Space Charge and Quantum Corrections in Free Electron Laser Evolution

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Abstract

Effects producing gain dilution in Free Electron Laser devices are well documented. We develop here a unified point of view allowing the introduction of space charge effects, along with the gain deterioration due to inhomogeneous broadening contributions and discuss the relevant interplay. We outline future developments and comment on the possibility of including in the formalism effects of quantum mechanical nature.

Keywords: Free Electron Laser 78A60; Charged Beam Propagation 81V99; Space Charge 78A99; Quantum FEL 37N20; X-Ray FEL 81V19.

1. Introduction

The space charge contributions to Free-Electron Laser (*FEL*) dynamics have been discussed in the past in a number of authoritative papers [1, 2, 3, 4, 5], which have clarified the physical mechanisms underlying the relevant bunching spoiling effects and the consequent gain degradation.

Shih and Yariv [6] employed a single particle model and quantified the space (SC) charge detrimental effect in terms of the relativistic plasma frequency. The usefulness of their point of view stems from the fact that the afore mentioned quantity can be framed within the same context of the inhomogeneous broadening parameters [7, 8, 9, 10, 11, 12, 13, 14, 15, 16], used to specify the gain reduction induced by the electron beam relative energy spread and emittances. Even though the physical mechanisms underlying the two effects are not the same, the use of similar parameters allows a straightforward comparison of the relative detrimental consequences. We will comment on the relevant distinguishing physical features in the final section of the paper.

In Refs. [5, 6] the authors treat the low gain case and write the gain function as

$$G\left(\phi,\mu_Q\right) = 2\pi g_0 \frac{\phi}{\left(\phi^2 - \mu_Q{}^2\right)^2}\left(2 - 2\cos(\phi)\cos(\mu_Q) - \left(\frac{\mu_Q}{\phi} + \frac{\phi}{\mu_Q}\right)\sin(\mu_Q)\sin(\phi)\right),$$

$$\phi = 2\pi N \, \frac{\omega_0 - \omega}{\omega_0},$$

(1)

where g_0 is the small signal gain coefficient, ϕ is the detuning parameter and

$$\mu_Q = 2\,N\left(\frac{\Omega_p \lambda_u}{2\,c}\right), \qquad \Omega_p^2 = \frac{e^2 \overline{N}_e}{\varepsilon_0 m_e \gamma^3},$$

(2)

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with N, λ_u being the number of undulator periods and period length respectively, $\overline{N}_e$ is the electron number density per unit volume, m_e is the electron mass and γ is the relativistic factor. A more appropriate way, for our purposes, of expressing μ_Q is the following

$$\mu_Q = \frac{2\,N}{\gamma}\left[\alpha\,\frac{E_f}{E_e}\,\frac{\lambda}{4\,\pi\,\varepsilon}\,\frac{\dot{N}_e\lambda_u^2}{c\beta_T}\right]^{\frac{1}{2}} \tag{3}$$

with

$$
\begin{aligned}
&\dot{N}_e = \frac{N_e}{\sqrt{2\,\pi}\sigma_\tau}, & &E_f = \hbar\omega, \\
&\lambda \equiv \text{FEL wavelength}, & &E_e = m_e\gamma\,c^2, \\
&\alpha \equiv \text{fine structure constant}, & &\beta_T \equiv \text{transverse twiss parameter}, \\
&\sigma_\tau \equiv \text{electron bunch time duration} & &\varepsilon \equiv \text{transvers beam emittance}.
\end{aligned}
\tag{4}
$$

The physical content of μ_Q is clear, it states indeed that the SC induced effects decreases with increasing beam energy and increase for smaller σ_τ (namely for large peak currents) and smaller β_T decreasing transverse dimensions, hence larger density current.

