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A 3D-printable machine for conics and oblique trajectories

Pietro Milici[a], Frédérique Plantevin[b] and Massimo Salvi[c]

[a]Machines4Math, Agrate Brianza (MB), Italy;
[b]Laboratoire de Mathématiques de Bretagne Atlantique–CNRS UMR 6205, Univ Brest, France;
[c]Ministero Istruzione, Università e Ricerca, Italy

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ABSTRACT
We propose an original machine that traces conics and some transcendental curves (oblique trajectories of confocal conics) by the solution of inverse tangent problems. For such a machine, we also provide the 3D-printable model to be used as an intriguing supplement for geometry, calculus, or ODE classes.

Classification: 97I40, 97U60, 51-03

KEYWORDS
Conics, tangent, slope fields, isogonal trajectories, tractional motion, 3D printer

1. Introduction

In 1637 *Géométrie*, Descartes introduced a general method to adopt polynomial algebra to solve geometrical problems. We have to keep in mind that the Cartesian method still relied on geometric constructions, and curves were accepted when continuously traced by ideal machines (exactness problem), as highlighted in [6]. Although the French philosopher gave some examples, the *Géométrie* did not provide a well-defined class of machines for algebraic constructions. A general theory for such constructions can be considered that of linkages, fixed-length rods jointed each other. The fundamental linkage theorem, "Kempe's universality" (proved in [14] even though with some flaws), states that any bounded part of a planar algebraic curve can be traced by a linkage (if we do not consider intersection problems due to physical rods).

Descartes' foundation limited geometry to algebraic curves. However, already in the second half of the 17th century, this boundary was no longer widely accepted by mathematicians, who looked for an appropriate geometrical legitimation to introduce non-algebraic curves. Among other less powerful geometrical methods, a general problem that originated a broad class of transcendental curves was the *inverse tangent problem* (that, in a modern setting, involves the geometrical solution of differential equations). Although to find an object tangent to a given curve while satisfying specific properties (the direct tangent problem) is present at least since classical Greek geometry, it was only after Descartes that mathematicians tried to consider new curves given the properties that their tangents have to satisfy. The first documented appearance of an inverse tangent problem is the "constant-subtangent problem" studied by Florimond Debeaune in 1638 [29]. In 1672 the architect Claude Perrault proposed a first operative construction of a curve given its tangent: a new curve, the tractrix, is introduced as the trace of a pocket watch on a plane while moving the endpoint of its chain along a straight line. The role of traction made such constructions termed "tractional." Until the first half of the 18th century, tractional constructions interested many influential mathematicians, from Leibniz to Euler (see [5], [31] or also, for a brief introduction, [7]). Quite curiously, it seems that the geometrical principle behind tractional constructions hides something interesting yet [9].

To mechanically solve an inverse tangent problem, we have to constrain a point to move along a direction. Instead of using a chain watch, this can be better accomplished by something that guides the direction of the motion, like the blade of a pizza-cutter or the front wheel of a bike (Figure 1).

CONTACT Pietro Milici. Email: p.milici@gmail.com

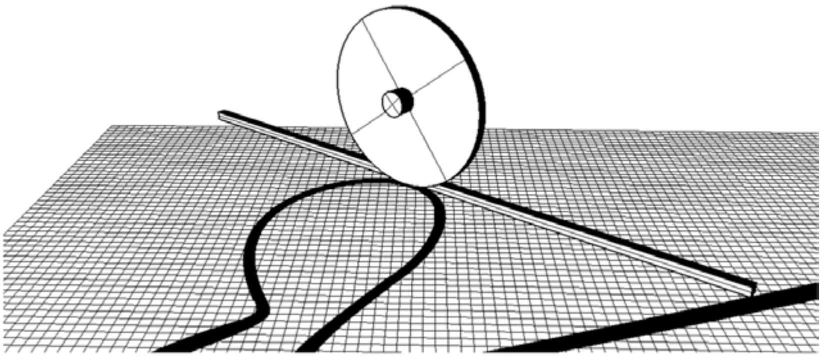

Figure 1. Considering a wheel rolling on a curve, the direction of the wheel (in the image represented by a bar) is the tangent to the curve.

Specifically, Descartes' machines pose "position constraints," i.e., according to mechanics, *holonomic* constraints (analytically, they pose relations between the position variables without any reference to the speed or the direction of the motion). On the other side, tractional motion poses "direction constraints" that analytically involve the derivatives, i.e., are *non-holonomic.* The constructive boundaries of tractional machines (linkages extended by wheels, firstly introduced in [19]) appeared only recently in [22] as a differential extension of Kempe's universality theorem. In this new setting, the language of polynomial algebra is extended by differential algebra, and the limit of constructions is no longer given by curves but by functions. The class of functions generable by tractional machines coincides with the algebraic differential functions, the solutions of non-trivial polynomials $P(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$. Tractional machines generate new interest in foundations [20] and didactical laboratory activities [18].

This paper proposes an original machine (invented by the first author) that traces conics and some transcendental curves (oblique trajectories of confocal conics) by the solution of inverse tangent problems. For such a machine, we also provide the 3D-printable models to be used as an intriguing supplement for laboratory activities. Therefore, the intended readers are instructors (or their best students) of geometry, calculus, and ODEs, including science and engineering courses.

2. Machines for conic sections

Machines to construct curves were historically relevant in geometry not only for practical/artistic aims but also for foundational/theoretical purposes (as the "exactness problem of geometric constructions" of Descartes [6] or Leibniz [4] in the early modern period). Specifically, we can note how the same curves can be traced by very different devices, each one physically implementing a specific geometric property. Therefore, finding new ways of constructing old curves can be interesting because other machines may recall different mathematical contents. In education, that can help foster a unifying vision of various mathematical topics [1].

After the straight line and the circle, the conic sections are possibly the most familiar shapes that we encounter in our everyday experience; the enlightenment of a flashlight on a wall embodies a circular cone cut by a plane. While already studied in the 4th century B.C. by Greek geometers such as Menaechmus, the most influential systematic works on conic sections and their properties are ascribed to Apollonius of Perga (around 200 B.C.). A conics-drawing device is the *perfect compass*, known to mathematicians of the 17th century but dating back to the 10th-century Arab mathematician Al Quhi ([26], [10]). As visible in Figure 2[1], consider a fixed axis OA and a telescopically-extendable rod OP constrained to keep constant the angle β with the axis. While OP rotates around its axis OA, OP moves on a circular cone. Furthermore, the rod OP can telescopically change its length. By keeping the tracer P in contact with a horizontal plane, the machine plots the intersection of a plane with a circular cone. The trace can be any conic section: introducing the angle α as the inclination of the axis OA and the plane, if $\alpha = \beta$ the result is a parabola, if $\alpha > \beta$ an ellipse (a circumference if $\alpha = \pi/2$), otherwise a hyperbola when $\alpha < \beta$.

