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Multiple solutions for a nonlinear Dirichlet problem driven by degenerate p-Laplacian

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ABSTRACT

The paper develops a variational approach for a parametric quasilinear elliptic equation driven by degenerate p-Laplacian. Note that the use of variational methods allows to obtain multiple solutions. In particular, here, the existence of at least three solutions is established under an appropriate growth condition of the nonlinear term.

1. Introduction

In this paper we study the following nonlinear elliptic problem driven by a degenerate p-Laplacian

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

The problem (P) is defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N \geq 1$, $1 < p < \infty$. The leading differential part of the equation in (P) is the differential operator $\Delta_{a,p}(u) = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)$, called degenerate p-Laplacian with a weight $a \in L^1_{loc}(\Omega)$.

We assume that the weight $a : \overline{\Omega} \to [0, +\infty[$ is a measurable function such that:

$$a^{-s} \in L^1(\Omega) \quad \text{for some } s \in \left]\frac{N}{p}, +\infty\right[\cap \left[\frac{1}{p-1}, +\infty\right[, \tag{1}$$

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and denote by $p_s = \frac{ps}{s+1}$ the exponent associated to s. We point out that if $a = 1$ we get back to elliptic problem driven by p-Laplacian operator. In many boundary elliptic equations singularities or degenerations can appear, they can be caused by introducing the coefficients into the terms of the equations. For instance we can modify the p-Laplacian operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ introducing a coefficient a obtain the differential operator $\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)$. If the coefficient a satisfies $a \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$ a.e. in Ω, the associated problem is studied in the classical Sobolev space $W^{1,p}(\Omega)$, if these conditions are not satisfied because a is unbounded or a is only positive a.e. in Ω, singularities or degenerations arise in the problem. We can study the associated problem using the weighed Sobolev space $W^{1,p}(a,\Omega)$. We refer to [6] (see also [11]) and references therein for an overview of the topic, in which also various examples of weights are given.

Recently, many authors have studied degenerated differential equations. In particular, in [7], using the compactness of an imbedding for weighted Sobolev spaces, the authors study some weaker assumptions to obtain the pseudo-monotonicity of nonlinear degenerate or singular operators. In [8] the authors study a quasilinear homogeneous eigenvalue problem with degenerate p-Laplacian, whose weights are negative powers of the distance from the boundary of the domain. In [12] the authors use a sub and supersolution approach to obtain the existence and location of at least one solution for a problem with degenerate p-Laplacian and convection term. In [14] the authors obtain existence of positive or compactly supported radial ground states for a quasilinear singular elliptic equation with weights; the approach is based on the Mountain Pass Theorem and the constrained minimization method. In [16], using a variational approach, the authors prove the existence of solutions for a class of degenerate semilinear elliptic systems with measurable, unbounded and nonnegative weights.

Degenerate and singular problems may be found in several applications, for instance, in the study of behavior of turbulent fluid flows, in induction heating and in oceanography, in which many problems are described in domains whose depth vanish on the shore ([1,9]) or in the electrical conductivity imaging ([13]).

The aim of this paper is to obtain existence and multiplicity results for problems driven by degenerate p-Laplacian operator using variational methods and critical points theorem. In this respect the functional setting is adapted to appropriate weighted Sobolev space, where the associated exponent p_s plays a crucial role. It is worth noting that our variational approach allows to obtain multiple solutions on the contrary of other methods, such as the upper and lower solution or the fixed point methods, which can study only the existence.

The paper is arranged as follows. Our main result (Theorem 3.4) ensures the existence of at least three solutions under a sign hypothesis on the primitive of the nonlinearity and a suitable algebraic condition (see assumption (ii)). As a special case, we obtain Theorem 3.5 where the algebraic condition is replaced by a $(p-1)$-sublinear condition at zero. Finally, the existence of at least three nonnegative solutions for an autonomous case is pointed out (see Theorem 3.7) and an example is presented.

2. Prelimiaries

In this section we introduce the weighted Sobolev space that we use throughout the paper and we present some of their properties. We assume that $p > 1$ and denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p.

