Comparison of the P_r-integral with Burkill's integrals and some applications to trigonometric series

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Abstract

It is proved that the P_r-integral [9] which recovers a function from its derivative defined in the space L^r, $1 \leq r < \infty$, is properly included in Burkill's trigonometric CP- and SCP-integrals. As an application to harmonic analysis, a de La Vallée-Poussin-type theorem for the P_r-integral is obtained: convergence nearly everywhere of a trigonometric series to a P_r-integrable function f implies that this series is the P_r-Fourier series of f.

Keywords:
Non-absolute integral, derivative in L^r, Perron-type integral, Cesaro-Perron integral, trigonometric series, Fourier coefficients

1. Introduction

In this paper we consider a Perron-type integral, the P_r-integral, which was introduced by L. Gordon [9] to solve a problem of recovering a function from its derivative defined in the space L^r, $1 \leq r < \infty$. Originally a version of this L^r-derivative appeared in an earlier paper [7] by Calderon and Zygmund and was used to obtain some estimates for solutions of elliptic partial differential equations. Note that the problem of recovering a function from its L^r-derivative can be solved also by a Kurzweil-Henstock-type integral,

*The work of Valentin Skvortsov was partially supported by the Russian Foundation for Basic Research, grant 20-01-00584.

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the HK_r-integral, which was introduced by Musial and Sagher in [11] (see its equivalent definitions in [13] and in [15]) and which turned out to strictly include the P_r-integral (see [12]). In [14] Musial and Tulone developed a dual to the space of HK_r-integrable functions. Here we are interested in comparing the P_r-integral with other integrals of Perron type.

The P_r-integral belongs to the family of non-absolute generalizations of the Lebesgue integral. Another class of non-absolute integrals of great importance in harmonic analysis are the so-called trigonometric integrals which serve to recover coefficients of trigonometric series from their sums by generalized Fourier formulas. This theory started with a rather cumbersome Denjoy construction [8] called totalization T_{2s} and was based on transfinite induction, but the same coefficients problem was solved later in a much simpler way by defining Perron-type integrals. Among them, Burkill's Symmetric Cesaro-Perron integral [5], (SCP-integral for short) seems to be most useful and easy to deal with. The more narrow CP-integral, defined by Burkill earlier in [3], solves the coefficients problem under some additional assumptions.

L. Gordon in [9] (Theorems 19 and 20) gave an application of the L^r-derivative and his P_r-integral to trigonometric series, proving that P_r-Fourier series of a function f is (C, 1)-summable to f. Having in mind some further application to trigonometric series, we start by comparing this integral with the two Burkill trigonometric integrals. In Section 3 of the present paper we compare the P_r-integral with the CP-integral. It is not difficult to show that the P_r-integral is included in the CP-integral and therefore also in the SCP-integral. Our main result in Section 3 is Theorem 3.4, which states that this inclusion is strict.

The inclusion of the P_r-integral in the SCP-integral and consistency of those integrals allow us to prove in Section 4 Theorem 4.4, a de La Vallée-Poussin-type theorem for the P_r-integral: convergence nearly everywhere of a trigonometric series to a P_r-integrable function f implies that this series is the P_r-Fourier series of f.

2. Preliminaries

2.1. A Perron-type integral in the metric L^r

Throughout this paper we assume that $r \geq 1$ and we work on the closed interval $[a, b]$. We recall the definitions of the L^r-derivates, the L^r-derivative and the P_r-integral introduced in [9].

Definition 2.1. Let $f \in L^r[a,b]$. We define the *upper right L^r-derivate* of f at x, denoted by $D_r^+ f(x)$, to be the greatest lower bound of all α such that

$$\left(\frac{1}{h}\int_0^h [f(x+t)-f(x)-\alpha t]_+^r \, dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+. \tag{1}$$

If no real number α satisfies (1), we set $D_r^+ f(x) = +\infty$. If (1) holds for every real number α, we set $D_r^+ f(x) = -\infty$.

