Asymptotic Analysis -1 (2021) 1–19
DOI 10.3233/ASY-211710
IOS Press

Asymptotics for third-order nonlinear differential equations: Non-oscillatory and oscillatory cases

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Abstract. We discuss a third-order differential equation, involving a general form of nonlinearity. We obtain results describing how suitable coefficient functions determine the asymptotic and (non-)oscillatory behavior of solutions. We use comparison technique with first-order differential equations together with the Kusano–Naito's and Philos' approaches.

Keywords: Nonlinear differential equation, oscillation and non-oscillation, asymptotic behavior, comparison technique, third-order differential equation

1. Introduction and preliminaries

In this paper we study the third-order nonlinear differential equation of the form

$$\begin{cases} (a(t)w''(t))' + w(t)A(w^2(t), t) = 0, \\ a(t) > 0, \qquad a'(t) \geqslant 0, \qquad A(z, t) > 0, \qquad t \geqslant t_0 > 0, \qquad z > 0, \end{cases} \tag{1}$$

where the first term means the weighted operator driving the equation, and the second term means the general form of involved nonlinearity. We need some regularities on coefficient functions, that is $a \in C^1([t_0, \infty), \mathbb{R}_+)$, and $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ is monotone with respect to its first variable.

By a solution of (1) we mean a function $w \in C^2([t_w, \infty), \mathbb{R})$, $t_w \geqslant t_0$, which has the property $aw'' \in C^1([t_w, \infty), \mathbb{R})$, and satisfies (1) on $[t_w, \infty)$. As usual, we are interested in those solutions w of (1) with $\sup\{|w(t)| : t \geqslant t_w\} > 0$ (that is, the solutions are non-trivial). So, we say that (1) is "oscillatory" whenever all its solutions are oscillatory (that is, they have arbitrarily large zeros). On the other hand, if $w \in C^2([t_w, \infty), \mathbb{R})$ is definitively positive (or negative), then (1) is "non-oscillatory".

The theory of higher-order differential equations originates by the classical theory of first- and second-order ordinary differential equations. A comprehensive study is provided by the book of Ladde–Lakshmikantham–Zhang [12]. For additional mathematical background, we also refer to the books of

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Agarwal–Grace–O'Regan [1] and Hale [7]. The interest for this kind of extended theory is strongly motivated by its usefulness in dealing with real nonlinear phenomena. Indeed, the higher-order differential equations during the last decades provided good models of problems in physics, engineering and economic processes. We point out the attention of the reader to the papers of Džurina [4], Zhang–Agarwal–Bohner–Li [17], Zhang–Li–Saker [18] (higher-order equations), and Baculíková–Džurina [3], Fišnarová–Mařík [5], Kusano–Naito [10] (second-order equations), and Baculíková–Džurina [2], Li–Rogovchenko [13] (third-order equations).

Here, we continue this study with the motivation to provide precise information on the (non-)oscillatory behavior of solutions to (1). The main idea is given in the pioneering papers of Nehari [14] and Philos [15]. In the first one, the author establishes oscillation and non-oscillation criteria for a second-order differential equation of the form

$$w''(t) + q(t)w(t) = 0, \quad q \in C\big([t_0, \infty), \mathbb{R}_+\big), t \geqslant t_0 > 0.$$

In addition, the author provides asymptotic estimates for the number of zeros. In the second paper, Philos focuses on non-oscillation criteria for n-th order general retarded differential equations of the form

$$\begin{cases} (-1)^n w^{(n)}(t) - f(t, x(\delta_1(t)), \dots, x(\delta_k(t))) = 0, & f \in C([t_0, \infty) \times [0, \infty)^k, \mathbb{R}), \\ \delta_1, \dots, \delta_k \in C((t_0, \infty), \mathbb{R}), \qquad \delta_1, \dots, \delta_k \to +\infty & \text{as } t \to +\infty, t \geqslant t_0 > 0, \end{cases}$$

with additional regularities on f (that is, $f(t, z_1, \dots, z_k)$ is increasing in each of $z_1, \dots, z_k$, and is a positive function). In particular, in [15] the existence of a positive solution to the above equation is obtained starting from positive solutions of suitable differential inequalities. We will also use this argument in establishing a result of this paper.

Another key-tool in obtaining our results is the comparison technique, where the oscillatory behavior of solutions to (1) is obtained developing a reasoning process which leads to contradiction with the known oscillatory behavior of some first-order differential equations. Some nice recent contributions in this direction are the above cited references [5,13].