The weighted Sobolev space is defined as

$$W^{1,p}(a,\Omega) = \{u \in L^p(\Omega) \;:\; a^{\frac{1}{p}}|\nabla u| \in L^p(\Omega)\},$$

endowed with the following norm

$$\|u\|_{1,p,a} = \left(\int_\Omega |u|^p dx + \int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{2}$$

Taking (1) into account one has $a^{-\frac{1}{p-1}} \in L^1(\Omega)$ (see [12]), then $W^{1,p}(a,\Omega)$ is a uniformly convex (hence reflexive) Banach space and $C_0^\infty(\Omega)$ is a subset. We denote by $W_0^{1,p}(a,\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,a}$.

On $W_0^{1,p}(a,\Omega)$ we consider the following norm

$$\|u\| = \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}},$$

which is equivalent to (2).

From (1) the following embedding is continuous

$$W_0^{1,p}(a,\Omega) \hookrightarrow W_0^{1,p_s}(\Omega).$$

Moreover, using Hölder's inequality we obtain

$$\int_\Omega |\nabla u|^{p_s} dx = \int_\Omega a^{-\frac{p_s}{p}} a^{\frac{p_s}{p}} |\nabla u|^{p_s} dx$$
$$\leq \left(\int_\Omega a|\nabla u|^p dx\right)^{\frac{p_s}{p}} \left(\int_\Omega a^{-s} dx\right)^{\frac{1}{s+1}} \quad \text{for all } u \in W_0^{1,p}(a,\Omega).$$

So, we have

$$\|u\|_{W_0^{1,p_s}(\Omega)} \leq \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} \|u\| \quad \text{for all } u \in W_0^{1,p}(a,\Omega). \tag{3}$$

Where $\|\cdot\|_{L^1(\Omega)}$ is the usual norm in $L^1(\Omega)$.

We assume that $1 < p_s < N$ and consider the critical exponent $p_s^* = \frac{p_s N}{N-p_s}$. Then the Sobolev embedding theorem ensures that the embedding $W_0^{1,p}(a,\Omega) \to L^q(\Omega)$ is continuous for $1 \leq q \leq p_s^*$ and by Rellich-Kondrochow compact embedding theorem the embedding $W_0^{1,p}(a,\Omega) \to L^q(\Omega)$ is compact for $1 \leq q < p_s^*$. It is known that, for every $u \in W_0^{1,p_s}(\Omega)$ there exists a constant $T \in \mathbb{R}_+$ such that

$$\|u\|_{L^{p_s^*}(\Omega)} \leq T\|u\|_{W_0^{1,p_s}(\Omega)}, \tag{4}$$

where T is the constant determined by Talenti (see [15])

$$T \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{p_s}} \left(\frac{p_s - 1}{N - p_s}\right)^{1-\frac{1}{p_s}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p_s}\right)\Gamma\left(1+N-\frac{N}{p_s}\right)}\right)^{\frac{1}{N}},$$

Γ is the Euler function.

Form (3) and (4) and Hölder's inequality, we obtain

$$\|u\|_{L^q(\Omega)} \leq T\|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-q}{p_s^* q}} \|u\|, \tag{5}$$

for every $1 \leq q \leq p_s^*$ and $u \in W_0^{1,p}(a,\Omega)$. We observe that, since $s > \frac{N}{p}$, we have $p < p_s^*$ so the (5) also holds for $q = p$.

By $\left(W_0^{1,p}(a,\Omega)\right)^*$ we denote the topological dual space of $W_0^{1,p}(a,\Omega)$ and by $\langle\cdot,\cdot\rangle$ the duality brancket. The negative degenerate p-Laplacian, with weight $a \in L^1_{loc}(\Omega)$ such that (1) holds, is the operator $A = -\Delta_{a,p} : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx. \tag{6}$$

Important properties of negative degenerate p-Laplacian are listed in the next proposition and for the proof we refer to [12].

Proposition 2.1. *The operator $A : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by (6) is a coercive operator, that is $\lim_{\|u\|\to+\infty} \frac{\langle A(u),u\rangle}{\|u\|} = +\infty$, and it has the S_+ property i.e. any sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ that satisfies $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ is strongly convergent.*

Finally, we recall a three critical points theorem obtained in [3] (see also [2]).