We define the *lower right L^r-derivate*, $D_{+,r} f(x)$, the *upper left L^r-derivate*, $D_r^- f(x)$, and the *lower left L^r-derivate*, $D_{-,r} f(x)$, in a similar manner.

Definition 2.2. We define the *upper (two-sided) L^r-derivate* as follows:

$$\overline{D}_r f(x) = \max\left\{D_r^+ f(x), D_r^- f(x)\right\}.$$

Similarly we define the *lower (two-sided) L^r-derivate* as follows:

$$\underline{D}_r f(x) = \min\left\{D_{+,r} f(x), D_{-,r} f(x)\right\}.$$

Definition 2.3. Let $f \in L^r[a,b]$. If $\overline{D}_r f(x)$ and $\underline{D}_r f(x)$ are the same real number, i.e., if all four L^r-derivates are equal and finite, then we say that f is *L^r-differentiable* at x. The common value, denoted by $f_r'(x)$, is the *L^r-derivative* of f at x.

If f is L^r-differentiable at x, then $f_r'(x)$ is the unique real number α such that

$$\left(\frac{1}{h}\int_{-h}^h |f(x+t)-f(x)-\alpha t|^r \, dt\right)^{\frac{1}{r}} = o(h).$$

It is clear that if a function f is differentiable at a point x then it is also L^r-differentiable at the same point and $f_r'(x) = f'(x)$. The converse is not true. It is easy to construct an example of an L^r-differentiable at some point x function which is even not continuous at x containing in any neighborhood of x sufficiently tall thin spikes.

Definition 2.4. A function $F \in L^r[a,b]$ is said to be *L^r-continuous* at $x \in [a,b]$ if

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} |F(y)-F(x)|^r \, dy = 0.$$

If F is L^r-continuous for all $x \in E$, we say that F is L^r*-continuous on* E. It can easily be shown by Chebyshev's inequality that if a function is L^1-continuous, then it is approximately continuous.

Definition 2.5. Suppose f is a function defined a.e. on $[a, b]$. A finite-valued function $\psi \in L^r[a, b]$ is said to be an L^r*-major function* of f if

1. $\psi(a) = 0$,
2. ψ is L^r-continuous on $[a, b]$,
3. $\underline{D}_r \psi(x) \geq f(x)$ a.e., and
4. except for at most a countable subset of $[a, b]$, we have

$$-\infty \neq \underline{D}_r \psi(x). \tag{2}$$

A function ϕ is an L^r*-minor function* of f if $-\phi$ is an L^r-major function of $-f$.

It was proved in [9] that for any L^r-major function ψ and any L^r-minor function ϕ of f, the function $\psi - \phi$ is non-decreasing on $[a, b]$. This property allows us to define the Perron-type P_r-integral in a standard way:

Definition 2.6. [9] Suppose f is a function defined a.e. on $[a, b]$. If it has at least one pair of L^r-major function and L^r-minor function and if $\inf \psi(b)$ taken over all L^r-major functions of f equals $\sup \phi(b)$ taken over all L^r-minor functions of f, then the common value, denoted by

$$(P_r) \int_a^b f$$

is called the P_r*-integral* of f on $[a, b]$, and f is said to be P_r*-integrable* on $[a, b]$.

Remark 2.7. We cannot avoid the exceptional set for the inequality (2) in the definition of L^r-major and L^r-minor functions without losing the so-called Hake property of the P_r-integral (see Example 1, §7 in [9]). This fact shows also that, in contrast to the classical case (see [10]), the requirement that the inequality (2) must hold everywhere leads to an integral that is more narrow than the original L. Gordon integral.

Note that if a function F is L^r-continuous everywhere and L^r-differentiable nearly everywhere then it is both a major and a minor function for its derivative and so the derivative is P_r- integrable with $F(x) - F(a)$ being its indefinite P_r-integral.

