Science & Education
https://doi.org/10.1007/s11191-024-00547-7

ARTICLE

Exploring a New Geometric-mechanical Artefact for Calculus

Michela Maschietto[1] · Pietro Milici[2]

Accepted: 1 July 2024

Abstract

We introduce a geometric-mechanical artefact designed for laboratory activities related to Calculus topics (3D models and construction instructions are freely available online). With new capabilities and a new design, this instrument adopts some mechanisms historically introduced to solve inverse tangent problems (that analytically correspond to solving differential equations). By such an instrument, besides materially revealing the tangent to a curve (tangent mode), it is possible to trace the graph of exponential functions and parabolas starting from the geometrical properties of their tangent (curvigraph mode). Furthermore, one can perform transformations as derivatives and integrals (transformation mode). Our research project aims to study the use of this artefact mainly for secondary school students. In this paper, we present the analysis of its semiotic potential, referring to the instrumental approach and the Theory of Semiotic Mediation. We also focus on a secondary school teacher manipulating the artefact to identify exploration processes and gestures of usage. The analysis supports the choice of starting the exploration in the tangent mode and suggests that the artefact fosters the emergence of the idea of the tangent line.

1 Introduction

The history of mathematics offers several examples of the intertwined relationships between artefacts and the development of mathematical ideas over the centuries. On the other hand, research in mathematics education is widely interested in the educational use of material and digital tools for mathematics teaching and learning (Monaghan et al., 2016). This interest has increased in the last decade, supported by the emergence of theoretical frameworks for planning experimentation and analysing results and processes. Even if digital tools are more studied because they offer new possibilities for approaching and representing mathematical concepts (Calder et al., 2018), some research focuses on the use

✉ Pietro Milici
pietro.milici@unipa.it

Michela Maschietto
michela.maschietto@unimore.it

1 Department of Education and Human Sciences, University of Modena E Reggio Emilia, viale Timavo, 93, 42121 Reggio Emilia, Italy

2 Department of Mathematics and Computer Science, University of Palermo, via Archirafi, 34, 90123 Palermo, Italy

Published online: 19 July 2024

of both digital and material tools (for instance, Maschietto & Soury-Lavergne, 2013). Nevertheless, the interest in material artefacts, such as mathematical machines (Bartolini Bussi & Maschietto, 2006), remains in mathematics education.

In this paper, we introduce a material artefact issued from the history of mathematics that can be constructed by digital fabrication tools (e.g., collaborating with a FabLab) following the instructions and the 3D models freely available at https://www.thingiverse.com/thing:6619300. Even though such a tool is new (patented, designed, and constructed by the second author), it embodies the historical ideas related to the mechanical implementation of the solution of inverse tangent problems. Here, as a first provisional name, we refer to it as the "Multipurpose Tangent Solver," shortened in MTS. Our research project concerns the study of this artefact for laboratory activities concerning Calculus, above all with secondary school students. As the first step of this project, we propose the MTS to a secondary school teacher expert in the educational use of material tools for geometry (i.e., the mathematical machines preserved at the University of Modena e Reggio Emilia; Bartolini Bussi & Maschietto, 2006); we are interested in analysing how this teacher explores the new artefact. In this sense, this paper can be considered a case study (Stake, 1995).

After presenting the historical setting and description of the new artefact, this paper introduces the theoretical background framing the use of tools in mathematics education. Finally, it carries out the analysis of the exploration by a secondary school teacher. Even though such a case is very specific, the results obtained shed some light on the expected behaviour and represent the starting point for more general activities.

2 Historical Setting

Regarding the conception of Calculus, we are obliged to think of the physical approach on the basis of Newton's method of *fluxions* and *fluents*. Though Newton considered the foundational role of constructions by machines for algebraic curves (cf. Blåsjö, 2023), we have no clue that he thought of mechanical tools in relation to Calculus, probably because his main analytical tools (infinite series) have no direct counterpart in finite mechanics. For such a reason, we follow Leibniz's more geometrical approach. From this perspective, the geometrical resolution of differential equations by mechanical tools constitutes an exciting chapter in the history and philosophy of science and technology. Indeed, both theoretical and actual machines (cf. Figure 1) formed a relevant example deeply involving the mathematical development, the design, and the practical construction of instruments in a common perspective. The starting geometrical problem is to construct a curve given the properties of its tangent, the so-called "inverse tangent problem" (as general references, see Bos, 1988; Tournès, 2009 or also, for a brief introduction, Crippa & Milici, 2019).

2.1 Background

Since the seventeenth century, the development of algebra as an analytic tool for geometric problems posed a foundational problem for mathematicians: How could such solutions and constructions be accepted in the dominant geometric paradigm of the time? There was a need for a general class of curves beyond circles and segments, and previous geometers introduced curves as traces of ideal planar machines. The conditions such machines had to satisfy to be considered "geometrical" constitute the "exactness issue." As well exposed in Bos (2001), such a quest got the first widely accepted answer by *La Géométrie* of Descartes

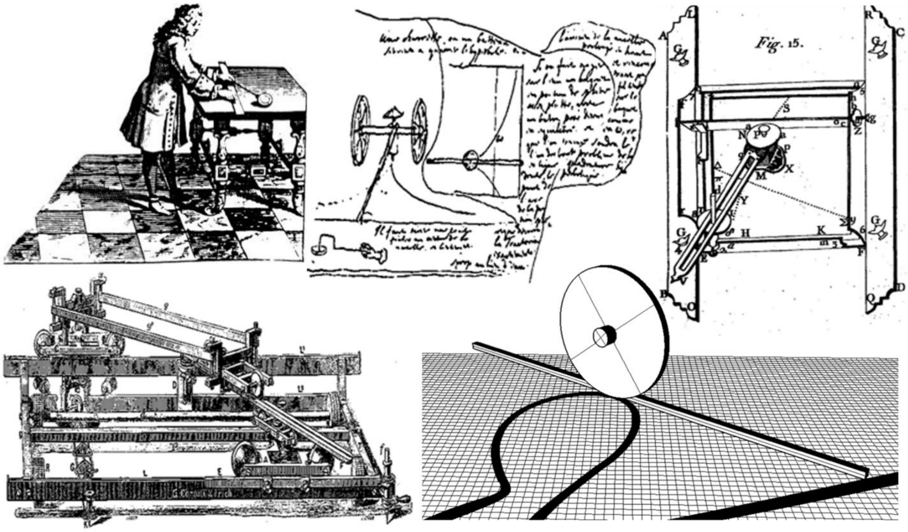

Fig. 1 Various mechanisms solving inverse tangent problems

(1637). Specifically, Descartes proposed a class of machines tracing algebraic curves as a synthetic part of his method and polynomial algebra as an analytical tool to simplify the problems. That determined a dualism between "geometric" (in modern terms: algebraic) and "mechanic" (transcendental) curves, being only the first ones considered acceptable in a Cartesian setting. Soon after the spreading of Descartes' canon, mathematicians got habituated to polynomials, which became no longer a crucial step for problem-solving but directly the solution.

In that period, there were only a few transcendental curves (e.g., the Archimedean spiral or the quadratrix), but a new general method to generate curves beyond Cartesian limits was about to emerge. A problem proposed in Paris by the architect Claude Perrault in the 1670 s provided the first insight (cf. the top left of Fig. 1, published in Poleni, 1729). The problem was easily described: if we move the end of a chain watch along a line slowly enough to avoid inertia, what curve does the watch describe? Cartesian tools appeared powerless to answer.

2.2 Foundation

In Perrault's setting, the traction is crucial. That provided the name "tractional motion," and the traced curve was called "tractrix." Nevertheless, the passage from problems to constructions is not trivial: using a chain watch moving on a table is hard to consider ideal enough as a mathematical construction. The first step was to refine the problem to become a "pure geometrical problem," using the words of Huygens (1693). Indeed, the key is that the direction of the taut chain is always tangent to the tractrix. Such a tangent problem radically differs from the classical ones. At least since the Classical Greek period, tangent problems involved finding the tangents to given curves satisfying specific properties. The problem is inverted now: we know the tangent properties and seek the curve with these tangent lines. That is the rise of "inverse tangent problems."

However, how can inverse tangent problems be concretely implemented? The reference to a heavy load having friction on a plane (as the watch) involved physical issues far from the domain of pure geometry, e.g. the unwanted role of velocity (because of the inertia that the load acquires) or the non-perfect horizontality of the plane. In 1693, Huygens provided an instrumental legitimacy to the tractrix: the curve is geometrical because it guides the direction/tangent of the curve and can be traced as precise and straightforward as circles from compasses. In doing that, Huygens designed instruments for the tractrix using a rigid rod instead of a string to work both in traction and compression. Some of these sketches are visible in the top centre of Fig. 1 (Bos, 1988). In the same year of Huygens' publication, other relevant players in the early development of Calculus (Jean and Jacques Bernoulli, L'Hôpital) paid attention to tractional motion. Among them, Leibniz (1693) proposed a general canon of geometric constructions beyond Cartesian limits: mathematicians do not have to restrict geometry to algebraic curves. Still, they should include any curve traced by a continuous and uniquely defined motion (Crippa, 2020). Suitable ideal machines constituted the foundational insights that brought Leibniz to justify his *Infinitesimal Calculus* geometrically (Blåsjö, 2017).

To sum up, even though the habituation of mathematicians to consider polynomials as solutions made obsolete the interest in machines for algebraic curves, machines regained importance at the end of the seventeenth century, this time to justify the construction of transcendental curves.

2.3 Expositions and Teaching

The first generation of mathematicians working on tractional motion focused on the ideal role machines could assume to justify the construction/introduction of certain transcendental curves. The vision in the first half of the eighteenth century rapidly evolved: curves like the tractrix or the logarithmic were well-known and generally accepted. Thus, the interest was no longer foundational but in real machines tracing analytic curves.