Theorem 2.2. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gateaux differentiable functionals with Φ bounded below. Assume $\Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $\Phi(\tilde{u}) > r$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{7}$$

and, for each

$$\lambda \in \Lambda = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies the (PS)-condition.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least three critical points.

3. Main results

Throughout the sequel, we assume that the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function i.e. $f(\cdot,t)$ is measurable for every $t \in \mathbb{R}$, $f(x,\cdot)$ is continuous for almost every $x \in \Omega$ and satisfies the subcritical growth condition:

(H) There exist two nonnegative constants a_1, a_2 and $\beta \in [1,p]$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{\beta-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Put $F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in \Omega \times \mathbb{R}$.

From (H) follows

$$|F(x,t)| \leq a_1|t| + \frac{a_2}{\beta}|t|^\beta \text{ for every } (x,t) \in \Omega \times \mathbb{R}. \tag{8}$$

We consider the C^1-functionals Φ, $\Psi : W_0^{1,p}(a,\Omega) \to \mathbb{R}$, defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p, \tag{9}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx, \tag{10}$$

for all $u \in W_0^{1,p}(a,\Omega)$.

The Gâteaux derivative of the functional Φ at point $u \in W_0^{1,p}(a,\Omega)$ is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx,$$

and it is the negative degenerated p-Laplacian operator i.e.

$$\langle \Phi'(u), v\rangle = \langle A(u), v\rangle$$

for every $v \in W_0^{1,p}(a,\Omega)$.

The Gâteaux derivative of the functional Ψ at point $u \in W_0^{1,p}(a,\Omega)$ is given by

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in W_0^{1,p}(a,\Omega)$.

Lemma 3.1. *If $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ such that $u_n \rightharpoonup u$ and hypothesis (H) holds, up a subsequence one has*

$$\lim_{n\to+\infty} \langle \Psi'(u_n), u_n - u\rangle = 0.$$

Proof. Hypothesis (H), Hölder's inequality and the compact embedding theorem enable us to assert that

$$|\int_\Omega f(x,u_n)(u_n-u)dx| \leq a_1\|u_n-u\|_{L^1(\Omega)} + a_2\|u_n\|^{\beta-1}_{L^\beta(\Omega)}\|u_n-u\|_{L^\beta(\Omega)} \to 0 \text{ for } n \to 0. \quad \square$$

Lemma 3.2. *The functional $\Psi : W_0^{1,p}(a,\Omega) \to \mathbb{R}$, is sequentially weakly continuous in $W_0^{1,p}(a,\Omega)$ and its Gâteaux derivative is compact.*

Proof. We observe that

$$\Psi' : W_0^{1,p}(a,\Omega) \to \left(W_0^{1,p}(a,\Omega)\right)^*$$

is compact because it is the composition of the mapping $u \to f(x,u)$, taking value in $L^{p'}(\Omega)$ (which is continuous and bounded by hypothesis (H)) and the linear embedding $L^{p'}(\Omega) \hookrightarrow \left(W_0^{1,p}(a,\Omega)\right)^*$ (which is compact, because it is the adjoint of the compact embedding $W_0^{1,p}(a,\Omega) \hookrightarrow L^p(\Omega)$).

Finally, since $W_0^{1,p}(a,\Omega)$ is reflexive, Ψ is sequentially weakly continuous (see [17, Corollary 41.9]). $\square$

We recall that a weak solution of problem (P) is a function $u \in W_0^{1,p}(a,\Omega)$ such that

$$\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx = \lambda\int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in W_0^{1,p}(a,\Omega)$.

We observe that, under hypothesis (H) the integral in the right side of weak definition of solution exists.

Put $I_\lambda = \Phi - \lambda\Psi$ we can observe that the critical points of I_λ are the weak solutions of (P).

Remark 3.3. Since Φ is continuous and convex on $W_0^{1,p}(a,\Omega)$, then Φ is sequentially weakly lower semicontinuous (see [5, Remark 6]) then, taking into account of Lemma 3.2, we obtain that I_λ is sequentially weakly lower semicontinuous for every $\lambda > 0$.