It is straightforwardly checked that, for vanishing μ_Q, the gain function (1) reduces to the usual anti-symmetric form

$$\lim_{\mu_Q\to 0}\frac{G(\phi,\mu_Q)}{g_0} = -\pi\,\partial_\phi\left[\frac{\sin\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)}\right]^2 \tag{5}$$

The relevant consequences on the gain curve (a reduction of the peak and a broadening) are shown in Fig. 1, where we have reported $\frac{G(\phi,\mu_Q)}{g_0}$ vs. ϕ for different values of μ_Q.
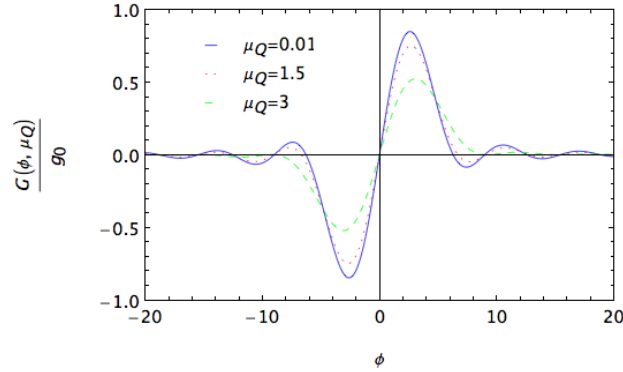

Figure 1: Gain divided g_0 vs. ϕ for different values of μ_Q.

The presence of a non vanishing μ_Q determines a broadening and a shift of the gain line, which is not dissimilar from that due to the beam relative energy spread. The behavior of the maximum gain vs. μ_Q is shown in Fig. 2 and is reproduced by the scaling relation

$$G^*(\mu_Q) \simeq 0.848g_0\exp\left(-\frac{\mu_Q^2}{18} + 1.15\cdot 10^{-4}\mu_Q^5\right). \tag{6}$$

It should be noted that Eq. (6) has not been derived on the basis of any physical assumption. The fitting formula has been chosen to get the best approximation with the numerical data. A Lorentzian type fitting formula, albeit less accurate, can also be exploited (See Section 4 for more details).
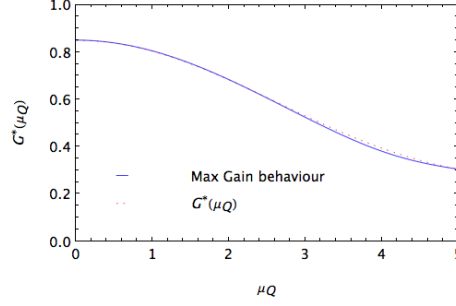
Figure 2: Comparison between maximum gain behaviour vs. μ_Q using numerical calculations for $\phi = 0.01$ and scaling formula in Eq. (6).

We have closely followed the treatment of Shih and Yariv of Ref. [6] and therefore we are restricted to the low gain case.

In the forthcoming sections we show how the study can be extended in various directions.

1. We will determine a fairly simple way of including in the gain formula the combined detrimental effects of space charge and relative energy spread.
2. We exploit the same formalism to go beyond the low gain approximation, used so far, and extend the treatment to the high gain regime, by a suitable modification of the relevant small signal integral equation.
3. We eventually continue the discussion by proposing a modification to the equations ruling the *FEL* small signal dynamics, which allows the inclusion of the quantum corrections [17, 18]

2. Low gain regime: Space charge and relative energy spread contribution

The gain in Eq. (1) can be expressed in a different form more useful for our purposes such that the space charge effect can be displayed separately. We note indeed that it can be cast in the alternative way [6]

$$G\left(\phi, \mu_Q\right) = \frac{\pi g_0}{2\mu_Q}\left[\left(\frac{\sin\left(\frac{\phi_-}{2}\right)}{\frac{\phi_-}{2}}\right)^2 - \left(\frac{\sin\left(\frac{\phi_+}{2}\right)}{\frac{\phi_+}{2}}\right)^2\right], \qquad \phi_\mp = \phi \mp \mu_Q. \tag{7}$$

According to the previous identity the gain function, including μ_Q corrections, has been expressed as the balance between two different emission processes. From the physical point of view it accounts for a kind of Raman effect including the frequency shift associated with the emission and absorption of a plasma oscillation (see also Ref. [19] where this aspect of the problem is treated with high accuracy).

