On Independence and Compound and Iterated Conditionals

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Abstract. Understanding the logic of uncertain conditionals is a key problem in the new paradigm psychology of reasoning and related fields. We investigate conjunctions of conditionals, iterated conditionals, and independence within the theory of logical operations on conditionals, where compound conditionals are suitably defined as conditional random quantities. We show how conjunctions of conditionals and conditionals which feature conditionals in the antecedent and the consequent can be rationally interpreted. In particular, we study the behavior of such objects under different logical constraints, by also considering a kind of "independence" property. Unlike alternative approaches, in our framework we avoid counterintuitive consequences, which is necessary for understanding, or improving, human and artificial rationality in general.

Keywords: Coherence · Compound and iterated conditionals · Conditional random quantities · Conditional bets · Independence · Probabilistic reasoning

1 Introduction

Understanding reasoning about conditionals is key to understanding human rationality and artificial cognition, and knowledge of independence is essential to human rationality and artificial cognition. In this paper, we will consider how compound and iterated conditionals can be used to convey information about independence. We take a probabilistic approach to this topic. Probabilistic analyses have recently become popular in artificial intelligence, philosophy, and in the psychology of reasoning.

In this framework, the basic question is how to evaluate the probability of a conditional *if H then A*. Many philosophers and psychologists interpret the

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probability of a conditional *if H then A* as the conditional probability $P(A|H)$, known as *Adams's Thesis* ([1]), also called *the Equation* (see, e.g., [13,18,59,60]), or *Conditional Probability Hypothesis (CPH)* ([8,47,57]). Many psychological experiments confirm this hypothesis. People understand the probability of a conditional to be the conditional probability (e.g., [21,24,44,45,48,49,51,54]). For psychological experiments on the Equation and three-valued logics see also [2,3,55] A central problem is reasoning about sentences which include conditionals, like iterated (also called "nested"; see, e.g., [20,37,41]) conditionals or compounds of conditionals (i.e., conjunctions featuring (non-material) conditionals in their conjuncts). As David Lewis has shown with his triviality results ([42]), such objects cannot be represented simply by conditional probability. In our theory of logical operations among conditionals we define compound conditionals as suitable conditional random quantities (see e.g., [30,33]); this allows us to avoid such triviality results ([26,52,57,58]). In the subjective theory of de Finetti, given two events A and C, with $A \neq \emptyset$, the probability $P(C|A)$ of the conditional event $C|A$, for an agent, measures their degree of belief that C will be true, by assuming that A is true. The conditional event $C|A$ is looked at as a three-valued object which is *true* when A and C are both true (i.e., AC is true), *false* when A is true and C is false (i.e., $A\overline{C}$ is true), and *void* when A is false (i.e., $\overline{A}$ is true). In numerical terms its indicator, still denoted by $C|A$, takes the value 1, or 0, or $P(C|A)$, according to whether the conditional event is true, or false, or void, respectively. Then, by setting $P(C|A) = x$, the indicator of $C|A$ can be represented as

$$C|A = AC + x\overline{A} = \begin{cases} 1, \text{ if } AC \text{ is true,} \\ 0, \text{ if } A\overline{C} \text{ is true,} \\ x, \text{ if } \overline{A} \text{ is true.} \end{cases} \tag{1}$$

Notice that the prevision of the indicator, denoted by $\mathbb{P}(C|A)$, coincides with $P(C|A)$. Indeed, as $P(AC) = P(C|A)P(A)$, one has

$$\begin{aligned} \mathbb{P}(C|A) &= 1 \cdot P(AC) + 0 \cdot P(\overline{A}C) + P(C|A) \cdot P(\overline{A}) \\ &= P(C|A) \cdot P(A) + P(C|A) \cdot P(\overline{A}) = P(C|A). \end{aligned}$$

Of special interest are *conjunctions of conditionals*, like $(A|H) \wedge (B|K)$, and *iterated conditionals*, like $(B|K)|(A|H)$. Such objects, as $A|H$ and $B|K$, may not only be true or false but also void, and must be defined, in order to preserve some basic logical and probabilistic properties, on more than three conditions and hence cannot typically be conditional events any longer ([7,25,35,36,56]). They are defined in the setting of coherence as suitable conditional random quantities.

There are long-standing disagreements about conjunctions of conditionals and how to analyse them in natural language ([6,17,43]): how should conjunctions of the form *if A then C* & *if not- A then C* be interpreted, where "&" denotes a natural language "and". These conjunctions can be used to "sum up" the valid inference of Dilemma: inferring C from a proof of C from A and a proof of C from not-A. An acceptable analysis of *if A then C* & *if not- A then C* must have the result that C validly follows from it. We will show below in the formal section of this paper that our account does have this result.

Another important use of these conjunctions is to indicate that two propositions or events are independent. Consider:

(S1) If your children are vaccinated (H), they will not get autism (A).

We could use (S1) in certain contexts to indicate, pragmatically, that developing autism is independent of having a vaccination. To be as clear and explicit as possible about this fact, doctors could assert this conjunction for the benefit of parents whose children did not have autism at birth:

(S2) If your children are vaccinated (H), they will not get autism (A), and if they are not vaccinated (not-H), they will not get autism (A).

After all, a use of (S1) could (mistakenly) be interpreted as stating that the vaccination prevents autism, but (S2) makes it perfectly clear that (actually) independence is being conveyed. A formal analysis will be given in Sect. 2.4 and in Sect. 3.1.

One recent view, inferentialism (see, e.g., [16,38,61]), of conditionals like (S1) is that they are somehow "non-standard". A "standard" conditional *if A then C* is supposed to be one supported by a deductive, inductive, abduction, or other relation between A and C ([14,15]). As the term "inferentialism" suggests, this account of conditionals is based on the notion that the conditional is like an inference from A to C in which A supplies a positive reason for believing C. If we call a conditional for which such a relation holds between A and C a *dependence conditional*, and a conditional, like (S1), in which A and C are independent an *independence conditional*, then inferentialism states that only dependence conditionals are "standard" conditionals, and that independence conditionals are "non-standard", implying that conjunctions of independence conditionals are "non-standard" as well. We, however, can see nothing non-standard about independence conditionals like (S1) or conjunctions of them like (S2), or non-standard about Dilemma inferences for that matter, and we will focus on conditionals in the form *(if A then C)* & *(if not-A then C)* in this paper, treating them as standard conditionals that sometimes have the very important pragmatic role of indicating that A and C are independent (for critical comments on inferentialism, see [4,9,10,40,48,51]).