[1] Image taken from the page of the site of the Associazione Macchine Matematiche (AMM) `http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=136&Itemid=216&lang=en`, where an explicative animation is also provided.

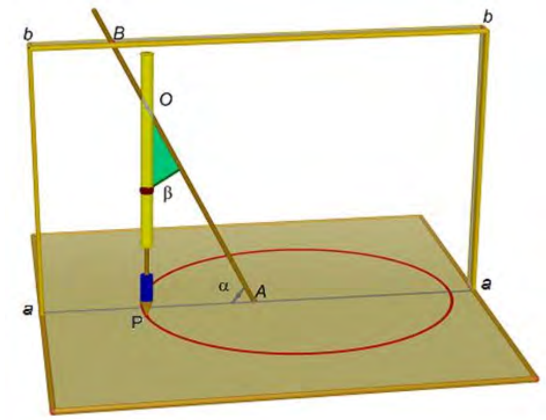

Figure 2. The perfect compass.

Although the perfect compass traces all conic sections, many machines draw specific conics; besides the compass for circles, many tools drawing ellipses, parabolas, and hyperbolas have been proposed since antiquity. Many science museums exhibit these machines with historical samples, like the National Museum of American History[2], or with machines reconstructed for didactical pathways, like the Associazione Macchine Matematiche (shortly: AMM)[3] of Modena, Italy.

Some constructions directly embody the definition of the traced curve (in the gardener's ellipse, the thread keeps constant the sum of the distances from the foci); others require a more in-depth analysis. Below, we spend a few words on a class of machines that, by adopting straight rods and guides, show the tangent line while tracing the conic. Such machines have in common the use of a deformable rhombus, whose sides can rotate while keeping jointed the final edges.

In the left of Figure 3, we can observe an ellipsograph. Given the foci O and B, the point D moves along the director circle centered in O. By the articulated rhombus $ABCD$, we can consider the intersection P of $\overline{OD}$ and $\overrightarrow{AC}$ (C slides on $\overrightarrow{AC}$). To prove that P defines the sought ellipse (i.e., $OP + PB$ is constant), note that $PD = PB$ (the triangles APB and APD are congruent) hence $OP + PB = OP + PD = OD$. Note also that $\overrightarrow{AC}$, the external bisector of the rays $\overrightarrow{PB}$ and $\overrightarrow{PO}$, is tangent to the ellipse in P. Indeed, no point Q on $\overrightarrow{AC}$ different from P would intersect the ellipse because $OQ + QB > OP + PB$: considering the triangle inequality for OQD, $OQ + QB = OQ + QD > OD = OP + PB$.

Similar reasoning applies to the other cases of Figure 3: for hyperbolas, $PA = PC$ because the triangles CPD and APD are congruent, hence $PO - PA = CO$ is constant; for parabolas, $PC = PA$ and $\overrightarrow{CP}$ is perpendicular to r. Note that the construction involves the director circle relative to a focus for the ellipse and the hyperbola; for the parabola, such a circle degenerates in a line, the directrix r (for some geometrical references see [15] or [8]).

These machines not only trace conics, but a rod also gives the direction of the tangent line in the tracing point. For any point of the curve, the ellipse tangent is the external bisector of the angle from the foci, while the hyperbola tangent is the internal bisector. The parabola has only one finite focus and the other one, according to projective geometry, is a point at infinity that defines the direction of the symmetry axis. In this case, the tangent bisects the angle between the rays by connecting the finite focus and the line passing through the focus at infinity. In Figure 3, the tangent is the external bisector of $\overrightarrow{PA}$ and $\overrightarrow{Pa}$.

3. Machines for dynamic slope fields

Graphical representations of slope fields involve the simultaneous drawing of directions at many points in the plane. This representation is a static one; by machines, we can extend this idea to dynamic slope fields.

[2] https://americanhistory.si.edu/collections/object-groups/ellipsographs
[3] http://www.macchinematematiche.org/

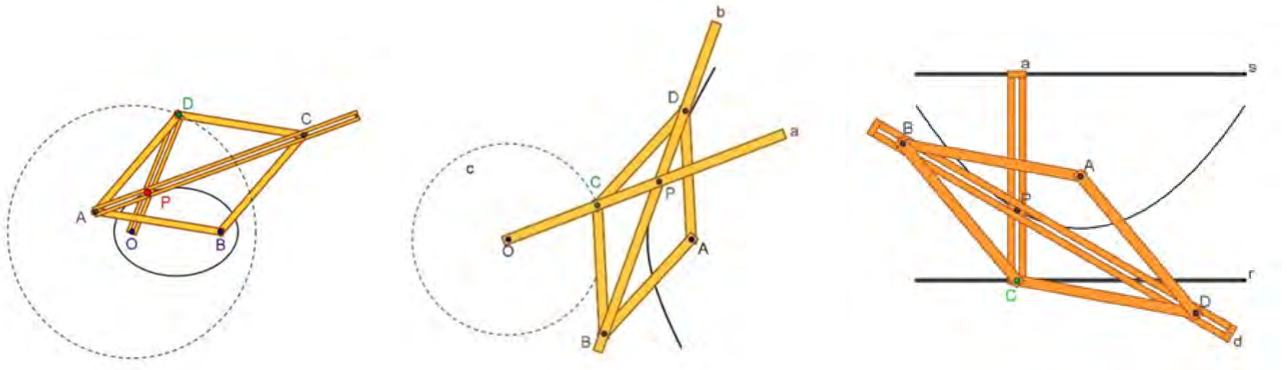

Figure 3. Machines tracing conics by "articulated rhombi:" ellipse (left), hyperbola (centre), parabola (right). Images from the AMM's site.

Indeed, by considering a rod r with a point P marked on it, P freely moving on the plane, we can generate a slope field by a mechanism that links the inclination of r to the position of P. Such an inclination dynamically defines a slope field over the plane.

To consider the rod as a tangent line to the sought curve defines an inverse tangent problem. Starting from an initial point, an orbit can be found by introducing a wheel in P with direction r. These finite tools permit constructing a geometrically exact, not iteratively approximated solution, although subject to physical errors.

In the following sections, by considering slope fields as the mathematical objects generated by tractional machines (that happens if the analytical counterpart is a first-order ODE), we introduce the slope fields that our new machine solves.

