambda^n 1_{A_{\mathbb{K}}})^{-1}$, it follows that

$$1_{A_{\mathbb{K}}} = (a - \lambda 1_{A_{\mathbb{K}}})(b(a, \lambda)(a^n - \lambda^n 1_{A_{\mathbb{K}}})^{-1}),$$

that is, $\exists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}$, whence $\lambda \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$.

For the proof of (6), first of all, we recall the usual correspondence $p(X) \to p(a) \in A_{\mathbb{K}}$ between (abstract) polynomials and polynomial functions (that it is bijective if $A_{\mathbb{K}}$ is an infinite integral algebra); moreover, in a commutative ring, if $a = \prod_{i=1}^{n} a_i$, $n \geq 2$, then a is invertible if and only if each a_i is invertible.

We establish the statement for each $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$ because, otherwise, we will have a trivial case. Furthermore, the assertion is true if $p(X)$ is a constant polynomial, so that let $\partial eg\ p(X) = n \geq 1$ with $p(X) = \sum_{i=0}^{n} a_i X^i \in \mathbb{K}[X]$; hence, if $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$, then we have formally

$$p(X) - p(\lambda)1_{\mathbb{K}[X]} = \sum_{i=0}^{n} a_i X^i - \left(\sum_{i=0}^{n} a_i \lambda^i\right)1_{\mathbb{K}[X]} = \sum_{i=1}^{n} a_i (X - \lambda 1_{\mathbb{K}[X]})^i$$

$$= (X - \lambda 1_{\mathbb{K}[X]}) \sum_{i=1}^{n} a_i (X - \lambda 1_{\mathbb{K}[X]})^{i-1} = (X - \lambda 1_{\mathbb{K}[X]}) q(X)$$

so that $p(a) - p(\lambda) = (a - \lambda 1_{A_{\mathbb{K}}}) q(a) \in A_{\mathbb{K}}$, and since $\nexists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}$, it follows that $\nexists (p(a) - p(\lambda)1_{A_{\mathbb{K}}})^{-1}$, hence $p(\lambda) \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(p(a))$, that is, $p(\sigma_{A_{\mathbb{K}},\mathbb{K}}(a)) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(p(a))$.

Conversely, if $\mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(p(a))$, let us set $q(X) = p(X) - \mu 1_{\mathbb{K}[X]} \in \mathbb{K}[X]$. Because of the algebraic closure of $\mathbb{K}$, we have $q(X) = c \prod_{i=1}^{n} (X - \lambda_i)$ for some $c, \lambda_i \in \mathbb{K}$, $\forall i$, so that $q(a) = p(a) - \mu 1_{\mathbb{K}} = c \prod_{i=1}^{n} (a - \lambda_i 1_{\mathbb{K}})$, and since $\nexists (p(a) - \mu \cdot 1_{\mathbb{K}})^{-1}$, it follows that there exists, at least, one $i \in \{1, ..., n\}$ such that $\nexists (a - \lambda_i 1_{\mathbb{K}})^{-1}$, hence (in $\mathbb{K}$) must be $a - \lambda_i = 0_{\mathbb{K}}$ for such i, whence $p(a) - \mu 1_{\mathbb{K}} = 0_{\mathbb{K}}$, that is, $\mu = p(a) \in p(\sigma_{A_{\mathbb{K}},\mathbb{K}}(a))$, and thus $\sigma_{A_{\mathbb{K}},\mathbb{K}}(p(a)) \subseteq p(\sigma_{A_{\mathbb{K}},\mathbb{K}}(a))$.

Finally, if we fix, arbitrarily, $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$, let $\chi = f/g \in \mathbb{K}(X)$ be an arbitrary rational function regular on $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$. Hence, if $\mu = f(\lambda)/g(\lambda) \in \chi(\sigma_{A_{\mathbb{K}},\mathbb{K}}(a))$, since

$$\chi(a) - \mu 1_{A_{\mathbb{K}}} = \frac{f(a)}{g(a)} - \frac{f(\lambda)}{g(\lambda)} 1_{A_{\mathbb{K}}} = \left(\frac{f(a)g(\lambda) - f(\lambda)g(a)}{g(a)g(\lambda)}\right)^{-1},$$

it follows that $\nexists(\chi(a) - \mu 1_{A_{\mathbb{K}}})^{-1}$ if and only if $\nexists(f(a)g(\lambda) - f(\lambda)g(a))^{-1}$, that is, if and only if $\nexists g(a)(g(\lambda)f(a)/g(a) - f(\lambda)1_{A_{\mathbb{K}}})^{-1}$, hence if and only if $f(\lambda) \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(g(\lambda)\chi(a))$, that is, by (4), if and only if $f(\lambda)/g(\lambda) \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(\chi(a))$.

VI. *If* $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]$, *then* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$ *for each* $f \in \mathbb{K}[X_1, ..., X_n]$.

First proof. We recall that the group of units of $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]$, say $G(A_{\mathbb{K}})$, is $\mathbb{K}$.

If $f = a \in \mathbb{K}$ is a constant, then $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(f) = \{a\} \neq \varnothing$. If f is not constant, then $\partial eg\ f > 0$ with $f \notin G(A_{\mathbb{K}})(= \mathbb{K})$, that is, f is not invertible and so is also $f - a1_{A_{\mathbb{K}}}$ for each $a \in \mathbb{K}$, whence $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(f) = \mathbb{K}$. In any case, we have $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$ for each $f \in \mathbb{K}[X_1, ..., X_n]$.

Second proof[11]**.** Let $A'_{\mathbb{K}} = K[[X_1, ..., X_n]]$ be the linear unitary commutative $\mathbb{K}$-algebra of the formal power series in the variables $X_1, ..., X_n$. Then $A_{\mathbb{K}} \subseteq A'_{\mathbb{K}}$, so that, by **V**, (2), we have $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(f)$ for each $f \in A_{\mathbb{K}}$. On the other hand, if $f = 0$, then $\sigma_{A'_{\mathbb{K}}, \mathbb{K}} = \{0\} \neq \varnothing$; instead, if $f \neq 0$ is not a invertible polynomial in $A'_{\mathbb{K}}$, then, at least, $0 \in \sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f)$ so that $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$ in this case, whereas, if $f \neq 0$ is an invertible polynomial in $A'_{\mathbb{K}}$, then it follows that the zero-degree term of f, namely, a_0, must be nonzero, whence $f - \lambda 1_{A'_{\mathbb{K}}}$ is a polynomial without constant term if $\lambda = a_0$, so that $f - a_0 1_{A'_{\mathbb{K}}}$ is not invertible in $A'_{\mathbb{K}}$, and thus at least $a_0 \in \sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f)$, so that $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$ is also true. In conclusion, in any case, we have $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$ for each $f \in A_{\mathbb{K}}$, and then $\varnothing \neq \sigma_{A'_{\mathbb{K}}, \mathbb{K}}(f) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(f)$, $f \in A_{\mathbb{K}}$ implies $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(f) \neq \varnothing$, $f \in A_{\mathbb{K}}$, as required.

Note 1. The result in **VI** holds true if $\mathbb{K}$ is any integral domain.

Note 2. In **IX**, we shall prove a more general result for some linear unitary commutative $\mathbb{K}$-algebra $A_{\mathbb{K}}$.

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[11]The principle of the method of this second proof, lies on the possible invertibility of a no constant polynomial in $A'_{\mathbb{K}}$, whose group of units is more large than $G(A_{\mathbb{K}}) = \mathbb{K}$.

VII. *Let* $A_{\mathbb{K}}$ *be an arbitrary linear unitary commutative* $\mathbb{K}$*-algebra, and* $a \in A_{\mathbb{K}}$. *If there exists a* $\phi \in \Delta(A_{\mathbb{K}})$ *such that* $\phi(a) = \lambda \in \mathbb{K}$, *then* $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$.

If a is not invertible, then, at least, $0 \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, so that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \varnothing$ and, then it is allowed to consider the following argument: if $\lambda \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, then $\exists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$, hence $\phi(a - \lambda 1_{A_{\mathbb{K}}}) \neq 0$, $\phi \in \Delta(A_{\mathbb{K}})$, because otherwise

$$1_{\mathbb{K}} = \phi(1_{A_{\mathbb{K}}}) = \phi((a - \lambda 1_{A_{\mathbb{K}}})(a - \lambda 1_{A_{\mathbb{K}}})^{-1})$$

$$= \phi(a - \lambda 1_{A_{\mathbb{K}}})\phi((a - \lambda 1_{A_{\mathbb{K}}})^{-1}) = 0_{\mathbb{K}},$$

being $\phi(a - \lambda 1_{A_{\mathbb{K}}}) = 0_{\mathbb{K}}$ for, at least, one $\phi \in \Delta(A_{\mathbb{K}})$. Therefore, it must be $\phi(a - \lambda 1_{A_{\mathbb{K}}}) \neq 0$ for each $\phi \in \Delta(A_{\mathbb{K}})$, that is, $\phi(a) \neq \phi(\lambda 1_{A_{\mathbb{K}}}) = \lambda$, against to the hypothesis. Instead, if $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \varnothing$, then $(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ exists for each[12] $\lambda \in \mathbb{K}$, whence arguing likewise as above, we obtain a contradiction. Therefore, $\nexists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ for, at least, one $\lambda \in \mathbb{K}$, whence $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, and thus $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \varnothing$.

