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On an Umbral Treatment of Gegenbauer, Legendre and Jacobi Polynomials

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Abstract

Special polynomials, ascribed to the family of Gegenbauer, Legendre, and Jacobi and of their associated forms, can be expressed in an operational way, which allows a high degree of flexibility for the formulation of the relevant theory. We develop a point of view based on an umbral type formalism, exploited in the past, to study some aspects of the theory of special functions, in general, and in particular those of Bessel functions. We propose a fairly general analysis, allowing a transparent link between different forms of special polynomials .

Keywords: Gegenbauer, Legendre and Jacobi Polynomials, Umbral Calculus, Integral Transforms, Operatorational Methods

1 Introduction

In the following we develop a novel method regarding the study of the properties of the Gegenbauer polynomials [1] and of their generalized forms. We use an operational formalism of umbral nature, which will be shown to guarantee significant simplifications of the underlying algebraic technicalities. We introduce the main concepts, associated with the technique we are going to deal with, starting from a very simple example.

We consider indeed the elementary function

$$e^{(\nu)}(x) = (1+x)^{-\nu} \tag{1.1}$$

with ν being any rational or complex number with $Re(\nu) > 0$, according to the use of standard Laplace transform identities, the function (1.1) can be rewritten as

$$e^{(\nu)}(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-s} s^{\nu-1} e^{-sx} ds \tag{1.2}$$

Following ref. [7] we use the operational rule[1]

$$(\alpha)^{x\,\partial_x} f(x) = f(\alpha\, x) \tag{1.3}$$

to write

$$e^{(\nu)}(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-s} s^{\nu-1} s^{x\,\partial_x} e^{-x} ds \tag{1.4}$$

We note that we can define the effect of the Laplace transform on the exponential function according to the following identities

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)} \int_0^\infty e^{-s} s^{\nu-1} s^{x\,\partial_x} e^{-x} ds = \left(\frac{1}{\Gamma(\nu)} \int_0^\infty e^{-s} s^{\nu-1} s^{x\,\partial_x} ds \right) e^{-x} = \\
&= \hat{\Gamma}_\nu e^{-x} = \sum_{r=0}^\infty \frac{(-1)^r}{r!} (\nu)_r\, x^r, \\
&\hat{\Gamma}_\nu = \frac{\Gamma(\nu + x\,\partial_x)}{\Gamma(\nu)}, \\
&(\nu)_r = \frac{\Gamma(\nu + r)}{\Gamma(\nu)}
\end{aligned} \tag{1.5}$$

[1] By Setting $(\alpha)^{x\,\partial_x} = e^{\ln(\alpha)\,x\,\partial_x}$ and by making the change of variables $x = e^t$ we get $(\alpha)^{x\,\partial_x} f(x) = e^{\ln(\alpha)\,\partial_t} f(e^t) = f(e^{t+\ln(\alpha)})$ finally going back to the original variable we end up with eq. (1.3).

The procedure of bringing the exponential outside the sign of integration, thus defining the operator $\hat{\Gamma}_\nu$, is allowed only for the values of $|x| < 1$, for which the series, containing the Pochhammer symbol $(\nu)_r$, converges. The series appearing in (1.5) is recognized as the Newton binomial, even though obtained in an "involved", albeit useful way, for the purposes of the present paper. In the spirit of the umbral calculus [10] we will reinterpret the function $e^{(\nu)}(x)$ as an ordinary exponential function, by introducing the operator $\hat{\nu}$, defined as[2]

$$\hat{\nu}^r \varphi_0 = (\nu)_r \tag{1.6}$$

accordingly, we can cast the function $e^{(\nu)}(x)$in the form

$$e^{(\nu)}(x) = e^{-\hat{\nu}\, x} \varphi_0 \tag{1.7}$$

thus formally treating it as an exponential (namely a transcendental function) even though the series (1.5) has a limited range of convergence $|x| < 1$. As explained below, we will take advantage from the previous exponential umbral restyling of the function in (1.1) to construct a new formalism useful for the study of various family o special polynomials.

By keeping the derivative of both sides of eq. (1.7) , with respect to the x variable, using the ordinary rules of calculus, we obtain the identity[3]

$$\begin{aligned} \partial_x e^{(\nu)}(x) &= \left(\partial_x e^{-\hat{\nu}\, x}\right)\ \varphi_0 = -\hat{\nu}\, e^{-\hat{\nu}\, x}\varphi_0 = \sum_{r=0}^{\infty}(-1)^{r+1}\hat{\nu}^{r+1}\frac{x^r}{r!}\,\varphi_0 = \\ &= -\sum_{r=0}^{\infty}(-1)^r\,(\nu)_{r+1}\,\frac{x^r}{r!} = -\nu\, e^{(\nu+1)}(x) \end{aligned} \tag{1.8}$$

which follows as a consequence of

$$(\nu)_{r+1} = \nu\ (\nu+1)_r \tag{1.9}$$

and more generally

$$(\nu)_{r+m} = (\nu)_m(\nu+m)_r \tag{1.10}$$

We can accordingly state the rule

[2]The adjective "umbral" in the sense of ref. [10] may not be fully correct, a comment on the appropriate framing of this term will be presented in the concluding section.

[3]The same identity can be obtained from eq. (1.2) which yields

$$\partial_x e^{(\nu)}(x) = -\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-s}s^\nu e^{-sx}ds = -\frac{\Gamma(\nu+1)}{\Gamma(\nu)}e^{(\nu+1)}(x)$$

The umbral identity we have derived is not limited by any convergence restriction

$$\begin{aligned} &\left(\partial_x^m e^{-\hat{\nu}\,x}\right)\ \varphi_0 = (-1)^m \left(\hat{\nu}^{\,m} e^{-\hat{\nu}\,x}\right)\ \varphi_0 = \\ &= (-1)^m \,(\nu)_m\, e^{(\nu+m)}(x) \end{aligned} \tag{1.11}$$

Before proceeding further it is worth clarifying a point, which will be more thoroughly treated in the concluding section. Even though the formalism we have developed allows to treat not trivial functions in terms of elementary exponential functions, some properties like the semigroup identities associated with the exponential case are not easily associated with $e^{(\nu)}(x)$. We find indeed that, albeit the following chain of identities is correct

$$e^{(\nu)}(x+y) = e^{-\hat{\nu}\,(x+y)}\varphi_0 = e^{-\hat{\nu}\,x}e^{-\hat{\nu}\,y}\varphi_0 \tag{1.12}$$

it is also true that

$$e^{(\nu)}(x+y) \neq e^{(\nu)}(x)\, e^{(\nu)}(y) \tag{1.13}$$

To overcome such an apparently paradoxical conclusion, we clarify that the concept of semi-group has to be properly framed within the appropriate algebraic context. In defining the semigroup and, thereby, the associated identities, the corresponding binary operations between x and y need to be defined.