Put

$$R := \sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega), \tag{11}$$

we observe that there is $x_0 \in \Omega$ such that the N-dimensional ball of radius R, denoted by $B(x_0,R)$, is contained in Ω.

Put

$$\mathcal{B} := B\left(x_0,R\right)\setminus B\left(x_0,\frac{R}{2}\right),$$

and

$$\kappa = \frac{2^pT^p\left\|a^{-s}\right\|_{L^1(\Omega)}^{\frac{1}{s}}}{R^p|\Omega|^{\frac{p}{p_s^*}}}\left\|a\right\|_{L^1(\mathcal{B})}, \tag{12}$$

where $|\Omega|$ denotes the measure of Lebesgue of Ω.

Now, we present our main result.

Theorem 3.4. *Assume that* (H) *holds and that there are two positive constants* c *and* d, *with* $c < \kappa^{\frac{1}{p}}d$, *such that*

(i) $F(x,t) \geq 0$, *for all a.e.* $x \in B\left(x_0,R\right)$ *and for every* $t \in [0,d]$.

(ii)

$$a_1c^{1-p} + \frac{a_2}{\beta}c^{\beta-p} < \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)dx}{\kappa|\Omega|d^p},$$

where a_1, a_2, β *and* κ *are given by* (H) *and* (12), *respectively.*

Then, for each $\lambda \in \tilde{\Lambda} = \left]\frac{2^p\|a\|_{L^1(\mathcal{B})}d^p}{pR^p\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)dx}, \frac{1}{pT^p\|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}}|\Omega|^{\frac{p_s^*-p}{p_s^*}}\left(a_1c^{1-p}+\frac{a_2}{\beta}c^{\beta-p}\right)}\right[$, *problem* ($P$) *has at least three weak solutions.*

Proof. Our aim is to apply Theorem 2.2. To this end put Φ and Ψ as in (9) and (10). The functionals Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.2. On the other hand, $\inf\limits_{u\in W_0^{1,p}(a,\Omega)} \Phi(u) =$

$\Phi(0) = \Psi(0) = 0$. So, our aim is to verify condition (7) of Theorem 2.2. Fix $\lambda \in \tilde{\Lambda}$, from (ii) we observe that one has $\tilde{\Lambda} \neq \emptyset$.

Define

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \dfrac{2d}{R}\left(R - |x - x_0|\right) & \text{if } x \in \mathcal{B}, \\ d & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Clearly, $\tilde{u} \in W_0^{1,p}(a, \Omega)$ and taking into account (12) we have

$$\Phi(\tilde{u}) = \frac{1}{p}\left(\frac{2d}{R}\right)^p \int_{\mathcal{B}} a(x)dx = \frac{2^p \|a\|_{L^1(\mathcal{B})}}{pR^p} d^p. \tag{13}$$

On the other hand, from (i) one has

$$\Psi(\tilde{u}) \geq \int_{B\left(x_0, \frac{R}{2}\right)} F(x, d)\, dx,$$

and hence, we obtain

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{pR^p}{2^p \|a\|_{L^1(\mathcal{B})}} \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x, d)\, dx}{d^p}. \tag{14}$$

Put $r = \frac{1}{p}\left(\frac{|\Omega|^{\frac{1}{p_s^*}}}{T\|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}}} c\right)^p$. From $c < \kappa^{\frac{1}{p}} d$ one has $\Phi(\tilde{u}) > r$; from (5) and (8) we obtain

$$\Psi(u) \leq a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{\beta}\|u\|_{L^\beta(\Omega)}^\beta \leq a_1 T \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-1}{p_s^*}} \|u\| + \frac{a_2}{\beta} T^\beta \|a^{-s}\|_{L^1(\Omega)}^{\frac{\beta}{ps}} |\Omega|^{\frac{p_s^*-\beta}{p_s^*}} \|u\|^\beta. \tag{15}$$

Taking (9) into account, for all $u \in W_0^{1,p}(a, \Omega)$ such that $u \in \Phi^{-1}\left(]-\infty, r]\right)$, one has that $\|u\| \leq (pr)^{\frac{1}{p}}$ and

$$\Phi^{-1}\left(]-\infty, r]\right) \subseteq \left\{u \in W_0^{1,p}(a, \Omega) : \|u\| \leq (pr)^{\frac{1}{p}}\right\}. \tag{16}$$