Giovanni Poleni was an emblem of this period of technological flourishing. Mathematician, physician, astronomer, and interested in engineering studies, Poleni (1729) proposed the machines for the logarithmic curve (cf. top right of Fig. 1) and for the tractrix, which were also built and used for exhibitions and teaching (for the reconstruction of these machines, see Milici & Plantevin, 2022). In 1739, Poleni inaugurated a laboratory of experimental physics (*Teatro di filosofia sperimentale*) at the University of Padua, which was unanimously praised by his contemporaries. He took the opportunity to set up one of the first university courses based on laboratory experiments (previously, he had students come to his home in his personal scientific cabinet). For this reason, Poleni paid attention to the design of his tractional machines: they were no longer only theoretical machines but also practical instruments guaranteeing adequate usability and accuracy. Poleni had a disciple, Giambatista Suardi, who continued his work as a designer of mathematical instruments. Among other ideas, Suardi (1752) simplified and technically perfected the tractional instruments for the tractrix and the logarithmic curve.

In these machines, a "wheel" played the role of driving the direction of the curve, like the blade of a pizza cutter or the front wheel of a bike. As visible in the bottom right of Fig. 1 (image from Milici, 2015), considering a wheel rolling on a curve, the direction of the rolling wheel is tangent to the curve (in the image, the direction is represented by a bar). The passage from the dragged weight of late 17th construction to the wheel is not a purely technical improvement. Indeed, dragged weight and wheel implement different mathematical contents that coincide only in simple cases (Dawson et al., 2021). Note that the first introduction of the wheel in inverse-tangent constructions precedes Poleni. Indeed, quite far from the academic environment, an English schoolteacher invented the machines for the hyperbola quadrature and the tractrix (Perks, 1706; Perks, 1714–1716).

2.4 Theoretical Achievements

Poleni's, 1729 publication made a strong impression on the geometers of his time. After reading it, Euler (1736) imagined tractional motion constructions to solve differential equations that he could not integrate in finite form by algebraic methods. Among other results, Euler succeeded in integrating the famous Riccati equation using tractional motion, which had remained unsolved for a long time. From there, an epistolary dialogue continued between Euler and Poleni for more than 4 years, revealing interesting interactions between theoretical and practical concerns.

Following on from Euler's work, a son of Jacopo Riccati (the mathematician of the Riccati equation) demonstrated that it is possible to exactly integrate any differential equation $y'=f(x,y)$ using tractional motion (even though he did not explicitly specify anything about the set of admissible functions f). From a theoretical point of view, Riccati (1752) was the culmination of the ancient problem-solving tradition through the construction of curves. Descartes had shown how to construct algebraic curves by a simple continuous motion using linkages. Riccati, for his part, established that many transcendental curves can be generated by a simple continuous motion from the differential equations that define them (cf. Tournès, 2009).

2.5 Practical Instruments

After the middle of the eighteenth century, the theory and practice of tractional motion rapidly fell into oblivion. The geometric paradigm was no longer dominant, and new mathematicians were interested in exploring mainly the analytical counterpart of infinitesimal analysis. However, in the late nineteenth and early twentieth century, a new generation of mathematicians and engineers re-invented essentially Poleni's solutions, increasing only the complexity of these instruments. Such instruments, called *integraphs*, were used neither to give a foundation nor to explain certain concepts. They were adopted to solve differential equations formally untreatable and kept their market until the widespread adoption of digital computation in the second half of the twentieth century. Thus, precision and efficacy were given much more care. An *integraph* is visible in the bottom left of Fig. 1 (Abdank-Abakanowicz, 1886). In the first half of the twentieth century, larger machines were built by connecting several integraphs to solve differential systems or higher-order differential equations.

A vast collection of integraphs is described in Pascal (1914). Ernesto Pascal created a mathematics seminar at the University of Naples and annexed a laboratory to each mathematics chair. These structures, probably inspired by what he had seen in Germany during his stay in Göttingen, significantly contributed to boosting research in his university and creating links between pure and applied mathematics. In addition, in terms of teaching, these laboratories allowed professors to complete their theoretical courses by having students work on problems using mathematical instruments and models. Similarly to Poleni, Pascal designed and realised various integraphs, collected them in his laboratory (*Gabinetto della cattedra di analisi infinitesimale*), and used them to teach Calculus.

2.6 Analogue Computation

Tractional motion has also recently gained interest from the perspective of theoretical analogue computation. Unlike digital computation, analogue computation does not deal with symbolic manipulation but with changing continuous quantities. From the computational perspective, we can consider Descartes' *Géométrie* (Section 2.1) as the proposal of a "balance" between geometric constructions (synthesis) and symbolic manipulation (analysis) with the introduction of suitable ideal machines. In modern terms, that is a balance between analogue and symbolic computation. The synthetic part of Descartes' program was extended by tractional motion, even though the class of the acceptable machines was not historically defined; the analytic part was soon extended by introducing non-finitary methods (infinite sums and infinitesimal elements). As proposed by Milici (2015), an interesting question is if it is possible to extend Descartes' balance in a finitist way.

The first step in this direction was to define a broad class of ideal machines beyond Cartesian ones. The machines of Descartes may be considered linkages, which Kempe (1876) showed to be algebraically complete in his *Universality Theorem*: for any finite part of an arbitrary algebraic plane curve, we can ideally construct a linkage drawing it. To overcome Cartesian limits and exhaustively define machines for tractional motion, Milici (2012) proposed the "tractional motion machines" as an extension of linkages with a tool named "wheel" able to drive the direction of a point.

Some computational, historical, and philosophical answers about tractional machines appeared in Milici (2020). Indeed, unlike infinitesimal analysis, an analysis without infinitary objects has been extended beyond the limits of algebraic polynomials by early twentieth-century *differential algebra*. Differential algebra (Ritt, 1950) today is a branch of computer algebra in which differential algebraic structures (such as rings and fields) are equipped with the operation of derivation. Informally, the indeterminates can be considered not numbers but continuous (and differentiable) functions. With these extensions, it is possible to define a new convergence of tractional motion machines (analogue computation), differential algebra (symbolic manipulations) and a well-determined class of mathematical objects (*differentially algebraic functions*, i.e. solutions of polynomial equations in the function and its derivatives) that gives scope for a constructive foundation of (a part of) Infinitesimal Calculus without the conceptual need of infinity. A *differential universality theorem* provides this balance and the subsequent definition of the constructive limits of tractional motion (Milici, 2020).

3 The New Artefact

In this work, we introduce the MTS (Multipurpose Tangent Solver), a new artefact that collects the legacy of the instruments for the inverse tangent problem. It integrates the eighteenth-century "curve-tracing" perspective with the subsequent "transformation" perspective. We realised the MTS with typical FabLab tools (laser cutting, 3D printing, CNC milling; see https://www.thingiverse.com/thing:6619300 also for images and videos). According to the numbering on the left of Fig. 2, it is made up of the following components:

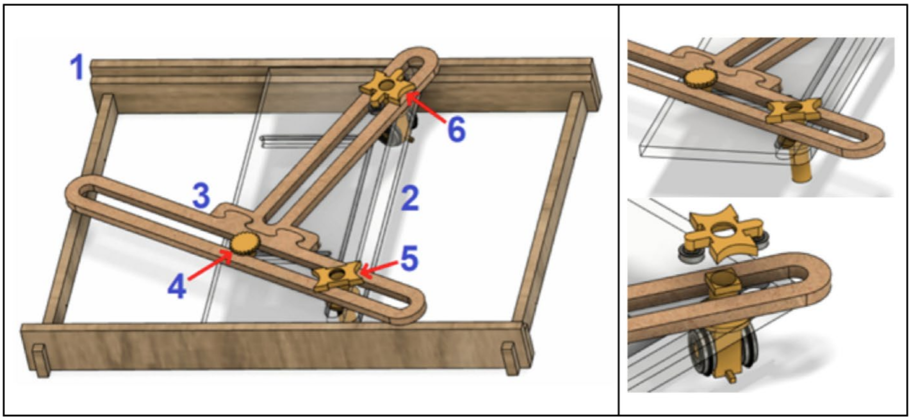

Fig. 2 The MTS: its components and some details

(1) *The frame*. It allows the plate [2] to slide along its long sides. Under it, a paper sheet is posed to leave traces.
(2) *The plate*. It is a rectangular piece of transparent plexiglass with three guides carved out, the two little ones perpendicular to the big one (a little guide, not well visible in the figure, stands below the peg [4]).
(3) *Two rods*. They play as linear guides to be put on the plate (pieces [4], [5], and [6] can slide inside). The rods can be joined to form a T in which a rod is the perpendicular bisector of the other; to shorten, we call the first "height rod" and the second "base rod" (as if they were the typical representation of the base and height of an isosceles triangle). Using the base or height is irrelevant when using a single rod.
(4) *The peg*. After fixing it in a point of the two little guides of the plate [2], it slides inside a rod [3].
(5) *The positional pointer*. It is a pen holder that can slide inside a rod [3] and the big guide of the plate [2] (see also Fig. 2, top right).
(6) *The directional pointer*. It is a pointer with, on its bottom, two parallel wheels (they can rotate at different speeds when touching the paper sheet). As the positional pointer, it can slide in a rod [3] and the big guide of the plate [2]. It has a cap to constrain the direction of the rod through which it passes to be parallel or perpendicular to the direction of the wheels (see also Fig. 2, bottom right).

Pointers have a hole that can be used as a viewfinder (to move the pointer along a curve) and to put a marker (to make the pointer leave a trace). Note that the square head of the directional pointer allows the top cap to be right-angle rotated. In the positional pointer, however, the orientation of the top cap is irrelevant.

In the next section, we outline the primary references for analysing this artefact from an educational perspective.

4 Theoretical Background

In this section, we outline the main theoretical references for our study concerning the use of artefacts in mathematics education and the meaning of tangent lines in approaching Calculus.

4.1 Artefacts in Mathematics Education

The study of this new artefact is carried out within a semiotic perspective, sharing the assumption that "sign and representations play an essential role in mathematics" (Hoffmann, 2006, p. 279) and "Mathematical cognition is mediated by representations. Mathematical activity is performed by means of visible signs, and by interpreting and transforming signs we develop mathematical knowledge" (Hoffmann, 2006, p.279). According to research carried out to frame the work with artefacts within an educational perspective, we mainly adopt two components for our theoretical background: the instrumental approach (Rabardel, 1995/2002) and the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). We present these two references in this section.