The associative binary operation (ABO) $e^{x+y} = e^x e^y = e^y e^x$ is a consequence of the fact that $(x+y)^n = \sum_{r=0}^{n} \binom{r}{n} x^{n-r} y^r$. This means that if we modify the Newton bynomial as

$$\begin{aligned} &(x \oplus_\nu y)^n = \sum_{r=0}^{n} \binom{n}{r} \left\{ \begin{array}{c} (\nu)_n \\ (\nu)_r \end{array} \right\}^{-1} x^{n-r} y^r, \\ &\left\{ \begin{array}{c} (\nu)_m \\ (\nu)_p \end{array} \right\} = \frac{(\nu)_m}{(\nu)_{m-p}\,(\nu)_p} = \frac{B(\nu+m,\,\nu)}{B(\nu+m-p,\nu+p)}, \\ &B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \equiv \text{Euler}\, B\, \text{Function} \end{aligned} \tag{1.14}$$

we can define the corresponding ABO

$$\begin{aligned} &e^{(\nu)}(y)\, e^{(\nu)}(x) = \sum_{r=0}^{\infty} \frac{(\nu)_r}{r!}(-y)^r \sum_{s=0}^{\infty} \frac{(\nu)_s}{s!}(-x)^s = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,(\nu)_n \left(\sum_{r=0}^{n} \binom{n}{r} \left\{ \begin{array}{c} (\nu)_n \\ (\nu)_r \end{array} \right\}^{-1} x^{n-r} y^r \right) = \\ &= e^{(\nu)}(x \oplus_\nu y) \end{aligned} \tag{1.15}$$

Accordingly we conclude that the proper environment for the algebraic semi-group property of the umbral exponential discussed in this section is the use of associative operations of the type (1.15).

The reliability of the formalism we are developing can be further checked by deriving integrals involving the pseudo Gaussian function

$$e^{(\nu)}(x^2) = e^{-\hat{\nu}\, x^2}\varphi_0 = (1+x^2)^{-\nu} \tag{1.16}$$

and according to the rules we have stipulated along with the properties of the ordinary Gaussian function, we can state that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{(\nu)}(x^2)\, dx \;= \int_{-\infty}^{+\infty} e^{-\hat{\nu}\, x^2}\, dx\, \varphi_0 = \\
&= \sqrt{\frac{\pi}{\hat{\nu}}}\,\varphi_0 = \sqrt{\pi}\,(\nu)_{-\frac{1}{2}} = \sqrt{\pi}\frac{\Gamma\left(\nu-\dfrac{1}{2}\right)}{\Gamma\,(\nu)}, \\
&Re(\nu) > \frac{1}{2}
\end{aligned} \tag{1.17}$$

This (well known) result is a byproduct of the outlined technique, but it could be also derived as a consequence of the Ramanujan Master Theorem (RMT)[4] [9].

It must be stressed that the integral in eq. (1.17) is extended to all the real axis and therefore the umbral representation should be representative of the function on the r.h.s. of eq. (1.16) and not of the relevant series expansion $\sum_{r=0}^{\infty}(-1)^r\,(\nu)_r\,\frac{x^{2\,r}}{r!}$, having radius of convergence $|x|<1$.

To clarify this point, we note that, by exploiting again the Laplace transform method, we can alternatively write the integral (1.17) as

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^{\;\nu}}\,dx = \int_{-\infty}^{+\infty} dx\left[\frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-s}s^{\nu-1}e^{-s\,x^2}ds\right] = \\
&= \frac{\sqrt{\pi}}{\Gamma(\nu)}\int_0^{\infty} e^{-s}s^{\nu-\frac{3}{2}}ds = \sqrt{\pi}\frac{\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma\,(\nu)}
\end{aligned} \tag{1.18}$$

which justifies the use of the previously stated umbral rules.

[4]The RMT may be formulated as follows: "If a function $f(x)$ admits an expansion $f(x) = \sum_{n=0}^{\infty}\dfrac{\phi(n)(-x)^n}{n!}$ in a neighborhood of $x=0$ then $\int_0^{\infty} f(x)x^{\nu-1}dx = \Gamma(\nu)\phi(-\nu)$".

By pushing further the formalism, we can take advantage from the wealth of properties of Gaussian integrals, by getting e.g.

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{(\nu)}(a\,x^2 + i\,b\,x)\,dx = \sqrt{\frac{\pi}{a\hat{\nu}}} e^{-\hat{\nu}\frac{b^2}{4\,a}}\,\varphi_0 = \\
&= \sqrt{\frac{\pi}{a}}\,\frac{\Gamma\left(\nu - \frac{1}{2}\right)}{\Gamma(\nu)}\,\frac{1}{\left(1 + \frac{b^2}{4\,a}\right)^{\nu-\frac{1}{2}}}, \\
&Re(\nu) > \frac{1}{2},\ Re(a) > 0
\end{aligned} \tag{1.19}$$

Let us now consider a further application of the previous procedure, by keeping the successive derivatives (with respect to the variable x) of the pseudo Gaussian function, namely

$$e_n^{(\nu)}(x^2) = \partial_x^n e^{(\nu)}(x^2) \tag{1.20}$$

We take advantage from the analogy with the properties of ordinary Gaussians and from the associated identity [9]

$$\begin{aligned}
&\partial_x^n e^{a\,x^2} = H_n(2\,a\,x,\ a)\,e^{a\,x^2}, \\
&H_n(\xi,\ \eta) = n!\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{\xi^{n-2\,r}\eta^r}{(n-2\,r)!\,r!}
\end{aligned} \tag{1.21}$$

where $H_n(\xi,\ \eta)$ are two variable Hermite-Kampé de Fériét (H-KdF) polynomials [2] with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\xi,\ \eta) = e^{\xi\,t + \eta\,t^2} \tag{1.22}$$