Let us consider an example that we can more easily make probability judgments about. (This example is derived from [17], and developed with Simone Sebben for psychological experiments.) Suppose two people are on the way to an airport in a car. They have enough time to make their flight as long as their car does not break down. There is a probability of 0.5 that the car will break down, and a probability of 0.5 that the passenger will cross their fingers. But the driver trusts the car, and not at all finger crossing, and says to the passenger:

(S3) If you cross your fingers (F), then we will be in time (T), and if you do not cross your fingers ($\bar{F}$), then we will be in time.

Following Lance ([39]), we could attempt to reason about (S3) in the following way. The probability that the car will break down is 0.5. Suppose first the car does not break down. Then (S3) is supposedly "true". Suppose second the car breaks down. Then (S3) is supposedly "false". Thus, (S3) is supposedly "true" half the time and has a probability of 0.5. The problem for Lance with this way of reasoning is that he also wants to hold the Conditional Probability Hypothesis that the probability of a natural language conditional, $P(\text{if } A \text{ then } C)$, is the conditional probability of C *given* A. As we have already remarked, there is certainly much to recommend this hypothesis $P(\text{if } A \text{ then } C) = P(C|A)$, which can be traced back to de Finetti ([11,12]). There are strong grounds for it in philosophical logic ([18]) and in experiments in the psychology of reasoning ([46,48,50,51]). By this relation in this example, $P(\text{if } F \text{ then } T) = P(T|F) = 0.5$. Lance thus claims about examples like this that $P((\text{if } F \text{ then } T) \,\&\, (\text{if not-}F \text{ then } T)) = P(\text{if } F \text{ then } T) = P(T|F)$.

However, Lewis ([42]) proved that $P(\text{if } A \text{ then } C) = P(C|A)$ cannot be combined with claiming that these conditionals are simply true or false, as in Lance's attempted reasoning. Cantwell ([6]) has also proven that there is a serious problems with Lance's claims about conjunctions like (S3) and holding that $P(\text{if } F \text{ then } T) = P(T|F)$ and $P((\text{if } F \text{ then } T) \,\&\, (\text{if not-}F \text{ then } T)) = P(\text{if } F \text{ then } T) = P(T|F)$. The underlying problem for Lance can be illustrated by an instance of a conditional that is not a conjunction of conditionals:

(S4) If you cross your fingers (F) or we will be in time (T), then we will be in time (T).

Let B be the car breaks down, and consider the case in which the possibilities below have equal probabilities $\frac{1}{4}$, with $P(BT) = P(\overline{B}\,\overline{T}) = 0$:

$$BF\overline{T},\ B\overline{F}\overline{T},\ \overline{B}FT,\ \overline{B}\,\overline{F}T.$$

Then by Lance's reasoning, (S4) is "true" 50% of the time, because (S4) is true when $\overline{B}FT$ is true, and when $\overline{B}\,\overline{F}T$ is true, since in both cases it has a true antecedent and a true consequent; moreover (S4) is false when $BF\overline{T}$ is true, because it has a true antecedent and a false consequent, and (S4) is also false when $B\overline{F}\overline{T}$ is true, since in this case it is not true for Lance, who only classifies conditionals as true or false. Then, for Lance the conditional (S4), which coincides in our framework with the conditional event $T|(F \vee T)$, "should" have a probability of 0.5. However,

$$P(T|(F \vee T)) = \frac{P(T)}{P(F \vee T)} = \frac{P(\overline{B}FT) + P(\overline{B}\,\overline{F}T)}{P(BF\overline{T}) + P(\overline{B}FT) + P(\overline{B}\,\overline{F}T)} = \frac{2}{3}.$$

Edgington knows that, given Lewis's proof, conditionals cannot be simply true or false in the actual world in general, and she follows Bradley ([5]) in holding that a conditional *if A then C* can only be simply true when $A \,\&\, C$ is true, and otherwise *if A then C* can only be made true by pairs of possible worlds ([19]). For example, suppose not-F holds in the actual world. Then in

Bradley's account, *if* F *then* T would be made "true" by the actual world and one alternative world where $F \,\&\, T$ was true and made "false" by another alternative world where F & not-T was true. We might express this account informally by saying that *if* F *then* T is true relative to one alternative to the actual world and false relative to another. People may judge that one of these alternatives is more probable than the other, or they may judge them equally probable.

Our approach is different. It goes back to de Finetti ([11,12]) and compares the assertion of an indicative conditional, like *if* F *then* T, closely to a conditional bet, *if* F *then I bet that* T. In our analysis, this indicative conditional is true, and the conditional bet is won, when $F\&T$ holds, the indicative conditional is false, and the conditional bet is lost, when F & not-T holds. The indicative conditional and the conditional bet are "void" when not-F holds. We will present our analysis formally below, showing that it implies in general that the expected value, or prevision in de Finetti's terms, of *if* A *then* C is $P(C|A)$. Our account, which properly manages void cases and where the Import-Export principle is not valid, avoids Lewis' triviality results ([30,57]).

We investigate conjunctions of conditionals *if* H *then* A & *if* K *then* B formally below, within an extended account in betting terms. In this extended account, we can say informally that *if* H *then* A & *if* K *then* B is fully true when $AHBK$ holds, is partly true when $\overline{H}BK$ holds, or $\overline{K}AH$ holds, is false when $\overline{A}H \vee \overline{B}K$ holds, and is void when $\overline{H}\,\overline{K}$ holds. In our account, the sentence (S3), *if* F *then* T & *if not-F* *then* T, has a prevision, i.e., an expected value, of .25 (see Sect. 2.4 for a formal analysis). Our account can be compared to McGee's account ([43]) to a limited extent, but there are, as well, differences between ours and McGee's; for instance, we do not require the general validity of the Import-Export principle.