The instrumental approach is rooted in research on the use of tools in human activity, and it is widely used in mathematics education research; it pays attention to the distinction between artefact and instrument. The former is a material or abstract object already produced by human activity; the latter is a mixed entity with components from the artefact and the user's utilisation schemes developed during activities. The process leading to the development of the instrument to perform specific tasks is called instrumental genesis. It consists of instrumentation (concerning the emergence and development of the utilisation schemes) and instrumentalisation (concerning the emergence and evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints). Utilisation schemes are of particular interest from a mathematics education perspective because they are strongly related to the construction of knowledge.

The Theory of Semiotic Mediation (TSM) has been elaborated on and applied to mathematics education by Bartolini Bussi and Mariotti (2008) from a Vygostkian perspective. The fundamental idea is that a teacher uses a specific artefact as a tool of semiotic mediation for constructing mathematical meanings. In this framework, a crucial notion is the semiotic potential of an artefact: its analysis is fundamental for the conception of tasks for students and is at the basis of the teacher's action with students. The semiotic potential of an artefact is defined as a double semiotic link: on the one hand, the link of the artefact established with the student's personal meanings and signs emerging when using it to accomplish a task (tasks must be suitable for students); on the other hand, the link of the artefact with the mathematical meanings evoked by its use and recognisable as mathematics by an expert. The analysis of the semiotic potential considers three components (Bartolini Bussi & Mariotti, 2008): historical references, mathematical content, and utilisation schemes. Following the instrumental approach, the last component concerns the analysis of the possible utilisation schemes to perform a task and the related meanings (personal and mathematical ones) emerging during the task resolution.

Activities for students are organised within didactical cycles, including group activities with the artefact, individual activities, and collective mathematical discussions. In research on mathematical machines, students' activities start with exploring the chosen artefact, which engages students in a rich semiotic activity. This exploration is often structured following four main questions (Bartolini Bussi et al., 2011) that can be proposed in worksheets depending on students' level and educational aims. These questions integrate some elements from the instrumental approach:

(1) *How is the machine made? (TSM-Q1)* This kind of question fosters students to describe the physical structure of the machine, detecting physical components and their spatial relations. It aims to support instrumental genesis and can be more detailed by asking what the machine's components are or what geometrical figures are represented by the machine's linkage, if any. The aim is to identify signs relevant to the subsequent questions and support the geometrical representation of the machine (for instance, a peg or a pointer can represent a point; a rod can represent a segment or straight line).
(2) *What does the machine make? (TSM-Q2)* Two processes must be activated to answer this question. The first process concerns how to use the machine to achieve a specific aim (e.g., to draw some figures); it is essential to move some pegs or rods of the machine and trace their trajectory. It supports the emergence of user utilisation schemes and gestures of usage. The second process concerns the production of conjectures on mathematics evoked by the artefact; this process is stimulated by the physical structure of the machine and by the analysis of the relationships between the drawn figures to detect invariant properties. In this case, the emergence of mathematical meanings is fostered: new signs are produced, and the relationships among them can be identified (Arzarello & Sabena, 2011).
(3) *Why does it make it? (TSM-Q3)* Students are expected to argue the conjectures produced through question TSM-Q2, up to constructing a proof (depending on students and class level). Arguments usually draw on the mathematical properties identified since the answers to question TSM-Q2.
(4) *What could happen if…? (TSM-Q4)* This final task fosters the solution of open-ended problems and connects problem-posing and problem-solving activities.

In this paper, we consider the analysis of the semiotic potential of the MTS. From our perspective, the four questions described above can provide an outline for analysing material artefacts for mathematics learning/teaching and their semiotic potential. Indeed, these questions are strictly related to the analysis of the mathematical content and instrumental genesis. Regarding the conception of tasks for students, essential in the Theory of Semiotic Mediation, the organisation of classroom activities within didactical cycles and the role of the teacher are not considered here because this paper does not deal with an experimental phase with students.

We are interested in paying attention to the components for the analysis of the semiotic potential of an artefact. In the Theory of Semiotic Mediation, the teacher starts his/her role as a cultural mediator in a discussion about mathematical meanings and signs produced by the subjects in interacting with an artefact while carrying out a specific task. There are also gestures among the signs emerging in the first phase of working with the artefact (mainly concerning TSM-Q1 and TSM-Q2). In the analysis of the educational use of an arithmetical artefact (a numerical machine called "pascaline Zero + 1"), Maschietto (2015) pays attention to gestures of usage emerging from the manipulation of the artefact, their relationships, and the evolution in solving strategies.

These gestures are linked to the subject's knowledge and the construction of his/her utilisation schemes. Still, they can also be precisely solicited by the features of the artefact and by an exploratory task seen as an open task that allows the subject to perform various actions. Based on gestures, Yoon et al. (2011) explore how people use gestures to support the construction and communication of their mathematical understanding. In particular, the authors investigate how a person's physical gesture space becomes endowed with mathematical meaning and how a subject can interpret gestures mathematically. Using gestures to communicate and create new gestures can support a new understanding of mathematical concepts (Arzarello & Edwards, 2005; Radford et al., 2005).

In this perspective, we consider it relevant to pay attention to the affordances of the artefact, i.e., the possibility for action which the environment offers to a subject according to his/her ability, but also to the constraints on action (Bril, 2015) in the thread of Gibson's construct of affordances and constraints to mediated action (Gibson, 1977). At this step of our study of a new machine, the focus on affordances could help identify what artefact exploration might foster. At the same time, it becomes relevant to specify design choices from the perspective of the user's actions. Monaghan and Trouche (2016, p. 391) claim that "each tool has affordances and constraints regarding what might be learnt in using each tool to do the task. The person-task aspect of this dialectic has several dimensions, including the person's engagement in the task and the ability of the person to realise the task." Affordances are also central in Chiappini (2013), which considers cultural affordances. In their reflection on the instrumental approach from an embodied perspective, Shvarts et al., (2021, pp. 455 f.) hypothesise that "conceptual understanding in mathematics is related to active recognition of artefacts' affordances or recreating of the artefacts in the course of instrumentalisation within fulfilling students' intentionality." They also "propose to see instrumentalisation as recognition and/or creation of affordances and instrumentation as development of body potentialities."

At this step of our study of a new machine, the focus on affordances could help identify what the exploration of the artefact could foster. At the same time, it becomes relevant to specify design choices from the perspective of the user's actions.

4.2 The Meaning of the Tangent Line

Several studies have been carried out on the mathematical content to which the machine refers, both in secondary school and university. Before learning Calculus, algebraic and geometrical objects are mainly considered from two points of view: a global point of view and a pointwise point of view. The former is supported by an iconic visualisation (Duval, 2005) of the graphical representation of the function. For instance, the functions of one real variable are considered global entities associated with an algebraic expression or a graph and pointwise correspondences or covariations between two quantities. Since the beginning, the study of Calculus has added a local point of view to these perspectives: we need to consider what happens to functions at a specific value and in its neighbourhoods (Chorlay, 2007; Maschietto, 2008). In the graphic register, a local point of view draws attention to what happens around a chosen point. The relationships among these points of view ("global/local game" in Maschietto, 2008) must be considered in teaching Calculus. Among the concepts of Calculus, research focuses on derivative/tangent and integral. Tall (2012) identifies the cognitive roots for differentiation and integration in terms of the "local straightness" of a curve (the property of some graphical representations to seem straighter and straighter when we zoom around their points, which students have to construct against previous tangent knowledge, cf. Castela, 1995) and area under the graph (it appears as a new meaning of area). He further clarifies that local straightness involves a perceptive level related to a primitive human perception of visual aspects and evokes a global approach deeply related to how an individual looks along the graph and apprehends changes in slope. Otherwise, “local linearity” (the property of some functions to be approximated by a linear function in the neighbourhood of a point) focuses on the idea of the best linear approximation of a function at a point, expressed formally. According to him, the cognitive root for the derivative is thus based on the fact that the graph of a differentiable function at a particular point looks straight if sufficiently magnified. These ideas were later developed in subsequent research (Biza, 2011; Maschietto, 2008).

Concerning Calculus, Tall and Vinner (1981) present the analysis of students’ cognitive functioning in terms of concept image (describing the total cognitive structure that is associated with the concept, which includes all the mental pictures and related properties and processes) and concept definition (it is a form of words used to specify that concept). Some research confirms a certain distance between concept images and concept definitions in students; they also stress that the different elements of a concept image “were often not conceptually connected in students, who were then unaware of the contradictions among them” (Przenioslo, 2004, p.129). Other research deepens the analysis of students’ concept images. Bingolbali and Monaghan (2008) consider the construct of concept image/concept definition within a social-cultural perspective, pointing out that the experiences grounding the concept image must be contextualised. The authors compare the development of concept images of the derivative for mathematics and engineering students: starting from a similar condition, after their university courses, mathematics students do better on questions regarding tangents, while engineering students do better in rate of change questions. It seems crucial not only to connect concept images and concept definition but also to enrich concept images. From our perspective, the MTS could support this.

In general, the construction of the concepts of Calculus is relevant not only in mathematics but also in other disciplines, like physics, with additional meanings and applications. Indeed, several studies in physics education show that students lack basic knowledge of Calculus (Stoffels et al., 2022).

5 Research Questions

As stated above, our research aims to construct educational activities with the MTS for approaching Calculus, mainly in secondary school. This paper is interested in analysing an expert’s artefact exploration.

The first research question concerns the potentiality and affordances of the machine: *Which is its semiotic potential? Which affordances can be identified in this machine?*

To formulate the subsequent research question, we introduce three ways the MTS can be assembled: tangent, curvigraph, and transformation modes. We deepen these modes in the next section. In activities with students, starting from the simpler version of the artefact (tangent mode) and later exploring the more complex versions (transformation and curvigraph mode) seems reasonable. However, we chose to start from the exploration of the transformation mode to analyse, besides the semiotic potential and the affordances related to the specific activity, how an expert can explore this mode without the need for a previous explicit focus on the wheel for the tangent. Thus, the second research question concerns the exploration processes by a mathematics teacher in two settings:

– The static setting, in which the teacher cannot move the machine: *Which components of the machine do emerge (instrumentalisation)? What movements do we expect? Why?*

– The dynamic setting, in which the teacher can act on the machine: *Which are the utilisation schemes that emerge in exploring the artefact? What kind of semiotic activity can we observe? Are specific gestures detectable?*

The third question concerns the emergence of mathematical meanings: *How are the invariant elements identified/detected? When and how does the idea of the tangent line emerge?*

6 Empirical Research Methodology

According to the Theory of Semiotic Mediation, we first analyse the MTS following the components presented in the "Theoretical background" section. Then, we describe the structure of the experimental session for its exploration by an expert.