Therefore, by just adapting eqs. (1.7, 1.21) to the pseudo-Gaussian case, we find

$$\begin{aligned}
&e_n^{(\nu)}(x^2) = H_n(-2\,\hat{\nu}\,x,\ -\hat{\nu})\,e^{-\hat{\nu}\,x^2}\varphi_0 = \\
&= (-1)^n n!\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r(2\,x)^{n-2\,r}}{(n-2\,r)!\,r!}(\hat{\nu}^{n-r}e^{-\hat{\nu}\,x^2})\,\varphi_0
\end{aligned} \tag{1.23}$$

On account of eq. (1.11) we note that

$$\left(\hat{\nu}^{n-r} e^{-\hat{\nu}\, x^2}\right) \varphi_0 = (\nu)_{n-r}\, e^{(\nu+n-r)}(x^2) \tag{1.24}$$

If we now introduce the two variable polynomials

$$K_n^{(\nu)}(\xi,\, \eta) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(\nu)_{n-r}\, \xi^{n-2\,r} \eta^r}{(n-2\,r)!\, r!} \tag{1.25}$$

we can recast eq. (1.23) in the non operatorial form

$$e_n^{(\nu)}(x^2) = (-1)^n K_n^{(\nu)}\left(\frac{2x}{1+x^2}, -\frac{1}{1+x^2}\right) \frac{1}{(1+x^2)^\nu} \tag{1.26}$$

For $\xi = 2x$, $y = -1$ the polynomials (1.25) reduce to the ordinary Gegenbauer polynomials, namely

$$K_n^{(\nu)}(2\,x,\, -1) = n!\, C_n^{(\nu)}(x) \tag{1.27}$$

Furthermore the identity

$$(-1)^n\, e^{(-\nu)}(x^2)\, e_n^{(\nu)}(x^2) = K_n^{(\nu)}\left(\frac{2x}{1+x^2}, -\frac{1}{1+x^2}\right) \tag{1.28}$$

can be viewed as the associated Rodriguez formula [1].

It is also worth stressing that the use of the relation

$$\partial_x^n e^{a\,x^2+b\,x} = H_n(2\,a\,x+b,\, a)\, e^{a\,x^2+b\,x} \tag{1.29}$$

And the same rules, stated before, yields the result

$$\begin{aligned} &e_n^{(\nu)}(a\,x^2+b\,x) = \\ &= (-1)^n K_n^{(\nu)}\left(\frac{2\,a\,x+b}{1+a\,x^2+b\,x}, -\frac{a}{1+a\,x^2+b\,x}\right) \frac{1}{(1+a\,x^2+b\,x)^\nu} \end{aligned} \tag{1.30}$$

The results we have presented in this introduction discloses one of the advantages of the formalism, which allows the derivation of the properties of Gegenbauer polynomials from those of Hermite. Further consequences of this point of view will be discussed in the following sections.

2 The Gegenbauer polynomials and their generalized forms

As already stressed the procedure we have followed so far seems to be tailor suited to treat the Gegenbauer polynomials and the relevant generalized forms, which can be introduced by means of the generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}K_n^{(\nu,\,m)}(-\xi,\,-\eta)=e^{(\nu)}(\xi\,t+\eta\,t^m)=\frac{1}{(1+\xi\,t+\eta\,t^m)^{\nu}},\qquad |\,t\,|<1; \tag{2.1}$$

according to our formalism, can easily be identified with the help of higher order two variable H-KdF polynomials [2]

$$\begin{aligned}
&K_n^{(\nu,\,m)}(\xi,\,\eta)=H_n^{(m)}(\hat\nu\,\xi,\,\hat\nu\,\eta)\,\varphi_0,\\
&H_n^{(m)}(x,y)=n!\sum_{r=0}^{\left[\frac{n}{m}\right]}\frac{x^{n-m\,r}y^r}{(n-m\,r)!\,r!}\\
&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(m)}(x,y)=e^{xt+yt^m}
\end{aligned}\tag{2.2}$$

The repeated derivatives of functions like $e^{(\nu)}(ax^m+bx)$ can be expressed by using the properties of the higher order Hermite Kampé de Fériét polynomials and of their generalized forms. The use of the following identity involving multivariable Hermite polynomials (see ref. [3])

$$\begin{aligned}
&\partial_x^n e^{P(x)}=H_n^{(m,m-1,\,\dots\,,2)}\left(P'(x),\,\frac{P''(x)}{2},\,\frac{P'''(x)}{3!},\,\dots\,,\,\frac{P^{(m)}(x)}{m!}\right)e^{P(x)},\\
&P(x)=a\,x^m+b\,x
\end{aligned}\tag{2.3}$$

Can be exploited, along with eq. (2.3), to get (see also eq. (1.23))

$$\begin{aligned}
&\partial_x^n e^{(\nu)}(a\,x^m+b\,x)=e_n^{(\nu)}(a\,x^m+b\,x)=\\
&=H_n^{(m,\,m-1,\,\dots\,,2)}\left(-\hat\nu\,P'(x),\,-\hat\nu\frac{P''(x)}{2},\,-\hat\nu\frac{P'''(x)}{3!},\dots,\,-\hat\nu\frac{P^{(m)}(x)}{m!}\right)e^{-\hat\nu\,P(x)}\varphi_0=\\
&=K_n^{(\nu,\,m,\,m-1,\,\dots\,,2)}\left(-\frac{P'(x)}{P(x)+1},\,-\frac{P''(x)}{2(P(x)+1)},\,-\frac{P'''(x)}{3!(P(x)+1)},\dots,\,-\frac{P^{(m)}(x)}{m!(P(x)+1)}\right)\frac{1}{(P(x)+1)^{\nu}}
\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}
&K_n^{(\nu,\,m,\,m-1,\,\dots\,,2)}(x_1,x_2,x_3,\dots,x_m)=\frac{1}{\Gamma(\nu)}\int_0^{\infty}e^{-s}s^{\nu-1}H_n^{(\,m,\,m-1,\,\dots\,,2)}(x_1s,x_2s,x_3s,\dots,x_ms)ds,\\
&H_n^{(m,\,m-1,\,\dots\,,2)}(x_1,x_2,x_3,\dots,x_m)=n!\sum_{r=0}^{\left[\frac{n}{m}\right]}\frac{x_m^rH_{n-mr}^{(m-1,\,\dots\,,2)}(x_1,\dots,x_{m-1})}{(n-mr)!r!}\\
&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(\{m\})}(\{x\})=e^{\sum_{s=1}^{m}x_st^s},\\
&\{m\}=m,m-1,\,\dots\,,2;\qquad \{x\}=x_1,x_2,\,\dots\,,x_m
\end{aligned}\tag{2.5}$$