The following result is known (see [9, Theorem 14]):

Proposition 2.8. *If $1 \leq r \leq q < \infty$, then any P_q-integrable function is P_r-integrable and the values of the integrals are equal.*

Note that if $r < q$ then the inclusion of the P_q-integral into the P_r-integral is strict. A related example can be constructed using the strict inclusion of the space $L^q[a, b]$ into the space $L^r[a, b]$.

2.2. The Cesàro-Perron Integral

All definitions in this subsection were introduced in [3] (see also [4, page 46]). In the two next definitions a function F is supposed to exist a.e. on $[a, b]$ and to be P-integrable (i.e., Perron integrable in the classical sense, see [16]) in some neighborhood of $x \in [a, b]$.

Definition 2.9. We say F is *Cesàro continuous* (or *C-continuous*) at $x \in [a, b]$ if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} F(t)\, dt = F(x)$$

Definition 2.10. We define the (ordinary) *Cesàro derivative* of F at x, $\mathrm{CD}F(x)$, as

$$\lim_{h \to 0} \frac{2}{h^2} \int_x^{x+h} (F(t) - F(x))\, dt$$

In a similar manner we define the *upper Cesàro derivative* of F at x, $\overline{\mathrm{CD}}F(x)$, and the *lower Cesàro derivative* of F at x, $\underline{\mathrm{CD}}F(x)$ as the corresponding upper and lower limits.

Definition 2.11. A function $M : [a, b] \to \mathbb{R}$ is said to be a *CP-major function* of f if

1. M is C-continuous at every $x \in [a, b]$,
2. $\underline{\mathrm{CD}}M(x) > -\infty$ for every $x \in [a, b]$, except possibly for an at most countable subset of $[a, b]$, and
3. $\underline{\mathrm{CD}}M(x) \geq f(x)$ for almost every $x \in [a, b]$.

The function m is said to be a *CP-minor function* of f if $-m$ is a CP-major function of $-f$.

For any CP-major function M and any CP-minor function m of f, the function $M - m$ is non-decreasing on $[a, b]$. This justifies the following

Definition 2.12. We say a function f, defined a.e. on $[a, b]$, is *CP-integrable* on $[a, b]$ if it has at least one pair of CP-major function and CP-minor function and $\inf M(b) = \sup m(b)$ where the infimum and supremum are taken respectively over all CP-major functions M and over all CP-minor functions m of f. Then the common value, denoted by

$$(CP) \int_a^b f$$

is called the *CP-integral* of f on $[a, b]$, and f is said to be *CP-integrable* on $[a, b]$.

2.3. The Symmetric Cesàro-Perron Integral

The notion of SCP-integral (Symmetric Cesàro-Perron Integral) was introduced in [5]. In what follows, $[a, b]$ is a fixed closed interval. A *basis* B *in* $[a, b]$ is a subset of $[a, b]$ of full measure, i.e., $|B| = b - a$, under an additional assumption $a, b \in B$.

Definition 2.13. Suppose a function F exists a.e. on $[a, b]$ and is P-integrable in some neighborhood of $x \in [a, b]$. We say F is *symmetric Cesàro continuous* (or *SC-continuous*) at x if

$$\lim_{h \to 0^+} \frac{1}{h} \left(\int_x^{x+h} F(t)\, dt - \int_{x-h}^{x} F(t)\, dt \right) = 0.$$

Definition 2.14. We define the *symmetric Cesàro derivative* of F at x, $\mathrm{SCD}F(x)$, as

$$\lim_{h \to 0^+} \frac{1}{h^2} \left(\int_x^{x+h} F(t)\, dt - \int_{x-h}^{x} F(t)\, dt \right).$$

In a similar manner we define the *upper symmetric Cesàro derivative* of F at x, $\overline{\mathrm{SCD}}F(x)$, and the *lower symmetric Cesàro derivative* of F at x, $\underline{\mathrm{SCD}}F(x)$, as the corresponding upper and lower limits.