However, we shall return on this result (see Remark 7), in view of a possible partial converse to it.

Remark 4. If we set $A^{*}_{\mathbb{K}}(a) \doteq \{\phi(a);\ \phi \in \Delta(A_{\mathbb{K}})\}$, then, by **VII**, it follows that $A^{*}_{\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$. Therefore, **VII** may be used as a criterion for the spectrum to be nonempty (see **VIII**), when $\Delta(A_{\mathbb{K}})$ is non-void. For instance, if $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]$, since the (polynomial) evaluation map $f \xrightarrow{\zeta_k} \zeta_k(f) = f(k)$, $f \in \mathbb{K}[X_1, ..., X_n]$, is an element of $\Delta(A_{\mathbb{K}})$, the equation $\phi(f) = \lambda \in \mathbb{K}$ becomes $\zeta_k(f) = f(k) = \lambda$, that is, $f(k) - \lambda = 0$, hence $g_{\lambda}(k) = 0$ with $g_{\lambda} = f - \lambda \in \mathbb{K}[X_1, ..., X_n]$; because of the algebraic closure of $\mathbb{K}$, the equation $g_{\lambda}(k) = 0$ has always, at least, one solution in $k \in \mathbb{K}$ for each λ (see **VI**).

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[12]In particular, for $\lambda = 0$, necessarily, it follows that $\exists a^{-1}$.

Remark 5. If I is an arbitrary maximal ideal of $\mathbb{K}[X_1, ..., X_n]$, then it follows that $\mathbb{A} = \mathbb{K}[X_1, ..., X_n]/I = \mathbb{K}[\overline{X}_1, ..., \overline{X}_n]$ is a finitely generated field extension of $\mathbb{K}$, with $\overline{X}_i = \pi(X_i)$, $i = 1, ..., n$, where $\pi : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}[X_1, ..., X_n]/I$ is the quotient epimorphism. Therefore, it is possible to consider the UFD $\mathbb{A}[X_1, ..., X_n]$, which is a linear unitary commutative $\mathbb{A}$-algebra, and, also, by the restriction of scalars, a linear unitary commutative $\mathbb{K}$-algebra (via σ of Remark 1); thus it is also possible to consider $\mathbb{K}[X_1, ..., X_n]$ as a sub-UFD of $\mathbb{A}[X_1, ..., X_n]$. Let $A'_{\mathbb{K}}$ be the linear unitary commutative $\mathbb{K}$-algebra obtained considering $\mathbb{A}[X_1, ..., X_n]$ as a $\mathbb{K}$-algebra, as said above.

Let $A'_{\mathbb{A}}$ be the linear unitary commutative $\mathbb{A}$-algebra $\mathbb{A}[X_1, ..., X_n]$, and let $A'^*_{\mathbb{A}}$ be the dual $\mathbb{A}$-linear space of $A'_{\mathbb{A}}$.

The map

$$\zeta : \mathbb{A} \to \Delta(A'_{\mathbb{A}}), \quad k \rightsquigarrow \zeta_k, \ \forall k \in \mathbb{A},$$

defined via the (polynomial) evaluation map $\zeta_k(f) = f(k) \in \mathbb{A}$ for each $f \in A'_{\mathbb{A}}$, proves that there exists $\phi \in \Delta(A'_{\mathbb{A}})$: for instance, $\phi = \zeta_k$ for $k \in \mathbb{A}$.

Later on (see **VIII**), it is important, above all, to verify, in $A'_{\mathbb{K}}$, if there exists, at least, one $\phi \in \Delta(A'_{\mathbb{K}})$, where

$$\Delta(A'_{\mathbb{K}}) = \{\phi; \phi : A'_{\mathbb{K}} \to \mathbb{K} \text{ multiplicative linear functional}\} (\subseteq A'^*_{\mathbb{K}} \setminus \{0\})$$

is the set of characters of $A'_{\mathbb{K}}$.

To this end, since I is a maximal ideal of $\mathbb{K}[X_1, ..., X_n]$, we have that I is a proper ideal of it, so that $\exists f \in \mathbb{K}[X_1, ..., X_n] \setminus I$, with $f \neq 0$; if J is the maximal ideal containing $\langle f \rangle$, then we have that I and J are comaximal distinct ideals (by $f \notin I$), so that $\mathbb{K}[X_1, ..., X_n] = I + J$. Therefore, it is possible to consider the map $\phi : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}$ such that $\phi(I) = \phi(I \cap J) = 0_{\mathbb{K}}$ and $\phi(f) = f(0) = a_0 \in \mathbb{K}$ for each $f \in J \setminus I$, where a_0 is the zero-degree term of the polynomial f. Then it is immediate to verify that ϕ is a non-zero multiplicative linear functional on $A'_{\mathbb{K}}$ if we extend ϕ from $\mathbb{K}[X_1, ..., X_n]$ to $\mathbb{A}[X_1, ..., X_n]$ setting ϕ also zero on $\mathbb{A}[X_1, ..., X_n] \setminus \mathbb{K}[X_1, ..., X_n]$. In short, we have $\phi \in \Delta(A'_{\mathbb{K}})$.

VIII (Weak Nullstellensatz). *If I is any proper ideal of $\mathbb{K}[X_1, ..., X_n]$, then* $V(I) \neq \emptyset$.

If J is the maximal ideal containing I, then we have $V(J) \subseteq V(I)$, so that $(V(J) \neq \emptyset) \Rightarrow (V(I) \neq \emptyset)$, and, then it is possible to consider I to be maximal. Then $A = \mathbb{K}[X_1, ..., X_n]/I$ is a field (often denoted by $\mathbb{A}$ as a scalar field - see Remark 1), hence a linear unitary commutative $\mathbb{A}$-algebra on itself, namely, $\mathbb{A}_{\mathbb{A}}$. But, according to Remark 1, A is, also, a linear unitary commutative $\mathbb{K}$-algebra, namely, $A_{\mathbb{K}}$. We are interested in $\sigma_{A_{\mathbb{K}}, \mathbb{K}}([f])$, rather than in $\sigma_{A_{\mathbb{A}}, \mathbb{A}}([f])$, for $[f] \in A$, and, for this, we observe the following. If $[f] = f + I \in A_{\mathbb{K}} \setminus \{0\}$, then we have[13], for $\lambda \in \mathbb{K}$, that $(f + I) - \lambda 1_{A_{\mathbb{K}}} = (f + I) - \lambda(1_{\mathbb{K}} + I) = (f - \lambda 1_{\mathbb{K}}) + I$ is not invertible if and only if it is zero (in a field, as A), that is, if and only if $f - \lambda 1_{\mathbb{K}} \in I$, with $f \notin I$ because $[f] \neq 0$ in A. Therefore, it is required to find $\lambda \in \mathbb{K} \setminus \{0\}$ such that[14] $f - \lambda 1_{\mathbb{K}} \in I$, if we wish to prove that[15] $\sigma_{A_{\mathbb{K}}, \mathbb{K}}([f]) \neq \emptyset$.

Now, taking into account Remarks 1 and 5 $(\text{with } n = 1)$, we consider $\mathbb{A}[X]$, with the field $\mathbb{A} = \mathbb{K}[X_1, ..., X_n]/I$ as a linear unitary commutative $\mathbb{K}$-algebra, say $A'_{\mathbb{K}}$. Therefore, since there exists $\phi \in \Delta(A'_{\mathbb{K}})$ by **VII**, it follows that $\sigma_{A'_{\mathbb{K}}, \mathbb{K}}([f]) \neq \emptyset$ for each[16] $[f] \in A'_{\mathbb{K}}$, whereas, by **V** (2), it follows that $\emptyset \neq \sigma_{A'_{\mathbb{K}}, \mathbb{K}}([f]) \subseteq \sigma_{A_{\mathbb{K}}, \mathbb{K}}([f])$ implies $\sigma_{A_{\mathbb{K}}, \mathbb{K}}([f]) \neq \emptyset$ for each $[f] \in A_{\mathbb{K}}$, if we consider $A_{\mathbb{K}}$ as a linear unitary commutative $\mathbb{K}$-algebra which is a sub-algebra of $A'_{\mathbb{K}} (= \mathbb{A}[X]$ as $\mathbb{K}$-algebra).

Then $\sigma_{A_{\mathbb{K}}, \mathbb{K}}([f]) \neq \emptyset$ for each $[f] \in A_{\mathbb{K}} \setminus \{0\}$, with $0 \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}([f])$ because,

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[13] $1_{A_{\mathbb{K}}}$ is the unit of $A = \mathbb{K}[X_1, ..., X_n]/I$ as $\mathbb{K}$-algebra, whereas $1_{\mathbb{K}}$ is the unit of $\mathbb{K}$, the same of $\mathbb{K}[X_1, ..., X_n]$.

[14] It is necessarily $\lambda \neq 0$ because $f \notin I$.