The same method allows some progress in the derivation of Gegenbauer generating functions and indeed we find

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}K_{n+l}^{(\nu)}(-\xi,\,-\eta)=H_l(-\hat{\nu}\,(\xi+2\,\eta\,t),\,-\hat{\nu}\,\eta)\,e^{-\hat{\nu}\,\left(\xi\,t+\eta\,t^2\right)}\varphi_0=\\
&=\frac{(-1)^l K_l^{(\nu)}\left(\dfrac{(\xi+2\,\eta\,t)}{1+\xi\,t+\eta\,t^2},\,-\dfrac{\eta}{1+\xi\,t+\eta\,t^2}\right)}{(1+\xi\,t+\eta\,t^2)^{\nu}},\\
&\mid t\mid<\left|\frac{\xi-\sqrt{\xi^2-4\eta}}{2\eta}\right|
\end{aligned} \tag{2.6}$$

which can be easily derived from the corresponding case of the Hermite polynomials [3]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+l}(x,y)=H_l(x+2yt,y)e^{xt+yt^2} \tag{2.7}$$

All the previous results can be obtained by the use of the Laplace transform method. The integral transforms are indeed not an alternative, but the rigorous support of the umbral methods we are developing. Further comments on this point will be presented in the concluding section.

3 The Jacobi Polynomials

In the previous section we have exploited a, seemingly, powerful tool to deal with a plethora of problems concerning the theory of special functions, whose relevant technicalities can accordingly be reduced be to straightforward exercises in elementary calculus.

The theory of Laguerre polynomials has been profitably reformulated by defining them as an ordinary Newton binomial including an appropriate umbral operator [4], namely

$$\begin{aligned}
&L_n(x,\,y)=(y-\hat{c}\,x)^n\,\varphi_0,\\
&\hat{c}^{\alpha}\varphi_0=\frac{1}{\Gamma(\alpha+1)}
\end{aligned} \tag{3.1}$$

The genesis of the operator $\hat{c}$ can be traced back to the theory of differintegral operators [8] and indeed

$$\hat{c}^{\alpha} \to \partial_{\eta}^{-\alpha} \tag{3.2}$$

And the "vacuum" φ_0 is the space spanned by the variable η itself. According to such a point of view the second of eq. (3.1) should be understood as[5]

$$\hat{c}^{m}\varphi_0 \to \partial_{\eta}^{-m}1 = \int_0^{\eta} d\eta_1 ... \int_0^{\eta_{m-1}} d\eta_m \;|_{\eta=1} = \frac{1}{\Gamma(m+1)} \tag{3.3}$$

If we forget all the intermediate steps in the previous chain of equalities and keep the "rule" as expressed in the second line of eq. (3.1) we find, after expanding the Newton binomial

$$\begin{aligned} L_n(x,\, y) &= \sum_{s=0}^{n} \binom{n}{s} (-x)^s y^{\,n-s} \hat{c}^{\,s} \varphi_0 = \\ &= \sum_{s=0}^{n} (-1)^s \frac{n!\, y^{\,n-s} x^s}{(s!)^2\, (n-s)!} \end{aligned} \tag{3.4}$$

which (*for y=1*) is the usual series definition of Laguerre polynomials.

Let us now take a step further, by introducing the following family of two variable polynomials

$$\frac{1}{n!} R_n^{(\alpha,\,\beta)}(\xi,\, \eta) = \hat{c}_1^{\alpha} \hat{c}_2^{\beta} \left[\hat{c}_1 \xi + \hat{c}_2 \eta\right]^n \varphi_{\,1,0}\, \varphi_{\,2,0} \tag{3.5}$$

where the operators $\hat{c}$ labelled by two different index act on two different vacua as

$$\begin{aligned} &\hat{c}_1^{\nu} \hat{c}_2^{\mu} \varphi_{\,1,0}\, \varphi_{\,2,0} = (\hat{c}_1^{\nu} \varphi_{\,1,0})\, (\hat{c}_2^{\mu} \varphi_{\,2,0}) = \\ &= \frac{1}{\Gamma\,(\,\nu+1)} . \frac{1}{\Gamma\,(\,\mu+1)} \end{aligned} \tag{3.6}$$

According to the previous definition we obtain the following explicit form for the polynomials defined in eq. (3.5)

$$R_n^{(\alpha,\,\beta)}(\xi,\, \eta) = (n!)^2 \sum_{s=0}^{n} \frac{\xi^{\,n-s} \eta^{\,s}}{[(n-s)!\,]\;\, s!\, \Gamma(n-s+\alpha+1)\, \Gamma(s+\beta+1)} \tag{3.7}$$

[5] For non- integer values of the exponent of $\hat{c}$ we remind that differentigral operators with non- integer exponents are defined through Laplace type transforms

The relevant properties can easily be derived by the use of elementary algebraic manipulations. It is indeed easily checked that

$$\begin{aligned}
&\frac{1}{(n+1)!} R_{n+1}^{(\alpha,\,\beta)}(\xi,\,\eta) = [\hat{c}_1\xi + \hat{c}_2\eta]\ \hat{c}_1^{\alpha}\hat{c}_2^{\beta}\,[\hat{c}_1\xi + \hat{c}_2\eta]^{n}\ \varphi_{1,0}\varphi_{2,0} = \\
&= \frac{1}{n!}\left(\xi\, R_n^{(\alpha+1,\,\beta)}(\xi,\,\eta) + \eta\, R_n^{(\alpha,\,\beta+1)}(\xi,\,\eta)\right)
\end{aligned} \tag{3.8}$$

and that

$$\begin{aligned}
\partial_\xi R_n^{(\alpha,\,\beta)}(\xi,\,\eta) &= n^2\, R_{n-1}^{(\alpha+1,\,\beta)}(\xi,\,\eta), \\
\partial_\eta R_n^{(\alpha,\,\beta)}(\xi,\,\eta) &= n^2\, R_{n-1}^{(\alpha,\,\beta+1)}(\xi,\,\eta)
\end{aligned} \tag{3.9}$$