6.1 Semiotic Potential

The analysis of semiotic potential considers three components (Bartolini Bussi & Mariotti, 2008): historical references, mathematical content, and utilisation schemes. The "Historical setting" section has already dealt with the first point. We analyse mathematical content and utilisation schemes in this section.

MTS's components have been designed to be assembled differently for different activities. In all these cases, one puts and slides the plate on the external guides of the frame. The directional pointer ([6], Fig. 2) is placed in the big slot of the plate, and at least one rod is used above the plate through the pointer. There are three modes:

(1) *Tangent mode:* Curve-tracing or curve-following free mode with only the plate and the directional pointer with a rod (Fig. 3).
(2) *Curvigraph mode*: Including the peg on the plate and making the rod pass through it (Fig. 5), the MTS allows tracing some curves.
(3) *Transformation mode*: Using two rods and two pointers (Fig. 7), the MTS works as an integraph to construct derivatives and antiderivatives graphically.

6.1.1 The Tangent Mode

The *tangent mode* allows one to experiment with a gesture of usage and obtain the machine's first signs. The directional pointer has two parallel wheels, so the gesture of usage is to grasp and move the pointer by its cap (Fig. 3). In doing that, one has to make the wheels roll without slipping; to move along a given trajectory, the pointer must be displaced back and forward and rotated during the motion (corresponding to a change in the direction).

If a subject wants to trace a curve with the pointer, he/she can insert a marker in the hole and move the pointer. The role of the rotation to change the direction radically differs from tracing a curve freehand or under constraints involving only the position (and not the direction), as in the gardener's ellipse.

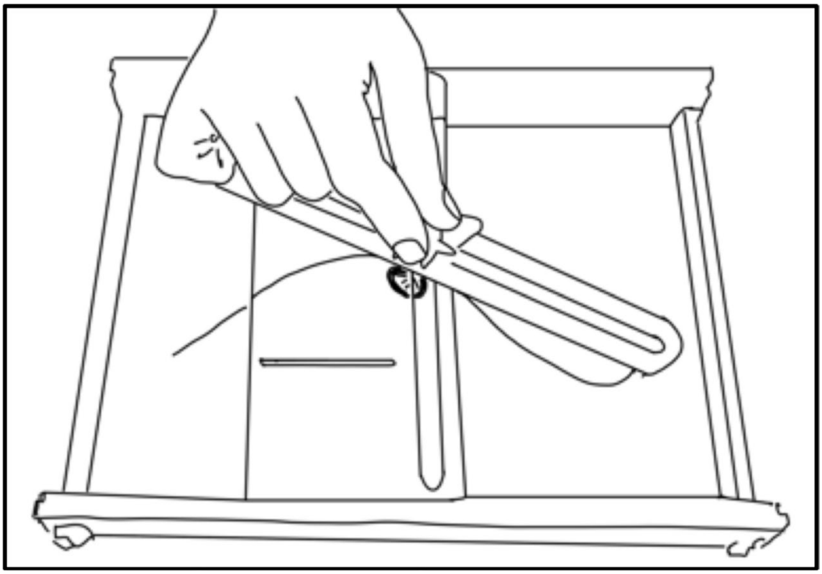

Fig. 3 The tangent mode for tracing and/or following curves

The wheels' direction has a solid mathematical meaning: it corresponds to the direction of the line tangent to the traced curve at the pointer's position. We need some remarks to justify in a geometrical way that the direction of a wheel that rotates without slipping is related to the tangent. As visible in Fig. 4, consider an *n*-sided regular polygon inscribed in a fixed circle *C*. Rotate the polygon side-by-side on the plane and make the two vertices of the bottom side lie on a given curve. If we call "direction" the intersection of the rotating figure with the base plane, the direction is a secant to the given curve. If we consider polygons with more and more sides, the figure tends to the circle C, and its direction tends to the limit to the secants, i.e., the tangent to the curve.

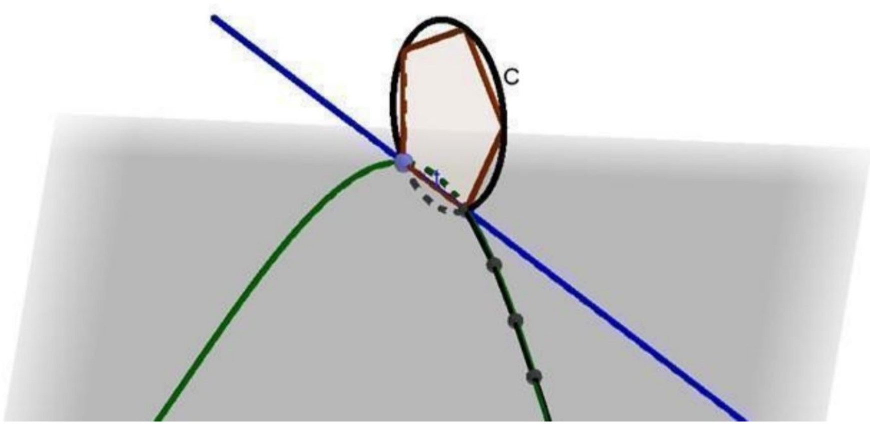

Fig. 4 A polygon rotates along a curve

The pointer has two wheels with parallel directions, but wheels can have different rotation speeds, like segways and hoverboards. By basic kinematic reasoning, the direction of the pointer is parallel to the wheels' directions (such a direction is represented by the little "wings" at the base of the pointer, cf. right bottom of Fig. 2). Therefore, the centre of the pointer moves as having a single wheel rotating in the middle of the two wheels.

This mode can support the emergence of the idea of the tangent line to the curve. In this case, the visualisation can be considered iconic (Duval, 2005), global, and dynamic, in which the objects are represented not only in the graphic register but also by a "hybrid representation" matching graphic and material ones. Furthermore, the hand grasping the pointer can prompt one to focus on the point, fostering a local point of view. The rod plays a new semiotic role compared to the representation of the straight-line tangent to a curve at one point (as often happens in graphical representations, cf. Maschietto, 2008): the machine seems to offer a new graphic-material representation. For this, the tangent mode could be accessible to secondary school students who have approached the meaning of tangent, for instance, concerning circles and conics sections, or it could be used to suggest the iconic visualisation of the tangent to a curve according to its meaning in Calculus.

6.1.2 The Curvigraph Mode

For the *curvigraph mode*, the machine's structure consists of the plate with the peg, one rod, and the directional pointer, as represented in Fig. 5. If a subject wants to trace a curve with the pointer (P in Fig. 5), he/she has to insert a marker in the hole and drag the pointer by grasping it. The pointer will move along a trajectory determined by the initial point.

In this mode, the machine traces exponentials and parabolas; it acts as the machine introduced by Salvi and Milici (2013) and later studied in experimental activities by Maschietto et al. (2019). Naming A the peg, P the directional pointer, and B the projection of A on the big guide of the plate, the distance AB is an invariant. For the parabola (left of Fig. 5), the wheels' direction is perpendicular to PA; for the exponential (right of Fig. 5), the wheels' direction is PA. By introducing a Cartesian system with the x-axis passing through A and B, as in Fig. 6, we have that the machine imposes as a parameter the constant length AB as the subtangent/subnormal of the traced curves (in Fig. 6, we also represent the points in another configuration A'B'P'). Assuming AB as a unit to simplify the calculations, the point P traces the translated exponentials $y = \pm e^{x+m}$ (represented in blue), which are orthogonal to parabolas $x = y^2/2 + n$ (represented in red). From a geometrical perspective, the orthogonal trajectories to the pencil of translated exponentials are

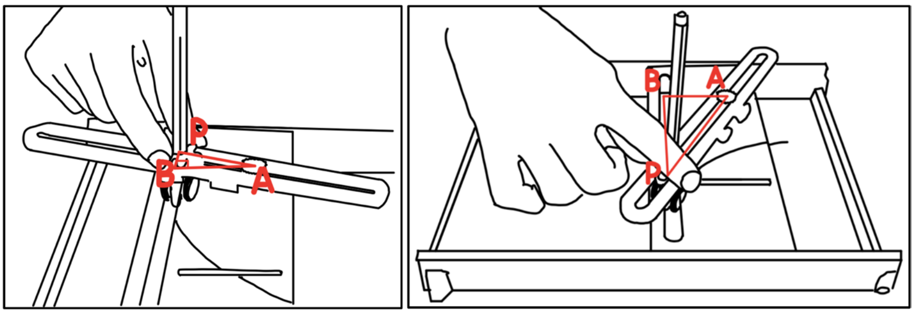

Fig. 5 The curvigraph mode: parabola (left), exponential (right)

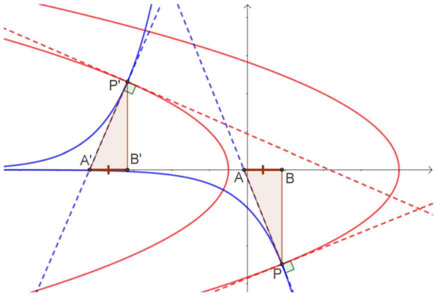

Fig. 6 The translated exponentials are orthogonal to parabolas

parabolas because exponentials have constant subtangents, and parabolas have constant subnormals. Note that the initial condition determining the parameters *m* and *n* is posed by the position of the directional pointer when the wheels of the pointer come in contact with the sheet of paper below.