[15] See footnote [7]

[16] Whence, by V, (3)(i), with $\mathbb{K}' = \mathbb{A}$, it follows that $\sigma_{A_{\mathbb{A}}, \mathbb{A}}([f]) \neq \emptyset$ for each $[f] \in A_{\mathbb{K}}$. Moreover, we observe that the support of $A_{\mathbb{K}}$ and the support of $A_{\mathbb{A}}$ are equal $(= A)$; likewise for $A'_{\mathbb{K}}$ and $A'_{\mathbb{A}}$, with common support $\mathbb{A}[X]$.

by $0 \neq [f] \in A_{\mathbb{K}}$ with $A(= \mathbb{A})$ field, it follows that $[f]$ is invertible, so that 0 cannot be an element of $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$; likewise, each $[f] \neq 0$ is invertible in the field A, so that it must be $[f] - \lambda 1_{A_{\mathbb{K}}} = 0$ for each $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ because, if it is nonzero, it would admit inverse in A whereas $\nexists([f] - \lambda 1_{A_{\mathbb{K}}})^{-1}$ since $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$; therefore, for any $[f] \in A_{\mathbb{K}} \setminus \{0\}$, we have $[f] = \lambda 1_{A_{\mathbb{K}}}$ for each $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$. Thus $([f] =)\lambda 1_{A_{\mathbb{K}}} = \lambda' 1_{A_{\mathbb{K}}}$, $\lambda, \lambda' \in \sigma_{A_{\mathbb{K}},\mathbb{K}}$, for which $\lambda = \lambda'$, whence $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ is a singleton for each $[f] \in A_{\mathbb{K}} \setminus \{0\}$; on the other hand, if $[f] = 0$, then we have $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = \{0\}$. Therefore, $\operatorname{card} \sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = 1$ for all $[f] \in A_{\mathbb{K}}$.

Thus, it is possible to consider the map

$$\psi : A_{\mathbb{K}} \to \mathbb{K}, \quad [f] \rightsquigarrow \lambda(\in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])), \quad \forall [f] \in A_{\mathbb{K}}$$

that it is well-defined because, for $[f] \neq 0$, we have the singleton $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = \{\lambda\}(\neq \{0\})$, whereas, if $[f] = 0$, then $\sigma_{A_{\mathbb{K}},\mathbb{K}}(0) = \{0\}$.

ψ is a monomorphism. In fact, we have $\psi([f] + [g]) = \lambda + \mu$, if $\psi([f]) = \lambda$ and $\psi([g]) = \mu$, because, as we will see, $\nexists(([f] + [g]) - (\lambda + \mu)1_{A_{\mathbb{K}}})^{-1}$ so that $\lambda + \mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f] + [g])$, and since $\operatorname{card} \sigma_{A_{\mathbb{K}},\mathbb{K}}([h]) = 1$ for all $[h] \in A_{\mathbb{K}}$, it follows that $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f] + [g]) = \{\lambda + \mu\}$; on the other hand, $\lambda + \mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f] + [g])$ because $\nexists([h] - \rho 1_{A_{\mathbb{K}}})^{-1}$ if and only if $h - \rho 1_{\mathbb{K}} \in I$, and since $\nexists([f] - \lambda 1_{A_{\mathbb{K}}})^{-1}$, $\nexists([g] - \mu 1_{A_{\mathbb{K}}})^{-1}$, we have $f - \lambda 1_{\mathbb{K}}$, $g - \mu 1_{\mathbb{K}} \in I$, that is, $(f + g) - (\lambda + \mu)1_{\mathbb{K}} \in I$, so that $\lambda + \mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f + g]) = \sigma_{A_{\mathbb{K}},\mathbb{K}}([f] + [g])$. Thus,

$$\psi([f] + [g]) = \lambda + \mu = \psi([f]) + \psi([g]).$$

Analogously, we have $\psi([f][g]) = \lambda\mu$ because $f - \lambda 1_{\mathbb{K}}$, $g - \mu 1_{\mathbb{K}} \in I$ implies $(f - \lambda 1_{\mathbb{K}})(g - \mu 1_{\mathbb{K}}) \in I$, that is,

$$\begin{aligned} fg - \mu f - \lambda g + \lambda\mu 1_{\mathbb{K}} &= fg - \lambda\mu 1_{\mathbb{K}} + 2\lambda\mu 1_{\mathbb{K}} - \mu f - \lambda g \\ &= (fg - \lambda\mu 1_{\mathbb{K}}) + \mu(f - \lambda 1_{\mathbb{K}}) + \lambda(g - \mu 1_{\mathbb{K}}) \in I, \end{aligned}$$

and since $f - \lambda 1_{\mathbb{K}},\ g - \mu 1_{\mathbb{K}} \in I,$ it follows that $\mu(f - \lambda 1_{\mathbb{K}}), \lambda(g - \mu 1_{\mathbb{K}}) \in I$ and thus $fg - \lambda\mu 1_{\mathbb{K}} \in I$, whence $\lambda\mu \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}([fg]) = \sigma_{A_{\mathbb{K}}, \mathbb{K}}([f][g])$, so that $\psi([fg]) = \psi([f][g]) = \lambda\mu = \psi([f])\psi([g])$.

Finally, if $\alpha \in \mathbb{K}\backslash\{0\}$, then $\psi(\alpha[f]) = \alpha\lambda$ because it is enough to observe that $(\alpha[f] - \alpha\lambda 1_{A_{\mathbb{K}}})^{-1} = \alpha^{-1}([f] - \lambda 1_{A_{\mathbb{K}}})^{-1}$, and that $(\alpha[f] - \alpha\lambda 1_{A_{\mathbb{K}}})^{-1}$ exists if and only if $([f] - \lambda 1_{A_{\mathbb{K}}})^{-1}$ exists, so that $\psi(\alpha[f]) = \alpha\lambda = \alpha\psi([f])$.

Finally, since $Ker\,\psi = \{[f]; [f] \in A_{\mathbb{K}}, \psi([f]) = 0\} = \{0\}$, it follows that ψ is injective[17].

Therefore, we have the immersion

$$\psi : A_{\mathbb{K}} = \mathbb{K}[X_1, \ldots, X_n]/I \hookrightarrow \mathbb{K} \tag{4}$$

that, combined with the immersion (see Remark 1)

$$\sigma : \mathbb{K} \hookrightarrow \mathbb{K}[X_1, \ldots, X_n]/I, \tag{5}$$

gives rise to an isomorphism of the type $\mathbb{K} \cong \mathbb{K}[X_1, \ldots, X_n]/I (= A_{\mathbb{K}})$[18].

On the other hand, for each $[f] = f + I \in A_{\mathbb{K}}\backslash\{0\}$, we have seen that $[f] - \lambda 1_{A_{\mathbb{K}}} = 0$ in $A_{\mathbb{K}}$, for all $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}([f])$, so that $(f + I) - \lambda(1_{\mathbb{K}} + I) = I$ (since $1_{A_{\mathbb{K}}} = 1_{\mathbb{K}} + I$ in $A_{\mathbb{K}}$). Then $(f - \lambda 1_{\mathbb{K}}) \in I$ for each $f \in \mathbb{K}[X_1, \ldots, X_n]$, and hence $f \in I + \mathbb{K}$ (for $\mathbb{K} = \{\lambda 1_{\mathbb{K}}; \lambda \in \mathbb{K}\} \subset \mathbb{K}[X_1, \ldots, X_n]$). By **I,** since $\mathbb{K} \subseteq \mathbb{K}[X_1, \ldots, X_n]$, it follows that $\mathbb{K}[X_1, \ldots, X_n] = I + \mathbb{K}$; then $X_i \in I + \mathbb{K}$, $i \in \{1, \ldots, n\}$, that is, $X_i = a_i + m_i$ with $a_i \in \mathbb{K}$ and $m_i \in I$ for all i, for which $X_i - a_i = m_i \in I$, and then $\langle X_1 - a_1, \ldots, X_n - a_n\rangle \subseteq I$. But $\langle X_1 - a_1, \ldots, X_n - a_n\rangle$ is a maximal ideal (by **III**), so that $\langle X_1 - a_1, \ldots, X_n - a_n\rangle = I$, and thus $V(I) = V(\langle X_1 - a_1, \ldots, X_n - a_n\rangle)$, whence $(a_1, \ldots, a_n) \in V(I)$, that is, $V(I) \neq \varnothing$.

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[17]Direct proof: if $\psi([f]) = \psi([g])$ for $[f], [g] \in A_{\mathbb{K}}$, then $\lambda = \mu$, whence $[f] = \lambda 1_{A_{\mathbb{K}}} = \mu 1_{A_{\mathbb{K}}} = [g]$, as required.

[18]From here, it is also possible to get the thesis continuing in the next Remark 6.

Remark 6. If we consider the quotient epimorphism

$$\pi : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}[X_1, ..., X_n]/I = \mathbb{K}[\overline{X}_1, ..., \overline{X}_n] = A_{\mathbb{K}} \tag{6}$$

with $\overline{X}_i = \pi(X_i)$, $i \in \{1, ..., n\}$, then by **VIII**, it follows that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(\overline{X}_i) \neq \varnothing$, that is, $\nexists((X_i + I) - (a_i + I)1_{A_{\mathbb{K}}})^{-1}$, for $a_i \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(\overline{X}_i)$, in the field $A_{\mathbb{K}} (= \mathbb{A})$, so that it must be $(X_i + I) - (a_i + I) = 0_{A_{\mathbb{K}}}$. Then $X_i - a_i \in I$, and hence $\langle X_1 - a_1, ..., X_n - a_n \rangle \subseteq I$, so that, by the maximality of $\langle X_1 - a_1, ..., X_n - a_n \rangle$ (see **III**), it follows that $I = \langle X_1 - a_1, ..., X_n - a_n \rangle$. From the latter, it is possible to understand the meaning of $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(\overline{X}_i)$ (via $a_i \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(\overline{X}_i)$, $\forall i$).