Furthermore we can determine its generating functions (g.f.) by the use analogous elementary procedures. We obtain for example

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} R_n^{(\alpha,\,\beta)}(\xi,\,\eta) = \hat{c}_1^{\alpha}\hat{c}_2^{\beta} e^{\,t\,(\hat{c}_1\xi + \hat{c}_2\eta)}\varphi_{1,0}\varphi_{2,0} \tag{3.10}$$

and, by noting that

$$\begin{aligned}
&\hat{c}^{\nu} e^{-\hat{c}\,x}\varphi_0 = \sum_{r=0}^{\infty} \frac{(-x)^r\,\hat{c}^{r+\nu}}{r!}\varphi_0 = C_\nu(x) \\
&C_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,\Gamma(r+\nu+1)}
\end{aligned} \tag{3.11}$$

where the functions $C_\nu(x)$ (namely the Tricomi-Bessel) are linked to the cylindrical Bessel functions by

$$C_\nu(x) = \left(\frac{1}{x}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{x}) \tag{3.12}$$

We can write the g.f. (3.10) in terms of a product of Bessel functions[6]

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} R_n^{(\alpha,\,\beta)}(\xi,\,\eta) = C_\alpha(-\xi\,t)\, C_\beta(-\eta\,t) = \\
&= \frac{1}{\sqrt{(\xi^{\,\alpha}\eta^{\beta})\, t^{\,\alpha+\beta}}}\ I_\alpha\left(2\sqrt{\xi\,t}\right)\ I_\beta\left(2\sqrt{\eta\,t}\right)
\end{aligned} \tag{3.13}$$

[6]Where $I_\nu(x) = (-i)^\nu J_\nu(ix)$ is the first kind modified Bessel function.

The polynomials $R_n^{(\alpha,\,\beta)}(\xi,\,\eta)$ can be used to define the ordinary Jacobi polynomials [1] through the identity

$$\begin{aligned} &P_n^{(\alpha,\,\beta)}(x) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2}\, R_n^{(\alpha,\,\beta)}\left(\xi(x),\,\eta(x)\right), \\ &\xi(x) = \frac{x-1}{2}, \qquad \eta(x) = \frac{1+x}{2} \end{aligned} \tag{3.14}$$

The relevant recurrences are obtained from eqs. (3.8), (3.9) and writes

$$\begin{aligned} (n+1)P_{n+1}^{(\alpha,\,\beta)}(x) =& \frac{1}{2}x\,\left[(n+\beta+1)\,P_n^{(\alpha+1,\,\beta)}(x) + (n+\alpha+1)\,P_n^{(\alpha,\,\beta+1)}(x)\right] + \\ &-\frac{1}{2}\left[(n+\beta+1)\,P_n^{(\alpha+1,\,\beta)}(x) - (n+\alpha+1)\,P_n^{(\alpha,\,\beta+1)}(x)\right] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\frac{d}{dx}P_n^{(\alpha,\,\beta)}(x) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{2\,(n-1)!^2}\left(R_{n-1}^{(\alpha+1,\,\beta)}(\xi(x),\,\eta(x)) + R_{n-1}^{(\alpha,\,\beta+1)}(\xi(x),\,\eta(x))\right) = \\ &= \frac{1}{2}\left[(n+\beta)P_{n-1}^{(\alpha+1,\,\beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\,\beta+1)}(x)\right] = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\,\beta+1)}(x) \end{aligned} \tag{3.16}$$

Finally the relevant generating function can be written as

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}P_n^{(\alpha,\,\beta)}(x) =& \left(\frac{2}{\sqrt{2(x-1)t}}\right)^{\alpha}\left(\frac{2}{\sqrt{2(x+1)t}}\right)^{\beta}\cdot \\ &\cdot I_\alpha(\sqrt{2\,(x-1)\,t})\, I_\beta\left(\sqrt{2\,(x+1)\,t}\right) \end{aligned} \tag{3.17}$$

We can now deduce further consequences from the previous umbral restyling of the Jacobi polynomials.

The index doubling ”theorem” can e.g. derived by noting that

$$\begin{aligned} &R_{2\,n}^{(\alpha,\,\beta)}(\xi,\,\eta) = (2n)!\,\hat{c}_1^{\alpha}\hat{c}_2^{\beta}\left[\hat{c}_1\xi + \hat{c}_2\eta\right]^n\left[\hat{c}_1\xi + \hat{c}_2\eta\right]^n\,\varphi_{1,0}\varphi_{2,0} = \\ &= \frac{(2\,n)!}{n!}\sum_{s=0}^{n}\binom{n}{s}\,\xi^{n-s}\eta^s R_n^{(n-s+\alpha,\,s+\beta)}(\xi,\,\eta) \end{aligned} \tag{3.18}$$

which, on account of eq. (3.14), yields

$$P_{2n}^{(\alpha,\beta)}(x) = \frac{n!}{(2n)!}\Gamma(2n+\alpha+1)\,\Gamma(2n+\beta+1)\sum_{s=0}^{n}\binom{n}{s}\,(\xi(x))^{n-s}\,(\eta(x))^{s}\cdot \cdot\frac{P_n^{(n-s+\alpha,\,s+\beta)}(x)}{\Gamma(2n-s+\alpha+1)\,\Gamma(n+s+\beta+1)} \tag{3.19}$$

Furthermore an analogous procedure yields the following argument duplication formula

$$P_n^{(\alpha,\beta)}(2x) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\sum_{s=0}^{n}\binom{n}{s}\left(\frac{x}{2}\right)^{s}\sum_{r=0}^{s}\binom{s}{r}\cdot \cdot\frac{(n-s)!P_{n-s}^{(s-r+\alpha,\,r+\beta)}(x)}{\Gamma(n-r+\alpha+1)\,\Gamma(n-s+r+\beta+1)} \tag{3.20}$$

It is evident that the method is so straightforward that all the previous identities can easily be generalized, as touched on in the concluding remarks.