This second mode can be proposed after identifying the meaning of the wheel about the tangent to the curve. Identifying the drawn curves as a parabola or an exponential could be first based on the iconic visualisation (Duval, 2005), based on a global point of view; in this case, the curve could be recognised by students who have already studied these functions and their graphical representations. However, even if the parabola and exponential function are studied at secondary school, their properties of subtangent/subnormal are not usually known. Furthermore, the machine does not directly provide information concerning the standard elements used in constructing these curves; for instance, the parabola is traced without involving the focus and directrix present in its metric definition. In addition, even if the tangent and the subtangent lines are embedded in the machine by rods (this could be seen in terms of affordances), it could be necessary to trace these lines to connect them with the machine's structure. The choice of the initial point becomes essential to trace the curves and support their identification, which needs a local point of view on the triangle ABP (Figs. 5 and 6) during the movement of the rod and the plate. As in tangent mode, tangent and sub-tangent lines are in the hybrid graphic-material representation. Finally, the orthogonality relationship between these two curves quoted above is not secondary school content. A potentially more interesting approach can be to set these constructions as geometric solutions of differential equations (slope fields). Nevertheless, in Italy, this content is usually studied only in universities (it belongs to the curricula of only a few types of secondary schools).

6.1.3 The Transformation Mode

Concerning the *transformation mode*, i.e., how to use the MTS to perform transformations, we use both the pointers as visible in Fig. 7 (and previously introduced in Fig. 2, as [5] and [6]). Specifically, after joining the two guides to form a T-shaped piece, we put the directional pointer D on the height rod and, on the base rod, the positional pointer P and the peg A. Pointers and the peg can slide along the T-shaped piece; therefore, we can consider the MTS as a "sliding-T integraph" in this mode. After drawing a curve, we can choose which pointer works as a viewfinder (following the input trajectory without the marker) and which is the tracer (drawing the output curve with the marker). In terms of action, the hand guides the viewfinder pointer, which determines the motion of the tracing pointer. To move the positional pointer or the directional implies a big difference in the related gesture

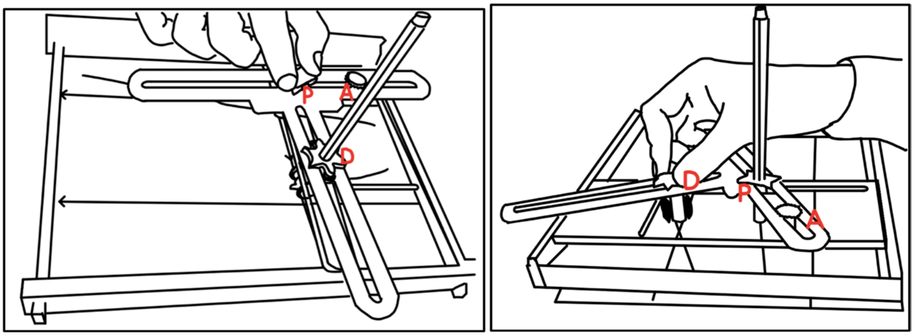

Fig. 7 The transformation mode: antiderivatives (left) and derivatives (right)

of usage. Indeed, guiding the directional pointer implies guiding its direction (as in the tangent mode).

In this transformation mode, the relation between the motion of the two pointers should emerge. A similar experience of functional dependency can be found in the pantographs for geometrical transformations (Bartolini Bussi & Maschietto, 2008), even though, in that case, the motion depends only on the position of the pointer and not also on the direction (in mechanical terms, pantograph's constraints are all holonomic, which differs from the MTS because of the wheels).

To understand why this machine is related to Calculus, let us introduce two reference frames such that abscissae correspond with the little guides (where one can fix the peg) on the transparent plate, and the two ordinate axes are superimposed. In Fig. 8, we represented in red the Cartesian axes related to the directional pointer D, and in blue the ones for the positional pointer P=(x, y): note that the abscissae of the two pointers coincide both in red and blue reference frames. We also take as a unit the distance between peg A and the big guide of the plane: thus, peg A has coordinates $(x-1, 0)$ in the blue reference frame, and the ordinate of P corresponds to the slope of the line PA (because $\Delta y/\Delta x = (y-0)/(x-(x-1)) = y$).

The direction of the pointer D, represented in Fig. 8 by the black segment centred in D, is set perpendicular to the height rod, which is perpendicular to the base rod; hence, PA is parallel to the direction of D. But if D follows the graph of a function *f*, its direction corresponds to the tangent to the curve; therefore, the slope of the line PA and the ordinate of P are the derivative *f '*.

To sum up, if we move D along the graph of a function in the red reference frame (remember that, to move D along a curve, we guide its direction), point P traces its derivative in the blue reference frame. Conversely, when P moves along the graph of a function in the blue reference frame, D traces one of its antiderivatives (according to the initial position of D). From a global point of view, the transformation mode provides two curves whose relationships need a local and pointwise point of view. Identifying the relationships between curves is an important task in Calculus (Arzarello & Sabena, 2011).

In this transformation mode, it is suitable to start using the machine with simple curves already traced on the paper, like straight lines (parallel and not to the x-axis), to foster the identification of relationships between them (TSM-Q2 and TSM-Q3). The need for two reference systems is essential for recognising the two curves and could be a sign that needs to be suggested.

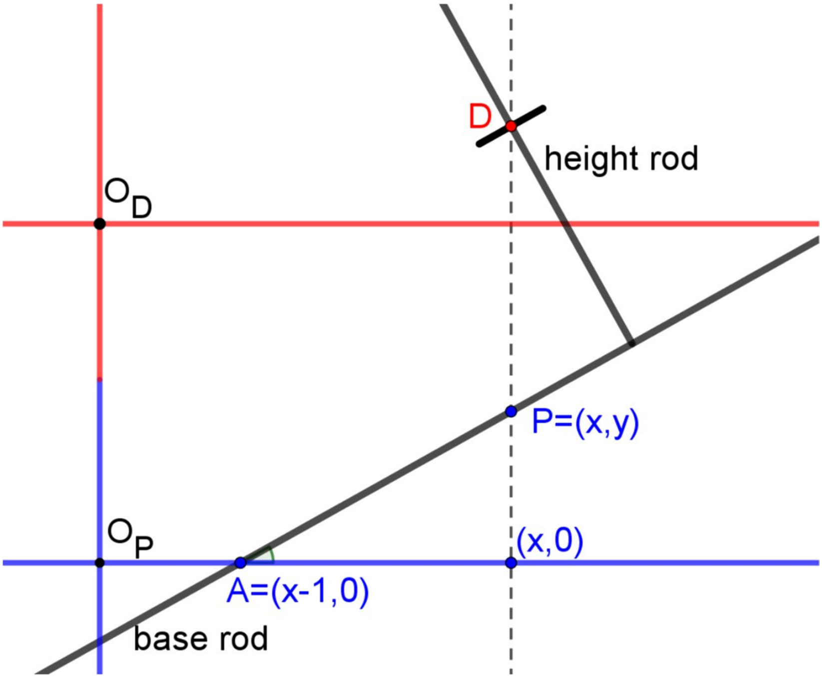

Fig. 8 Introduction of Cartesian systems in the transformation mode

In this section, the MTS has been analysed according to the components of the semiotic potential, trying to highlight some possible gestures of use and the traces/signs that the machine provides. This is based on research already conducted with other machines, some referring to the same mathematical content described in this article (integraphs, solvers of differential equations). Comparing the curvigraph and the transformation modes, even though tracing curves in the former mode is instrumentally simpler (only one pointer and one rod), the related mathematical contents are more complex than the one required in the latter mode. Indeed, the transformation mode mainly focuses on the fundamental theorem of Calculus; on the other side, the curvigraph mode analytically delves into the geometrical solution of differential equations.

The analysis carried out in this section represents a sort of a priori analysis of the machine and its use; in this way, it answers the first research question. To sum up, identifying the tangent in the first mode represents a crucial step in the mathematical understanding of the machine. The experimental part will allow us to deal with the exploratory processes, providing further elements for the construction of tasks for students and/or modifications of the artefact (in terms of gestures, feedback, and affordances).

6.2 Choices for the Exploration

To answer our second and third research questions, we proposed the exploration of the machine to a mathematics teacher (V). This choice is based on three main reasons:

(1) The mathematical knowledge embedded in the machine was part of her mathematical background.
(2) She had expertise in mathematics laboratories with mathematical machines for students; she collaborated with the Laboratory of Mathematical Machines of the University of Modena e Reggio Emilia and guided laboratory sessions with high secondary school students.
(3) She already knew our adopted theoretical framework.

The exploration was supported by one of the authors (P) of this paper; it was entirely videotaped. The exploration is split into two steps: in the beginning, the machine is provided in the transformation mode with all the components (Figs. 2 and 7), and then it is progressively simplified up to the tangent mode. In the first step, we planned two settings: a static setting, in which the teacher cannot act on the machine, and a dynamic setting, in which the teacher can move the component of the machine. In this paper, we focus just on tangents and derivatives/primitives because, as already written, the interpretation of the role of the rod as a tangent is a crucial element for understanding the mathematics embedded in the machine and its products (curves).

The analysis is based on the videotape and V's drawings made during the exploration.

7 Findings and Analysis of the Exploration

The analysis of the exploration is organised around two steps: the former corresponds to the manipulation of the transformation-mode machine, and the latter is related to the analysis of its use in the tangent mode.

7.1 First Step (Transformation Mode)

In this section, we analyse the beginning of the exploration of the machine when it is proposed in the transformation mode.

7.1.1 First Step — Static Setting: Describing the Machine

The exploration starts by asking for the description of the MTS in the transformation mode (Fig. 7) that corresponds to the question TSM-Q1: "How is the machine made?". Because of V's background, she began the description of the artefact without any further suggestion. Even if V asked to act on the machine, she was invited to describe the machine without moving its components in this step.

4. V: There is a plane [*the frame, 1 in* Fig. 2], there are two rails on which I suppose this structure [*the plate, 2 in* Fig. 2] can shift… Then there is a kind of set square … composed of two rods [*3 in* Fig. 2] forming a right angle. Inside these rods are sliders [*pointers 5 and 6, in* Fig. 2]. … Below, one of the two sliders [*6 in* Fig. 2] has some small wheels; the other does not.
5. P: Is there a third slider?
6. V: Ah, yes. The third slider [*the peg 6 in* Fig. 2] does not touch the plane, right? Two sliders touch the plane, the one with the little wheels and the one without wheels; on the contrary, the third slider does not.