3.1. Isogonal trajectories of ellipses

Fixed two points F_1 and F_2, the slope field defined by the external bisector of the rays connecting a general point to F_1 and F_2 is solved by ellipses of foci F_1, F_2. To trace the hyperbolas with foci F_1 and F_2, we should consider the internal bisectors instead of the external ones. Indeed, confocal ellipses and hyperbolas intersect orthogonally (however, note that by constructing hyperbolas as slope field trajectories, we can trace only one of the two branches of the curve, the one containing the starting point). In 1850, as a generalization, the Italian Gaspare Mainardi [16] posed and solved the problem of determining the oblique trajectories of a system of confocal ellipses (in that sense, the hyperbolas are the *orthogonal trajectories* of the family of confocal ellipses): some examples of such curves are shown in Figure 4[4]. To find out the curves intersecting every member of a given pencil of curves at a constant angle is the problem of isogonal trajectories, a typical problem of differential geometry. Later on, the Indian Sir Asutosh Mukherjee (sometimes anglicized to Mookerjee) remarkably simplified Mainardi's solution by introducing hyperbolic functions [23] (see also [28, pp. 589ff]). The system of curves, cutting a system of confocal ellipses at a constant angle α not multiple of $\pi/2$, is given by

$$x = c\cos(\phi)\cosh(n(\lambda+\phi)), \qquad y = c\sin(\phi)\sinh(n(\lambda+\phi)),$$

where $2c$ is the distance between the foci, n is $\tan\alpha$, and λ is an integrating constant. These symmetrical forms have been included by A. R. Forsyth in his very widespread textbook on differential equations [13, p. 146] as an exercise. By Mukherjee form we can easily evince that:

Lemma 1. *The isogonal trajectories are transcendental if and only if α is not a multiple of $\pi/2$ (i.e., $n \in \mathbb{R}\setminus\{0\}$).*

Proof. Being λ related to the initial value, every trajectory has to be considered while keeping λ constant at the changes of ϕ. Hence, out of the cases of ellipses and hyperbolas, for $\phi = 2k\pi$ (for any integer k), we can consider the points $(c\cosh(n(\lambda+\phi)), 0)$. Therefore, we have infinite intersections between the trajectory

[4]Images obtained using `https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html`.

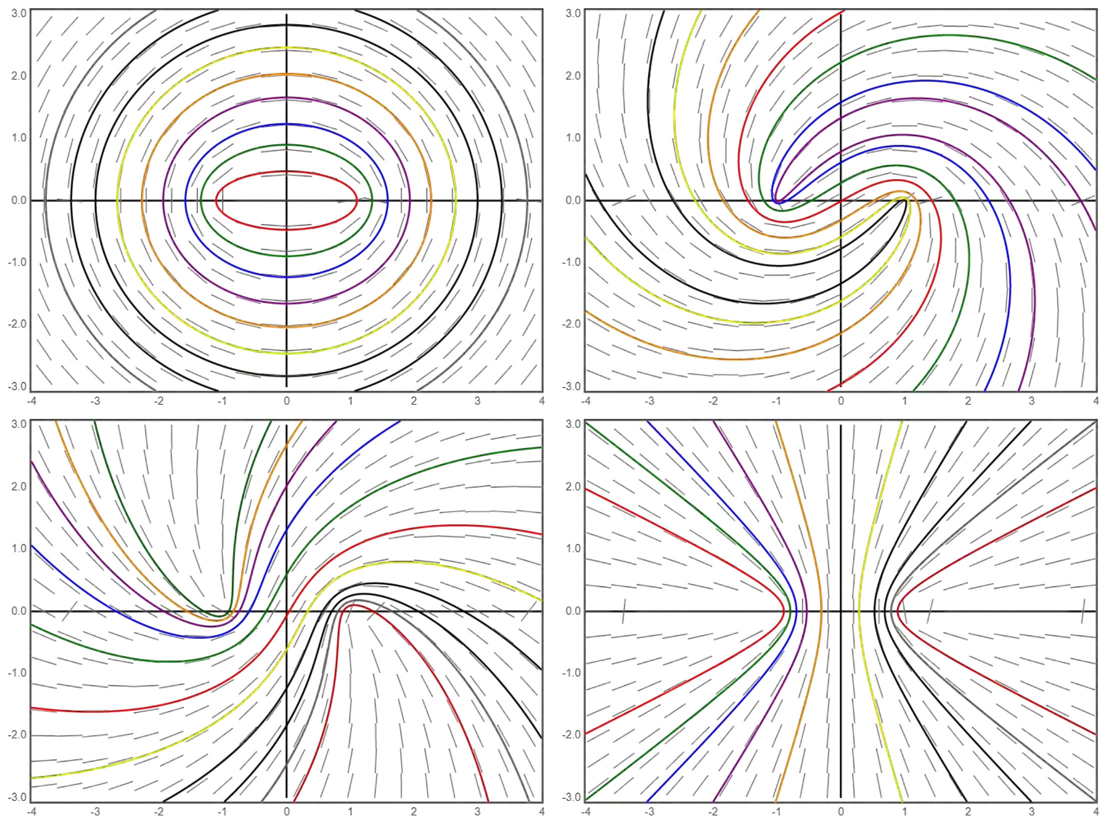

Figure 4. Isogonal trajectories of confocal ellipses (the foci are in $(\pm 1, 0)$) with the angles 0 (ellipses, top left), $\pi/6$ (top right), $\pi/3$ (bottom left), $\pi/2$ (hyperbolas, bottom right).

(that does not contain a straight line) and the x-axis. For Bézout's theorem, the solution curves other than ellipses and hyperbolas are transcendental. □

If the two foci coincide, ellipses degenerate in circles and hyperbolas in lines. Furthermore, an oblique trajectory has to keep constant the angle between the tangent and the radial direction; therefore, the sought curve is a logarithmic spiral, and a machine drawing it behaves as an equiangular compass [21]. From an asymptotic standpoint, far away from the foci, the behavior of oblique trajectories is somehow similar to logarithmic spirals (even when the two foci do not coincide).

3.2. Isogonal trajectories of parabolas

In the early 17th century, while conceiving machines for conical sections, Kepler imagined parabolas as ellipses with a focus at an infinite distance [12]. We intend to keep this idea and consider parabolas as bifocal conics with one focus at infinity in an oriented direction λ (oriented to define the side in which the parabola opens). As visible in Figure 5, the tangent at any point P of a parabola with (finite) focus F and oriented direction λ is the exterior bisector of $\angle FP\lambda$ (the angle formed by $\overrightarrow{PF}$ and $\overrightarrow{P\lambda}$, where the latter ray is parallel to and with the same orientation of λ).

Given the family of parabolas with the same focus and axis, we can consider the curves intersecting such a family at a constant angle α. Differently from the ellipse case, these trajectories are still parabolas, as we are going to show.

As visible in Figure 6, let us consider the fixed point F and the oriented direction λ. For any point P different from F, by adding an angle α to the exterior bisector h of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda}$, we obtain the line h_α: that defines a slope field whose solutions are the sought isogonal trajectories. Introducing the direction $\lambda_{2\alpha}$ as λ rotated by an angle 2α, we can prove the following lemma.