The last equation contains the wave intensity variation only, if phase variations need to be included, we take advantage from the following integral representation of the spontaneous emission line

$$S(\phi) = \left(\frac{\sin\left(\frac{\phi}{2}\right)}{\frac{\phi}{2}}\right)^2 = 2Re\left[E(\phi)\right], \qquad E(\phi) = \int_0^1 (1-\xi)e^{-i\phi\xi}d\xi. \tag{8}$$

The inclusion of the effects of the relative energy spread σ_ε occurs through the introduction of the parameter [7, 8, 9, 10, 11]

$$\mu_\varepsilon = 4N\sigma_\varepsilon. \tag{9}$$
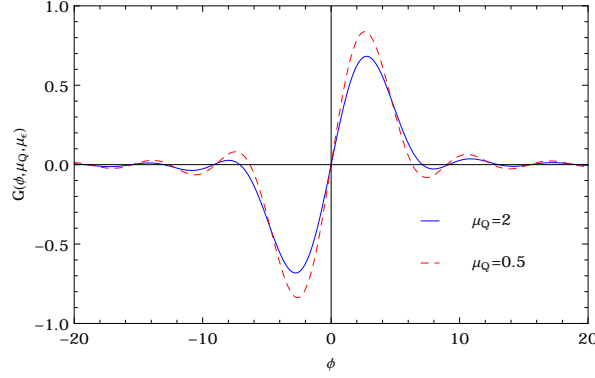
After convolving the line-width (8) on a Gaussian energy distribution we get

$$E(\phi, \mu_\varepsilon) = \int_0^1 (1-\xi)e^{-i\phi\xi - \frac{(\pi\mu_\varepsilon\,\xi)^2}{2}}d\xi \tag{10}$$
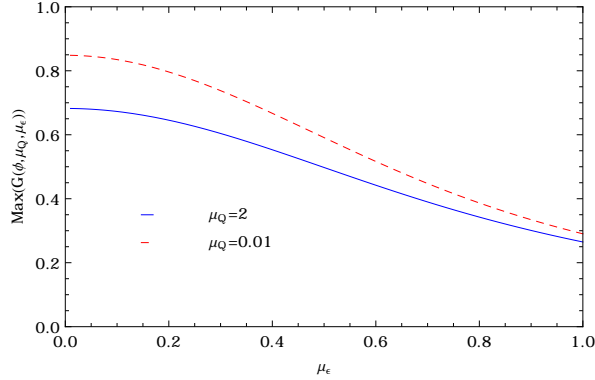
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The combined effect of $\mu_{\varepsilon,Q}$ parameters is eventually obtained as

$$G\left(\phi,\mu_Q,\mu_\varepsilon\right) = \frac{\pi\,g_0}{\mu_Q}Re\left[E(\phi_-,\mu_\varepsilon) - E(\phi_+,\mu_\varepsilon)\right]. \tag{11}$$

In Fig. 3 we have plotted the gain given by Eq. (11) vs. ϕ for different combination of $\mu_{Q,\varepsilon}$ and in Fig. 4 the maximum gain vs. μ_ε for different values of μ_Q.

Figure 3: Gain function vs. ϕ with $\mu_\varepsilon = 0.01$.

Figure 4: Maximum gain vs. μ_ε for different values of μ_Q.

From Fig. 3 it is worth noting that when the effect of energy spread increases (namely for larger values of μ_ε), the space charge gain dilution becomes less effective. Accordingly we find that a good parameterization of the combined contributions is

$$G^*\left(\mu_Q,\mu_\varepsilon\right) \simeq \frac{G^*\left(\dfrac{\mu_Q}{1 + 0.45\,\mu_\varepsilon^2}\right)}{1 + 1.7\,\mu_\varepsilon^2}. \tag{12}$$

The previous scaling relation (reducing for vanishing μ_Q to the ordinary scaling vs. the energy spread [7, 8, 9, 10, 11], [20]) shows that μ_ε and μ_Q cannot be straightforwardly disentangled.

We have already noted that the results obtained so far are limited to the low gain regime. The inclusion of high gain effects will be accomplished in the forthcoming section using an appropriate modification of the high gain small signal integral equation.