This is another proof of the fact that any maximal ideal of $\mathbb{K}[X_1, ..., X_n]$ is of the type $\langle X_1 - a_1, ..., X_n - a_n \rangle$, for certain $(a_1, ..., a_n) \in \mathbb{K}^n$.

Let $\mathbb{K}$ be an arbitrary field. If $A_{\mathbb{K}}$ is an arbitrary linear unitary commutative $\mathbb{K}$-algebra such that $A_{\mathbb{K}}/I \cong \mathbb{K}$ for every maximal ideal I of $A_{\mathbb{K}}$, then $A_{\mathbb{K}}$ is said to be a *spectral algebra*. Therefore, the Weak Nullstellensatz proves that $\mathbb{K}[X_1, ..., X_n]$ is a spectral algebra.

Let $\mathbb{K}$ be an arbitrary field of characteristic zero.

We recall that $A_{\mathbb{K}}$ is always a $\mathbb{K}$-linear space, so that we set (with abuse of notation) $[A_{\mathbb{K}} : \mathbb{K}] = \dim_{\mathbb{K}} A_{\mathbb{K}}$. Furthermore, if every non-zero element of $A_{\mathbb{K}}$ has a multiplicative inverse (that is, $A_{\mathbb{K}}$ is a field extension of $\mathbb{K}$), then we call $A_{\mathbb{K}}$, more specifically, a *division algebra* (according to B. L. Van der Waerden).

IX. *If $A_{\mathbb{K}}$ is a linear unitary commutative integral $\mathbb{K}$-algebra, and $\mathbb{K}$ is an arbitrary field of characteristic zero, then we have the following results*:

(1) *if $\mathbb{K}$ is not algebraically closed and $2 \leq [A_{\mathbb{K}} : \mathbb{K}] < \infty$, then there exists $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \varnothing$* (see final Note 3);

(2) *if $[A_{\mathbb{K}} : \mathbb{K}] = \infty$, $\mathbb{K}$ is not algebraically closed and $A_{\mathbb{K}}$ is a division algebra, then there exists an $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \varnothing$*;

(3) *if* $[A_{\mathbb{K}} : \mathbb{K}] = \infty$ *and* $A_{\mathbb{K}}$ *is not a division algebra, not finitely generated as* a $\mathbb{K}$ *-algebra*[19], *then* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset$ *for each* $a \in A_{\mathbb{K}}$, *whatever be the* $\mathbb{K}$.

(4) **(Generalized Gelfand-Mazur)** *if* $[A_{\mathbb{K}} : \mathbb{K}] = 1$, *then* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset$ *for each* $a \in A_{\mathbb{K}}$, *and* $A_{\mathbb{K}} \cong \mathbb{K}$, *whatever be the* $\mathbb{K}$ (see Remark 9).

If $A_{\mathbb{K}}$ *is a linear unitary commutative* $\mathbb{K}$*-algebra*, $\mathbb{K}$ *is an algebraically closed field and* $[A_{\mathbb{K}} : \mathbb{K}] < \operatorname{card} \mathbb{K}$, *then* $A_{\mathbb{K}}$ *is a spectral algebra*, *and therefore* $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset$ *for each* $a \in A_{\mathbb{K}}$.

Let us prove the first part of the theorem.

If $2 \leq [A_{\mathbb{K}} : \mathbb{K}] < \infty$, since $A_{\mathbb{K}}$ is an integral domain and a finite dimensional $\mathbb{K}$-linear space, it follows that it is a field (because $\varphi_a : x \to ax$, $x \in A_{\mathbb{K}}$ is an automorphism for each nonzero a) with $A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}} \neq \emptyset$ (by $[A_{\mathbb{K}} : \mathbb{K}] \geq 2$), so that for each $a \in A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}}$, we have $\exists (a - \lambda 1_{A_{\mathbb{K}}})^{-1} \in A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}}$, $\lambda \in \mathbb{K}$. Thus $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \emptyset$. Therefore, (1) is proved[20].

If $A_{\mathbb{K}}$ is a field, since $[A_{\mathbb{K}} : \mathbb{K}] = \infty$ and $\mathbb{K}$ is not algebraically closed, it follows that $A_{\mathbb{K}}$ is a proper field extension of $\mathbb{K}$, so that $A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}} \neq \emptyset$, and then $\exists (a - \lambda 1_{A_{\mathbb{K}}})^{-1} \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$, $\lambda \in \mathbb{K}$, $a \in A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}}$, that is, $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \emptyset$ for every $a \in A_{\mathbb{K}} \setminus \mathbb{K}1_{A_{\mathbb{K}}}$; whence[21] (2).

If $[A_{\mathbb{K}} : \mathbb{K}] = 1$, then we have that $A_{\mathbb{K}}$ is a trivial extension of $\mathbb{K}$, so that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = \{\lambda_a\}$ with $a = \lambda_a 1_{A_{\mathbb{K}}}$ for a unique $\lambda_a \in \mathbb{K}$ for every $a \in A_{\mathbb{K}}$; hence

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[19]Therefore, if $\mathbb{K}$ is not algebraically closed, then $A_{\mathbb{K}}$ may be a division algebra, finitely generated as $\mathbb{K}$-algebra. For instance, $\mathbb{K}(X)$ is a division algebra (finitely generated field extension of $\mathbb{K}$), not finitely generated as $\mathbb{K}$-algebra, for which (3) does not hold, as already seen.

[20]For instance, if $A_{\mathbb{K}} = \mathbb{C}_{\mathbb{R}}$, then $\sigma_{\mathbb{C}_{\mathbb{R}}, \mathbb{R}}(i) = \emptyset$.

[21]For instance, if $A_{\mathbb{K}} = \mathbb{R}_{\mathbb{Q}}$, then $[\mathbb{R} : \mathbb{Q}] = \infty$ and $\sigma_{\mathbb{R}_{\mathbb{Q}}, \mathbb{Q}}(e) = \emptyset$.

$\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \varnothing$, $a \in A_{\mathbb{K}}$, and thus (4) is proved, with $A_{\mathbb{K}} \overset{\psi}{\cong} \mathbb{K}$, where $\psi : a \to \lambda_a$, $a \in A_{\mathbb{K}}$.

For the proof of (3), assume it to be false.

Let $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) = \varnothing$ for, at least, one $a \in A_{\mathbb{K}}$, so that we have $\exists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$, $\forall \lambda \in \mathbb{K}$, whence it follows that, in particular, $a \in G(A_{\mathbb{K}})$. If $a = 1_{A_{\mathbb{K}}}$, then for $\lambda = 1_{\mathbb{K}}$, we arrive at an absurdity so that let $a \in G(A_{\mathbb{K}}) \backslash \{1_{A_{\mathbb{K}}}\}$; if $G(A_{\mathbb{K}}) = \{1_{A_{\mathbb{K}}}\}$, then we obtain again an absurdity, so we suppose $\varnothing \neq G(A_{\mathbb{K}}) \backslash \{1_{A_{\mathbb{K}}}\} \subset A_{\mathbb{K}}$.

Consider $A[[\lambda]]$, the linear unitary commutative integral $\mathbb{K}$-algebra of the formal power series in λ (considered, a priori, as an abstract theoretical variable), with coefficients in A(= support of $A_{\mathbb{K}}$). Since a is invertible, we have[22]

$$(a - \lambda 1_{A_{\mathbb{K}}})^{-1} = \sum_{n \in \mathbb{N}_0} a^{-(n+1)} \lambda^n, \quad (a^{-1} - \lambda 1_{A_{\mathbb{K}}})^{-1} = \sum_{n \in \mathbb{N}_0} a^{n+1} \lambda^n. \tag{7}$$

We endow $A[[\lambda]]$ with an adapted topology as follows. We recall that, if Ω is an abstract set (of indices), and A is a unitary commutative ring, then we denote the total algebra, generated by the monoid (of multi-indices) $\mathbb{N}_0^{\Omega}$ over A, by

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[22]In a power series ring $A[[T]]$, where A is an arbitrary unitary commutative ring, subsists the following *Neumann's expansion* $(1 - T)^{-1} = \sum_{n \in \mathbb{N}_0} T^n$. Moreover, we recall that, if we consider the norm

$$\nu : A[[\lambda]] \to \mathbb{N}_0, \quad \nu\left(\sum_{n \in \mathbb{N}_0} a_n \lambda^n\right) = \text{order of } \sum_{n \in \mathbb{N}_0} a_n \lambda^n,$$

then we may define the convergence

$$\sum_{n \in \mathbb{N}_0} a_n \lambda^n = \lim_{n \to \infty} \sum_{i=0}^{n} a_i \lambda^i \overset{def}{\Leftrightarrow} \nu\left(\sum_{n \in \mathbb{N}_0} a_n \lambda^n - \sum_{i=0}^{n} a_i \lambda^i\right) \geq n \tag{8}$$

definitively, from which it follows the result: if $\{\chi\}_{n \in \mathbb{N}_0}$ is a sequence of $A[[\lambda]]$, then $\sum_{n \in \mathbb{N}_0} \chi_n$ converges if and only if $\lim_{n \to \infty} \nu(\chi_n) = +\infty$, whence, by (8), also it follows that the general term of a convergent power series tends to zero (see [5]).