Definition 2.15. Let B be a basis in $[a, b]$. A function $M : B \to \mathbb{R}$ is said to be an *SCP-major function of f with respect to the basis B* if

1. M is SC-continuous at every $x \in [a, b]$,
2. M is C-continuous at every $x \in B$,
3. $M(a) = 0$,
4. $\underline{\text{SCD}}\ M(x) > -\infty$ everywhere on $[a, b]$, except possibly for an at most countable subset of $[a, b]$, and
5. $\underline{\text{SCD}}\ M(x) \geq f(x)$ for almost every $x \in [a, b]$.

The function m is said to be a *SCP-minor function of f with respect to a basis B* if $-m$ is a SCP-major function of $-f$ with respect to B.

It is shown in [5] that for any SCP-major function M and any SCP-minor function m of f, with respect to a basis B, the function $M - m$ is non-decreasing on B.

Definition 2.16. We say a function f, defined a.e. on $[a, b]$, is *SCP-integrable* on $[a, b]$ with respect to a basis B if it has at least one pair of SCP-major function and SCP-minor function and $\inf M(b) = \sup m(b)$ where the infimum and supremum are taken respectively over all SCP-major functions M and over all SCP-minor functions m of f with respect to the basis B. Then the common value, denoted by

$$F(x) = (SCP, B) \int_a^b f$$

defines the *SCP-integral* of f on $[a, b]$.

Note that if B_1 and B_2 are two bases in $[a, b]$ for each of which f is integrable, then the integrals are equal at least on $B_1 \cap B_2$. So talking about SCP-integrability of f we can often without ambiguity leave the basis unspecified. Moreover, for each SCP-integrable function, we can take as its basis of integration the set of all points where the indefinite integral is C-continuous (after correspondent extension).

An obvious relation of C-derivates and C-continuity to SC-derivates and SC-continuity imply

Proposition 2.17. *If a function is CP-integrable on an interval $[a, b]$ then it is also SCP-integrable on this interval with respect to the basis $B = [a, b]$ and the values of integrals coincide.*

3. Comparison of the P_r-integral with Burkill's integrals

We compare here the CP-integral (and by this also SCP-integral) with the P_r-integral.

By a direct computation we obtain

Lemma 3.1. *If a function is L_r-continuous at a point x then it is C-continuous at x.*

Lemma 3.2.

$$\underline{D}_1 F(x) \leq \underline{CD} F(x) \leq \overline{CD} F(x) \leq \overline{D}_1 F(x)\,.$$

The definitions of the CP- and P_r-integrals, Proposition 2.8, Lemmas 3.1 and 3.2 imply the following result, which in fact was noticed in [9, Remark 2, §5]:

Theorem 3.3. *If $f : [a, b] \to R$ is P_r-integrable then it is CP-integrable and the values of integrals coincide.*

We show now that this inclusion of the P_r-integral in the CP-integral is proper.

Theorem 3.4. *There exists a function which is CP-integrable on $[a, b]$ but which is not P_r-integrable on $[a, b]$.*

Proof. Due to Proposition 2.8, it is enough to prove the theorem for the case $r = 1$.

Let $P \subset [0, 1]$ be a symmetric Cantor-type set, defined by iteratively removing the central intervals (the so-called *contiguous intervals*) u_n of rank $n = 1, 2, \ldots$, having length $|u_n| = 3^{-n}2^{-n+2}$. The set which is left after removing all contiguous intervals up to rank n from $[0, 1]$ is constituted by 2^n segments (closed intervals) r_n, of length $3^{-n}2^{-n}$ each, which are called *segments of rank* n (note that r_n, as well as u_n, is the generic notation for all these segments and intervals of rank n). We can easily compute that $\mu(P) = 0$. Note that each u_n is the interval concentric with some segment r_{n-1} (we set $r_0 = [0, 1]$).