3. High Gain Regime and Space Charge Corrections

The FEL low gain condition is an approximation of the FEL theory, which corresponds to the physical conditions in which the field amplitude can be kept constant, during the beam wave interaction inside the undulator. The complex amplitude and complex phase variations are registered at the end of the undulator. These conditions have characterized the early FEL oscillator experiments, in which the small signal gain coefficient did not exceed few tens of percent. When high gain contributions become active, the field amplitude is ruled by a Volterra integro-differential equation with a memory kernel accounting for its self-consistent variation during the interaction.

Albeit the necessity for a more general formulation arouse very early [21], the relevant analytical treatment was undertaken in the early eighties [22, 23, 24] and lead to the understanding of the role of the so called FEL instability [25], which, on the other side, is a characteristic feature of all the existing free electron generators of coherent radiation [26].

In order to include the SC corrections in the high gain formalism and write the corresponding Volterra equation, we consider the time dependent complex function

$$E(\phi, \tau) = -\phi \int_0^\tau e^{-i\phi\xi} d\xi, \qquad \tau = \frac{z}{N\lambda_u} \tag{13}$$

along with its extension

$$K(\phi, \mu_Q, \tau) = \frac{E(\phi_-, \tau) - E(\phi_+, \tau)}{2\mu_Q}, \tag{14}$$

including the space charge parameter.

In correspondence of these quantities the high gain FEL equation writes

$$\begin{cases} \dfrac{\partial}{\partial\tau} a = i\pi g_0 \displaystyle\int_0^\tau K(\phi, \mu_Q, \tau') \, a(\tau - \tau') d\tau', \\ a(0) = 1 \end{cases} \tag{15}$$

where $a(\tau)$ is the FEL dimensionless Colson amplitude. Before proceeding further let us note that Eq. (15) can also be written as

$$\frac{\partial}{\partial\tau} a = i\pi g_0 \frac{e^{-\mu_Q \frac{\partial}{\partial\phi}} - e^{\mu_Q \frac{\partial}{\partial\phi}}}{2\mu_Q} \int_0^\tau E(\phi, \tau') \, a(\tau - \tau') d\tau' \tag{16}$$

for vanishing μ_Q we find

$$\lim_{\mu_Q \to 0} \frac{e^{-\mu_Q \frac{\partial}{\partial\phi}} - e^{\mu_Q \frac{\partial}{\partial\phi}}}{2\mu_Q} = -\frac{\partial}{\partial\phi} \tag{17}$$

and under this assumption Eq. (16) reduces to the "canonical" FEL high gain equation [27]

$$\frac{\partial}{\partial\tau} a = i\pi g_0 \int_0^\tau e^{-i\phi\tau'} \, \tau' a(\tau - \tau') d\tau'. \tag{18}$$

In Fig. 5 we have shown the gain function for values of the small signal gain parameter inducing high gain corrections ($g_0 = 2$) for cases with and without space charge effects. We have checked the correctness of the numerical procedure and we have found that the maximum gain exhibits the scaling vs. g_0 provided by the identity [28]

$$G^*(g_0) \simeq 0.848 \, g_0 + 0.19 \cdot g_0^2 + 4.23 \cdot 10^{-3} g_0^3 \tag{19}$$

The variables in Eq. (18) are more suitable for the low gain case. A more convenient form for the high regime is provided by

$$\partial_{\tilde{z}} \, a = \frac{i}{3\sqrt{3}} \int_0^{\tilde{z}} e^{-i\phi\tilde{z}'} \tilde{z}' a(\tilde{z} - \tilde{z}') d\tilde{z}', \qquad \tilde{z} = \frac{z}{L_g} \tag{20}$$

where

$$\tilde{\phi} = \frac{1}{2\rho\sqrt{3}}\frac{\omega_0 - \omega}{\omega_0}, \qquad\qquad L_g = \frac{\lambda_u}{4\pi\sqrt{3}\rho} \qquad\qquad (21)$$

with ρ being the Pierce parameter linked to the small signal gain coefficient by[1]