Five minutes from the beginning of the exploration, P asked about the system's degrees of freedom but got no direct answer from V. However, the question moved the focus to other machine components: V saw the holes in the pointers and highlighted the possibility of inserting a marker.

In this first part of the exploration, the verbalisation is difficult to follow because of pauses and unfinished sentences. Even if V knew how the exploration of a machine should be carried out, the components of this machine emerged little by little through the interaction with P. That makes us conclude that exploring the transformation mode is quite tricky, even for an expert, if introduced as the first activity, because there are too many semiotic components to be considered (for example, the peg [sentence #6] is mentioned by V only after P's question). Indeed, V is familiar with machines having simpler structures, mainly linkages without the wheel component (e.g., articulated quadrilaterals).

7.1.2 First Step — Static Setting: Identifying Movements and Constraints

Before V moved the machine for answering to the second question TSM-Q2, P asked how the machine could move and what the constraints were, in order to support V's instrumentalisation.

First, V made some conjectures about the relationships between the components.

10. V: So, I think that the structure [*the plate*] on the two rails moves vertically following the rails and … if this structure moves, […] this slider [*the positional pointer*] will move to the left, and this one [*the peg*] will move upwards, I suppose. […] Thus, the constraints… I think the machine [*the plate*] can probably be flipped because there is another groove here [*the little guide of the plate without the peg*]. You can flip it [*the plate*] by inserting this rod [*actually, she refers to the peg*] into this groove [*the little guide without the peg*], I suppose. The constraints are given by … these … small grooves; I guess the slider [*pointer*] can only shift in there [*in the big guide of the plate*] and go to the bottom …

In her answer [#10], we distinguish conjectures about movement in the first part of the answer ("So, … upwards, I suppose") and conjectures about constraints in the second part ("Thus, the constraints…"). As the components emerge gradually and are required to be related in the movement, the outcome of the corresponding actions also seems difficult to anticipate. This is emphasised by V's use of "I suppose" and her pauses in the answer.

7.1.3 First Step — Dynamic Setting: Acting on the Machine

Then, P proposes to put the hands on the machine without further instructions. V tested her conjectures directly by moving the transparent structure. In (#12), V verified the movement of the pointer with wheels, but V also formulated other conjectures on the role and meaning of other components. P supported V in her manipulation of the machine with some remarks on the movement of the pointer (#13) and some gestures of usage (use of the two hands, light pressure on the pointer).

12. V: [*V moves the plate up and down the frame and keeps all the other components free, as visible in the top left of* Fig. 9] I thought that while sliding [*the plate*], the rod would have moved [*with respect to the plate*]. Instead, only the wheeled slider moves

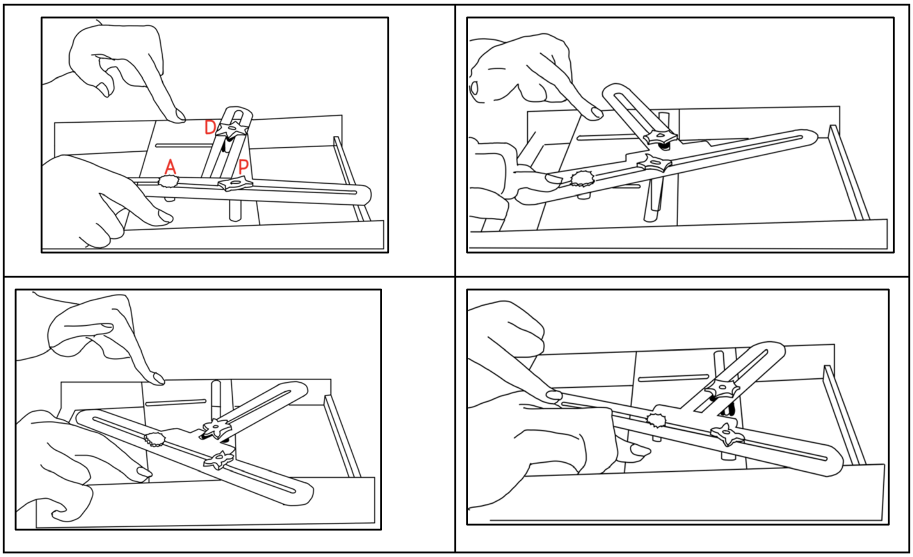

Fig. 9 Gestures in the manipulation of the third mode

[*she does not notice that the rods also move, even though the direction AP stays constant*].
13. P: Notice that the wheeled slider is now moving. If it should not move, that's because the machine is just a prototype, and one has to… [*gesture of pressing down the directional pointer*].

Figure 9 shows some gestures made by V when exploring the machine. They appear different from those described in our analysis in the "Semiotic potential" section, because in this manipulation V makes attempts to look for gestures of usage also based on her previous experience. Indeed, in the analysis of the instrumental genesis, it is common to observe the adaptation of schemes related to other instruments (Rabardel, 1995/2002).

Following the movement of MTS components, the question of degrees of freedom can be asked (#15). Indeed, V had experienced different mathematical machines working on a plane, which can be broadly distinguished in curve tracers (e.g., machines tracing conic sections) and tools for planar transformations (e.g., pantographs). This question concerns TSM-Q2 for exploring an artefact ("What does the machine make?") because it allows us to characterise the machine as a curve drawer (the tracer has one degree of freedom) or a transformation instrument (the tracer has two d.o.f., representing a function between points of the plane).

15. P: OK, up to now, how many degrees of freedom have you found?
16. V: We understood that [*while sliding the plate*] it moves only this [*the directional pointer*] … Ah, that is true. Previously, we noticed that [*in the directional pointer*] there is a hole. Thus, I suppose that we can insert a pencil. However, there is also a hole in the slider without wheels. Thus, maybe, when I assemble the machine differently, one can switch the sliders… I do not know… To sum up, I would say that the

tracing point is this one [*the directional pointer*] because this point [*the positional slider*] … stays still [*with respect to the plate*]; hence, it traces a line parallel to the rails.

In her answer (#16), V assigned the role of a tracing pointer to the hole where it is possible to insert the pencil/marker. This element can be traced back to the analysis regarding affordances: an expert knows that a hole is a good candidate to contain a pencil in drawing machines. Having identified the tracing pointer, V implicitly attributed one degree of freedom to the system because she cannot get anything but a straight line. In this way, she interprets the sign produced by the machine as a curve and, according to her previous experience, considers the system as a curve tracer. She did not relate the mutual motion of the two pointers and, therefore, did not recognise the two degrees of freedom.

22. V: [*V moves the rods with the right hand by pivoting on the peg while keeping the plate fixed with the left hand, as visible in the top right of* Fig. 9] So when I move the square [*i.e., the perpendicular rods*], the two sliders can move. OK, I thought this slider [*the positional pointer*] was fixed [*on the rod*]; instead, it can slide. Moreover, in this case, the peg that does not touch the sheet is fixed [*on the plate*] whether I move the plate or the T [*the rods*]. […] I think it [*the peg*] can move in its groove [*the little groove where the peg is placed*].
23. P: There is a screw to fix the peg inside this groove.
24. V: OK. Then, after fixing it, everything else moves according to its position. If I move the square [*the rods*], then I obviously move the slider without wheels. [*V makes the positional slider reach its bottom position and start moving the plate up and down*] In this case, it [*the positional slider*] is at the end of the plate—maybe I reached a limit position [*also the directional pointer reached a limit position and got stuck, as visible in the bottom left of* Fig. 9]. [*V goes back to a not-limit position, as visible in the bottom right of* Fig. 9 *and V moves the plate up and down.*] However, the trace is always a straight line if the tracer is this [the positional pointer]. Maybe I can move them [*the rods and the plate*] simultaneously…

From this moment until the end of the first step, the exploration is influenced and partially compromised by the gestures of usage that V could not implement without several suggestions by P. For example, P drew a straight line on the sheet of paper under the machine and asked V to follow it with the directional pointer. While V verbalised the preparatory gestures to respond to that task, P specified a gesture of usage to follow the line with the directional pointer and to get the trace using the positional pointer, again emphasising tactile feedback from the machine. In this first part of the exploration, in some cases, V made conjectures about the role of some components of the machine; in other cases, she referred to her experiences with the pantographs (mathematical machines for geometrical plane transformations). V's instrumental genesis appears a complex process, even if it is supported by P. This suggests paying close attention to students' instrumental genesis, if and how previous experience could represent an obstacle (in the sense of Brousseau, 1976) to the exploration of the machine.

The analysis of the first step of the exploration highlights two aspects. The first one is related to the design of the machine since, as already mentioned, the machine appears too complex from a semiotic perspective. Moreover, the tactile feedback of the machine is very weak, especially if one does not know what to expect. The other aspect concerns the second research question, relating to the exploratory process by an expert. This analysis supports the choice of proposing the machine in the more straightforward tangent mode for secondary school activities and even with university students and teachers.

After this first step, P assembled the MTS in the tangent mode by removing the positional pointer and the peg. A single rod is put in the direction of the wheels of the directional pointer.

7.2 Second Step (Tangent Mode)

With the MTS in the tangent mode (Fig. 3), P traced a curve on the paper sheet and asked V to follow it.

7.2.1 Second Step — Manipulation and Gestures

In continuity with the gestures of the first step (while moving the directional pointer along a straight line), V directly grasped the pointer and followed the drawn curve (Fig. 10). Unlike the transformation mode, V looked confident while manipulating the artefact in this case. Another gesture of usage emerges, supporting her instrumental genesis: she placed the pointer in the starting point by positioning its hole upon the curve. The starting point is chosen by V on an extremity of the curve.

Other gestures are established during the description of the motions of the plate and the rods aiming to follow the curve (#68): shifting the plate and seeing the curve through the hole. P asked for the explication of a gesture important for the meaning the machine can mediate: first, he asked where to place the hands to follow the curve (#69), then he asked for the use of only one hand (#71). In this case, P supported V's instrumental genesis.

68. V: [*To move along the curve*] I move the plate and the square [*one rod*], following the curve by looking inside the hole […]
69. P: Where is it more convenient to put the hands to follow that curve?
70. V: On the slider, I think [Fig. 10].
71. P: I agree with you. So, is it necessary to use two hands, or can only one be enough?
72. V: Maybe only one; with the second hand, one can complicate the manipulation.