$A[[(X_i)_{i\in\Omega}]]$: it is the algebra of formal power series in the indeterminates X_i $(i \in \Omega)$ with coefficients in A, whose general element is of the type $u = \sum_{\nu\in\mathbb{N}_0^\Omega} a_\nu X^\nu$ with term $a_\nu X^\nu$ (X^ν in multi-index notation) of degree $|\nu| \geq 0$. By the well-known identification series-sequences, we have $A[[(X_i)_{i\in\Omega}]] \cong A^{\mathbb{N}_0^\Omega}$, where $A^{\mathbb{N}_0^\Omega}$ is the formal Cartesian product of many copies of A, that is, of the type

$$\prod_{i\in I} S_i \doteq \left\{\varphi;\ \varphi : I \to \bigcup_{i\in I} S_i,\ \varphi(i) \in S_i,\ \forall i \in I\right\},$$

where $I = \mathbb{N}_0^\Omega$ and $S_i = A$, $\forall i$. By the Axiom of Choice (Zermelo), $A^{\mathbb{N}_0^\Omega} \neq \emptyset$. Hence, for instance, it is possible to endow A with the discrete topology, and then $(A^{\mathbb{N}_0^\Omega} \cong) A[[(X_i)_{i\in\Omega}]]$ with the product topology that we will call the *canonical topology* (if $\operatorname{card}\Omega < \infty$, such a topology is discrete). In this topological space, it is possible to prove the following result (see [5, Chapter IV, Section 4, n. 2]).

X. *Let* Ξ *be an infinite set of indices and* $(u_j)_{j\in\Omega}$ *be a family of elements of* $A[[(X_i)_{i\in\Omega}]]$ *with* $u_j = \sum_\nu a_{j\nu} X^\nu$ *for each* $j \in \Xi$. *Then the following conditions are equivalent*:

1. *The family* $(u_j)_{j\in\Xi}$ *is summable in* $A[[(X_i)_{i\in\Omega}]]$.

2. $\lim_j u_j = 0$ *taken along the filter of complements of finite subsets of* Ξ (*cofinite filter*).

3. *For every* $\nu \in \mathbb{N}_0^\Omega$, $a_{j\nu} = 0$ *except for a finite number of indices* $j \in \Xi$.

When, *at least*, *one of these conditions holds*, *then the series* $u = \sum_{j\in\Xi} u_j$ *is equal to* $\sum_\nu a_\nu X^\nu$ *with* $a_\nu = \sum_{j\in\Xi} a_{j\nu}$ *for each* $\nu \in \mathbb{N}_0^\Omega$.

Remark. As an example of summable family, we consider the following: if $u \in A[[(X_i)_{i\in\Omega}]]$ with a_ν the coefficient of X^ν in u, then the family $(a_\nu X^\nu)_{\nu\in\mathbb{N}_0^\Omega}$ is summable with sum u (see, also, footnote [21]).

In our case, we consider $A[[\lambda]]$ as above with $\operatorname{card}\Omega = 1$ and $(X_i)_{i\in\Omega}$ reduced only to λ. By the convergence of the series of (7) for every $\lambda \in \mathbb{K}$, it follows that the general term of these, tends to 0 in the adapted topology of **X**, (2), whence also that

$$\lim_{n\to\infty} a^{-(n+1)}\lambda^n = \lim_{n\to\infty} a^{(n+1)}\lambda^n = 0 \tag{9}$$

in $\mathbb{N}_0$. On the other hand, $\operatorname{card}\mathbb{K} = \infty$ because $\mathbb{K}$ has characteristic zero, so that $\operatorname{card} A = \infty$, and therefore there is a natural continuous[23] bijection between the algebra of (abstract) polynomials of $A[\lambda](\subset A[[\lambda]])$, and the algebra of polynomial functions (of the type $f : \lambda \to f(\lambda) \in A, \forall\lambda \in A$), namely, $\mathcal{F}_{pol}(A, A)$. Thus, by (9), it follows that $a^{-(n+1)}\lambda^n \to 0$ and $a^{(n+1)}\lambda^n \to 0$ for every $\lambda \in A$, hence also for $\lambda = 1_A$, whence

$$\lim_{n\to\infty} a^{-(n+1)} = 0, \quad \lim_{n\to\infty} a^{(n+1)} = 0. \tag{10}$$

But, since λ is arbitrary for every $\lambda \neq 0$, we have

$$(a^{-1} - \lambda 1_A)^{-1} = -\lambda^{-1}a(a - \lambda^{-1}1_A)^{-1}, \quad \forall\lambda \in \mathbb{K}\setminus\{0\},$$

from which it follows that (10) must hold simultaneously, which is absurd since (in the integral domain A), we have $a^{-(n+1)}a^{(n+1)} = 1_A, \ \forall n \in \mathbb{N}_0$. Hence, (3) must be true.

We prove the second part of the theorem.

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[23]This continuity holds in the previous topologies of $A[[\lambda]]$ (where $A[\lambda]$ is dense), and in the discrete topology of the algebra of polynomial functions $\mathcal{F}_{pol}(A, A)(\subseteq \mathcal{F}(A, A))$ (induced by the discrete topology of $A^A \cong \mathcal{F}(A, A)$, where A^A must be understood as product set - see [9, Chapter II, Section 5, n. 2] - but endowed with the discrete topology). Furthermore, since we have excluded that A may be a division algebra not finitely generated as $\mathbb{K}$-algebra, the natural identification $A[\lambda] \cong \mathcal{F}_{pol}(A, A)$ holds true (for instance, this does not subsist if A is the (infinite) pure transcendental field $\mathbb{K}(X)$, since, because of the singularities of the rational functions, these do not be extended, as a functions, on the whole of A).

If I is an arbitrary maximal ideal of $A_{\mathbb{K}}$, then it follows that $\widetilde{A}_{\mathbb{K}} = A_{\mathbb{K}}/I$ is a field extension of $\mathbb{K}$. Suppose $\widetilde{A}_{\mathbb{K}} \neq \mathbb{K}$, and let $\bar{x} \in \widetilde{A}_{\mathbb{K}} \backslash \mathbb{K}$. Hence, $(\bar{x} - \lambda 1_{\widetilde{A}_{\mathbb{K}}})$ is invertible for every $\lambda \in \mathbb{K}$, that is, $\sigma_{\widetilde{A}_{\mathbb{K}}, \mathbb{K}}(\bar{x}) = \varnothing$, and then $\rho_{\widetilde{A}_{\mathbb{K}}, \mathbb{K}}(\bar{x}) = \mathbb{K}$. Therefore, since $n = [A_{\mathbb{K}} : \mathbb{K}] < \operatorname{card} \mathbb{K}$, if $\{x_1, \ldots, x_n\}$ is a system of generators for $A_{\mathbb{K}}$, it follows that $\{\bar{x}_i = x_i + I; i = 1, \ldots, n\}$ is a system of generators for $\widetilde{A}_{\mathbb{K}}$, so that $[\widetilde{A}_{\mathbb{K}} : \mathbb{K}] \leq n < \operatorname{card} \mathbb{K}$. Hence, $\{(\bar{x} - \lambda 1_{\widetilde{A}_{\mathbb{K}}})^{-1}; \lambda \in \mathbb{K}\}$ is a linearly dependent system because we have $\operatorname{card}\{(\bar{x} - \lambda 1_{\widetilde{A}_{\mathbb{K}}})^{-1}; \lambda \in \mathbb{K}\} = \operatorname{card} \rho_{\widetilde{A}_{\mathbb{K}}, \mathbb{K}}(\bar{x}) = \operatorname{card} \mathbb{K}$, whence

$$\sum_{i \in \Lambda} \mu_i (\bar{x} - \lambda_i 1_{\widetilde{A}_{\mathbb{K}}})^{-1} = 0_{\widetilde{A}_{\mathbb{K}}} \quad \text{for some } \mu_i \in \mathbb{K}, \; i \in \Lambda, \text{ not all zero,}$$

and for each finite set (of indices) Λ. Therefore, $\bar{x}$ is algebraic over $\mathbb{K}$, so that $\bar{x} \in \mathbb{K}$, which is impossible. Thus, $\widetilde{A}_{\mathbb{K}} \cong \mathbb{K}$, and we have the proof arguing as in that of **VIII**.

Note 3. If $\mathbb{K}$ is algebraically closed, then $1 < [A_{\mathbb{K}} : \mathbb{K}] < \infty$ cannot be true. In fact, since $A_{\mathbb{K}}$ is an integral domain and $1 < [A_{\mathbb{K}} : \mathbb{K}] < \infty$, $A_{\mathbb{K}}$ would be a field and, therefore, $A_{\mathbb{K}} \cong \mathbb{K}$ because $\mathbb{K}$ is algebraically closed. Hence $\dim_{\mathbb{K}} A_{\mathbb{K}} = 1$, as against $[A_{\mathbb{K}} : \mathbb{K}] > 1$.

Likewise, if $[A_{\mathbb{K}} : \mathbb{K}] = \infty$, then $\mathbb{K}$ is algebraically closed and $A_{\mathbb{K}}$ is a division algebra. Instead, if $\mathbb{K}$ is algebraically closed and $A_{\mathbb{K}}$ is a division algebra, then (see 4 of **IX**) $A_{\mathbb{K}} \cong \mathbb{K}$, and so, in this case, we have again $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \varnothing$, $a \in A_{\mathbb{K}}$.