$$\rho = \frac{(\pi g_0)^{\frac{1}{3}}}{4\pi N} \qquad\qquad (22)$$

and L_g being the gain length which specifies the growth rate of the high gain FEL's operating in single pass configuration.
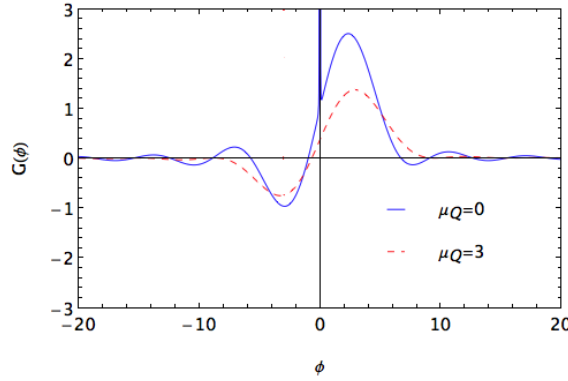

Figure 5: Gain vs. ϕ for $g_0 = 2$.

Also, in the limit of high gain, the presence of space charge contributions yields a gain distortion not dissimilar from those induced by the energy spread inhomogeneous broadening.

Non ideal beam qualities determines an increase of L_g and of the saturation length as well. Within this respect relative energy spread becomes harmful if the condition $\sigma_\varepsilon \le \frac{\rho}{2}$ is not fulfilled. In the high gain regime the parameter controlling the inhomogeneous effect is $\tilde{\mu}_\varepsilon = \frac{2\sigma_\varepsilon}{\rho}$ and the corresponding quantity for the SC is

$$\tilde{\mu}_Q = \frac{1}{\gamma\rho}\left[\alpha\,\frac{E_f}{E_e}\,\frac{\lambda}{4\pi\varepsilon}\,\frac{\dot{N}_e\lambda_u^2}{c\beta_T}\right]^{\frac{1}{2}}. \qquad\qquad (23)$$

An analogous quantity[2] has been introduced in Ref. [29], it has been normalized using the same criterion for the quantities marking the inhomogeneous broadening effects and quoting the authors of [29] "*the SC parameter is scaled to be twice the plasma phase advance over a one-dimensional gain length*".
We can now comment on the possibility that these effects be observed in an actual experimental configuration, or, saying better, whether they may induce negative sizeable effects like the increase of the saturation length in e.g. FEL designed as sources of bright hard X-ray beams. They demand for high charges, short bunches and high quality beams. A paradigmatic example is provided by the European $X-FEL$, which is foreseen to produce photons up to the angstrom wavelength, with a beam of a 17.5 GeV produced by a linear accelerator [30]. The high current intensity is achieved by compressing bunches bearing a charge of 250 and 500 pC. To this aim the electron beam line incorporates three vertical bunch compressors of C type [30].

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[1] It should be noted that, being $g_0 \propto N^3$, there is no dependence in Eq. (20) on the number of periods of the undulator.

[2] In Ref. [29] the SC parameter is defined by replacing γ^3 in the second of Eqs. (2) with $\gamma\gamma_z^2$, where $\gamma_z = \dfrac{\gamma}{\sqrt{1 + \frac{K^2}{2}}}$.

It is evident that the major problem in handling this type of beams is the control the stability of the compression and the qualities of the electron bunch after the compression.

The analysis of these problems goes beyond the scope of this paper, we assume therefore that compressing a 500 pc to hundreds of fs does not create additional problems in terms of energy spread and emittance (for more substantive comments see Ref. [30]).