Lemma 2. *The line h_α is the exterior bisector of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda_{2\alpha}}$.*

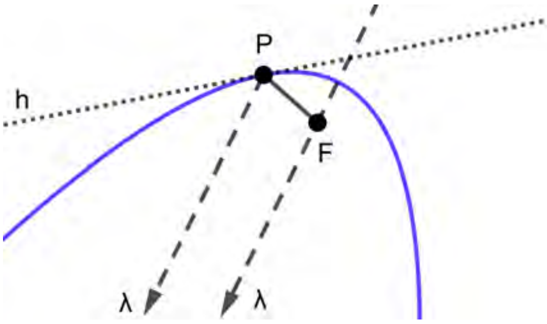

Figure 5. Parabola defined given a finite focus F, an oriented direction λ and passing through a point P. The external bisector h of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda}$ is tangent to the parabola.

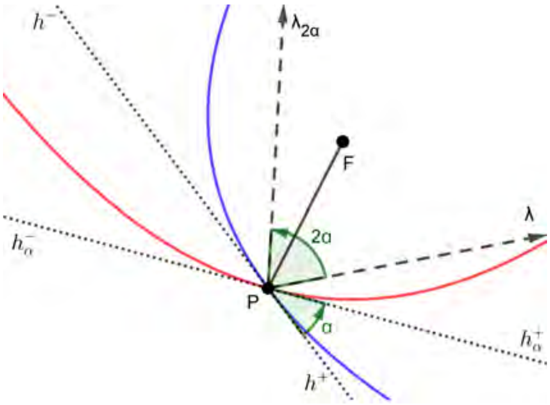

Figure 6. The red parabola with focus F and direction $\lambda_{2\alpha}$ (i.e., λ rotated of the angle 2α) solves the slope field generated by adding an angle α to the exterior bisector h of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda}$. This curve intersects at an angle α the family of parabolas of focus F and direction λ (in blue we can see such a parabola passing through P).

Proof. First of all, some remarks. Given a line s passing through a point O, consider its opposite directions s^+ and s^-. Such a line s is the external bisector of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if and only if the angle $\angle s^+OA$ is equal to $\angle s^-OB$. In our case, orient h and h_α such that $\angle h_\alpha^+PF = \angle h^+PF - \alpha$, as in Figure 6. Since h is the external bisector of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda}$, it holds $\angle h^+PF = \angle h^-P\lambda$. For construction, $\angle h_\alpha^-P\lambda_{2\alpha} = \angle h^-P\lambda_{2\alpha} + \alpha = (\angle h^-P\lambda - 2\alpha) + \alpha = \angle h^-P\lambda - \alpha = \angle h^+PF - \alpha = \angle h_\alpha^+PF$, hence $\angle h_\alpha^-P\lambda_{2\alpha} = \angle h_\alpha^+PF$. Thus, h_α is the exterior bisector of $\overrightarrow{PF}$ and $\overrightarrow{P\lambda_{2\alpha}}$. □

By this lemma, it follows that the slope field defined by h_α is solved by the parabolas of focus F and direction $\lambda_{2\alpha}$. Therefore, all the isogonal trajectories of the family of confocal parabolas (with finite focus F and oriented direction λ) are parabolas.

4. The new machine

4.1. The bisector mechanism

To solve the slope fields of Section 3, we need a mechanism to bisect the rays connecting the tracing point and the foci (finite or at infinity). Such a simple mechanism is visible in Figure 7 (exploded view of the model realized with Autodesk Fusion 360) and Figure 8 (3D-printed sample). The STL files necessary to 3D-print such a machine are available online at the link `https://www.thingiverse.com/thing:4006566`.

Adopting the components naming of Figure 7, the little gear transmits the motion around the central cylinder between the top and the bottom gears. Hence, considering the 2D projection of the mechanism on a base plane, any rotation of the top gear causes an opposite rotation of the bottom gear, keeping unchanged their bisector. Therefore, the angle α between the rods bisector and the direction of the wheel is constant.

Note that the number of gears' teeth is not essential. The bisector can also be implemented by two flat cylinders whose rotation is transmitted by rubber wheels, as visible in Figure 9 and Figure 10 (however, that implies problems related to possible slipping). Such a 3D model is available at `https://www.thingiverse.`

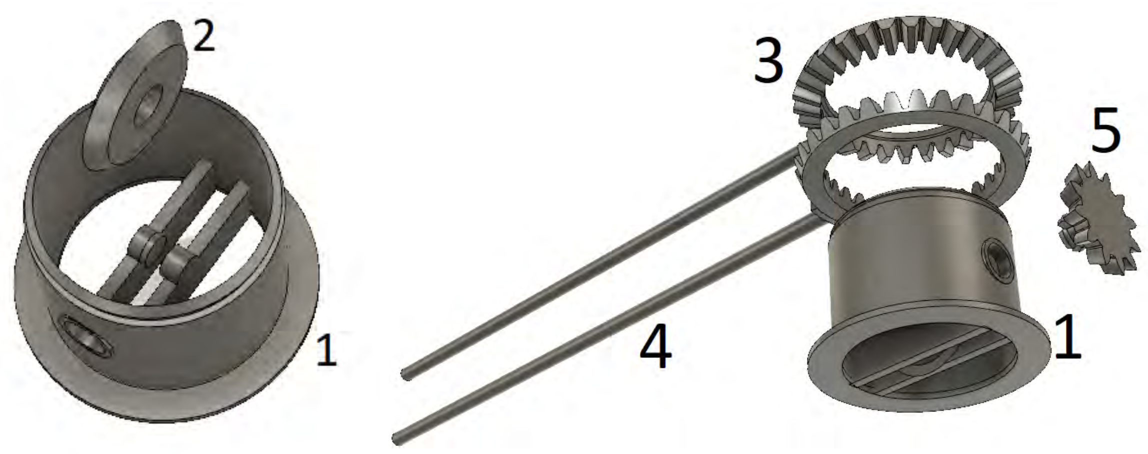

Figure 7. Expanded view of the machine with gears. Left: the wheel (2) is put and constrained to roll inside the central cylinder (1). Right: the two gears with rods stay on the top (3) and the bottom (4) of the central cylinder (1); the little gear (5) transmits the motion between other gears while inserted in the hole of the central cylinder (1).

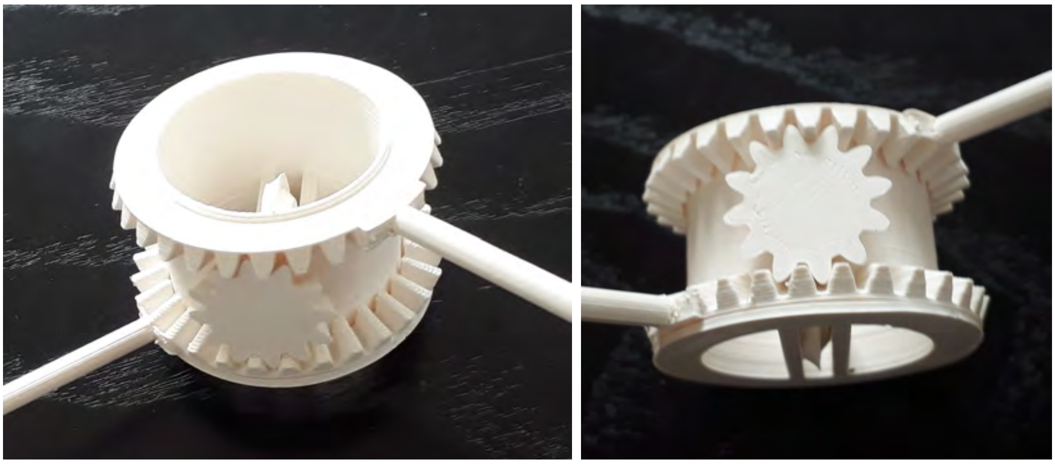

Figure 8. Main mechanism: the bisector (possibly added by a constant angle as explained in Section 4.3) works thanks to three conical gears. The axis of the little conical gear determines the plane of the wheel that touches the plane.