The associated Laguerre polynomials [1], defined, within the present context, as

$$\begin{aligned} L_n^{(\alpha)}(x,\,y) &= \frac{\Gamma(n+\alpha+1)}{n!}\,\Lambda_n^{(\alpha)}(x,\,y),\\ \Lambda_n^{(\alpha)}(x,\,y) &= \hat{c}^{\alpha}(y-\hat{c}\,x)^{n}\varphi_0 =\\ &= n!\sum_{r=0}^{n}\frac{(-x)^{r}y^{n-r}}{r!\,\Gamma(r+\alpha+1)\,(n-r)!} \end{aligned} \tag{3.21}$$

evidently further confirm the mutual link with the Jacobi family are closely linked and we find indeed that

$$\begin{aligned} &R_n^{(\alpha,\beta)}(\xi,\eta) = (n!)^2\sum_{s=0}^{n}\frac{(-1)^s L_{n-s}^{(\alpha)}(-\xi,\xi+\eta)\;L_s^{(\beta)}(\eta,\xi+\eta)}{\Gamma(n-s+\alpha+1)\Gamma(s+\beta+1)},\\ &P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\;\hat{c}_1^{\alpha}\hat{c}_2^{\beta}\left[\left(y+\hat{c}_1\frac{x-1}{2}\right)-\left(y-\hat{c}_2\frac{x+1}{2}\right)\right]^{n}\varphi_{1,0}\varphi_{2,0} =\\ &= \Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\sum_{s=0}^{n}\frac{(-1)^s L_{n-s}^{(\alpha)}\left(\frac{1-x}{2},\,y\right)\;L_s^{(\beta)}\left(\frac{x+1}{2},\,y\right)}{\Gamma(n-s+\alpha+1)\,\Gamma(s+\beta+1)} \end{aligned} \tag{3.22}$$

In these introductory remarks we have covered some of the properties of Jacobi polynomials by employing a minimal computational effort, we have fixed the formalism we are going to use and have provided an idea of the consequences which can be drawn by means of these methods.

4 The Legendre Polynomials

The Legendre polynomials are a particular of Jacobi [1] and can be identified as

$$P_n(x) = P_n^{(0,\,0)}(x) = R_n^{(0,0)}(\xi, \eta) \tag{4.1}$$

Their properties can be therefore derived as a consequence of those of the R polynomials in the particular case of $\alpha = \beta = 0$. Let us therefore go back to eq. (3.22) and note that

$$\begin{aligned} &\frac{1}{n!}P_n(x) = \left[\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right]^n \varphi_{1,0}\varphi_{2,0}, \\ &\frac{1}{n!}P_n(0) = \left(-\frac{\hat{c}_1}{2} + \frac{\hat{c}_2}{2}\right)^n \varphi_{1,0}\varphi_{2,0} = \frac{1}{n!}R_n\left(-\frac{1}{2}, \frac{1}{2}\right) = \\ &= \frac{(-1)^n}{2^n n!}\sum_{s=0}^{n}(-1)^s\left(\frac{n!}{s!\,(n-s)!}\right)^2, \\ &P_n(1) = R_n(0,\,1) = 1, \quad P_n(-1) = R_n(-1,\,0) = (-1)^n \end{aligned} \tag{4.2}$$

The use of the auxiliary polynomials R_n is a fairly important tool to state further identities, as e.g.

$$\begin{aligned} &P_n(\lambda\, x) = n! \left[\hat{c}_1 \frac{\lambda\, x-1}{2} + \hat{c}_2 \frac{\lambda\, x+1}{2}\right]^n \varphi_{1,0}\varphi_{2,0} = \\ &= n! \left[\lambda\left(\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right) + \hat{c}_1 \frac{\lambda-1}{2} + \hat{c}_2 \frac{-\lambda+1}{2}\right]^n \varphi_{1,0}\varphi_{2,0} = \\ &= (n!)^2 \sum_{s=0}^{n} \lambda^{n-s} \sum_{r=0}^{s} \frac{\xi(\lambda)^{s-r}\eta(-\lambda)^r P_{n-s}^{(s-r,\,r)}(x)}{(s-r)!r!(n-r)!(n-s+r)!} \end{aligned} \tag{4.3}$$

Furthermore we obtain

$$\begin{aligned} P_{n+m}(x) &= (n+m)!\left[\hat{c}_1\frac{x-1}{2}+\hat{c}_2\frac{x+1}{2}\right]^{n+m}\varphi_{1,0}\varphi_{2,0} = \\ &= n!m!\sum_{s=0}^{m}\binom{n+m}{s}\frac{\xi(x)^{m-s}\eta(x)^s P_n^{(m-s,\,s)}(x)}{(m-s)!(n+s)!} \end{aligned} \tag{4.4}$$

and

$$P_n(x+y) = (n!)^2\sum_{s=0}^{n}\left(\frac{y}{2}\right)^s\sum_{r=0}^{s}\frac{P_{n-s}^{(s-r,\,r)}(x)}{(s-r)!r!(n-r)!(n-s+r)!} \tag{4.5}$$

The previous identity cannot be considered an "addition theorem" in the strict sense, but rather a Taylor series expansion.

The next step will be the derivation of the differential equation satisfied by the Legendre polynomials.

By the use of eqs. (3.15), (3.16) we end up with

$$\begin{aligned} n\,P_{n-1}(x) &= \left[(1-x^2)\,\frac{d}{dx}+n\,x\right]\,P_n(x) \\ (n+1)P_{n+1}(x) &= \left\{(2\,n+1)\;x-\left[(1-x^2)\,\frac{d}{dx}+n\,x\right]\right\}\,P_n(x) \end{aligned} \tag{4.6}$$

by combining the previous recurrences, we can introduce the following operators

$$\begin{aligned} \hat{N}_- &= (1-x^2)\,\frac{d}{dx}+\hat{n}\,x, \\ \hat{N}_+ &= -(1-x^2)\,\frac{d}{dx}+(\hat{n}+1\,)\,x \end{aligned} \tag{4.7}$$

Defined in such a way that

$$\begin{aligned} \hat{N}_- P_n(x) &= n\,P_{n-1}(x), \\ \hat{N}_+ P_n(x) &= (n+1)\,P_{n+1}(x) \end{aligned} \tag{4.8}$$