Then, for each $u \in \Phi^{-1}\left(]-\infty, r]\right)$ and owing to (15) and (16), we have that

$$\begin{aligned} \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} &\leq \frac{\sup\limits_{\|u\| \leq (pr)^{\frac{1}{p}}} \Psi(u)}{r} \\ &\leq \frac{\sup\limits_{\|u\| \leq (pr)^{\frac{1}{p}}} \left(a_1 T \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-1}{p_s^*}} \|u\| + \frac{a_2}{\beta} T^\beta \|a^{-s}\|_{L^1(\Omega)}^{\frac{\beta}{ps}} |\Omega|^{\frac{p_s^*-\beta}{p_s^*}} \|u\|^\beta\right)}{r} \\ &\leq a_1 T \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-1}{p_s^*}} p^{\frac{1}{p}} r^{\frac{1-p}{p}} + \frac{a_2}{\beta} T^\beta \|a^{-s}\|_{L^1(\Omega)}^{\frac{\beta}{ps}} |\Omega|^{\frac{p_s^*-\beta}{p_s^*}} p^{\frac{\beta}{p}} r^{\frac{\beta-p}{p}} \\ &= pT^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}} \left[a_1 \left(\frac{T^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}}}{|\Omega|^{\frac{p}{p_s^*}}} pr\right)^{\frac{1-p}{p}} + \frac{a_2}{\beta}\left(\frac{T^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}}}{|\Omega|^{\frac{p}{p_s^*}}} pr\right)^{\frac{\beta-p}{p}}\right] \end{aligned}$$

$$= pT^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}} \left(a_1 c^{1-p} + \frac{a_2}{\beta} c^{\beta-p} \right),$$

that is

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \leq pT^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}} \left(a_1 c^{1-p} + \frac{a_2}{\beta} c^{\beta-p} \right). \tag{17}$$

Therefore, from (ii), (14) and (17) we obtain condition (7) of Theorem 2.2.

From (15) follows that, for each $u \in W_0^{1,p}(a,\Omega)$, one has

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \geq \frac{1}{p}\|u\|^p - \lambda \left(a_1 T \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-1}{p_s^*}} \|u\| + \frac{a_2}{\beta} T^\beta \|a^{-s}\|_{L^1(\Omega)}^{\frac{\beta}{ps}} |\Omega|^{\frac{p_s^*-\beta}{p_s^*}} \|u\|^\beta . \right). \tag{18}$$

If $\beta \in [1,p[$ from (18) we have $\lim_{\|u\|\to+\infty} \frac{I_\lambda(u)}{\|u\|} = +\infty$ for every $\lambda \geq 0$. If $\beta = p$, condition (18) becomes

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \geq \frac{1}{p} \left(1 - \lambda a_2 T^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}} \right) \|u\|^p - \lambda a_1 T \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{ps}} |\Omega|^{\frac{p_s^*-1}{p_s^*}} \|u\|,$$

that implies $\lim_{\|u\|\to+\infty} \frac{I_\lambda(u)}{\|u\|} = +\infty$ being

$$\lambda < \frac{1}{pT^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}} \left(a_1 c^{1-p} + \frac{a_2}{p} \right)} \leq \frac{1}{a_2 T^p \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{s}} |\Omega|^{\frac{p_s^*-p}{p_s^*}}}.$$

Then, fix $\lambda \in \tilde{\Lambda}$, I_λ is coercive. Since I_λ is sequentially weakly lower semicontinuous (see Remark 3.3) and coercive, it follows that I_λ is bounded from below. Now, we are going to prove that I_λ satisfies the (PS)-condition. Consider a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded, and $\{I'_\lambda(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in $\left(W_0^{1,p}(a,\Omega)\right)^*$. Since I_λ is coercive, the sequence $\{u_n\}$ is bounded. Then, since $W_0^{1,p}(a,\Omega)$ is a reflexive Banach space, up to a subsequence, we have that $u_n \rightharpoonup u$. Hence

$$\langle \Phi'(u_n), u_n - u \rangle = \langle I'_\lambda(u_n), u_n - u \rangle + \lambda \langle \Psi'(u_n), u_n - u \rangle,$$

and, taking into account of Lemma 3.2, we have

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0.$$

By the S_+-property of Φ' in $W_0^{1,p}(a,\Omega)$ (see Proposition 2.1) we have that $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. Hence I_λ fulfills (PS)-condition.