Finally, we will consider the iterated conditional sentence

(S5) If we will be in time (T) when you do not cross your fingers ($\overline{F}$), then we will be in time when you cross your fingers (F).

We will show that the "intuitive" assertion "*the probability of the sentence* ($S5$) *coincides with* $P(T|F)$ " in our formalism is correct. This result is another way of looking at the "independence" of the sentences *"if* F *then* T" and *"if not-F then* T".

The paper is organized as follows. In Sect. 2 we give an interpretation of the possible values of the conjunction $(A|H) \wedge (B|K)$ of two conditional events $A|H$ and $B|K$. We illustrate a real world application within the context of multiple bets. We also recall the iterated conditional $(B|K)|(A|H)$. Then, we consider the compound conditionals $(A|H) \wedge (A|\overline{H})$ and $(A|H)|(A|\overline{H})$ in order to formalize the intuitions on the sentences (S3) and (S5), by discussing the aspects of "independence". In Sect. 3, we recall a general notion of iterated conditional and we study the particular objects $A|[(A|H) \wedge (A|\overline{H})]$ and $A|(A|H)$, in order to examine the *p-validity* of the inferences from (S2) to A, and from (S1) to A. Finally, in Sect. 4 we give some conclusions.

2 Conjunctions and Iterated Conditionals

In this section we first give an interpretation of the possible values of conjoined conditionals, which is in agreement with [6,43]. Then, we recall the notion of iterated conditionals. In particular, in order to correctly formalize the intuition about the sentences ($S3$) and ($S5$), we analyze the conjunction $(A|H) \wedge (A|\overline{H})$ and the iterated conditional $(A|H)|(A|\overline{H})$.

2.1 Interpretation of the Possible Values of $(A|H) \wedge (B|K)$

In our approach the conjunction of two conditional events $A|H$ and $B|K$ is a conditional random quantity, defined in the setting of coherence as

$$(A|H) \wedge (B|K) = \begin{cases} 1, \text{ if } AHBK \text{ is true,} \\ 0, \text{ if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ P(A|H), \text{ if } \overline{H}BK \text{ is true,} \\ P(B|K), \text{ if } AH\overline{K} \text{ is true,} \\ \mathbb{P}[(A|H) \wedge (B|K)], \text{ if } \overline{H}\,\overline{K} \text{ is true.} \end{cases} \tag{2}$$

In more explicit terms, by setting $P(A|H) = x$, $P(B|K) = y$, it holds that

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + yAH\overline{K})|(H \vee K),$$

with

$$\mathbb{P}[(A|H) \wedge (B|K)] = \\ = P(AHBK|(H \vee K)) + xP(\overline{H}BK|(H \vee K)) + yP(AH\overline{K}|(H \vee K)).$$

Notice that by coherence ([30, Theorem 7])

$$\max\{x + y - 1, 0\} \leq \mathbb{P}[(A|H) \wedge (B|K)] \leq \min\{x, y\}. \tag{3}$$

Intuitively, in agreement with [6,43], we could say that

- when $AHBK$ is true, i.e., both conditional events $A|H$ and $B|K$ are true, then the conjunction is "true", and its numerical value is 1
- when $\overline{A}H \vee \overline{B}K$ is true, that is at least a conditional event is false, the conjunction is "false", and its numerical value is 0
- when $\overline{H}BK$ is true, that is $A|H$ is void and $B|K$ is true, the conjunction is "partly true", and its numerical value is the conditional probability $P(A|H)$,
- when $AH\overline{K}$ is true, that is $A|H$ is true and $B|K$ is void, the conjunction is "partly true", and its numerical value is the conditional probability $P(B|K)$,
- when $\overline{H}\overline{K}$ is true, that is both conditional events are void, the conjunction is "void", and its numerical value is its prevision $\mathbb{P}[(A|H) \wedge (B|K)]$.

Then, in a bet on $(A|H) \wedge (B|K)$, the prevision $\mathbb{P}[(A|H) \wedge (B|K)]$ is the amount you agree to pay in order to receive the random win $(A|H) \wedge (B|K)$, that is

- when $AHBK$ is true, you receive 1 (you "win"),

– when $\overline{A}H \vee \overline{B}K$ is true, you receive 0 (you "lose"),
– when $\overline{H}BK$ is true, you receive x, (you "partly win"),
– when $AH\overline{K}$ is true, you receive y, (you "partly win"),
– when $\overline{H}\,\overline{K}$ is true, you receive back the amount you paid $\mathbb{P}[(A|H) \wedge (B|K)]$.

We observe that $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$, i.e. conjunction satisfies the commutativity property; moreover, when $H = K$, the conjunction $(A|H) \wedge (B|H)$ reduces to the conditional event $AB|H$.

2.2 A Real World Application of the Conjunction $(A|H) \wedge (B|K)$

In this section, based on an example given in [22], we illustrate an instance of a real world interpretation of the conjunction $(A|H) \wedge (B|K)$ in terms of the return of a double-bet. Let us consider two football matches. For each (valid) match the possible outcomes are: *Home win*, *Draw*, and *Away win*. Let us consider two single bets on the two events

A = *The outcome of match 1 is Home win*,
B = *The outcome of match 2 is Away win*.

In a single bet, when a match is not being played or abandoned (i.e., it is not valid) the bet is cancelled and the stake will be refunded (in this case the bet is called off). Let us define the events H=*the match 1 is valid*, and K = *the match 2 is valid*. Then, actually, we have to consider two bets on the conditional events:

$A|H$ = *The outcome of match 1 is Home win, given that it is valid*,
$B|K$ = *The outcome of match 2 is Away win, given that it is valid*.