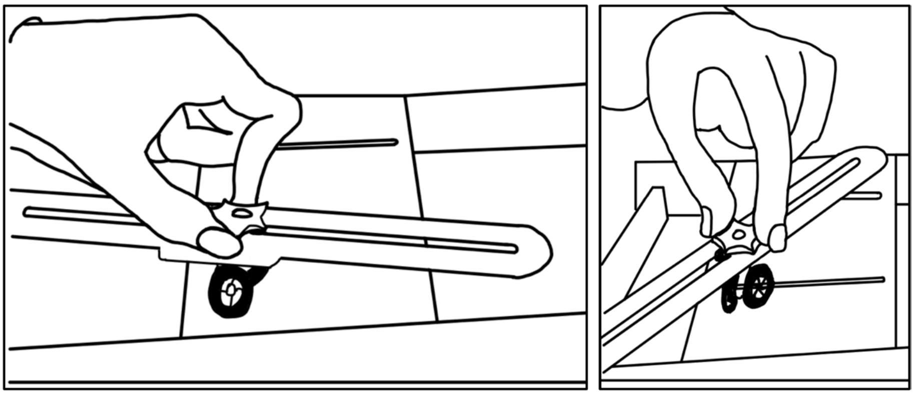

Fig. 10 Gestures in the manipulation of the tangent mode

It is interesting that, differently from the transformation mode, in this simpler case, V autonomously asked herself the role of the plate. She recognised that, in this mode, it worked only to keep the pointer up. Indeed, we are using none of the two little guides (the ones perpendicular to the big one, in which one can fix the peg). From the perspective of affordances, that can constitute an obstacle when analysing this mode (there are unused elements that confuse the user). That provided a clue to improve the design of newer prototypes to avoid the plate in tangent mode.

7.2.2 Second Step — The Emergence of the Idea of the Tangent Line

As introduced in the "Semiotic potential" section, the relation between the direction of the wheel and the tangent is crucial. After V moved the pointer, as visible in Fig. 10, P posed some questions to focus on the gestures of usage.

79. P: How can we follow the traced curve?
80. V: By guiding with our hands this slider.
81. P: Yes, and we not only translate it, but we also make another kind of motion…
82. V: Sure, we must rotate it. Wait, I will try to formulate a sensible sentence. To sum up, like before [*in the transformation mode, while moving along a straight line*], [*I put the directional pointer*] in the initial position, then I follow [*the curve*], and I have […] both to translate and rotate it [*the directional pointer*]. Obviously, that depends on the provided curve.

The controlled movement of following a curve (#82) suggests paying attention to the wheel and the relationship between the rod and the curve.

83. P: What does this rod represent? With respect to the drawn curve?
84. V: At each point, it is the tangent line…
85. P: Yes.
86. V Yes, at each point of the curve, it is the tangent line at the curve in the point.

When the attention is focused on only one element, the tangent line emerges (#84, 86) by matching the curve drawn on the paper with the rod as a unique sign. The iconic visualisation of the tangent line seems to support its identification, even if V refers to a local point of view (#84: "at each point"). Of course, this meaning emerges if the subject, as V, has this concept image. This can answer our third research question.

8 Conclusions

This paper focused on analysing a new material artefact, the Multipurpose Tangent Solver (MTS, cf. https://www.thingiverse.com/thing:6619300), constructed on historical sources and related to Calculus. Our research project aims to study this artefact for proposing it to secondary school students; this requires gathering elements on MTS use, constraints, and manipulation for constructing appropriate tasks. As the first step of our project, we were interested in analysing how a secondary school teacher explores this artefact. From a semiotic perspective, our analysis of the artefact and exploration process was based on the theoretical background of the instrumental approach and the Theory of Semiotic Mediation. In this first step, we asked three research questions.

The first research question asked which is the semiotic potential of the artefact. It is a kind of a priori analysis within the chosen theoretical frameworks. Three modes for using the MTS were identified, as well as some crucial elements that should be considered in the conception of tasks, particularly regarding the link to students' personal meanings. The tangent mode appears interesting for secondary school because it offers the material representation of the tangent and can support a local point of view by following a curve when grasping the pointer.

The second research question focused on the exploration processes during the use of the artefact: we decided to observe and analyse them in an exploration task with a teacher expert in mathematics and instruments (i.e., mathematical machines). We chose to start from the exploration of the transformation mode to test whether an expert can easily follow the exploration without previously focusing on the relation between the wheel direction and the tangent. The answer was negative: the exploration was tricky for the interviewed expert. The wheel's role was not grasped naturally; wheels were seen as one of the many components and did not get any particular focus. Likewise, detecting the constraints requires a non-trivial effort to produce conjectures and tests. Similar conclusions are coherent with Maschietto et al. (2019), which tested another decomposable artefact for solving inverse tangent problems. The results of this analysis support the idea that the exploration of the artefact must start with the tangent mode. The analysis of the expert's exploration furnished suggestions regarding the design of the artefact, eliminating as much as possible the components not strictly evoking a mathematical meaning. For example, the little guides in the plate made V focus on possible modes without an interesting mathematical counterpart (cf. begin of "First step – Static setting: describing the machine" section, #10). Furthermore, using the plate in the tangent mode introduced the problem of guiding this component as an additional gesture (cf. end of "Second step – Manipulation and gestures" section). However, the main design problem was related to the need to keep the plate pressed to avoid the sliding of the pointer's wheels (cf. "First step – Static setting: describing the machine" section, #13). Successive design improvements are visible at www.machines4math.com

The third research question asked if and when the invariants and the idea of the tangent line emerged during the exploration of the artefact. The transformation mode proposed at the beginning of the exploration did not effectively generate such an emergence, as evinced in the discussion on the second research question above. Differently, the tangent mode fostered the interpretation of the rod in terms of the tangent line. In other words, this mode seems to provide an iconic visualisation of the tangent line to the curve in a point. Even though such a point constitutes only the starting point of the constructions available with the MTS, the relation between the direction of the wheels and the tangent is crucial for understanding the mathematics embedded in the machine and its behaviour in the curvigraph and transformation mode. This suggests exploiting the link between the tangent line and the inverse problem within an education perspective; it can be approached at school by starting from the simplest mode and then moving to more complex modes.

The results obtained in our research will represent the starting point for the next steps, which include materially improving the artefact and focusing on the construction of worksheets for secondary school students, in line with the research on the educational use of mathematical machines (Bartolini Bussi & Maschietto, 2008). Another future perspective is to study MTS's potential in consolidating the related mathematical meanings with university students.

In conclusion, we must emphasise the role of historical and epistemological components, which are highly important in sustaining the students' interest. Besides introducing new voices from history and mathematicians into the teaching of Calculus, the MTS allows new experiences related to the meaning of the tangent line and its role in constructing curves and transformations. Interestingly, our approach can also be adopted in a purely synthetic geometrical way without referring to formulas. Historical and epistemological reflections can also be integrated by reviving suitable experiences (e.g., Perrault's construction of the tractrix) and reading excerpts of original manuscripts (e.g., Huygens, Leibniz, Poleni). In contrast with the abstraction of a topic analytically involving infinity, we hope to provide new ways to get students in touch with Calculus.

Author Contribution The authors contributed equally to the preparation of this paper.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

References

Abdank-Abakanowicz, B. (1886). *Les intégraphes, la courbe intégrale et ses applications*. Gauthier-Villars. https://ia800304.us.archive.org/10/items/lesintegraphes18abda/lesintegraphes18abda.pdf. Accessed 15 July 2024.

Arzarello, F., & Edwards, L. (2005). Gesture and the construction of mathematical meaning. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME conference* (Vol. 1, pp. 22–145). Melbourne University. https://www.emis.de/proceedings/PME29/PME29ResearchForums/PME29RFArzarelloEdwards.pdf. Accessed 15 July 2024.

Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics, 77*(2/3), 189–206. https://doi.org/10.1007/s10649-010-9280-3

Bartolini Bussi, M.G., & Maschietto, M. (2008). Machines as tools in teacher education. In D. Tirosh, & T. Wood (Eds.), *Tools and processes in mathematics teacher education. The international handbook of mathematics teacher education* (Vol. 2, pp. 183–208). Sense Publishers. ISBN: 978–90–8790–544–6.

Bartolini Bussi, M. G., Garuti, R., Martignone, F., & Maschietto, M. (2011). Tasks for teachers in the MMLAB-ER Project. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 127–130). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME35-2011-Ankara.zip. Accessed 15 July 2024.

BartoliniBussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge. https://doi.org/10.4324/9780203930236

BartoliniBussi, M. G., & Maschietto, M. (2006). *Macchine matematiche: Dalla storia alla scuola*. Springer. https://doi.org/10.1007/88-470-0403-9

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics, 68*, 19–35. https://doi.org/10.1007/s10649-007-9112-2

Biza, I. (2011). Students' evolving meaning about tangent line with the mediation of a dynamic geometry environment and an instructional example space. *Technology, Knowledge and Learning, 16*, 125–151. https://doi.org/10.1007/s10758-011-9180-3

Blåsjö, V. (2017). *Transcendental curves in the Leibnizian calculus.* Academic Press. https://www.sciencedirect.com/book/9780128132371/transcendental-curves-in-the-leibnizian-calculus. Accessed 15 July 2024.

Blåsjö, V. (2023). Newton on constructions in geometry. *Historia Mathematica, 65*, 14–29. https://doi.org/10.1016/j.hm.2023.09.002

Bos, H. J. (1988). Tractional motion and the legitimation of transcendental curves. *Centaurus, 31*(1), 9–62. https://doi.org/10.1111/j.1600-0498.1988.tb00714.x

Bos, H. J. (2001). *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*. Springer. https://doi.org/10.1007/978-1-4613-0087-8

Bril, B. (2015). Chapter 5. Learning to use tools: A functional approach to action. In L. Filliettaz & S. Billett (Eds.), *Francophone perspectives of learning through work* (pp. 95–118). Springer. https://doi.org/10.1007/978-3-319-18669-6_5

Brousseau, G. (1976). Les obstacles épistémologiques et les problèmes en mathématiques. In W. Vanhamme & J. Vanhamme (Eds.), *La problématique et l'enseignement de la mathématique. Comptes rendus de la XXVIIIe rencontre organisée par la CIEAEM* (pp. 101–117). Louvain-la-Neuve. https://hal.science/hal-00516569v2. Accessed 15 July 2024.