Finally, we observe that the condition $[A_{\mathbb{K}} : \mathbb{K}] < \operatorname{card} \mathbb{K}$ in the last part of the theorem, is satisfied by every linear unitary commutative $\mathbb{K}$-algebra having a (finite or) infinite system of generators Θ whose cardinality has an order of infinity less than of $\mathbb{K}$ (we recall that $\operatorname{card} \mathbb{K} = \infty$, since $\mathbb{K}$ is algebraically closed): for instance, may be $\operatorname{card} \Theta = \aleph_0$ and $\operatorname{card} \mathbb{K} = 2^{\aleph_0}$.

Then, if we take into account the order of infinity of $\operatorname{card} \mathbb{K}$, it is possible to extend the first part of **IX**, on the basis of its second part.

Remark 7. Under the hypothesis and notations of **VIII**, by (4), it follows that $\psi \in A_{\mathbb{K}}^{*}\setminus\{0\}$ is a non-zero[24] multiplicative linear functional. We write this, more specifically, as ψ_I.

Let $A_{\mathbb{K}}$ be a linear unitary commutative integral $\mathbb{K}$-algebra that satisfies **IX**, (3). Let $\Delta(A_{\mathbb{K}})(\subseteq A_{\mathbb{K}}^{*}\setminus\{0\}),$ the set of characters of $A_{\mathbb{K}}$.

Each $\phi \in \Delta(A_{\mathbb{K}})$ is surjective and $\dim_{\mathbb{K}} \mathbb{K} = 1,$ so that $A_{\mathbb{K}}/Ker\,\phi \cong \mathbb{K},$ and then $Ker\,\phi$ is a maximal ideal of $A_{\mathbb{K}}$. Conversely, let M be an arbitrary maximal ideal of $A_{\mathbb{K}}$. Putting $\widetilde{A}_{\mathbb{K}} = A_{\mathbb{K}}/M,$ we obtain that $\widetilde{A}_{\mathbb{K}}$ is a field, therefore also (by the restrictions of the scalars) a linear unitary commutative $\mathbb{K}$-algebra. Using **VIII** (via (3)) of **IX**, we have $\sigma_{\widetilde{A}_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset,$ with $\operatorname{card}\sigma_{\widetilde{A}_{\mathbb{K}},\mathbb{K}}(a) = 1,$ for each $a \in \widetilde{A}_{\mathbb{K}}$. Analogously, we may define the monomorphism $\psi_M : \widetilde{A}_{\mathbb{K}} \to \mathbb{K}$ like ψ_I of (4), and, if $\pi : A_{\mathbb{K}} \to \widetilde{A}_{\mathbb{K}}$ is the quotient epimorphism, then $\phi_M = \psi_M \circ \pi : A_{\mathbb{K}} \to \mathbb{K}$ is a multiplicative linear functional with $Ker\,\phi_M = M$, so that $\psi_M \in \Delta(A_{\mathbb{K}}),$ since $Ker\,\phi_M = M$ is a proper ideal, whence $\phi_M \in \Delta(A_{\mathbb{K}})$. Furthermore, since $M = Ker\,\phi_M$ with ϕ_M surjective, it follows that ψ is also surjective, so that $\widetilde{A}_{\mathbb{K}} \cong \mathbb{K}$.

Therefore, from this last result, it follows that any $\mathbb{K}$-algebra satisfying **IX**, (3), is a spectral algebras.

Furthermore, if $\mathcal{M}(A_{\mathbb{K}})$ denotes the set[25] of all maximal ideals of such an $A_{\mathbb{K}}$ (often called *maximal spectrum* of $A_{\mathbb{K}}$), then it follows the existence of a bijective map between $\Delta(A_{\mathbb{K}})$ and $\mathcal{M}(A_{\mathbb{K}})$, namely, $\xi : \Delta(A_{\mathbb{K}}) \to \mathcal{M}(A_{\mathbb{K}}),$ given by $\phi \rightsquigarrow Ker\,\phi$.

Thus for such $\mathbb{K}$-algebras of **IX**, (3), it is possible to have the converse of the result in **VII** because, if $a \in A_{\mathbb{K}}$ and $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a),$ then $(a - \lambda 1_{A_{\mathbb{K}}})$ is not

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[24]Since ψ is injective.

[25]It is non-void because $A_{\mathbb{K}}$ is not a field, so that there exists, at least, one maximal ideal containing $\{0_{A_{\mathbb{K}}}\}$.

invertible, and therefore, it belongs to the proper[26] ideal $J_\lambda = \langle a - \lambda 1_{A_{\mathbb{K}}} \rangle$. Let $I_\lambda \in \mathcal{M}(A_{\mathbb{K}})$ be the maximal ideal containing J_λ; if $\phi_\lambda = \xi^{-1}(I_\lambda) \in \Delta(A_{\mathbb{K}})$, then $Ker\, \phi_\lambda = I_\lambda \supseteq J_\lambda$, so that $(a - \lambda 1_{A_{\mathbb{K}}}) \in Ker\, \phi_\lambda$, that is, $\phi_\lambda(a - \lambda 1_{A_{\mathbb{K}}}) = 0_{\mathbb{K}}$, whence $\phi_\lambda(a) = \lambda$, with ϕ_λ non-zero because $I_\lambda = Ker\, \phi_\lambda$ is a proper ideal. Now, we prove that such ϕ_λ is unique, because, if I'_λ is another maximal ideal containing J_λ, then $\phi'_\lambda = \xi^{-1}(I'_\lambda)$ may be, a priori, distinct by ϕ_λ, and then we will prove that $\phi_\lambda = \phi'_\lambda$. For this consider $A_{\mathbb{K}}/I_\lambda$ and we put $\bar{x} = x + I_\lambda$, $x \in A_{\mathbb{K}}$. If $x \notin I_\lambda$, then $Y = \langle x \rangle_{\mathbb{K}} + I_\lambda = \mathbb{K}x + I_\lambda$ is an ideal of $A_{\mathbb{K}}$ with $I_\lambda \subset Y$ (since $x \notin I_\lambda$), so that it must be $Y = A_{\mathbb{K}}$ by the maximality of I_λ, hence $1_{A_{\mathbb{K}}} \in Y$, that is, $1_{A_{\mathbb{K}}} = ax + y'$ for some $a \in \mathbb{K}\backslash\{0_{\mathbb{K}}\}$ (since $1_{A_{\mathbb{K}}} \notin I_\lambda$) and $y' \in I_\lambda$, whence $1_{A_{\mathbb{K}}/I_\lambda} = a\bar{x}$, and then $a^{-1} \in \sigma_{A_{\mathbb{K}}/I_\lambda, \mathbb{K}}(\bar{x})$; but $A_{\mathbb{K}}/I_\lambda$ is a field, so that $(\bar{x} - \mu 1_{A_{\mathbb{K}}/I_\lambda})^{-1}$ does not exist if and only if $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(0) \neq \emptyset$, that is, if and only if $x - \mu 1_{A_{\mathbb{K}}} \in I_\lambda$, whence $\bar{x} = \mu 1_{A_{\mathbb{K}}/I_\lambda}$. But also $\bar{x} = \lambda 1_{A_{\mathbb{K}}/I_\lambda} (= a^{-1} 1_{A_{\mathbb{K}}/I_\lambda})$ because $x - \lambda 1_{A_{\mathbb{K}}} \in J_\lambda$, so that $\mu = \lambda (= a^{-1})$, hence $x = \lambda 1_{A_{\mathbb{K}}} + y$ for some $y \in J_\lambda$. Thus for every $x \in A_{\mathbb{K}}$, we have $x = \lambda 1_{A_{\mathbb{K}}} + y$ for some $y \in J_\lambda$. Therefore, if $Ker\, \phi'_\lambda = I'_\lambda \supseteq J_\lambda$, then we have $J_\lambda \subseteq I_\lambda \cap I'_\lambda$ so that, by $x = \lambda 1_{A_{\mathbb{K}}} + y$, $y \in J_\lambda$,

$$\phi_\lambda(x) = \phi_\lambda(\lambda 1_{A_{\mathbb{K}}} + y) = \lambda 1_{A_{\mathbb{K}}} = \phi'_\lambda(\lambda 1_{A_{\mathbb{K}}} + y) = \phi'_\lambda(x), \quad \forall x \in A_{\mathbb{K}},$$

and hence $\phi_\lambda = \phi'_\lambda$.

Thus, from Remark 4, we conclude that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = A^*_{\mathbb{K}}(a)$, for each $a \in A_{\mathbb{K}}$, if $\Delta(A_{\mathbb{K}}) \neq \emptyset$.

Let $A_{\mathbb{K}}$ be a linear unitary commutative integral $\mathbb{K}$-algebra that satisfies **IX**,

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[26] J_λ is a proper ideal because it contains the non-invertible element $(a - \lambda 1_{A_{\mathbb{K}}})$: in fact, otherwise, it would be $1_{A_{\mathbb{K}}} \in J$, so that $1_{A_{\mathbb{K}}} = f(a - \lambda 1_{A_{\mathbb{K}}})$ for some $f \in A_{\mathbb{K}}$, against the non-invertibility of $(a - \lambda 1_{A_{\mathbb{K}}})$.