In a bet on $A|H$, with $P(A|H) = x \in [0, 1]$, for every s_1, you agree to pay xs_1 and to receive $s_1(AH + x\overline{H})$, that is you receive s_1, or 0, or xs_1, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true (bet called off). In equivalent terms, by setting $Q_1 = \frac{1}{x}$ (when $x \neq 0$) and $r_1 = xs_1$, as $s_1 = r_1Q_1$, in a bet on $A|H$ you agree to pay r_1 and to receive $Q_1r_1(AH + \frac{1}{Q_1}\overline{H}) = Q_1r_1AH + r_1\overline{H}$. Similarly, in a bet on $B|K$ with $P(B|K) = y$, by setting $Q_2 = \frac{1}{y}$ and $r_2 = ys_2$, you agree to pay ys_2 and to receive $s_2(BK + y\overline{K})$, or equivalently as $s_2 = r_2Q_2$, you agree to pay r_2 and to receive $Q_2r_2BK + r_2\overline{K}$.

A *double-bet* is a linked series of the two single bets, where the return from one bet is automatically staked on the other bet. Then, if one of the two single bets is void, the double-bet continues on the remaining bet (the double-bet becomes a single bet). In our approach, the return of a double-bet can be related to the conjunction $(A|H) \wedge (B|K)$, when its prevision, $z = \mathbb{P}[(A|H) \wedge (B|K)]$, coincides with the product xy (notice that the product xy satisfies the inequalities in (3)). Based on (2), we observe that

$$(A|H) \wedge (B|K) = AHBK + x\overline{H}BK + y\overline{K}AH + z\overline{H}\,\overline{K}. \quad (4)$$

Moreover, when $z = xy$ it holds that

$$\begin{aligned}(A|H) \wedge (B|K) &= AHBK + x\overline{H}BK + y\overline{K}AH + xy\overline{H}\,\overline{K} \\ &= (AH + x\overline{H})(BK + y\overline{K}) = (A|H) \cdot (B|K).\end{aligned} \tag{5}$$

In a bet on $(A|H) \wedge (B|K)$, for every s, you agree to pay zs and to receive the random win $s(AHBK + x\overline{H}BK + y\overline{K}AH + z\overline{H}\,\overline{K})$. In particular, when $z = xy = \frac{1}{Q_1 Q_2}$, by setting $r = xys$, that is $s = Q_1 Q_2 r$, you agree to pay the amount r and to receive

$$\begin{aligned}&Q_1Q_2r(AHBK + \tfrac{1}{Q_1}\overline{H}BK + \tfrac{1}{Q_2}\overline{K}AH + \tfrac{1}{Q_1Q_2}\overline{H}\,\overline{K}) \\ &= Q_1Q_2rAHBK + Q_2r\overline{H}BK + Q_1rAH\overline{K} + r\overline{H}\overline{K} \\ &= r(Q_1AH + \overline{H})(Q_2BK + \overline{K}),\end{aligned}$$

that is to receive

$$\begin{cases} Q_1Q_2r & \text{(win),} \quad \text{if } AHBK \text{ is true ,} \\ 0 & \text{(lose),} \quad \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ Q_2r & \text{(partly win),} \quad \text{if } \overline{H}BK \text{ is true,} \\ Q_1r & \text{(partly win),} \quad \text{if } AH\overline{K} \text{ is true,} \\ r & \text{(bet called off),} \quad \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

When a match is cancelled, the double-bet reverts to a single bet on the remaining match, with the double-bet void when both matches are cancelled, in which case your stake will be refunded.

Based on this normative analysis, new psychological predictions can easily be derived for future experiments: E.g., what exchange price would participants assign in a betting situation according to the described conditions? It is psychologically interesting to investigate whether people assume (as usually being done by bookmakers) or do not assume "independence" among the conditional events in such cases.

2.3 The Conjunction $(A|H) \wedge (A|K)$

When $B = A$ formula (2) becomes

$$(A|H) \wedge (A|K) = \begin{cases} 1, \text{ if } AHK \text{ is true,} \\ 0, \text{ if } \overline{A}H \vee \overline{A}K \text{ is true,} \\ P(A|H), \text{ if } A\overline{H}K \text{ is true,} \\ P(A|K), \text{ if } AH\overline{K} \text{ is true,} \\ \mathbb{P}[(A|H) \wedge (A|K)], \text{ if } \overline{H}\,\overline{K} \text{ is true,} \end{cases} \tag{6}$$

with $\mathbb{P}[(A|H) \wedge (A|K)]$ satisfying the inequalities ([33, Theorem 9])

$$P(A|H)P(A|K) \leq \mathbb{P}[(A|H) \wedge (A|K)] \leq \min\{P(A|H), P(A|K)\}. \tag{7}$$

Moreover, when $HK = \emptyset$, it holds that $\mathbb{P}[(A|H) \wedge (B|K) = P(A|H)P(B|K)]$ and, in particular, $\mathbb{P}[(A|H) \wedge (A|K) = P(A|H)P(A|K)]$, as shown in the next section.

2.4 The Conjunction $(A|H) \wedge (A|\overline{H})$

The conjunction $(A|H) \wedge (B|K)$, when $B = A$ and $K = \overline{H}$, reduces to the conjunction $(A|H) \wedge (A|\overline{H})$. By setting $P(A|H) = x$ and $P(A|\overline{H}) = y$, from (2) one has

$$(A|H) \wedge (A|\overline{H}) = (A|H) \cdot (A|\overline{H}) = \begin{cases} 0, & \text{if } \overline{A}H \vee \overline{A}\,\overline{H} \text{ is true}, \\ x, & \text{if } A\overline{H} \text{ is true}, \\ y, & \text{if } AH \text{ is true}. \end{cases}$$

Then $(A|H) \wedge (A|\overline{H}) = xA\overline{H} + yAH$. Moreover,

$$\begin{aligned} &\mathbb{P}[(A|H) \wedge (A|\overline{H})] = x\,P(A\overline{H}) + y\,P(AH) \\ &= x\,P(A|\overline{H})P(\overline{H}) + y\,P(A|H)P(H) = xy\,P(\overline{H}) + xy\,P(H) = xy\,. \end{aligned} \tag{8}$$

In particular, $\mathbb{P}[(A|H) \wedge (A|\overline{H})] = 1$ if and only if $P(A|H) = P(A|\overline{H}) = 1$. Moreover, in this particular case it holds that $(A|H) \wedge (A|\overline{H}) = A\overline{H} + AH = A$, with $P(A) = P(A|H)P(H) + P(A|\overline{H})P(\overline{H}) = P(H) + P(\overline{H}) = 1$.