Let v_n be the interval concentric with u_n, all of them of the same length

$$|v_n| = 6^{-n}|u_n| = 3^{-2n}2^{-2n+2}. \tag{3}$$

Now we define a function F which is the indefinite CP-integral of its CP-derivative. We put $F(x) = 0$ outside of the union of intervals v_n of all rank, i.e., on the set P and on each set $u_n \setminus v_n$.

To define F on v_n we subdivide each v_n into 4^n sub-intervals of the same length. Let e_n be the left endpoint of a fixed v_n. For $1 \leq k \leq 4^n/2$ let

$$v_{n,k}^+ = [e_n + (2k-2)|v_n|/4^n, e_n + (2k-1)|v_n|/4^n]$$

and let

$$v_{n,k}^- = [e_n + (2k-1)|v_n|/4^n, e_n + 2k|v_n|/4^n].$$

Define $F(x) = 2^n$ if $x \in v_{n,k}^+$ and $F(x) = -2^n$ if $x \in v_{n,k}^-$. Let v_n^+ be the union of the $v_{n,k}^+$ and let v_n^- be the union of the $v_{n,k}^-$. We then have

$$|v_{n,k}^+| = |v_{n,k}^-| = |v_n|/(4^n) = 6^{-n}4^{-n}|u_n| = 3^{-2n}2^{-2n+2}4^{-n}. \tag{4}$$

and

$$|v_n^+| = |v_n^-| = |v_n|/2 = 3^{-2n}2^{-2n+1}. \tag{5}$$

The function F must be made to be differentiable on u_n. It is clear how to make it smooth changing it in small neighborhoods of each point of discontinuity of F without influencing further estimations, ensuring, for instance, that the average of F on $v_{n,k}$ should stay about the same. We keep the same notation F for the modified function, but to simplify computation we shall allow ourselves to treat it as if it has its original constant values on all sub-intervals of v_n and on $u_n \setminus v_n$. Now we have

$$\int_{v_n^+} F = -\int_{v_n^-} F = \frac{2}{3^{2n}2^n} \tag{6}$$

and

$$\int_{u_n} |F| = \frac{4}{3^{2n}2^n} \tag{7}$$

Summing (7) over all intervals u_n of all rank and recalling that the complement to the union of all these intervals is P, i.e., a set of measure zero, we get

$$\int_0^1 |F| = \sum_{n=1}^{\infty} 2^{n-1}\frac{4}{3^{2n}2^n} = \frac{1}{4}.$$

Hence, $F \in L[0,1]$. Note that F is differentiable a.e.

Let

$$f(x) = \begin{cases} F'(x) & \text{at } x \in [0,1] \setminus P, \\ 0 & \text{at } x \in P. \end{cases}$$

We will first show that $\mathrm{CD}F(x) = 0$ at each point of the set P, and therefore, F is the indefinite CP-integral of its CP-derivative f. Then we will show that f is not P_1-integrable.

Having fixed $x \in P$ take any $\varepsilon > 0$ and choose n to be such that $2^{-n} < \varepsilon/9$. It is clear that for any $h > 0$

$$\int_x^{x+h} F(t)\, dt \geq 0.$$

We shall show that

$$\int_x^{x+h} F(t)\, dt < \varepsilon h^2/2.$$

If $x+h$ is not in v_n for any n, then $\int_x^{x+h} F(t)\, dt = 0$. Therefore we assume that $x+h \in v_n$ for some n. It follows that $h > |u_n|/3$, and that $x+h \in v_{n,k}^+$ or $x+h \in v_{n,k}^-$ for some k. In either case we have that

$$\int_x^{x+h} F(t)\, dt \leq \int_{v_{n,k}^+} 2^n\, dt = 2^n |v_{n,k}^+| = 12^{-n}|u_n|$$

$$= |u_n|^2 2^{-n-2} < \varepsilon h^2/2.$$

A similar argument holds for $h < 0$. This proves that $\mathrm{CD}F(x) = 0$.