All assumptions of Theorem 2.2 are verified, then for each $\lambda \in \tilde{\Lambda} \subset \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)} \right[$ problem (P) admits at least three weak solutions and the proof is complete. □

A result of multiplicity solutions of problem (P) is obtained, assuming that the nonlinearity is $(p-1)$-sublinear at zero.

Theorem 3.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, satisfying (H) with $1 \leq \beta < p$ and assume that there is a positive constant d, such that $F(x,t) \geq 0$, for a.e. $x \in B(x_0,R)$ and for every $t \in [0,d]$ and $\int_{B(x_0,\frac{R}{2})} F(x,d)dx > 0$, moreover*

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{p-1}}=0 \quad \textit{uniformly a.e. in } \Omega \tag{19}$$

where x_0 and R are given by (11).

Then, for every $\lambda > \frac{2^p\|a\|_{L^1(\mathcal{B})}d^p}{pR^p\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)dx}$, problem (P) admits at least three weak solutions.

Proof. Arguing as [10, Lemma 3.3] and [4, Theorem 3.3] we obtain that assumption (19) ensures that

$$\lim_{r\to 0^+}\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}=0. \tag{20}$$

Let $\tilde{u}\in W_0^p(a,\Omega)$ be as in Theorem 3.4, then (14) holds. For each $\bar{\lambda} > \frac{2^p\|a\|_{L^1(\mathcal{B})}d^p}{pR^p\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)dx}$ from (20) there exists $\bar{r}\in\,]0,\Phi(\tilde{u})[$ small enough such that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,\bar{r}])}\Psi(u)}{\bar{r}}\le\frac{1}{\bar{\lambda}}<\frac{pR^p\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)dx}{2^p\|a\|_{L^1(\mathcal{B})}d^p}<\frac{\Psi(\tilde{u})}{\Psi(\tilde{u})}.$$

Then, condition (7) of Theorem 2.2 is satisfied.

Moreover, in the proof of Theorem 3.4 we have already showed that, when $1\le\beta<p$, the functional I_λ is coercive, bounded from below and satisfies the (PS)-condition for each $\lambda\ge 0$.

Since of all assumptions of Theorem 2.2 are verified, for each $\lambda > \frac{2^p\|a\|_{L^1(\mathcal{B})}d^p}{pR^p\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)dx}$, problem (P) admits at least three weak solutions. □

Remark 3.6. We observe that assumption (19) implies that zero function is a solution of the problem (P). Therefore, Theorem 3.5 ensures, in particular, that problem (P) admits at least two distinct non-zero weak solutions.

We point out an autonomous case and we assume that the nonlinear term has a $(p-1)$-sublinear growth at zero and at infinity. We consider the following problem

$$\begin{cases}-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)=\lambda f(u) & \text{in }\Omega,\\ \\ u=0 & \text{on }\partial\Omega,\end{cases} \tag{P_1}$$

and put $F(t)=\int_0^t f(s)ds$.

Theorem 3.7. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(t)\ge 0$ for every $t\in[0,+\infty[$, with $f\not\equiv 0$. Moreover, we suppose that*

$$\lim_{|t|\to+\infty}\frac{f(t)}{|t|^{p-1}}=0, \tag{21}$$

and

$$\lim_{t\to 0}\frac{f(t)}{|t|^{p-1}}=0. \tag{22}$$

Put $\lambda^*=\frac{2^p\|a\|_{L^1(\mathcal{B})}}{pR^p|B(x_0,\frac{R}{2})|}\inf\limits_{d>0}\{\dfrac{d^p}{F(d)}\ :F(d)>0\}$.

Then, for every $\lambda > \lambda^*$, the problem (P_1) admits at least three nonnegative weak solutions.