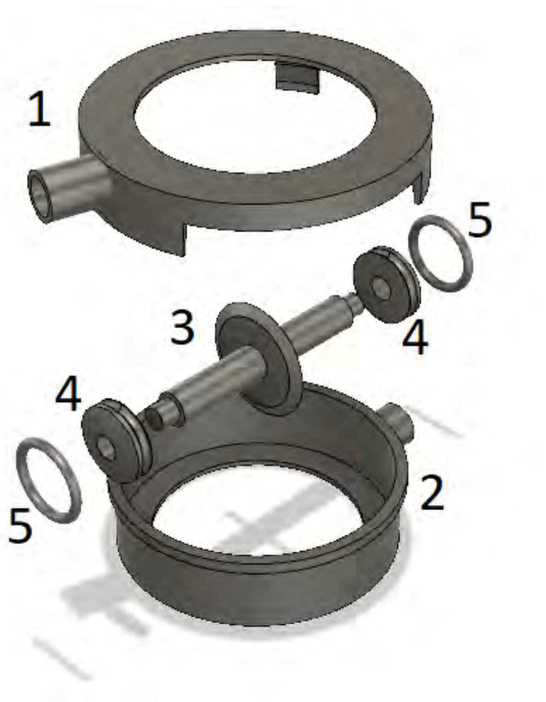

Figure 9. Components of the bisector mechanism with tires. The rods (not shown in the image) have to be inserted in the holes of the top (1) and bottom (2) components. Note that a click mechanism allows them to rotate (the top is external to the bottom element) while keeping the wheeled-axis (3) inside. Furthermore, two little pieces (4) are necessary to hold the O-rings (5): while the top and the bottom components are kept together by the click mechanism, the pressure grants (with a certain degree of precision) that the O-rings roll without slipping on the horizontal parts of (1) and (2).

`com/thing:4006702`.

After imposing rods to pass through the foci (the aim of Section 4.2), the wheel can generate conics and oblique trajectories. Considering the difference between machines posing direction-properties and position-properties (recalling Section 1, non-holonomic vs. holonomic constraints), differently from the devices of Section 2, the practical use of our mechanism requires particular attention to the composition of the base substrate. Indeed, to improve the resistance to lateral forces and, in the meanwhile, to make the wheel easily mark its path, we suggest the adoption of a foam sheet. On it, the curve appears as a furrow carved out by the wheel.

Figure 10. A bisector machine using rubber wheels to avoid gears. The two rubber wheels share the same axis but can rotate independently.

Figure 11. Left: model of a peg for a finite focus. Center: a 3D printed peg, including inside a drawing pin (glued in the bottom hollow). Right: bisector mechanism with two pegs constraining the passage of the rods.

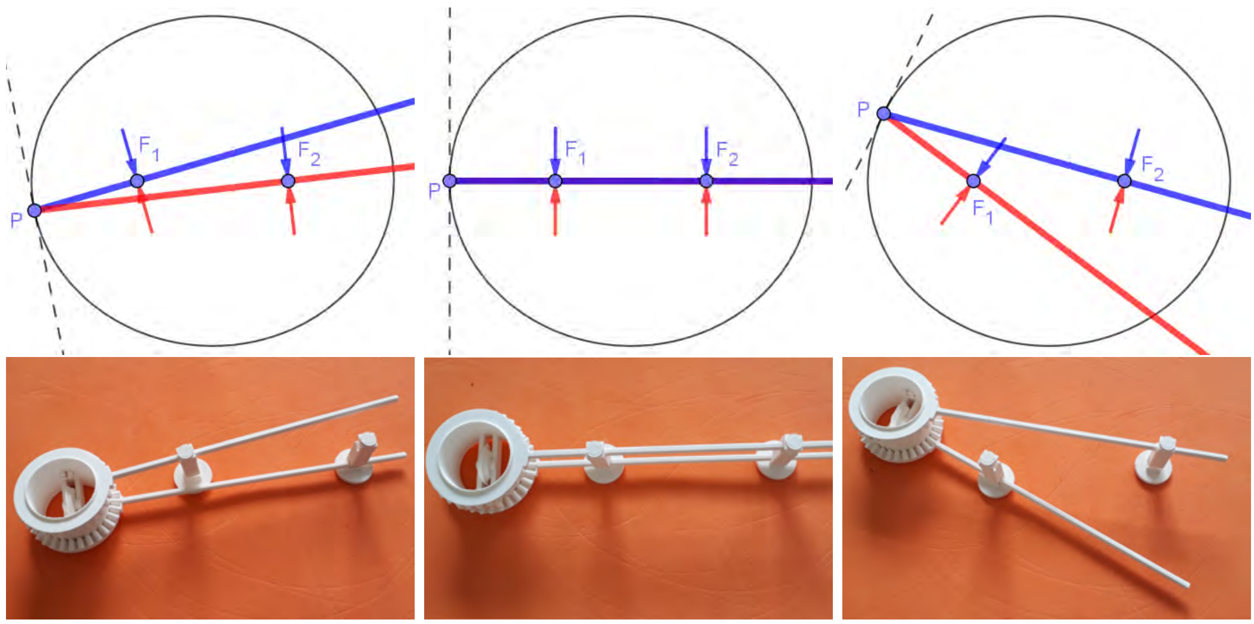

Figure 12. Properly set, our pegs allow to trace an ellipse continuously. We indicate the direction in which the rod can come in the carving by an arrow. To distinguish between upper and lower levels (both for rods and carvings), we color the first by blue and the latter by red.
In this sequence of images (top=simulations, bottom=photos): while the wheel moves upward, rods switch their foci. In the first step, the upper blue rod passes through F_1 and the lower red one through F_2; in the third step, the rods inverted their foci. The switching is possible because the carvings are external to $\angle F_1PF_2$ (both the blue arrow on the blue rod and the red arrow on the red rod have to be external to the angle $\angle F_1PF_2$).