Before entering further into the discussion whether the space charge effects may have some relevance in high FEL devices by keeping in consideration the previously quoted $X - FEL$ parameters, we note that $\tilde{\mu}_Q \simeq 3$ may slightly modify the FEL intensity growth along the undulator, as reported in Fig. 6.
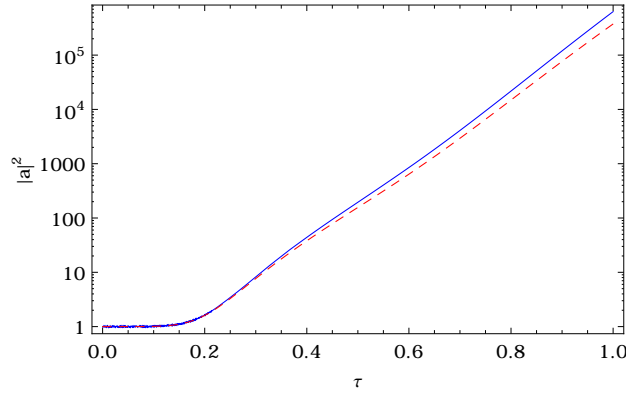
Figure 6: Dimensionless intensity growth $|a|^2$ vs. τ for $\mu_Q = 0.24$ (blue) and $\mu_Q = 3$ (red dashed) with $g_0 = 200$.

If we assume that the produced X-ray-beam is diffraction limited (namely $\varepsilon \simeq \frac{\lambda}{4\pi}$) we can write

$$\tilde{\mu}_Q = \frac{5 \cdot 10^{-6}}{\gamma \rho} \left[\frac{E_f}{E_e} \frac{\dot{N}_e \lambda_u^2}{\beta_T} \right]^{\frac{1}{2}}. \tag{24}$$

Imposing the condition $\tilde{\mu}_Q > 3$ to appreciate some space charge distortions to the FEL dynamics, we obtain the following condition on the peak beam current

$$\hat{I} = e \dot{N}_e > 5.76 \cdot 10^{11} \, (\gamma \rho)^2 \, \frac{\beta_T}{\lambda_u^2} \frac{E_e}{E_f} \tag{25}$$

Keeping for example $E_e \simeq 15 \, GeV$, $E_f \simeq 12.4 \, keV$, $\lambda_u = 0.04 \, m$, $\beta_T = 8 \, m$ and assuming $\gamma \rho \simeq 5$ we get $\hat{I} > 5 \, kA$. According to Ref. [30] operating conditions with 10 kA are foreseen, by considering such an extreme value we can assume values of $\tilde{\mu}_Q$ on the order of 5 which (as reported in Fig. 5) might allow quite a significant increase of the gain length induced by space charge effects. In the following section we will provide further comments on SC impact on FEL device and on their interplay with other effects including those of quantum nature.

4. Final considerations

In the previous parts of the article, we have discussed a fairly simple method to embed space charge and inhomogeneous broadening effects into a straightforward procedure. The method we have adopted to extend the analysis to the high gain seems to rely on a heuristic recipe, because we just exploited the function $E(\phi, \tau)$ to derive the high gain integral equation. Albeit we have checked the consistency of the procedure with previous results, in absence of a more rigorous mathematical tool there are still elements which makes our analysis doubtful.

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We have therefore checked our results by making a comparison with two previous papers in which the problem is addressed reducing the high gain equation to a third order differential equation, which in our notation reads

$$a''' + 2i\tilde{\phi}\,a'' + (\tilde{\mu}_Q^2 - \tilde{\phi}^2)a' = \frac{i}{3\sqrt{3}}\,a \tag{26}$$

where the apices denote derivatives taken with respect to $\tilde{z}$.

The use of the procedure envisaged in Ref. [31] allows to transform Eq. (26) into an integro-differential equation. We note indeed that we can factorize the differential operators on the l.h.s. as

$$(\partial_{\tilde{z}} + i\phi_+)\,(\partial_{\tilde{z}} + i\phi_-)\,(\partial_{\tilde{z}}a) = \frac{i}{3\sqrt{3}}\,a \tag{27}$$

which can also be written as

$$\partial_{\tilde{z}}a = \frac{i}{3\sqrt{3}}\hat{K}a, \qquad \hat{K} = \left[(\partial_{\tilde{z}} + i\phi_+)\,(\partial_{\tilde{z}} + i\phi_-)\right]^{-1}. \tag{28}$$

The operator $\hat{K}$ is easily shown to be an integral operator, we note indeed that

$$\left[(\partial_{\tilde{z}} + i\phi_+)\,(\partial_{\tilde{z}} + i\phi_-)\right]^{-1} = -\frac{i}{2\tilde{\mu}_Q}\left(\frac{1}{\partial_{\tilde{z}} + i\phi_+} - \frac{1}{\partial_{\tilde{z}} + i\phi_-}\right) \tag{29}$$

The use of standard techniques of Laplace transform eventually reduces its action on the dimensionless amplitude a to the integral form in Eq. (15) .