Calder, N., Larkin, K., & Sinclair, N. (Eds.). (2018). *Using mobile technologies in the teaching and learning of mathematics*. Springer. https://doi.org/10.1007/978-3-319-90179-4

Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures. *Recherches en Didactique des Mathématiques*, *15*(1), 7–47. https://revue-rdm.com/2005/apprendre-avec-et-contre-ses/. Accessed 15 July 2024.

Chiappini, G. (2013). Cultural affordances of digital artifacts in the teaching and learning of mathematics. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching (ICTMT 11)* (pp. 95–100). Università degli Studi di Bari. ISBN: 978–88–6629–000–1

Chorlay, R. (2007). La multiplicité des points de vue en Analyse élémentaire comme construit historique. In E. Barbin & D. Bénard (Eds.), *Histoire des mathématiques: rigueurs, erreurs, raisonnements* (pp. 203– 227). INRP.

Crippa, D., & Milici, P. (2019). A relationship between the tractrix and logarithmic curves with mechanical applications. *The Mathematical Intelligencer, 41*(4), 29–34. https://doi.org/10.1007/s00283-019-09895-7

Crippa, D. (2020). One string attached: Geometrical exactness in Leibniz's Parisian manuscripts. In V. De Risi (Ed.) *Leibniz and the structure of sciences: Modern perspectives on the history of logic, mathematics, epistemology* (pp. 203–252), Boston studies in the philosophy and history of science 337, Springer. https://doi.org/10.1007/978-3-030-25572-5_6

Dawson, R., Milici, P., & Plantevin, F. (2021). Gardener's hyperbolas and the dragged-point principle. *American Mathematical Monthly, 128*(10), 911–921. https://doi.org/10.1080/00029890.2021.1982634

Descartes, R. (1637). *Discours de la méthode pour bien conduire sa raison, et chercher la verité dans les sciences. Plus la Dioptrique. Les Meteores. Et la Geometrie. Qui sont des essais de cette Methode*. Leyde: Maire. https://gallica.bnf.fr/ark:/12148/btv1b86069594. Accessed 15 July 2024.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de Didactique et de Sciences Cognitives*, *10*, 5–53. https://publimath.univ-irem.fr/numerisation/ST/IST05010/IST05010.pdf. Accessed 15 July 2024.

Euler, L. (1736). De constructione æquationum ope motus tractorii aliisque ad methodum tangentium inversam pertinentibus. *Commentarii academiæ scientiarum Petropolitanæ*, *8*, 66–85. https://www.biodiversitylibrary.org/item/38573#page/86/mode/1up. Accessed 15 July 2024.

Gibson, J. J. (1977). *The ecological approach to visual perception*. Taylor & Francis. https://doi.org/10.4324/9781315740218

Hoffmann, M.H.G. (2006). What is a "semiotic perspective", and what could it be? Some comments on the contributions to this Special Issue. *Educational studies in mathematics, 61 (½), semiotic perspectives in mathematics education: A PME special issue*, 279–291. https://doi.org/10.1007/s10649-006-1456-5

Huygens, C. (1693). Letter to H. Basnage de Beauval, February 1693. Printed in *Histoire des ouvrages des savants* (or *Journal de Rotterdam*), 244–257. https://play.google.com/books/reader?id=KHcPAAAAQAAJ&pg=GBS.PA244. Accessed 15 July 2024.

Kempe, A.B. (1876). On a general method of describing plane curves of the nth degree by linkwork.*Proceedings of the London Mathematical Society*, *7*, 213–216.https://doi.org/10.1112/plms/s1-7.1.213

Leibniz, G.W. (1693). Supplementum geometriæ dimensoriæ seu generalissima omnium tetragonismorum effectio per motum: Similiterque multiplex constructio lineæ ex data tangentium conditione. *Acta Eruditorum* 385–392 [Translated from Math. Schriften 5: 294–301]. https://archive.org/details/s1id13206590/page/385. Accessed 15 July 2024.

Maschietto, M. (2008). Graphic calculators and micro-straightness: Analysis of a didactical engineering. *International Journal of Computers for Mathematical Learning, 13*, 207–230. https://doi.org/10.1007/s10758-008-9141-7

Maschietto, M. (2015). The arithmetical machine Zero +1 in mathematics laboratory: Instrumental genesis and semiotic mediation. *International Journal of Sciences and Mathematics Education, 13*, 121–144. https://doi.org/10.1007/s10763-013-9493-x

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artifacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM—the International Journal on Mathematics Education, 45*(7), 959–971. https://doi.org/10.1007/s11858-013-0533-3

Maschietto, M., Milici, P., & Tournès, D. (2019). Semiotic potential of a tractional machine: A first analysis. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European society for research in mathematics education (CERME 11)* (pp. 2133–2140). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.science/hal-02421865/document. Accessed 15 July 2024.

Milici, P., & Plantevin, F. (2022). Reconstructing Poleni's machines for transcendental curves, *Bulletin of the Scientific Instrument Society*, (155), 34–37. https://scientificinstrumentsociety.org/bulletin-155-december-2022. Accessed 15 July 2024.

Milici, P. (2020). A differential extension of Descartes' foundational approach: A new balance between symbolic and analog computation. *Computability, 9*(1), 51–83. https://doi.org/10.3233/COM-180208

Milici, P. (2012). Tractional motion machines extend GPAC-generable functions. *International Journal of Unconventional Computing*, *8*(3), 221–233. https://www.oldcitypublishing.com/journals/ijuc-home/ijuc-issue-contents/ijuc-volume-8-number-3-2012/ijuc-8-3-p-221-233/. Accessed 15 July 2024.

Milici, P. (2015). A geometrical constructive approach to infinitesimal analysis: Epistemological potential and boundaries of tractional motion. In G. Lolli, M. Panza, & G. Venturi (Eds.), *From logic to practice* (pp. 3–21). Boston studies in the philosophy and history of science 308. Springer. https://doi.org/10.1007/978-3-319-10434-8_1

Monaghan, J., & Trouche, L. (2016). Chapter 17. Tasks and digital tools. In J. Monaghan, L. Trouche, & J.M. Borwein (Eds.), *Tools and mathematics*. *Instruments for learning* (pp. 391–415). Springer. https://doi.org/10.1007/978-3-319-02396-0_17

Monaghan, J., Trouche, J., & Borwein, J. M. (2016). *Tools and mathematics*. Springer. https://doi.org/10.1007/978-3-319-02396-0

Pascal, E. (1914). *I miei integrafi per equazioni differenziali*. Napoli: B. Pellerano. ISBN: 9781429702485.

Perks, J. (1706). The construction and properties of a new quadratrix to the hyperbola. *Philosophical Transactions*, *25*, 2253–2262. https://doi.org/10.1098/rstl.1706.0017

Perks, J. (1714–1716). An easy mechanical way to divide the nautical meridian line in Mercator's projection, with an account of the relation of the same meridian line to the curva catenaria, *Philosophical Transactions*, *29*(345), 331–339. https://doi.org/10.1098/rstl.1714.0038

Poleni, J. (1729). *Epistolarum mathematicarum fasciculus*. Patavii: Typographia Seminarii. https://play.google.com/books/reader?id=BymYyFnKwYgC. Accessed 15 July 2024.

Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics, 55*, 103–132. https://doi.org/10.1023/B:EDUC.0000017667.70982.05

Rabardel, P. (1995/2002). *Les hommes et les technologies. Une approche cognitives des instruments contemporains.* Paris: Armand Colin. [English version: (2002). People and technology. A cognitive approach to contemporary instruments. https://hal.archives-ouvertes.fr/file/index/docid/1020705/filename/people_and_technology.pdf. Accessed 15 July 2024.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME Conference* (Vol. 4, pp. 113–120). Melbourne University. https://www.emis.de/proceedings/PME29/PME29RRPapers/PME29Vol4RadfordEtAl.pdf. Accessed 15 July 2024.

Riccati, V. (1752). *De usu motus tractorii in constructione æquationum differentialium commentarius*. Bononiæ: Ex typographia Lælii a Vulpe. https://play.google.com/books/reader?id=doR7K66neu8C. Accessed 15 July 2024.

Ritt, J.F. (1950). *Differential Algebra.* Colloquium Publications, Vol. 33, American Mathematical Society. ISBN: 978–0–8218–3205–9.

Salvi, M., & Milici, P. (2013). Laboratorio di matematica in classe: Due nuove macchine per problemi nel continuo e nel discreto. *Quaderni di Ricerca in Didattica (Mathematics)*, *23*, 15–24. https://sites.unipa.it/grim/QRDM_%20Salvi_Milici_23_2013.pdf. Accessed 15 July 2024.

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107*, 447–469. https://doi.org/10.1007/s10649-021-10053-0

Stake, R. E. (1995). *The art of case study research*. Sage Publications. ISBN: 0–8039–5767-X

Stoffels, G., Witzke, I., & Holten, K. (2022). Comparison: Differential Calculus through applications. In F. Dilling, & S. F. Kraus (eds.), *Comparison of mathematics and physics education II* (pp. 227–242). MINTUS – Beiträge zur mathematisch-naturwissenschaftlichen Bildung. Springer. https://doi.org/10.1007/978-3-658-36415-1_17

Suardi, G. (1752). *Nuovi istromenti per la descrizione di diverse curve antiche e moderne e di molto altre, che servir possono alla speculazione de' geometri, ed all'uso de' pratici*. Brescia: Rizzardi. https://wellcomecollection.org/works/dhpgeu4a/items. Accessed 15 July 2024.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169. https://doi.org/10.1007/BF00305619

Tall, D. (2012). *A sensible approach to Calculus*. From David Tall website: https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2012z-sensible-calculus.pdf. Accessed 15 July 2024.

Tournès, D. (2009). *La construction tractionnelle des équations différentielles*. Blanchard. ISBN: 978–2–85367–247–4

Yoon, C., Thomas, M., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics, 78*, 371–393. https://doi.org/10.1007/s10649-011-9329-y