(3). If we set

$$\hat{a}(\phi) \doteq \phi(a), \quad \phi \in \Delta(A_{\mathbb{K}}) \text{ and } a \in A_{\mathbb{K}},$$

then we can define the following correspondence, called *Gelfand's transformation* (or the *canonical functional representation* of $A_{\mathbb{K}}$) :

$$\mathcal{G} : A_{\mathbb{K}} \to \mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K}), \quad a \rightsquigarrow \mathcal{G}(a) = \hat{a}, \ \forall a \in A_{\mathbb{K}},$$

where $\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})$ denotes the usual linear unitary commutative $\mathbb{K}$-algebra of maps from $\Delta(A_{\mathbb{K}})$ to $\mathbb{K}$; $\mathcal{G}(a) = \hat{a}$ is called the *Gelfand's transform* of a. It is, also, possible to consider $\mathcal{G}$ as the result of the following semi-duality pairing $(a, \phi) \to \langle a, \phi \rangle = \phi(a)$.

That $\mathcal{G}$ is a ring homomorphism follows from

$$(\mathcal{G}(\lambda a + \mu b))(\phi) = \phi(\lambda a + \mu b) = \lambda\phi(a) + \mu\phi(b)$$

$$= (\lambda \hat{a} + \mu \hat{b})(\phi) = (\lambda\mathcal{G}(a) + \mu\mathcal{G}(b))(\phi), \quad \phi \in \Delta(A_{\mathbb{K}})$$

and

$$(\mathcal{G}(ab))(\phi) = \phi(ab) = \phi(a)\phi(b) = (\mathcal{G}(a)(\phi))(\mathcal{G}(b)(\phi)), \quad \phi \in \Delta(A_{\mathbb{K}}).$$

Therefore, $\mathcal{G}(A_{\mathbb{K}})$ is a $\mathbb{K}$-subalgebra of $\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})$, whose elements separate the points of $\mathcal{M}(A_{\mathbb{K}})$: indeed, by $\mathcal{M}(A_{\mathbb{K}}) \cong \Delta(A_{\mathbb{K}})$, if $M_1, M_2 \in \mathcal{M}(A_{\mathbb{K}})$ with $M_i = \xi(\phi_i)$, $\phi_i \in \Delta(A_{\mathbb{K}})$, $i = 1, 2$, and $M_1 \neq M_2$, then $M_i = Ker\,\phi_i$, $i = 1, 2$ and there is an $a \in M_1 \setminus M_2$ such that $\hat{a}(\phi_1) = 0$ and $\hat{a}(\phi_2) \neq 0$, so that, if we put (with abuse of notation) $\hat{a}(\phi) = \hat{a}(M)$ and $M = \xi(\phi)$, we have, at least, $\hat{a}(M_1) = 0$ and $\hat{a}(M_2) \neq 0$.

Then, as a result of a partial converse of **VII**, we have

XI (Wiener-Gelfand). *Let* $A_{\mathbb{K}}$ *be a linear unitary commutative integral* $\mathbb{K}$*-algebra satisfying* **IX**, (3), *that is*, *not a division algebra*. *Then*

(1) $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) = A_{\mathbb{K}}^{*}(a) = \mathcal{G}(a)(\Delta(A_{\mathbb{K}})) = \operatorname{Im}\mathcal{G}(a)$ *for each* $a \in A_{\mathbb{K}}$.

(2) $a \in A_{\mathbb{K}} \setminus \{0\}$ *is invertible if and only if* $\hat{a}(\phi) \neq 0$ *for each* $\phi \in \Delta(A_{\mathbb{K}})$.

(3) $a \in A_{\mathbb{K}} \setminus \{0\}$ *is invertible if and only if* $\hat{a}$ *is invertible in* $\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})$ *with* $\hat{a}^{-1} = 1/\hat{a}$.

(4) $Ker\, \mathcal{G} = \bigcap_{M \in \mathcal{M}(A_{\mathbb{K}})} M = \operatorname{rad} A_{\mathbb{K}}$.

(5) **(Spectral mapping theorem)** $\sigma_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K}), \mathbb{K}}(\mathcal{G}(a)) = \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$ *for each* $a \in A_{\mathbb{K}}$.

Indeed, if $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, then there is a $\phi_\lambda \in \Delta(A_{\mathbb{K}})$ such that $Ker\, \phi_\lambda \supseteq \langle a - \lambda 1_{A_{\mathbb{K}}} \rangle$, whence $a - \lambda 1_{A_{\mathbb{K}}} \in Ker\, \phi_\lambda$, that is, $\hat{a}(\phi_\lambda) = \phi_\lambda(a) = \lambda \in \operatorname{Im} \mathcal{G}(a)$, and then $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \subseteq \operatorname{Im} \mathcal{G}(a)$. Therefore, (1) follows from **VII**.

Then $a \in A_{\mathbb{K}} \setminus \{0\}$ is invertible if and only if $0 \notin \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, that is, by (1), if and only if $0 \notin \operatorname{Im} \mathcal{G}(a)$, hence if and only if $\hat{a}(\phi) = \phi(a) \neq 0$ for every $\phi \in \Delta(A_{\mathbb{K}})$. Therefore, (2) is proved. Furthermore, (3) follows immediately from (2).

As regard (4), because of the bijective correspondence $\xi : \phi \to Ker\, \phi$, between $\Delta(A_{\mathbb{K}})$ and $\mathcal{M}(A_{\mathbb{K}})$, we have $a \in Ker\, \mathcal{G}$ if and only if $\hat{a} \equiv 0$, that is, if and only if $\hat{a}(\phi) = 0$, $\phi \in \Delta(A_{\mathbb{K}})$, hence if and only if $a \in Ker\, \phi$, $\phi \in \Delta(A_{\mathbb{K}})$, therefore, if and only if $a \in \bigcap_{\phi \in \Delta(A_{\mathbb{K}})} Ker\, \phi \stackrel{\xi}{=} \bigcap_{M \in \mathcal{M}(A_{\mathbb{K}})} M = \operatorname{rad} A_{\mathbb{K}}$ (the latter being the so-called *Jacobson's radical of* $A_{\mathbb{K}}$).

Finally, we establish (5) for each $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}}, \mathbb{K}}(a) \neq \emptyset$, since, otherwise, the assertion is plainly true. If $\lambda \in \sigma_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K}), \mathbb{K}}(\mathcal{G}(a))$, then $\nexists (\mathcal{G}(a) - \lambda 1_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})})^{-1}$, hence it is zero, at least, at one point of its domain, that is, there exists $\phi \in \Delta(A_{\mathbb{K}})$ such that $(\mathcal{G}(a) - \lambda 1_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})})(\phi) = \phi(a - \lambda 1_{A_{\mathbb{K}}}) = 0$, whence $\phi(a) = \lambda$, and then (by **VII**) $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$. Conversely, let $\lambda \in \sigma_{A_{\mathbb{K}}, \mathbb{K}}(a)$, and therefore, there exists an $M \in \mathcal{M}(A_{\mathbb{K}}) \cong \Delta(A_{\mathbb{K}})$ such that $(a - \lambda 1_{A_{\mathbb{K}}})^{-1} \in M$, that is, $(a - \lambda 1_{A_{\mathbb{K}}}) \in M = \xi^{-1}(\phi)$ for a unique $\phi \in \Delta(A_{\mathbb{K}})$, whence $\phi(a - \lambda 1_{A_{\mathbb{K}}}) = 0$, that is, $(\mathcal{G}(a) - \lambda 1_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})})(\phi) = 0$, and therefore $(\mathcal{G}(a) - \lambda 1_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})})$ is not invertible, and so $\lambda \in \sigma_{\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K}), \mathbb{K}}(\mathcal{G}(a))$.

If $\operatorname{rad} A_{\mathbb{K}} = \{0_{A_{\mathbb{K}}}\}$, we say that $A_{\mathbb{K}}$ is *semi-simple*, then $\mathcal{G}$ is injective if and only if $A_{\mathbb{K}}$ is semi-simple.

For semi-simple algebras, we have the isomorphism $A_{\mathbb{K}} \cong \mathcal{G}(A_{\mathbb{K}})$, with $\mathcal{G}(A_{\mathbb{K}})\,\mathbb{K}$-subalgebra of $\mathcal{F}(\Delta(A_{\mathbb{K}}), \mathbb{K})$.

Note 4. XI may be used as useful criterion for non-voidness, or not, of the spectrum of a given element, by means of the analysis of $\Delta(A_{\mathbb{K}})$, only when it is non-void. For instance, if $A_{\mathbb{K}}$ is a pure transcendental field extension of an arbitrary field of characteristic zero, say $\mathbb{K}$, then we have $A_{\mathbb{K}} = \mathbb{K}(X)$ and thus $\Delta(A_{\mathbb{K}}) = \varnothing$: indeed, if $\phi \in \Delta(A_{\mathbb{K}})$, $\phi : \mathbb{K}(X) \to \mathbb{K}$ would be a non-zero surjective multiplicative linear functional. Since $\mathbb{K}(X)$ is a field (hence with $\mathcal{M}(A_{\mathbb{K}}) = \varnothing$), ϕ must be injective, that is, ϕ would be an automorphism between $\mathbb{K}(X)$ and $\mathbb{K}$, which is impossible.