Where $\hat{n}$ is a kind of number operator "counting" the index of the Legendre polynomial, namely

$$\hat{n}\, P_{m+k}(x) = (m+k)\, P_{m+k}(x) \tag{4.9}$$

According to the previous definitions we find

$$\begin{aligned}
&\hat{N}_+\hat{N}_- P_n(x) = \left[-(1-x^2)\frac{d}{dx} + (\hat{n}+1)x\right]\left[(1-x^2)\frac{d}{dx} + \hat{n}x\right] P_n(x) = \\
&= \left[-(1-x^2)\,\frac{d}{dx} + n\,x\right]\,\left[(1-x^2)\,\frac{d}{dx} + n\,x\right]\, P_n(x) = n^2 P_n(x)
\end{aligned} \tag{4.10}$$

which explicitly yields the following second order equation satisfied by the Legendre polynomials written in the form

$$\left(\frac{d}{dx}(1-x^2)\,\frac{d}{dx}\right)\, P_n(x) + n\,(n+1)\, P_n(x) = 0 \tag{4.11}$$

In the forthcoming section we will extend the umbral formalism to make further progress by including the properties of the associated Legendre polynomials and the theory of Spherical Harmonics.

5 Final Comments

In these concluding remarks we will present further arguments supporting the effectiveness of the method we have proposed. Let us therefore consider the evaluation of the following repeated derivatives

$$\begin{aligned}
&F_m^{(\nu)}(x) = \left(\frac{d}{dx}\right)^m \left(\frac{e^{P(x)}}{(1+Q(x))^\nu}\right), \\
&P(x) = \alpha x^2 + \beta x, \\
&Q(x) = a x^2 + b x
\end{aligned} \tag{5.1}$$

The use of the umbral procedure allows a significant simplification of the relevant algebra, by setting indeed

$$\frac{e^{P(x)}}{(1+Q(x))^\nu} = e^{P(x)} e^{(\nu)}(Q(x)) = e^{P(x)-\hat{\nu}Q(x)}\varphi_0 = e^{(\alpha-\hat{\nu}a)x^2+(\beta-\hat{\nu}b)x}\varphi_0 \tag{5.2}$$

we find

$$F_m^{(\nu)}(x) = H_m\left(2(\alpha-\hat{\nu}a)x + (\beta-\hat{\nu}b), \alpha-\hat{\nu}a\right) e^{P(x)-\hat{\nu}Q(x)}\varphi_0 \tag{5.3}$$

The use of the so far developed rules yields

$$F_m^{(\nu)}(x) = m! \sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{1}{(m-2r)!r!} \sum_{s=0}^{m-2r} \binom{m-2r}{s} A^{m-2r-s} B^s \sum_{q=0}^{r} \binom{r}{q} (-1)^{s+q} \alpha^{r-q} a^q (\nu)_{s+q}\, e^{(\nu+s+q)}(Q(x)) e^{P(x)}$$
$$A = 2\alpha x + \beta; \qquad B = 2ax + b \tag{5.4}$$

or, in a more compact form

$$F_m^{(\nu)}(x) = \Omega_m^{(\nu)}\left(P'(x), \frac{P''(x)}{2}; \frac{Q'(x)}{(1+Q(x))}, \frac{Q''(x)}{2(1+Q(x))}\right) \frac{e^{P(x)}}{(1+Q(x))^{\nu}}$$
$$\Omega_m^{(\nu)}(x,y;u,z) = \sum_{s=0}^{m} \binom{m}{s} (-1)^s H_{m-s}(x,y) K_s^{(\nu)}(u,-z) \tag{5.5}$$

It is now worth to explore more accurately the role of the $K_n^{(\nu)}(.,.)$ polynomials discussed in the first part of the paper.
To this aim we consider the particular case for which $\nu = \frac{1}{2}$ for which

$$e_n^{(\frac{1}{2})}(x^2) = (-1)^n K_n^{(\frac{1}{2})}\left(\frac{2x}{1+x^2}, -\frac{1}{1+x^2}\right) \frac{1}{(1+x^2)^{\frac{1}{2}}} \tag{5.6}$$

furthermore, by recalling that

$$K_n^{(\frac{1}{2})}(a,b) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s} s^{-\frac{1}{2}} H_n(as, bs) ds \tag{5.7}$$

and that

$$H_n(x,y) = y^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{y}}, 1\right) \tag{5.8}$$

we can easily infer that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} K_n^{(\frac{1}{2})}(a,b) = \frac{1}{\sqrt{1 - at - bt^2}} \tag{5.9}$$

which is the generating function of Legendre polynomials for $a = 2x,\ b = -1$, moreover the use of the identity (5.8) yields

$$(1+x^2)^{\frac{n+1}{2}} e_n^{(\frac{1}{2})}(x^2) = (-1)^n n! P_n\left(\frac{x}{\sqrt{1+x^2}}\right) \tag{5.10}$$

which can be extended to the cases involving the generalized Legendre forms, as discussed elsewhere.

The same procedure can be applied to derive the following generating function for ordinary Legendre

$$\sum_{n=0}^{\infty}\binom{n+l}{l}t^n P_{n+l}(x)=\frac{P_l\left(\dfrac{x-t}{\sqrt{1-2xt+t^2}}\right)}{(1-2xt+t^2)^{\frac{l+1}{2}}} \tag{5.11}$$

which is also a particular case of eq. (2.6).