Proof. From (21) and (22) one has $\{\frac{d^p}{F(d)} : F(d) > 0\} \neq \emptyset$. Fix a positive number $\lambda > \lambda^*$. So, there is $d > 0$ such that $\lambda > \frac{2^p\|a\|_{L^1(\mathcal{B})}}{pR^p|B(x_0,\frac{R}{2})|}\frac{d^p}{F(d)}$.

We claim that I_λ is coercive for every $\lambda \geq 0$. We observe that from (21) we obtain

$$\lim_{|t|\to+\infty}\frac{F(t)}{|t|^p} = 0. \tag{23}$$

Indeed, fix $0 < \epsilon < \frac{1}{p\lambda T^p\|a^{-s}\|^{\frac{1}{s}}_{L^1(\Omega)}|\Omega|^{\frac{p_s^*-p}{p_s^*}}}$; from (23) there exists $c > 0$ such that

$$F(t) \leq \epsilon|t|^p + c,$$

for each $t \in \mathbb{R}$.

Then, from (5)

$$\Psi(u) = \int_\Omega F(u)dx \leq \int_\Omega (\epsilon|t|^p + c)\, dx = \epsilon\, \|u\|^p_{L^p(\Omega)} + c|\Omega| \leq \epsilon T^p\|a^{-s}\|^{\frac{1}{s}}_{L^1(\Omega)}|\Omega|^{\frac{p_s^*-p}{p_s^*}}\|u\|^p + c|\Omega|, \tag{24}$$

then

$$\begin{aligned} I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{p}\|u\|^p - \lambda\epsilon T^p\|a^{-s}\|^{\frac{1}{s}}_{L^1(\Omega)}|\Omega|^{\frac{p_s^*-p}{p_s^*}}\|u\|^p - \lambda c|\Omega| \\ &\geq \left[\frac{1}{p} - \lambda\epsilon T^p\|a^{-s}\|^{\frac{1}{s}}_{L^1(\Omega)}|\Omega|^{\frac{p_s^*-p}{p_s^*}}\right]\|u\|^p - \lambda c|\Omega|. \end{aligned}$$

This leads to the coercivity of I_λ. Moreover, I_λ is bounded from below, owing to the fact that it is coercive and sequentially weakly lower semicontinuous (see Remark 3.3). Finally, as we have just shown in the proof of Theorem 3.4, I_λ satisfies the (PS)-condition. Arguing as in the proof of Theorem 3.5, we prove that condition (7) of Theorem 2.2 holds.

All assumptions of Theorem 2.2 are verified, then, for each $\lambda > \lambda^*$, problem (P_1) admits at least three weak solutions.

We observe that by condition (22), the function $u = 0$ is solution of (P_1).

We claim that the others weak solutions are nonnegative. We use a classical truncation argument, put

$$f^+(t) = \begin{cases} f(0), & \text{if} \quad t < 0, \\ f(t), & \text{if} \quad t \geq 0. \end{cases} \tag{25}$$

We can apply the Theorem 3.5 to the function f^+, then the following problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f^+(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1^+}$$

admits at least three weak solutions for every $\lambda > \lambda^*$, and two of them are nonnegative since f^+ is nonnegative as standard arguments prove. So the solutions of problem (P_1^+) are also solutions of problem (P). The proof is finished. □

Now, we present an example in which our results can be applied.

Example 3.8. Put $N = 3$, $p = 2$, $\Omega = B(0, 1)$ and

$$a(x, y, z) = \sqrt[3]{x^2 + y^2 + z^2}.$$

Fix $s = 3$, then $p_s = \frac{3}{2}$, $p_s^* = 3$ and then $a^{-s} \in L^1(\Omega)$. Indeed

$$\left\|a^{-s}\right\|_{L^1(\Omega)} = \iiint\limits_{\Omega} \left|a^{-s}\right| dxdydz = \iiint\limits_{B(0,1)} \left[x^2 + y^2 + z^2\right]^{-1} dxdy = 4\pi.$$