4.2. Passage through the foci

To make a rod projection pass through a finite focus, one can use a vertical peg with a horizontal hole. If the peg can rotate around its center by a nail in a point F of the plane, the passage of the rod through the hole ensures the projection of the rod to pass through F. To drive the nail, we can put a base in cork, softwood or thick cardboard below the foam sheet. However, even to trace an ellipse, to constrain foci by cylinders with holes would imply problems; even assuming two rods at different heights, the lower rod can collide with the peg of the other rod. Thus, to trace conics without rod/peg collisions, we propose a kind of peg (Figure 11) that allows rods to switch foci when superimposed.

The peg has two carvings with opposite openings at different heights. Each carving allows a rod to enter and exit only from one side. Figure 12 shows how the switching works out, allowing to trace whole ellipses (in this case, the switch repeats every time the wheel axis is aligned with the foci). While such pegs permit to trace conics continually, such an idea does not solve rod/peg collision for general isogonal curves, as visible in Figure 13.

Recalling Section 3.2, parabolas' "focus at infinity" implies that one of the two rods has to translate. A hollowed cylinder can easily implement that; it has to rotate without slipping on the plane while rotating and sliding along the rod section. Figure 14 shows how to use a 3D-printed cylinder around the rod that

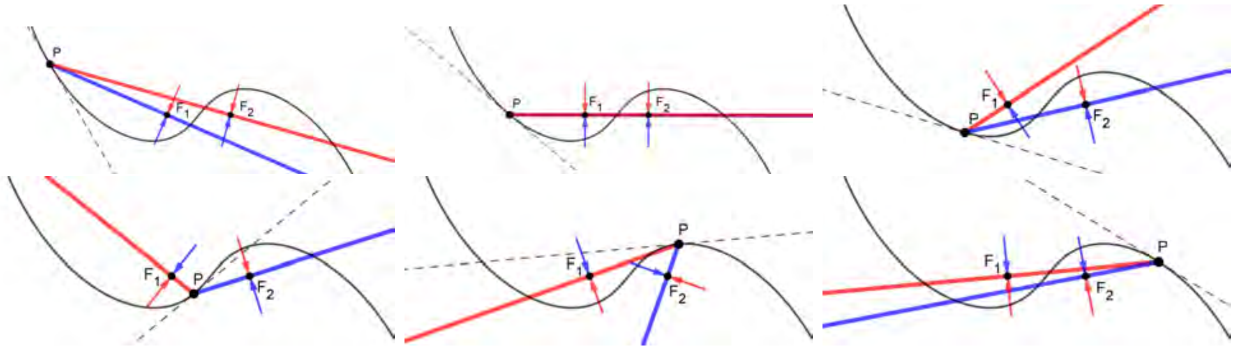

Figure 13. Some paths cannot be traced continuously. Following the represented isogonal to confocal ellipses ($c = 1, \lambda = \pi/2, n = -1.2$ in Mukherjee's formula of Section 3.1), the rods can switch the first time they overlap (second step), but not the second time (last step). In the end, the red rod meets the blue arrow in F_2, and vice-versa for the blue rod in F_1; thus, rods can no longer change their foci and collide with the pegs. Recalling Figure 12, the problem arises when P passes between the foci (steps 4, 5), making the carvings change from external to internal to $\angle F_1PF_2$.

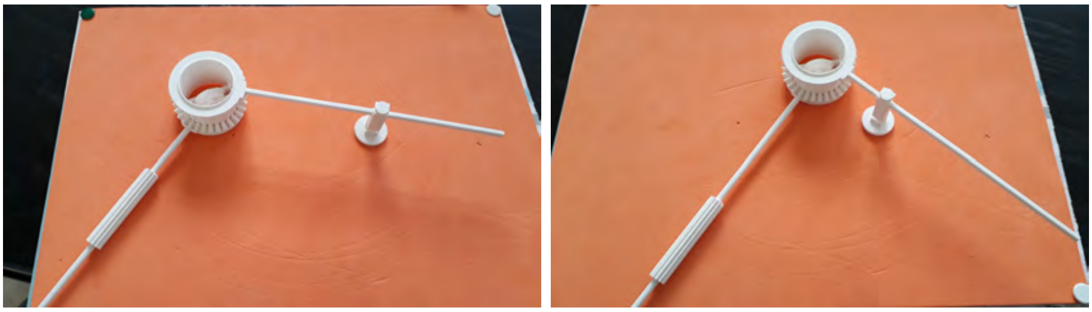

Figure 14. A cylinder rotating without slipping on the plane can impose a rod translation, useful to trace parabolas. The cylinder's outer part isn't smooth: it has many long thin teeth to prevent slipping.

has to pass through a focus at infinity.

We can note that even the parabola drawing can be afflicted by problems due to the collision between the translating rod and the peg (as in Figure 14 if the wheel moves on the right). That can be avoided adopting as wheel direction the internal bisector. In this case, according to the notation of Section 3.2, the ray $\overrightarrow{P\lambda}$ bumps the focus F in no configuration.

As a final remark, conics with both foci at infinity could be implemented by two cylinders keeping the rods moving in a parallel way. In this case, intractable by our components because we provided a cylinder only for the lower rod, the wheel direction would be constant, and the traced curve would be a straight line.

4.3. Operating instructions

We have to set three parameters to make our machine solve a specific slope field:

(1) the angle between the bisector of the rods and the direction of the wheel;
(2) the foci;
(3) the starting point.

The first parameter can be easily set by superimposing the rods. More precisely, we can follow the following steps:

- the little gear is taken away, leaving the two geared rods free to rotate around the central cylinder;
- the rods are set parallel, one above the other (like lancets of a clock at noon) by rotating top and bottom gears;
- the cylinder is rotated up to defining the desired angle between the superposed rods and the wheel direction;
- the little gear is put back on the cylinder lateral hole to keep unchanged the angle between the rods bisector and the wheel direction.

To set foci, we have to distinguish two cases. For a finite focus, we pin a peg of Figure 11 on the plane; for a focus at infinity, we put the cylinder of Figure 14 around the lower rod. If both foci are finite, to allow the rods switch, peg carvings have to be external to the angle defined by the two rods (cf. Figure12). If a focus is at infinity, we have to put the cylinder on the plane in the desired direction.

Let us summarize all the possibilities. With two pegs, when the wheel and the rod bisector are aligned, the trace is a hyperbola; when the wheel and the bisector are perpendicular, it is an ellipse; with any other angle, the curve is a transcendental isogonal trajectory. With one peg and the hollowed cylinder, the curve is always a parabola.

Finally, to choose the initial point, we simply press the wheel on a specific point of the foam sheet. This selects one trajectory in the slope field defined by the first two settings (angle and foci). To plot, we push the top of the cylinder both vertically (to press the wheel on the foam sheet) and horizontally (to make the wheel move forward) without preventing the top-gear rotation. While moving, the rods must remain inside the pegs (or, in the case of a focus at infinity, we have to rotate the teethed cylinder on the plane, avoiding its slipping). These conditions ensure the uniqueness of the motion. The expected curve appears as a furrow in the foam sheet.