Now we are to show that f is not P_1-integrable. If this were the case then, according to Theorem 3.3, F would be its indefinite P_1-integral with $F(x) = \sup m(x)$ where sup is taken over all P_1-minor functions of f. We obtain a contradiction with this assumption by showing that for any non-decreasing function R with $R(0) = 0$, the function $F - R$ is not a minor function because $D_1^+(F(x) - R(x)) = +\infty$ on an uncountable set. In fact this equality holds at any point of the set P, which is not a left endpoint of any contiguous interval to P. Let x be such a point. Recall that $F(x) = 0$ if $x \in P$. We are going to show that for any positive α

$$\limsup_{h \to 0} \frac{1}{h^2} \int_0^h [F(x+t) - R(x+t) + R(x) - \alpha t]_+ dt = +\infty. \tag{8}$$

We obtain this by choosing a decreasing sequence $h_k \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{h_k^2} \int_0^{h_k} [F(x+t) - R(x+t) + R(x) - \alpha t]_+ dt = +\infty. \tag{9}$$

Having in mind the way the point x is chosen, we can find an increasing sequence of natural numbers $n_k \to \infty$ such that $x \in r_{n_k} \subset r_{n_k-1}$, and r_{n_k} is the left of two segments of rank n_k which are subsets of r_{n_k-1}, i.e., r_{n_k} and r_{n_k-1} have common left endpoint. Indeed if x is the right endpoint of some contiguous interval to P then it is the common left endpoint of all r_n, starting with some n. If x is an endpoint of no contiguous interval and so of no segment r_n, then the above situation repeats infinitely many times.

So for a fixed x we have chosen the sequence $\{r_{n_k}\}$ with the above property. Let u_{n_k} be a contiguous interval of rank n_k which is concentric with r_{n_k-1}. Note that u_{n_k} is to the right of x. Take h_k so that $u_{n_k} \subset (x, x+h_k)$ and so that

$$h_k < |r_{n_k}| + |u_{n_k}| < 2|u_{n_k}| = 3^{-n_k} 2^{-n_k+3}. \tag{10}$$

We can assume that n_k are chosen so that $2^{n_k-1} > R(1) + |\alpha|$. Then $[F(x+t) - R(x+t) + R(x) - \alpha t]_+ > 2^{n_k-1}$ if $x+t \in v_{n_k}^+ \subset u_{n_k}$, and using (5) and (10) we obtain

$$\begin{aligned} &\frac{1}{h_k^2} \int_0^{h_k} [F(x+t) - R(x+t) + R(x) - \alpha t]_+ dt \geq \\ &\frac{1}{h_k^2} \int_{v_{n_k}^+} [F(x+t) - R(x+t) + R(x) - \alpha t]_+ dt > \\ &> \frac{2^{n_k-1}|v_{n_k}^+|}{h_k^2} > \frac{2^{n_k-1}(3^{-2n_k} 2^{-2n_k+1})}{3^{-2n_k} 2^{-2n_k+6}} = 2^{n_k-6}. \end{aligned}$$

This shows that (8) and (9) hold and that no real α satisfies (1) for the function $F - R$ at the considered x. Therefore $\overline{D}_r(F-R)(x) = +\infty$ on an uncountable set and $F - R$ is a L^1-minor function for no R. This proves that f has no L^1-minor function and so f is not P_1-integrable. □

4. Application to Fourier Series

We are using below the integration by parts formula for the P_r-integral

Theorem 4.1. *[9] Suppose that, on the interval [a,b], f is P_r-integrable, G is absolutely continuous and $g(x) := G'(x)$ is in $L^{r'}, r' = r/(r-1)$ (If $r = 1$, the condition on G becomes G is Lipschitz. Then fG is P_r-integrable on $[a, b]$ and if $F(x) = C + \int_a^x f\, dt$, then*

$$(P_r) \int_a^b fG\, dx = FG]_a^b - \int_a^b Fg\, dx,$$

where the integral on the right exists as a Lebesgue integral.