The previous analysis, applied with $A = T$ and $H = F$, allows to correctly formalize our intuition about the conjunction (S3), that is, from $P(T|F) = P(T|\overline{F}) = 0.5$ it follows that $\mathbb{P}[(T|F) \wedge (T|\overline{F})] = P(T|F)P(T|\overline{F}) = 0.25$.

Remark 1. Let us consider two conditional events $A|H$ and $B|K$, with $P(A|H) = P(B|K) = 0.5$. To the pair $(A|H, B|K)$ we can associate the four conditional constituents ([32]):

$$(A|H) \wedge (B|K),\ (A|H) \wedge (\overline{B}|K),\ (\overline{A}|H) \wedge (B|K),\ (\overline{A}|H) \wedge (\overline{B}|K),$$

which are such that

$$(A|H)\wedge(B|K)+(A|H)\wedge(\overline{B}|K)+(\overline{A}|H)\wedge(B|K)+(\overline{A}|H)\wedge(\overline{B}|K) = A|H+\overline{A}|H = 1.$$

Then,

$$\mathbb{P}[(A|H)\wedge(B|K)]+\mathbb{P}[(A|H)\wedge(\overline{B}|K)]+\mathbb{P}[(\overline{A}|H)\wedge(B|K)]+\mathbb{P}[(\overline{A}|H)\wedge(\overline{B}|K)] = 1.$$

In particular

$$P(A|H) = \mathbb{P}[(A|H) \wedge (B|K)] + \mathbb{P}[(A|H) \wedge (\overline{B}|K)], \tag{9}$$

and hence, if $\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)$, then $\mathbb{P}[(A|H) \wedge (\overline{B}|K)] = 0$. By applying the previous reasoning to the sentence (S3), with any (natural language) conjunction & satisfying (9), that is

$$P(T|F) = P[(T|F)\&(T|\overline{F})] + P[(T|F)\&(\overline{T}|\overline{F})],$$

from $P[(T|F)\&(T|\overline{F})] = P(T|F) = 0.5$ it would follow the unlikely result that $P[(T|F)\&(\overline{T}|\overline{F})] = 0$.

In our approach, when $HK = \emptyset$ it holds that

$$\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(A|H) \cdot (B|K)] = P(A|H)P(B|K);$$

see [28, Section 5], where this result is interpreted as a case of uncorrelation between two random quantities. Then, concerning (S3), we have that $\mathbb{P}[(T|F) \wedge (T|\overline{F})] = \mathbb{P}[(T|F) \wedge (\overline{T}|\overline{F})] = \mathbb{P}[(\overline{T}|F) \wedge (T|\overline{F})] = \mathbb{P}[(\overline{T}|F) \wedge (\overline{T}|\overline{F})] = 0.25$.

We note that the object $(A|H) \wedge (A|\overline{H})$ is relevant to connexive logic [62], as its negated version corresponds to the well-known connexive principle *Aristotle's Second Thesis* $({\sim}((H \rightarrow A) \wedge ({\sim}H \rightarrow A)))$. A systematic study on connexive principles within our framework of logical operations on conditionals has been done in [52], where it is shown that Aristotle's Second Thesis is not valid. This result follows by observing that the conjunction $(A|H) \wedge (A|\overline{H})$ is not constant and equal to 0. However, in a framework where the Aristotle's Second Thesis were valid it would follow, for instance, that the sentence (S3) should be always false. Moreover, experimental psychological data suggests that most people interpret Aristotle's Second Thesis as not valid [53].

2.5 The Iterated Conditional $(B|K)|(A|H)$

The notion of an iterated conditional $(B|K)|(A|H)$ is based on a structure like (1), that is $\Box|\bigcirc = \Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}$, where $\Box$ denotes $B|K$, $\bigcirc$ denotes $A|H$, $\Box \wedge \bigcirc$ is the conjunction $(B|K) \wedge (A|H)$, and where we set $\mathbb{P}(\Box|\bigcirc) = \mu$. In the framework of subjective probability $\mu = \mathbb{P}(\Box|\bigcirc)$ is the amount that you agree to pay, by knowing that you will receive the random quantity $\Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}$. Then, given two conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, and a coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$, the iterated conditional $(B|K)|(A|H)$ is defined as (see, e.g., [27,28,30]):

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\,\overline{A}|H = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\overline{B}K \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \overline{H}\overline{B}K \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{H}\overline{K} \text{ is true,} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \end{cases} \tag{10}$$

where

$$\mu = \mathbb{P}[(B|K)|(A|H)] = \mathbb{P}[(B|K) \wedge (A|H) + \mu\,\overline{A}|H] = z + \mu(1-x).$$

Then

$$z = \mathbb{P}[(B|K) \wedge (A|H)] = \mathbb{P}[(B|K)|(A|H)]P(A|H) = \mu x\,;$$

in particular, $z = \mu$ when $x = 1$. Notice that, when $x > 0$, a bet on $(B|K)|(A|H)$ is called off when $\overline{A}H \vee \overline{H}\,\overline{K}$ is true. When $x = 0$, the bet is called off when $\overline{A} \vee \overline{H}$ is true. Moreover, in the particular case where $H = K = \Omega$, it holds that $\mu = P(B|A)$ and $(B|\Omega)|(A|\Omega) = AB + \mu\overline{A} = AB + P(B|A)\overline{A} = B|A$.