5. Conclusions

In this paper, we introduced a new device to trace all the conics using the property of their tangents according to the position of the foci (on the plane or at infinity). Furthermore, the device can also trace transcendental curves (isogonal trajectories of confocal conics): such a property should have been considered weird by 17th-century mathematicians soon after the spread of Cartesian geometry and the related dualism between algebraic and transcendental curves. However, such behavior is not new: the more recent example is probably a tractional machine introduced in [27] and explored in [18] which, by the change of direction of a right angle, traces a parabola or an exponential. However, the first and more extraordinary example of a problem in which a variation in parameters cause algebraic or transcendental curves is the late 17th-century "Bernoulli's problem" [5, pp. 32-43]. This problem is a generalization of the tractrix: on the x-axis, consider the endpoint of a cord $(t, 0)$ whose length varies proportionally to t, let's say pt (where p is a constant). If the other endpoint of the cord imposes the tangent condition pointing to $(t, 0)$, we have that for rational values of p the curve is algebraic (if $p = 1$, we construct circular arcs); otherwise, the solution is transcendental. Jakob Bernoulli sketched a machine for such a problem.

Besides introducing a new machine, this paper aims to give everyone the chance to easily (and cheaply) get their sample, primarily for laboratory activities. The use of tractional machines in education is new in today's mathematical education, however it was already adopted in Italy by Giovanni Poleni in Padua (first half of the 18th century) and Ernesto Pascal in Naples (early 20th century). In particular, Poleni designed the first machines for tractrix and logarithmic curves [24] used in a pedagogical laboratory (his Cabinet of Experimental Philosophy).

Today, many (non-tractional) curve-drawing machines have been transposed in the classroom for laboratory activities (e.g. [2], [17]), with also interests from a historical and epistemological perspective. Indeed, since motion and manipulation are not so widely adopted in classroom teaching, the exploration of curve-tracing machines can offer engaging, unusual standpoints to the curricular mathematical contents. According to the framework of the theory of *semiotic mediation* [3], a fruitful exploration of a machine should start from a physical manipulation and then move to a conceptual and mathematical understanding of the artifact. The underlying cognitive theory is the *embodiment*, the idea that the human understanding is shaped by aspects of the whole body and not solely by the use of language and formalism [30], [25]. The didactical use of mathematical machines also overcomes pure-mathematics boundaries and reaches fields as art or architecture [11].

As evinced by the only (to our knowledge) experimental paper on tractional machines [18], it can be hard to internalize how the wheel guides the tangent. Therefore, especially for students that already explored other linkage machines, it can be more appropriate to propose the variant of Figures 15 and 16. This model implements the bisector by linkages (without gears) and traces only ellipses and hyperbolas (downloadable files are available online at `https://www.thingiverse.com/thing:4012950`). Unlike our precedent model, this one also requires hardware besides 3D-printed components (a metal base, magnets, sandpaper, screws and nuts, a toy-car wheel, a spring, chalk). Readers interested in details can contact

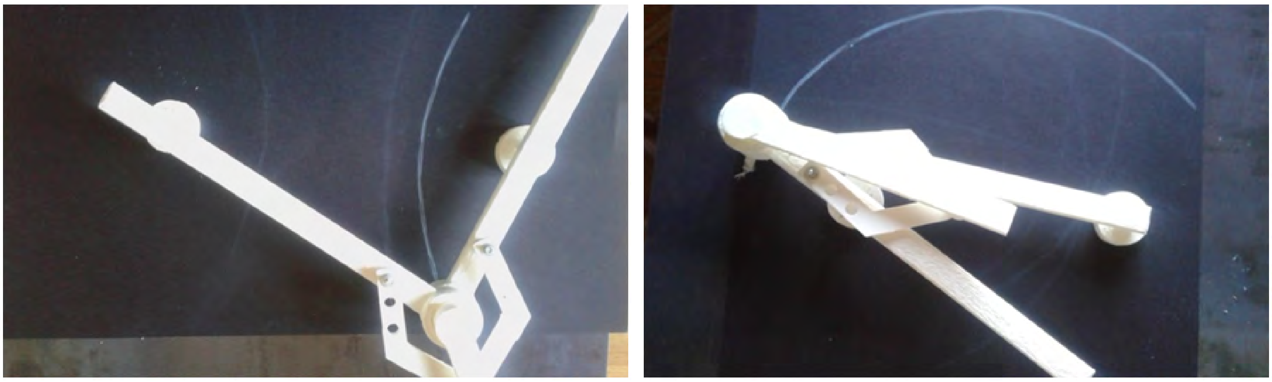

Figure 15. Simplified machine tracing a hyperbola (left) and an ellipse (right).

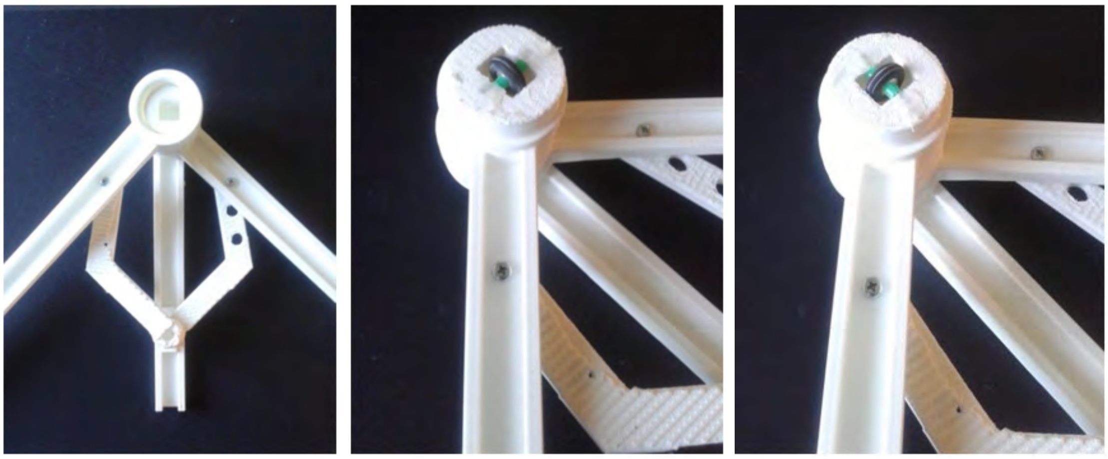

Figure 16. The simplified machine seen from below. Left: The bisector mechanism without the wheel (the little rods are not straight to permit a larger opening). In the other pictures, the wheel is put to trace a hyperbola (center) and an ellipse (right).

us for further information. In any case, it is probably possible to slightly modify such a model to have a pure 3D-printable variant, as the mathematical machines of the educational project "Vietato Non Toccare" (Forbidden Not To Touch) visible at `https://sites.google.com/view/vietatonontoccare`.