The SCP-integral was introduced in [5] with the principal object of proving the following

Theorem 4.2. *If the series*

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{11}$$

converges to a function $f(x)$ for all x, then there exists a 2π-periodic set A (i.e., $a \in A$ implies $a + 2\pi \in A$) of full measure on each compact interval on $\mathbb{R}$ such that function $f(x)$ and functions $f(x)\cos nx$ and $f(x)\sin nx$ for all $n = 0, 1, 2, \ldots,$ are SCP-integrable on $[\beta, \beta + 2\pi]$, for any $\beta \in A$, with respect to basis $B = A \cap [\beta, \beta + 2\pi]$, and

$$a_n = \frac{1}{\pi}(SCP, B) \int_\beta^{\beta+2\pi} f(x) \cos nx\, dx \quad n = 0, 1, 2, \ldots; \tag{12}$$

$$b_n = \frac{1}{\pi}(SCP, B) \int_\beta^{\beta+2\pi} f(x) \sin nx\, dx \quad n = 1, 2, \ldots. \tag{13}$$

The above formulas were obtained in [5] by an integration by parts result, the proof of which contained a flaw that was corrected later in [17] (see also [6] and [1]). Another correct proof, based on formal multiplication of trigonometric series, was given in [2].

We use below the following generalization of Theorem 4.2 which is a particular case of [5, Theorem 5.3 (i)], provided its proof is corrected according to [17] or [6]:

Theorem 4.3. *If the series (11) converges to a function $f(x)$ almost everywhere and the partial sums of (11) are bounded for each x except an countable set then the conclusion of Theorem 4.2 holds true, in particular formulas (12) and (13) hold with the same meaning for β and B.*

Now we apply this theorem together with Theorem 3.3 to obtain

Theorem 4.4. *If the series (11) converges to a function $f(x)$ almost everywhere, the partial sums of (11) are bounded for each x except on a countable set and f is P_r-integrable on $[0, 2\pi]$, then*

$$a_n = \frac{1}{\pi}(P_r)\int_0^{2\pi} f(x)\cos nx\, dx \qquad n = 0, 1, 2, \ldots; \tag{14}$$

$$b_n = \frac{1}{\pi}(P_r)\int_0^{2\pi} f(x)\sin nx\, dx \qquad n = 1, 2, \ldots. \tag{15}$$

Proof. By Theorem 4.3, the functions $f(x)\cos nx$ and $f(x)\sin nx$ for all $n = 0, 1, 2, \ldots$ are SCP-integrable on interval $[\beta, \beta + 2\pi]$ for any $\beta \in A$ with respect to the basis $B = A \cap [\beta, \beta + 2\pi]$ where A is defined by the series as in Theorems 4.2 and 4.3, and formulas (12) and (13) hold. At the same time by assumption and by Theorem 4.1 the functions $f(x)\cos nx$ and $f(x)\sin nx$ are P_r-integrable on $[0, 2\pi]$ for all n. So by Theorem 3.3 they are also CP-integrable on $[0, 2\pi]$. But the function f is periodic. This implies P_r- and CP-integrability of all those functions on any compact interval, in particular on interval $[\beta, \beta + 2\pi]$. Then Proposition 2.17 implies that f and all considered functions are SCP-integrable on any compact interval, in particular on $[0, 2\pi]$, with respect to the basis coinciding with the whole interval. This means that in our case we can put $\beta = 0$ in formulas (12) and (13). Hence, by the consistency of the P_r-, CP- and SCP-integrals, formulas (12) and (13) imply formulas (14) and (15) giving the desired result. □

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