We have already mentioned that the presence of the SC contributions produces changes similar to those of inhomogeneous broadening and indeed the most significant effect which can be drawn from Fig. 6 is an increase of the gain length, which can be cast in the form

$$L_g(\tilde{\mu}_Q) = \frac{\lambda_u}{4\pi\sqrt{3}\rho(\tilde{\mu}_Q)}, \qquad \rho(\tilde{\mu}_Q) = \frac{\rho}{1 + 0.974\left(\frac{\tilde{\mu}_Q}{16}\right)^2 + 0.983\left(\frac{\tilde{\mu}_Q}{16}\right)^4} \tag{30}$$

In the high regimethe use of a rational function appears more appropriate to get an accurate fit (a maximum relative error within 2%) of SC effect on the Pierce parameter [10]. An analogous fitting formula has been proposed in Ref. [29] using a power law scaling, in the spirit of Ming Xie parameterization [12, 13, 14].The use of the analytical form for the linear intensity growth, along with the redefinition of (30) of the gain length

$$I(z) = \frac{I_0}{9}\left[3 + 2\cosh\left(\frac{z}{L_g(\tilde{\mu}_Q)}\right) + 4\cos\left(\frac{\sqrt{3}}{2}\frac{z}{L_g(\tilde{\mu}_Q)}\right)\cosh\left(\frac{z}{2L_g(\tilde{\mu}_Q)}\right)\right] \tag{31}$$

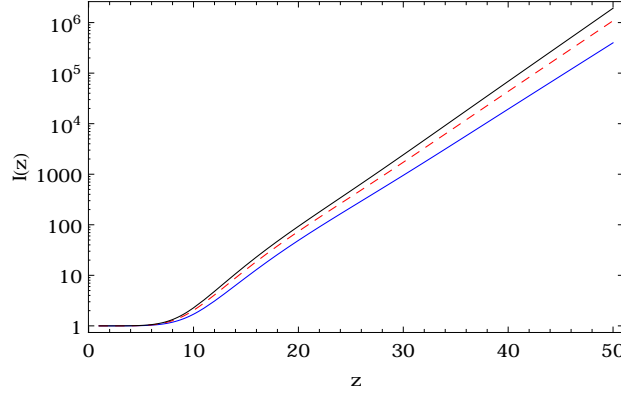yields a fairly accurate reproduction of the numerical results reported in Fig. 7.

We can however gain further insight by solving Eq. (26) with, e.g., $\phi = 0$. Assuming that the solution be of the exponential type $a \propto e^{i\lambda\tilde{z}}$, the exponents λ are obtained as the roots of the cubic equation

$$\lambda^3 - \tilde{\mu}_Q^2\lambda + \frac{1}{3\sqrt{3}} = 0. \tag{32}$$

The field driven by the fast growing root behaves like

$$a \propto e^{\Lambda\tilde{z}},$$

$$\Lambda = \frac{\sqrt{3}}{2}\sqrt[3]{-\frac{1}{6\sqrt{3}} + \sqrt{\frac{1}{\left(6\sqrt{3}\right)^2} - \frac{\tilde{\mu}_Q^6}{27}}} - \sqrt[3]{-\frac{1}{6\sqrt{3}} - \sqrt{\frac{1}{\left(6\sqrt{3}\right)^2} - \frac{\tilde{\mu}_Q^6}{27}}}, \qquad \tilde{\mu}_Q < \left(\frac{1}{4}\right)^{\frac{1}{6}}. \tag{33}$$

Figure 7: Dimensionless intensity growth vs. z for different values of $\tilde{\mu}_Q$ (0 black, 3 dashed red, 5 blue).