2.6 The Iterated Conditional $(A|H)|(A|\overline{H})$

We now examine the formal aspects related with the sentence $(S5)$. We observe that

$$(A|H)|(A|\overline{H}) = (A|H) \wedge (A|\overline{H}) + \mu(\overline{A}|\overline{H}) = \begin{cases} x, & \text{if } A\overline{H} \text{ is true,} \\ y + \mu(1-y), & \text{if } AH \text{ is true,} \\ \mu(1-y), & \text{if } \overline{A}H \text{ is true,} \\ \mu, & \text{if } \overline{A}\,\overline{H} \text{ is true.} \end{cases}$$

Then, by the linearity of prevision

$$\mu = xy + \mu(1-y),$$

that is $\mu y = xy$. Then $\mu = x$, when $y > 0$. Moreover, when $y = 0$, it can be verified that

$$(A|H)|(A|\overline{H}) = \begin{cases} x, & \text{if } A\overline{H} \text{ is true} \\ \mu, & \text{if } A\overline{H} \text{ is false} \end{cases} = xA\overline{H} + \mu(1 - A\overline{H}) \in \{x, \mu\}.$$

Hence, in a bet on $(A|H)|(A|\overline{H})$, we agree to pay μ, by receiving x, when $A\overline{H}$ is true, or by receiving back the paid amount μ, when $A\overline{H}$ is false (in this case the bet is called off). Then, by coherence, $\mu = x$. Therefore in all cases $\mathbb{P}[(A|H)|(A|\overline{H})] = P(A|H)$. By setting $A = T$ and $H = F$, we can apply the previous analysis to the sentence $(S5)$, by showing that our account is in agreement with the "intuitive" assertion that "*the probability of the sentence* $(S5)$ *coincides with* $P(T|F)$". Likewise, $\mathbb{P}[(A|\overline{H})|(A|H)] = P(A|\overline{H})$. In the context of connexive principles and compounds of conditionals ([52]), the objects $(A|\overline{H})|(A|H)$ and $(A|H)|(A|\overline{H})$ can be interpreted as the unnegated versions (i.e., the consequent of the main connective is not negated) of the *variations of Boethius' theses* $(H \to A) \to \neg(\neg H \to A)$ and $(\neg H \to A) \to \neg(H \to A)$, respectively (called (B3) and (B4) in [23]). In [52] these principles are not valid, which is consistent with the present context.

Remark 2. Notice that in the case of simple conditionals the relation $P(E|H) = P(E)$, when $P(E) \in]0,1[$, does not follow from some logical relation between E and H, but from probabilistic evaluations of the involved events (stochastic independence is a subjective relation). Indeed, in this case coherence allows any value in $[0,1]$ for $P(E|H)$. However, differently from the case of simple conditionals, concerning the particular object $(A|H)|(A|\overline{H})$ it holds that coherence always requires that $\mathbb{P}[(A|H)|(A|\overline{H})]$ must be equal to $P(A|H)$. Then, in this case $A|H$ is "independent" from $A|\overline{H}$ for any coherent evaluation.

Moreover in general, when $HK = \emptyset$, it holds that $\mathbb{P}[(A|H)|(B|K)] = P(A|H)$ and $\mathbb{P}[(B|K)|(A|H)] = P(B|K)$, with in particular $\mathbb{P}[(A|H)|(A|K)] = P(A|H)$ and $\mathbb{P}[(A|K)|(A|H)] = P(A|K)$. Therefore, when $HK = \emptyset$, $A|H$ and $B|K$ (in particular, $A|H$ and $A|K$) are "independent". Indeed, by setting

$$P(A|H) = x\,, \quad P(B|K) = y\,, \quad \mathbb{P}[(A|H) \wedge (B|K)] = z\,, \quad \mathbb{P}[(B|K)|(A|H)] = \mu\,,$$

and by recalling formula (10), it holds that $z = \mu x$. Moreover, under the hypothesis $HK = \emptyset$, it holds that $z = xy$ and hence, when $x > 0$, one has $\mu = y$.
If $x = 0$ (and hence $z = 0$), by observing that $AHBK = AH\overline{B}K = \emptyset$, formula (10) becomes

$$(B|K)|(A|H) = (A|H) \wedge (B|K) + \mu\, \overline{A}|H = \begin{cases} y, \text{ if } AH\overline{K} \text{ is true,} \\ \mu, \text{ if } AH\overline{K} \text{ is false.} \end{cases}$$

Then, by coherence, (when $x = 0$) it holds that $\mu = y$ and the iterated conditional $(B|K)|(A|H)$ is constant and coincides with $y = P(B|K)$.

Therefore, when $HK = \emptyset$, in all cases it holds that $\mathbb{P}[(B|K)|(A|H)] = P(B|K)$. Likewise, $\mathbb{P}[(A|H)|(B|K)] = P(A|H)$.

3 On General Iterated Conditionals

We recall that a family of conditional events $\mathcal{F} = \{E_i|H_i\,,\ i = 1, \ldots, n\}$ is *p-consistent* if and only if the assessment $(1, 1, \ldots, 1)$ on $\mathcal{F}$ is coherent. Moreover, a p-consistent family $\mathcal{F}$ *p-entails* a conditional event $E|H$ if and only if the unique coherent extension on $E|H$ of the assessment $(1, 1, \ldots, 1)$ on $\mathcal{F}$ is $P(E|H) = 1$ (see, e.g., [29]). We say that the inference from $\mathcal{F}$ to $E|H$ is *p-valid* if and only if $\mathcal{F}$ p-entails $E|H$. The characterization of p-entailment using the notion of conjunction has been given in [31]. In particular, given a p-consistent family $\mathcal{F}$ of n conditional events it holds that. $\mathcal{F}$ p-entails $E|H$ if and only if $\mathcal{C}(\mathcal{F}) \leq E|H$, where $\mathcal{C}(\mathcal{F}) = \bigwedge_{E_i|H_i \in \mathcal{F}} E_i|H_i$. Let two logically independent events A and H be given. Then, the assessment $P(A|H) = P(A|\overline{H}) = 1$ is coherent and the family $\{A|H, A|\overline{H}\}$ is p-consistent. As $P(A) = P(A|H)P(H) + P(A|\overline{H})P(\overline{H})$, from $P(A|H) = P(A|\overline{H}) = 1$ it follows that $P(A) = 1$; thus the inference from $\{A|H, A|\overline{H}\}$ to A is p-valid ([1]). In this section we show that this result is in agreement with the formalism of iterated conditionals. We first recall a general notion of iterated conditional given in ([34]). Let us consider two finite families of conditional events $\mathcal{F}_1$ and $\mathcal{F}_2$. We set

$$\mathcal{C}(\mathcal{F}_1) = \bigwedge_{E|H \in \mathcal{F}_1} E|H\,,\ \ \mathcal{C}(\mathcal{F}_2) = \bigwedge_{E|H \in \mathcal{F}_2} E|H\,,$$

and by definition ([31])

$$\mathcal{C}(\mathcal{F}_1) \wedge \mathcal{C}(\mathcal{F}_2) = \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) = \bigwedge_{E|H \in \mathcal{F}_1 \cup \mathcal{F}_2} E|H\,.$$