According to the above point of view, the $m-th$ derivative of the $P_n(x)$ can therefore be easily calculated, thus finding

$$\begin{aligned}\left(\frac{d}{dx}\right)^m P_n(x)&=\frac{2^m}{\sqrt{\pi}(n-m)!}\int_0^{\infty}e^{-s}s^{m-\frac{1}{2}}H_{n-m}(2xs,-s)ds=\\ &=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\left[\frac{n-m}{2}\right]}\frac{(-1)^r 2^{n-2r}x^{n-m-2r}\Gamma(n-r+\frac{1}{2})}{(n-m-2r)!r!}\end{aligned} \tag{5.12}$$

On the other side the successive derivatives of the Legendre polynomials can be obtained from (4.2) and yields the following link with the Jacobi polynomials

$$\begin{aligned}\left(\frac{d}{dx}\right)^m P_n(x)&=\frac{1}{2^m}\frac{(n!)^2}{(n-m)!}(\hat{c}_1+\hat{c}_2)^m\left[\hat{c}_1\frac{x-1}{2}+\hat{c}_2\frac{x+1}{2}\right]^{n-m}\varphi_{1,0}\varphi_{2,0}=\\ &=\frac{(n!)^2}{2^m}\sum_{s=0}^{m}\binom{m}{s}\frac{P_{n-m}^{(m-s,\,s)}(x)}{(n-s)!(n-m+s)!}\end{aligned} \tag{5.13}$$

We will discuss a further alternative formulation of the theory of Legendre polynomials, using a formalism touched in [8], which will be embedded with the technique developed in this note. We consider indeed the following family of polynomials

$$\Pi_n(x,\,y)=\left((\hat{\pi}_y+x)^n\varphi_{y,\,0}\right)\phi_0 \tag{5.14}$$

where

$$\begin{aligned}\left(\hat{\pi}_y^r\varphi_{y,\,0}\right)\phi_0&=\frac{y^{\frac{r}{2}}\,r!}{\Gamma\left(\frac{r}{2}+1\right)}\left|\cos\left(r\frac{\pi}{2}\right)\right|\hat{c}^{\frac{r}{2}}\phi_0=\\ &=\frac{y^{\frac{r}{2}}\,r!}{\Gamma\left(\frac{r}{2}+1\right)^2}\left|\cos\left(r\frac{\pi}{2}\right)\right|\end{aligned} \tag{5.15}$$

According to the above definition, we obtain the explicit expression for the Π polynomials as

$$\Pi_n(x,\, y) = H_n(x,\, \hat{c}\, y)\, \phi_0 = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2\,k} y^k}{(k!)^2\, (n-2\,k)!} \tag{5.16}$$

They are essentially hybrid Laguerre-Hermite polynomials satisfying the "heat" equation

$$\begin{aligned} &{}_L\hat{D}_y G(x,\, y) = -\partial_x^2 G\,(x, y)\,, \\ &{}_L\hat{D}_y = -\partial_y y\, \partial_y, \\ &G(x,\, 0) = x^n \end{aligned} \tag{5.17}$$

We have denoted by ${}_L\hat{D}_y$the Laguerre derivative [5, 6] and it is evident that the polynomials (5.16) can be defined through the following operational rule

$$\Pi_n(x,\, y) = C_0(-y\, \partial_x^2)\, x^n \tag{5.18}$$

and the Legendre polynomials can be identified with the particular case

$$P_n(x) = \Pi_n\left(x,\, -\frac{1-x^2}{4}\right) \tag{5.19}$$

Finally, from the previous identities we find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) = e^{x\,t - \hat{c}\frac{1-x^2}{4} t^2} \phi_0 = e^{x\,t} J_0\left[t\sqrt{1-x^2}\right] \tag{5.20}$$

We can now derive a further consequence from the above equation and from the umbral definition of the Legendre polynomials, according to which we find

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) = \sum_{n=0}^{\infty} t^n \left[\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right]^n \varphi_{1,0}\, \varphi_{2,0} = \\ &= \frac{1}{1 - t\, \left[\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right]} \varphi_{1,0}\, \varphi_{2,0} \end{aligned} \tag{5.21}$$

The use of standard Laplace transform identities yields

$$\begin{aligned} &\frac{1}{1 - t\, \left[\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right]}\, [\varphi_{1,0}\, \varphi_{2,0}] = \int_0^{\infty} e^{-s} e^{s\,t\left(\hat{c}_1 \frac{x-1}{2} + \hat{c}_2 \frac{x+1}{2}\right)} ds\, \varphi_{1,0}\, \varphi_{2,0} = \\ &= \int_0^{\infty} e^{-s} C_0\left(\frac{1-x}{2} s\, t\right)\, C_0\left(-\frac{1+x}{2} s\, t\right)\, ds \end{aligned} \tag{5.22}$$

which once confronted with (5.20) yields

$$\begin{aligned}&\int_0^\infty e^{-s} C_0\left(\frac{1-x}{2} s\,t\right) C_0\left(-\frac{1+x}{2} s\,t\right) ds = \\ &= e^{xt} J_0\left(t\sqrt{1-x^2}\right)\end{aligned} \tag{5.23}$$

Before closing this paper we will discuss some points left open since the introductory section and relevant to the legitimacy of our treatment based on the use of umbral operators. We must underline that

a) As already remarked elsewhere [7] the methods we have developed here and in previous papers are justified by the correctness of their results, rather than by the rigor of the relevant mathematical foundations

b) A well sound motivation can however be proposed along the lines discussed in ref. [7] where most of the rules relevant to the umbral operator algebra have been justified using the properties of the Borel transform.

c) As to the umbral definition of the Gegenbauer polynomials, proposed in this paper we can adopt an analogous point of view by noting that the use of the operational identity

$$M(a\,x,\,b\,y) = a^{x\,\partial_x} b^{y\,\partial_y} M(x,\,y) \tag{5.24}$$

based on the Euler dilation operator allows the following reshuffling of eq. (2.5)

$$K_n^{(\nu)}(\xi,\,\eta) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-s} s^{\nu-1} s^{\xi\,\partial_\xi + \eta\,\partial_\eta}\,ds\,H_n(\,\xi,\,\eta) \tag{5.25}$$

and the use of the properties of the Gamma function eventually leads to the following operational definition of the Gegenbauer polynomials

$$\begin{aligned}&K_n^{(\nu)}(\xi,\,\eta) = \hat{\Gamma}_\nu\,H_n(\xi,\,\eta) \\ &\hat{\Gamma}_\nu = \frac{\Gamma(\nu + \xi\,\partial_\xi + \eta\,\partial_\eta)}{\Gamma(\nu)}\end{aligned} \tag{5.26}$$

The operator $\hat{\Gamma}_\nu$ is therefore a differential realization of its umbral counterpart $\hat{\nu}$. In a forthcoming paper we will discuss the problem using this point of view , which offers some advantage for e.g. the derivation of the differential equation satisfied by the different polynomial families. In a forthcoming paper we will extend the procedure developed in this paper to further families of special functions and to the relevant application in classical and quantum electromagnetism.

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