Moreover, fix

$$\mathcal{B} = B\left(0, 1\right) \setminus B\left(0, \frac{1}{2}\right),$$

then

$$\|a\|_{L^1(\mathcal{B})} = \iiint\limits_{\mathcal{B}} |a(x, y, z)|\, dxdydz = \iiint\limits_{B(0,1)\setminus B(0,\frac{1}{2})} \sqrt[3]{x^2 + y^2 + z^2} dxdydz = \frac{2^2 \cdot 3(2^{\frac{11}{3}} - 1)}{2^{\frac{11}{3}} \cdot 11}\pi.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the following function

$$f(t) = \frac{(t^2 + 1)t^2}{e^{|t|}}.$$

We observe that

$$\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = \lim_{t\to +\infty} \frac{f(t)}{t^{p-1}} = 0,$$

$$F(d) = \int_0^d f(t)dt = \int_0^d \frac{(t^2 + 1)t^2}{e^{|t|}} dt = e^{-d}\left(-d^4 - 4d^3 - 13d^2 - 26d - 26\right) + 26,$$

and

$$\lambda^* = \frac{3^2(2^{\frac{11}{3}} - 1)}{2^{\frac{2}{3}} \cdot 11} \inf_{d>0} \frac{d^p}{F(d)} = \inf_{d>0} \frac{d^2}{e^{-d}\left(-d^4 - 4d^3 - 13d^2 - 26d - 26\right) + 26} < e.$$

Then, owing to Theorem 3.7, the problem

$$\begin{cases} -\mathrm{div}(\sqrt[3]{x^2 + y^2 + z^2}\ \nabla u) = \lambda \dfrac{(u^2 + 1)u^2}{e^{|u|}} & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least three nonnegative weak solutions for all $\lambda > e$.

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References

[1] D. Bresch, J. Lemoine, F. Guíllen-Gonzalez, A note on a degenerate elliptic equation with applications for lakes and seas, Electron. J. Differ. Equ. (45) (2004) 1–13.
[2] G. Bonanno, Relations between the mountain pass theorem and local minima, Adv. Nonlinear Anal. 1 (2012) 205–220.
[3] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal. 75 (5) (2012) 2992–3007.
[4] G. Bonanno, G. D'Aguì, R. Livrea, Triple solutions for nonlinear elliptic problems driven by a non-homogeneous operator, Nonlinear Anal. 197 (2020) 1–17.
[5] H. Brézis, Analyse fonctionelle - théorie et applications, Masson, Paris, 1983.
[6] P. Drabek, A. Kufner, F. Nicolosi, Quasilinear Eliptic Equations with Degenerations and Singularities, W. de Gruyter, Berlin-New York, 1997.
[7] P. Drábek, A. Kufner, V. Mustonen, Pseudo–monotonicity and degenerated or singular elliptic operators, Bull. Aust. Math. Soc. 58 (2) (1998) 213–221.
[8] P. Drábek, J. Hernández, Quasilinear eigenvalue problems with singular weights for the p-Laplacian, Ann. Mat. Pura Appl. (4) 198 (4) (2019) 1069–1086.
[9] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, L. Tartar, On a turbulent system with unbounded eddy viscosities, Nonlinear Anal. 52 (2003) 1051–1068.
[10] A. Kristály, H. Lisei, C. Varga, Multiple solutions for p-Laplacian type equations, Nonlinear Anal. 68 (2008) 1375–1381.
[11] A. Kufner, Weighted Sobolev Spaces, Translated from the Czech, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, NY, USA, 1985.
[12] D. Motreanu, E. Tornatore, Quasilinear Dirichlet problems with degenerated p-Laplacian and convection term, Mathematics 9 (2021) 139.
[13] M.Z. Nashed, A. Tamasan, Structural stability in a minimization problem and application to conductivity imaging, Inverse Probl. Imaging 5 (1) (2011) 219–236.
[14] P. Pucci, R. Servadei, Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 25 (3) (2008) 505–537.
[15] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[16] G. Zhang, Y. Wang, Some existence results for a class of degenerate semilinear elliptic systems, J. Math. Anal. Appl. 333 (2007) 904–918.
[17] E. Zeidler, Nonlinear Functional Analysis and Its Applications, vol. III, Variational Methods and Optimization, Springer–Verlag, New York, 1985.