Then, under the assumption that $\mathcal{C}(\mathcal{F}_1)$ eq0, the iterated conditional $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)$ is defined as

$$\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_1) \wedge \mathcal{C}(\mathcal{F}_2) + \mu(1 - \mathcal{C}(\mathcal{F}_1))$$

where $\mu = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)]$. In a bet on $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)$ the quantity μ is the amount to be paid in order to receive the random amount $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)$. The characterization of the p-entailment in terms of iterated conditionals has been given in [34] (see also [26]). More precisely, given a p-consistent family $\mathcal{F}$ of n conditional events it holds that $\mathcal{F}$ p-entails $E|H$ if and only if the iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ is constant and equal to 1.

3.1 On the Iterated Conditional $A|[(A|H) \wedge (A|\overline{H})]$

By setting $\mathcal{F}_1 = \{A|H, A|\overline{H}\}$ and $\mathcal{F}_2 = \{A\}$, we will show that the p-entailment from $\{A|H, A|\overline{H}\}$ to A can be characterized by the property that the iterated conditional $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1) = A|((A|H) \wedge (A|\overline{H}))$ is constant and coincides with 1 ([26,34]).

Let us consider the following complex conditional sentence:

(S6) If ((A when H) and (A when not-H)), then A,

which we represent by the iterated conditional $A|((A|H) \wedge (A|\overline{H}))$. We will show the p-validity of the inference (S6) from the conjunction "*A when H and A when not- H*" to the conclusion A.

We set $P(A|H) = x, P(A|\overline{H}) = y, \mathbb{P}[A|((A|H) \wedge (A|\overline{H}))] = \mu$. Then

$$A \wedge (A|H) \wedge (A|\overline{H}) = (A|H) \wedge (A|\overline{H}) = \begin{cases} 0, & \text{if } \overline{A} \text{ is true,} \\ y, & \text{if } AH \text{ is true,} \\ x, & \text{if } A\overline{H} \text{ is true,} \end{cases}$$

and

$$\mu\,[1 - (A|H) \wedge (A|\overline{H})] = \begin{cases} \mu, & \text{if } \overline{A} \text{ is true,} \\ \mu\,(1-y), & \text{if } AH \text{ is true,} \\ \mu\,(1-x), & \text{if } A\overline{H} \text{ is true.} \end{cases}$$

Therefore

$$A|((A|H) \wedge (A|\overline{H})) = (A|H) \wedge (A|\overline{H}) + \mu\,[1 - (A|H) \wedge (A|\overline{H})] = \\ = \begin{cases} \mu, & \text{if } \overline{A} \text{ is true,} \\ y + \mu\,(1-y), & \text{if } AH \text{ is true,} \\ x + \mu\,(1-x), & \text{if } A\overline{H} \text{ is true,} \end{cases}$$

where we assume that $((A|H) \wedge (A|\overline{H}))$ does not coincide with the constant 0, that is $(x, y) \neq (0, 0)$. By the linearity of prevision and by (8), it holds that

$$\mu = \mathbb{P}[A|((A|H) \wedge (A|\overline{H}))] = xy + \mu(1 - xy)$$

or equivalently $\mu xy = xy$. Then, $\mu = 1$ when $x > 0$ and $y > 0$. When $x = 0$ and $y > 0$, it holds that

$$A|((A|H) \wedge (A|\overline{H})) = \begin{cases} \mu, & \text{if } \overline{A} \vee A\overline{H} \text{ is true,} \\ y + \mu\,(1-y), & \text{if } AH \text{ is true.} \end{cases}$$

By coherence, $\mu = y + \mu\,(1-y)$, that is $\mu y = y$, and hence $\mu = 1$. Likewise, when $y = 0$ and $x > 0$, it follows that $\mu = 1$. Thus, $\mu = 1$ in all cases, and hence the iterated conditional $A|((A|H) \wedge (A|\overline{H}))$ is constant and coincides with 1.

By applying the previous analysis to the sentence (S2), we can infer the p-validity of the inference from "*if your children will not get autism (A), when they are vaccinated (H), and your children will not get autism (A), when they are not vaccinated* ($\overline{H}$)" to the conclusion "*your children will not get autism (A)*".

3.2 On the Two Iterated Conditionals $A|(A|H)$ and $A|(A|\overline{H})$

Notice that, differently from the iterated conditional $A|((A|H) \wedge (A|\overline{H}))$ which coincides with the constant 1, we can verify that both the iterated conditionals $A|(A|H)$ and $A|(A|\overline{H})$ do not coincide with the constant 1. In other words, both the inferences from $A|H$ to A and from $A|\overline{H}$ to A are not p-valid. Indeed, by setting $P(A|H) = x$ and $\mathbb{P}[A|(A|H)] = \eta$, it holds that

$$A|(A|H) = A \wedge (A|H) + \eta\,(1 - A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ x + \eta\,(1-x), & \text{if } A\overline{H} \text{ is true,} \\ \eta\,(1-x), & \text{if } \overline{A}\,\overline{H} \text{ is true,} \\ \eta, & \text{if } \overline{A}H \text{ is true,} \end{cases}$$

which, in general, does not coincide with the constant 1. For instance, if we evaluate $P(A|H) = P(A|\overline{H}) = P(H) = \frac{1}{2}$, it holds that

$$A|(A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \frac{1}{2} + \frac{1}{2}\,\eta, & \text{if } A\overline{H} \text{ is true,} \\ \frac{1}{2}\,\eta, & \text{if } \overline{A}\,\overline{H} \text{ is true,} \\ \eta, & \text{if } \overline{A}H \text{ is true,} \end{cases}$$

with

$$\eta = P(AH) + (\frac{1}{2} + \frac{1}{2}\,\eta)P(A\overline{H}) + \frac{1}{2}\,\eta P(\overline{A}\overline{H}) + \eta P(\overline{A}H) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\,\eta + \frac{1}{8}\,\eta + \frac{1}{4}\,\eta\,,$$

that is $\eta = \frac{3}{4}$. Therefore, the inference from the premise "if H then A" to the conclusion A is not p-valid. By recalling the sentence (S1), if we evaluate that the probability is high of getting autism (event A), given vaccination (event H), it does not follow that the probability of getting autism (event A) is high.