To conclude, we think that our machine can be proposed with different purposes at different levels: to high-school and university students, to math teachers, to professional mathematicians. According to the context, the exploration can naturally lead to deepening some related topics. In physics, tangents to conics offer a rich connection to light reflection on concentrating mirrors and parabolic antennas. In projective geometry, one could deepen the relationship between finite foci and points at infinity, passing from the affine to the projective plane.

Although there is a lot of work to be done in this direction, we hope that this manuscript will provide a basis for the diffusion of traction machines for hands-on activities (the 3D-printable files are freely available online). That's a way to give concreteness to some mathematical contents usually considered useful but abstract. As a motto, the aim is to touch the transcendence.

Disclosure statement

No potential conflict of interest was reported by the authors.

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References

[1] Bartolini Bussi, M. G. (2005). The meaning of conics: historical and didactical dimensions. In: Kilpatrick, J., Hoyles, C., Skovsmose, O., Valero, P. (Eds.), *Meaning in mathematics education*, pp. 39–60. Springer, New York, NY, pp. 39–60.

[2] Bartolini Bussi, M. G. & Maschietto, M. (2006). *Macchine matematiche. Dalla storia alla scuola.* Springer

[3] Bartolini Bussi, M. G. & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artifact and signs after a Vygotskian perspective. In English e al. (Eds.) *Handbook of International Research in Mathematics Education* (pp. 746-783). New York and London, Routlegde

[4] Blasjo, V. (2017). *Transcendental Curves in the Leibnizian Calculus.* Academic Press.

[5] Bos, H. J. (1988). Tractional motion and the legitimation of transcendental curves. *Centaurus*, 31(1), 9-62.

[6] Bos, H. J. (2001). *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction.* Springer Science & Business Media.

[7] Crippa, D. & Milici, P. (2019). A Relationship between the Tractrix and Logarithmic Curves with Mechanical Applications, *The Mathematical Intelligencer*, 41(4), 29-34, DOI 10.1007/s00283-019-09895-7.

[8] Courant, R. & Robbins, H. (1941). *What Is Mathematics?*, Oxford University Press

[9] Dawson, R., Milici, P. & Plantevin, F. (to appear, accepted on 2020.12.28) Gardener's hyperbolas and the dragged-point principle, *American Mathematical Monthly.* Pre-print: `hal.archives-ouvertes.fr/hal-03100561`

[10] De Vittori, T. (2011). The perfect compass: conics, movement and mathematics around the 10th century. In Barbin, E., Kronfellner, M and Tzanakis, C. (Eds) *History and epistemology in mathematics education. Proceedings of the 6th European Summer University*, Vienna, Austria. pp.539-548. `http://numerisation.univ-irem.fr/ACF/ACF11006/ACF11006.pdf`

[11] Farroni, L. & Magrone, P. (2016). A Multidisciplinary Approach to Teaching Mathematics and Architectural Representation: Historical Drawing Machines. Relations Between Mathematics and Drawing. *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics.* L. Radford, F. Furinghetti, T. Hausberger eds. Montpellier, IREM de Montpellier, pp. 641-651.

[12] Ferrara, F. & Maschietto, M. (2013). Are mathematics students thinking as Kepler? Conics and mathematical machines. In *Eight Congress of European Research in Mathematics Education (CERME8).* Ankara, Turkey: Middle East Technical University and ERME, pp. 635-644.

[13] Forsyth, A. R. (1948). *A treatise on differential equations*, reprinted 1929 6th ed., MacMillan & co., London. `https://archive.org/details/ATreatiseOnDifferentialEquations_650`

[14] Kempe, A. B. (1876). On a general method of describing plane curves of the n^{th} degree by linkwork. *Proc. London Math. Soc.*, 7, 213–216.

[15] Lebossé, C. & Hémery, C. (1961). *Géométrie.* Gabay, Paris.

[16] Mainardi G. (1850). Sulla integrazione delle equazioni differenziali, *Annali di Scienze Matematiche e Fisiche*, 1, 50-89, 251-255.

[17] Maschietto, M. & Bartolini Bussi, M. G. (2011). Mathematical machines: from history to mathematics classroom. In *Constructing knowledge for teaching secondary mathematics* (pp. 227-245). Springer, Boston, MA.

[18] Maschietto, M., Milici, P. & Tournès, D. (2019) Semiotic potential of a tractional machine: a first analysis, In: U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11)*, Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME, pp. 2133-2140.

[19] Milici, P. (2012) Tractional motion machines extend GPAC-generable functions. *International Journal of Unconventional Computing*, 8:3, 221-233.

[20] Milici, P. (2015). A geometrical constructive approach to infinitesimal analysis: Epistemological potential and boundaries of tractional motion. In: Lolli, G., Panza M., Venturi G., eds. *From Logic to*

Practice. Boston Studies in the Philosophy and History of Science 308. Cham, Switzerland: Springer, pp. 3–21.

[21] Milici, P. & Dawson, R. (2012), The Equiangular Compass, *The Mathematical Intelligencer*, 34(4), 63–67, DOI: 10.1007/s00283-012-9308-x.

[22] Milici, P. (2020). A differential extension of Descartes' foundational approach: A new balance between symbolic and analog computation, *Computability*, 9(1), pp. 51-83, DOI: 10.3233/COM-180208

[23] Mookerjee, A. (1887). On the differential equations of a trajectory, *J. Asiat. Soc. Bengal*, 56.2.1, 117–120.

[24] Poleni, G. (1729). *Epistolarum mathematicarum fasciculus*. Patavii: Typographia Seminarii.

[25] Radford L., Bardini C., Sabena C., Diallo P. & Simbagoye A. (2005). On embodiment, artifacts and signs: a semiotic-cultural perspective on mathematical thinking, in Chick H.L. & Vincent J.L. (eds.). *Proceedings of the 29 Conference of the international group for the psychology of mathematics education (PME 28)*, Vol.4, pp. 73-80, Norway: Bergen University College

[26] Rashed, R. (2003) Al-Qūhi and al-Sijzī on the perfect compass and the continuous drawing of conic sections, *Arabic sciences and philosophy*, Vol. 13, No. 1, pp. 9-43.

[27] Salvi, M. & Milici, P. (2013). Laboratorio di matematica in classe: due nuove macchine per problemi nel continuo e nel discreto, *Quaderni di Ricerca in Didattica (Mathematics)*, 23, 15-24.

[28] Sar, S. (2015). Asutosh Mukhopadhyay and his mathematical legacy. *Resonance*, 20(7), 575-604.

[29] Scriba C. J. (1961). Zur Lösung des 2. Debeauneschen Problems durch Descartes, *Archive for History of Exact Sciences*, Vol. 1, No. 4, pp. 406-419.

[30] Tall, D. (2013), *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*, Cambridge University Press

[31] Tournès, D. (2009), *La construction tractionnelle des équations différentielles*, Paris: Blanchard, 2009.