The gain length, affected by space charge effects, can therefore be defined as

$$L_g\left(\tilde{\mu}_Q\right) = \frac{L_g}{\Lambda} \tag{34}$$

which is consistent (albeit in a limited interval of $\tilde{\mu}_Q$) with what is expected from Eq. (30).
The inclusion of SC effects on the FEL dynamics has been based on the exam of the induced distortion of the gain function, within the context of a 1D analysis. The inclusion of diffractive effects can be accomplished by substituting the ρ parameter with its 3D counterpart, according to the Ming-Xie recipe or to that in [11]. The equations ruling the evolution of the beam transverse section (with respect to the longitudinal coordinate) is affected by a transverse Coulomb force and reads (for a round beam)

$$\sigma'' = \frac{\varepsilon}{\sigma^3} + \frac{1}{4}\frac{\Omega_p^2}{c^2}\frac{\sigma_0^2}{\sigma} \tag{35}$$

with σ_0 being the SC unperturbed beam section (absence of space charge contributions). The increase of the beam transverse dimension σ and divergence σ' contribute to the FEL gain detriment through a decrease of the current density and with a further inhomogeneous broadening effect associated with the SC induced emittance increase. The pivotal parameter allowing the quantifications of these further effects is $\tilde{\mu}_Q$ as shown in a forthcoming investigation.
The final point we like to touch is whether it is possible to include in this scheme the so called quantum corrections [17, 18, 32], which have raised recently interest, mainly for the works in [33, 34]. Albeit we will examine the interplay between classical and quantum parameter affecting FEL dynamics more thoroughly in a forthcoming note, we just discuss whether the procedure put forward so far is suitable to include this further contribution.

To this aim we remind that the relevance of quantum effect is measured by the parameter

$$\tilde{\mu}_q = \frac{\hbar\omega}{\rho m_e \gamma c^2} \tag{36}$$

which is recognized as quantity determining the ratio between the classical energy spread $\Delta\gamma m_e c^2$ and the energy of an emitted photon $\hbar\omega$ where the induced energy spread $\Delta\gamma \sim \rho\gamma$. It is obvious that when $\tilde{\mu}_Q < 1$ the discreteness of energy exchange becomes significant and the quantum effect becomes apparent, where a distinct transition between two energy levels is possible. For properly detuning of the system, each electron emits one photon at maximum. A physical counterpart of the Self-Amplified Spontaneous Emission ($SASE$) FEL operated in a quantum regime is the photons scattering from electrons in the Compton backscattering [35]. In a first approximation, considering only the number of photon growth along the undulator, we find that it is ruled by an equation of the type (26) with $\tilde{\mu}_q$ in place of $\tilde{\mu}_Q$ [17, 18, 35, 36].

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The gain process can accordingly be viewed as the balance between two processes regarding the emission and absorption of a photon of energy $\hbar\omega$.

The associated dispersion relation can therefore be written as

$$a''' + 2i\tilde{\phi}a'' + \left(\tilde{\mu}_Q^2 + \tilde{\mu}_q^2 - \tilde{\phi}^2\right)a' = \frac{i}{3\sqrt{3}}\,a. \tag{37}$$

Quantum and SC terms affect the FEL intensity evolution in the same way, which indicates that it is quite hard to disentangle the effects if one is just looking at the field intensity evolution. The possibility of observing effects of quantum nature is a very difficult task. Ad hoc designed experimental configurations, like those reported in [33, 35], should be accurately examined in order to disentangle the quantum contributions from all the others (homogeneous broadenings and SC) contributing to the FEL gain.

This point, even though needing substantive improvements, indicates that the formalism we have followed is suitable to address the inclusion of effects of different nature in FEL dynamics, within the same unifying context. The procedure, even though hampered by its 1-D nature, may provide a first and useful aid to get a feeling of their importance and whether they can be experimentally detected.

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Author Contributions

Conceptualization: G.D., H.F.; methodology: G.D., H.F.; data curation: G.D., S.L.; validation: G.D., H.F., S.L.; formal analysis: G.D., H.F., S.L.; writing - original draft preparation: G.D., H.F..; writing - review and editing: S.L. .

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