By a similar analysis, by setting $P(A|\overline{H}) = y$ and $\mathbb{P}[A|(A|\overline{H})] = \nu$, it holds that

$$A|(A|\overline{H}) = A \wedge (A|\overline{H}) + \nu\,(1 - A|\overline{H}) = \begin{cases} 1, & \text{if } A\overline{H} \text{ is true,} \\ y + \nu\,(1-y), & \text{if } AH \text{ is true,} \\ \nu\,(1-y), & \text{if } \overline{A}H \text{ is true,} \\ \nu, & \text{if } \overline{A}\overline{H} \text{ is true,} \end{cases}$$

which, in general, does not coincide with the constant 1. Therefore, the inference from the premise "if $\overline{H}$ then A" to the conclusion A is not p-valid. Recalling the sentence (S1), if we evaluate that the probability is high of getting autism (event A), given no vaccination (event $\overline{H}$), it does not follow that the probability of getting autism (event A) is high.

On the contrary, as shown in Sect. 3.1, if we evaluate that the probability of getting autism (event A), given vaccination (event H), and the probability of getting autism (event A), given no vaccination (event $\overline{H}$), are both high, then the probability of getting autism (event A) is high.

Remark 3. What about the iterated conditional $((A|H) \wedge (A|\overline{H}))|A$? It can be verified that the inference from the premise A to the conjunction $(A|H) \wedge (A|\overline{H})$

is not p-valid, that is from $P(A) = 1$ it does not follow that $\mathbb{P}[(A|H) \wedge (A|\overline{H})] = 1$. Indeed, the assessment $(1, y, 1)$ on $\{A|H, A|\overline{H}, H\}$, with $y < 1$, is coherent. Then $P(A) = P(A|H)P(H) + P(A|\overline{H})P(\overline{H}) = 1$, while $\mathbb{P}[(A|H) \wedge (A|\overline{H})] = y < 1$.

We also observe that, by setting $P(A|H) = x$, $P(A|\bar{H}) = y$, $\mathbb{P}[((A|H) \wedge (A|\bar{H}))|A] = \eta$,

$$((A|H) \wedge (A|\overline{H}))|A = (A|H) \wedge (A|\overline{H}) + \eta\,(1 - A) = \begin{cases} \eta, & \text{if } \overline{A} \text{ is true,} \\ y, & \text{if } AH \text{ is true,} \\ x, & \text{if } A\overline{H} \text{ is true,} \end{cases}$$

and hence, by coherence, η must belong to the convex hull of the set of values $\{x, y\}$, that is $\eta \in [\min\{x, y\}, \max\{x, y\}]$. Moreover, by the linearity of prevision, $\eta = xy + \eta(1 - P(A))$, that is $\eta P(A) = xy$. In particular, when $P(A) > 0$, by setting $P(H) = t$ it holds that

$$\eta = \mathbb{P}[((A|H) \wedge (A|\overline{H}))|A] = \frac{P((A|H) \wedge (A|\overline{H}))}{P(A)} = \frac{xy}{xt + y(1-t)},$$

with $\eta \in [x, y]$ when $x < y$, and $\eta \in [y, x]$ when $x > y$.

4 Conclusions

In this paper, we investigated conjunctions of conditionals, iterated conditionals, and (stochastic) independence within the theory of logical operations on conditionals (i.e., interpreted by suitable conditional random quantities). We presented a formal and a more intuitive approach compared to alternative approaches in the literature and also avoided Lewis' triviality results. Our probabilistic analysis also lays the groundwork for expanding the current domain of new paradigm psychology of reasoning from basic conditionals to complex conditional structures, like nested and compound conditionals.

We gave an interpretation of the possible values of the conjunction $(A|H) \wedge (B|K)$ in agreement with [6,43]. This interpretation shows why more than three values are needed. We also illustrated this situation through a real world application within the context of multiple bets from which psychological predictions for future experiments can easily be derived.

Then, we formalized intuitions about the sentences (S3) and (S5) by using the compound conditionals $(A|H) \wedge (A|\overline{H})$ and $(A|H)|(A|\overline{H})$. These two objects can be related to the non-validity of Aristotle's Second Thesis and of the variation (B4) of Boethius' Thesis, respectively. Connexivity is a desired property of inferentialist accounts of conditionals (since *if A then not-A* also violates basic inferentialist intuitions), but more work is needed to fully understand formally the relations of inferentialism and our account.

We also discussed "independence" in sentences (S3) and (S5), based on the coherence conditions $\mathbb{P}[(A|H) \wedge (A|\overline{H})] = P(A|H)P(A|\overline{H})$ and $\mathbb{P}[(A|H)|(A|\overline{H})] = P(A|H)$.

We recalled a general notion of iterated conditional $\mathcal{C}(\mathcal{F}_2)|\mathcal{C}(\mathcal{F}_1)$. In particular we showed that the object $A|[(A|H) \wedge (A|\overline{H})]$ is constant and equal to 1, by characterizing the p-validity of the inference from the premise $(A|H) \wedge (A|\overline{H})$ to the conclusion A. In our framework, this inference amounts to the inference from $\{A|H, A|\overline{H}\}$ to A.

Finally, concerning the sentence (S1), we discussed the p-invalidity of the inferences from $A|H$ to A, and from $A|\overline{H}$ to A, by studying the iterated conditionals $A|(A|H)$ and $A|(A|\overline{H})$, respectively.

The present paper contributes to the understanding of judgments about independence and human and artificial rationality. It also provides a rationality framework for future experimental studies, which are needed to investigate its psychological plausibility.

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