On the other hand, $\mathbb{K}(X)$ does not satisfy **IX**, (3), since $\mathbb{K}(X)$ is not finitely generated as $\mathbb{K}$-algebra, so that, for this **XI**, (1) does not hold. Nevertheless, **VII** is not in contrast with that, if $A_{\mathbb{K}} = \mathbb{K}(X)$, because, in this case, it is (see Remark 4) $A_{\mathbb{K}}^{*}(a) = \varnothing$, $a \in A_{\mathbb{K}}$ for $\Delta(\mathbb{K}(X)) = \varnothing$, but $\sigma_{\mathbb{K}(X),\mathbb{K}}(a) = \{a\}$, $\forall a \in \mathbb{K}$, and $\sigma_{\mathbb{K}(X),\mathbb{K}}(a) = \varnothing$, $a \in \mathbb{K}(X) \backslash \mathbb{K}$.

Due to the last results, if we define

$$\Omega(A_{\mathbb{K}}) \doteq \bigcup_{a \in A_{\mathbb{K}}} \sigma_{A_{\mathbb{K}},\mathbb{K}}(a) (\subseteq \mathbb{K})$$

as the *Gelfand's spectrum* of $A_{\mathbb{K}}$, then we may consider the bijective map

$$\varphi : \Omega(A_{\mathbb{K}}) \to \Delta(A_{\mathbb{K}}), \quad \lambda \rightsquigarrow \phi_{\lambda}, \quad \forall \lambda \in \Omega(A_{\mathbb{K}}),$$

making it possible to consider $\Delta(A_{\mathbb{K}}) (\overset{\varphi}{\cong} \Omega(A_{\mathbb{K}}))$ also as the Gelfand's spectrum of $A_{\mathbb{K}}$; but $\Delta(A_{\mathbb{K}}) \overset{\xi}{\cong} \mathcal{M}(A_{\mathbb{K}})$, so that $\mathcal{M}(A_{\mathbb{K}}) \cong \Omega(A_{\mathbb{K}})$.

XII. *Let* $A_{\mathbb{K}}$ *be a linear unitary commutative integral* $\mathbb{K}$*-algebra that fulfills*

IX, (3). *If* $a_i \in A_{\mathbb{K}}$, $i = 1, ..., n$, *then either there exists* $\phi \in \Delta(A_{\mathbb{K}})$ *such that* $\hat{a}_i(\phi) = 0$, $i = 1, ..., n$, *or there exists* $b_i \in A_{\mathbb{K}}$, $i = 1, ..., n$ *such that* $\sum_{i=1}^{n} b_i a_i = 1_{A_{\mathbb{K}}}$.

Indeed, if $\langle a_1, ..., a_n \rangle$ is a proper ideal of $A_{\mathbb{K}}$, then there exists, at least, one maximal ideal containing it, say M, so that let $\phi = \xi^{-1}(M) \in \Delta(A_{\mathbb{K}})$ with $M = Ker\,\phi$, whence $a_i \in M$, $\forall i$ implies $\phi(a_i) = \hat{a}_i(\phi) = 0$. Instead, if $\langle a_1, ..., a_n \rangle$ is a not proper ideal, then $\langle a_1, ..., a_n \rangle = A_{\mathbb{K}}$, so that $1_{A_{\mathbb{K}}} \in \langle a_1, ..., a_n \rangle$, from which it follows the second statement.

Remark 8. On the basis of the last results of the previous Remark 7, we introduce an application connected with elementary algebraic geometry.

Let $\mathbb{K}$ be algebraically closed. Let $V = V(J)$ be an affine variety, defined by the ideal $J \subseteq \mathbb{K}[X_1, ..., X_n]$, not reduced to only one point. Then, if $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]/I(V) = \Gamma(V)$ is the *coordinate ring* of V, with $J \subseteq I(V(J))$, then $\Gamma(V)$ is not a field[27], so that $\mathcal{M}(A_{\mathbb{K}}) \neq \emptyset$. Furthermore, $\Gamma(V)$ is a linear unitary commutative integral $\mathbb{K}$-algebra, with $[\Gamma(V) : \mathbb{K}] = \infty$, that it is not a division algebra, and so $A_{\mathbb{K}} = \Gamma(V)$ satisfies **IX**, (3) (see also Note 3).

We prove that $V(J) \overset{\iota}{\cong} \Delta(\Gamma(V))$. Indeed, if $P = (a_1, ..., a_n) \in V(J)$, then we have the (polynomial) evaluation map $\zeta_P : f \to f(P)$, $f \in \mathbb{K}[X_1, ..., X_n]$, hence there exists a unique monomorphism $\psi_P : \Gamma(V) \to \mathbb{K}$ such that $\zeta_P = \psi_p \circ \pi$, where $\pi : \mathbb{K}[X_1, ..., X_n] \to \Gamma(V)$ is the quotient epimorphism, so that we may associate to P the monomorphism $\psi_P \in \Delta(\Gamma(V))$, namely, let $\iota : P \to \psi_P$.

Conversely, if $\psi \in \Delta(\Gamma(V))$, then $\vartheta = \psi \circ \pi : \mathbb{K}[X_1, ..., X_n] \to \mathbb{K}$ is a homomorphism. Let $a_i = \vartheta(X_i)$, $i = 1, ..., n$, then $P = (a_1, ..., a_n)$. Therefore,

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[27]Indeed, it is known the equivalence of the following conditions: (1) $V(J)$ is a point; (2) $\Gamma(V) \cong \mathbb{K}$; (3) $[\Gamma(V) : \mathbb{K}] < \infty$. See [1, Problem 2-4].

$\zeta_P(f) = f(a_1, ..., a_n) = f(\vartheta(X_1), ..., \vartheta(X_n)) = \vartheta(f),$[28] so that $\vartheta = \zeta_P$, whence $\psi_P \circ \pi = \zeta_P = \vartheta = \psi \circ \pi$ for which, by the uniqueness of ψ_P, it follows that $\psi = \psi_P = \iota(P)$. Furthermore, for each $f \in I(V)$, we have $f(P) = f(a_1, ..., a_n)$ $= \vartheta(f) = (\psi \circ \pi)(f) = \psi(\pi(f)) = 0$ because $f \in I(V)$ and therefore, $\pi(f) = 0$, so that $P \in V(J)$. Hence, ι is bijective[29], and since it is immediate to verify that it is also a homomorphism, we conclude that $V(J) \overset{\iota}{\cong} \Delta(\Gamma(V))(\subseteq (\Gamma(V))^* \setminus \{0\})$.

On the other hand, as regards to what has been said above, we have $\Delta(\Gamma(V)) \overset{\xi}{\cong} \mathcal{M}(\Gamma(V))$, whence $V(J) \cong \mathcal{M}(\Gamma(V))$, achieving results similar to those described in [6, Chapter I, Exercises[30] 26, 27]. We prove this property, also taking into account the notation of [6, Exercises 26, 27].

We recall that $\Delta(A_{\mathbb{K}}) \overset{\xi}{\cong} \mathcal{M}(A_{\mathbb{K}})$ with $\xi : \phi \to Ker\,\phi$, so that we have simply $\mu = \xi \circ \iota : P \to Ker\,\phi_P = \mathfrak{m}_P$ for every $P \in V(J)$. Therefore, μ is bijective, as required.

Remark 9. Historically, the origin of the notion of spectrum of an element of an algebra may be recognized in a proof given by Weyl in [10, Chap. V, Part A, Section 7, p. 316 (of the Dover edition)], where he proves that the only division algebra of finite order, over an algebraically closed field, is this field itself.

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[28]Indeed, if $f(X_1, ..., X_n) = \sum_{\substack{i_j \geq 0 \\ j=1,...,n}} a_{i_1 \cdots i_n} \prod_{j=1}^{n} X_j^{i_j}$, then, since ϑ is a ring homomorphism, we have

$$f(\vartheta(X_1), ..., \vartheta(X_n)) = \sum_{\substack{i_j \geq 0 \\ j=1,...,n}} a_{i_1 \cdots i_n} \prod_{j=1}^{n} (\vartheta(X_j))^{i_j} = \vartheta\left(\sum_{\substack{i_j \geq 0 \\ j=1,...,n}} a_{i_1 \cdots i_n} \prod_{j=1}^{n} X_j^{i_j} \right) = \vartheta(f).$$

[29]Among other things, the bijectivity of ι implies that each character of $\Gamma(V)$ is a (polynomial) evaluation map.

[30]These exercises seem to be an initial link-point between Algebraic Geometry and Operator Algebras Theory.

On the other hand, the so-called *Gelfand-Mazur theorem* says that every complex Banach algebra, that is a division algebra, is (isometrically) isomorphic to $\mathbb{C}$. This theorem is truly one of the most fundamental theorems in the study of commutative Banach algebras. It was announced by Mazur in 1938 (see [11]) and proved by Gelfand in 1941 (see [12]). For various other proofs of this theorem, see [13].

Remark 10. By an evasive suggestion of Danilov (see [4, part II, Chapter 1, Section 1, Subsection 1.5]), about a presumed analogy (of enunciations) between the Weak Nullstellensatz and the Gelfand-Mazur theorem of the Theory of Operator Algebras (as, for example, exposed by [3, Chapter 1, Section 4]; see, also, [8]), we have outlined above results, taking into account few, but meaningful, problems proposed by Fulton [1], and two exercises of [6], together with some subjects drawn from the theory of abstract Banach algebras that could be extended in a purely algebraic setting.

On the other hand, in another context, Almira in [2] (and [7]), has given a proof of the Weak Nullstellensatz through both the Fundamental Theorem of Algebra (via Gröbner's division algorithm), and the Gelfand-Mazur theorem, but only in the case $\mathbb{K} = \mathbb{C}$, following a procedure completely different from that taken here.

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