Here, we look to a third-order differential equation involving a general form of nonlinearity, and hence we are aimed to investigate whether the choice of different nonlinearities influences the analysis in [13] and complement the results in [5]. We use the Philos' approach to differential inequalities [15], together with the comparison technique with first-order differential equations to establish an oscillatory criteria. We also discuss necessary and sufficient criteria describing how the properties of nonlinearity determine the (non-)oscillatory behavior of equation (1), in the sense of Kusano–Naito [11].

2. Hypotheses and auxiliary results

In this section we collect some relevant facts from the existing literature and auxiliary results. Also, we fix the notation. In the sequel we will assume the hypothesis:

(H_0) $\int_t^\infty \frac{1}{a(s)}\,ds = +\infty$ for all $t \geqslant t_0 > 0$.

A simple function $a \in C^1([t_0, \infty), \mathbb{R}_+)$ satisfying (H_0) is

$$a(t) = t \ln t \quad \text{for all } t \geqslant t_0 > 1.$$

We also need that the nonlinearity is as follows:

(H_1) $wA(w^2, t)$ is continuous in $\mathbb{R}_+ \times [t_0, \infty)$.

A particular case of $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ satisfying (H_1) is

$$A\big(w^2(t), t\big) = q(t)\big(w^2(t)\big)^{\beta}, \quad \text{for all } t \geqslant t_0 > 0, \text{ some } \beta > 0,$$

where $q \in C([t_0, \infty), \mathbb{R}_+)$.

Using a similar reasoning to the one in [2, Lemma 1] we start giving a lemma, about the cases that we will discuss here (that is, a "classification of positive solutions").

Lemma 2.1. *Let $w \in C^2([t_0, \infty), \mathbb{R})$ be a (eventually) positive solution of (1). Then, we have the following situations:*

(S_1) $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0$, $(a(t)w''(t))' \leqslant 0$,
(S_2) $w(t) > 0$, $w'(t) < 0$, $w''(t) > 0$, $(a(t)w''(t))' \leqslant 0$,

for $t \geqslant t_1$, where $t_1 \geqslant t_0$ is large enough.

Proof. Let w be a (eventually) positive solution to (1). By (1) and $A(z, t) > 0$ for all $t \geqslant t_0$, $z > 0$, we have:

$$\begin{aligned} &\big(a(t)w''(t)\big)' = -w(t)A\big(w^2(t), t\big) < 0, \quad \text{for all } t \geqslant t_0, \\ &\quad \Rightarrow \quad aw'' \text{ is decreasing and does not change sign definitively,} \\ &\quad \Rightarrow \quad w'' \text{ does not change sign definitively too.} \end{aligned}$$

So, we distinguish two cases: w'' is negative definitively, and w'' is positive definitively.

If we assume $w''(t) < 0$ for $t \geqslant t_1 \geqslant t_0$, then we can find a positive real number K with $a(t)w''(t) \leqslant -K < 0$. This leads to

$$\begin{aligned} &w''(t) \leqslant -\frac{K}{a(t)}, \\ &\quad \Rightarrow \quad \int_{t_1}^{t} w''(s)\,ds \leqslant -K\int_{t_1}^{t} \frac{1}{a(s)}\,ds \quad \big(\text{we integrate over } [t_1, t]\big), \\ &\quad \Rightarrow \quad w'(t) \leqslant w'(t_1) - K\int_{t_1}^{t} \frac{1}{a(s)}\,ds \\ &\quad \Rightarrow \quad w'(t) \to -\infty \quad \big(\text{since the last integral goes to } +\infty \text{ as } t \text{ goes to } +\infty, \text{ by } (H_0)\big). \end{aligned}$$

We deduce that $w'(t) < 0$ for $t \geqslant t_2 \geqslant t_1$ (large enough). Now, we observe that $w''(t) < 0$ and $w'(t) < 0$ for all $t \geqslant t_2$, imply $w(t) < 0$ too. So, we have a contradiction to the fact that w is positive. We conclude that w'' must be positive definitively.

Thus, we conclude easily that only the situations (S_1) and (S_2) may occur. $\square$

Remark 2.2. In both cases (S_1) and (S_2), we can find $c > 0$ and $\bar{t} \geqslant t_0$ such that

$$w(t) \leqslant c \int_{t_0}^{t} \frac{t-s}{a(s)}\, ds, \quad t \geqslant \bar{t}. \tag{2}$$

Since aw'' is decreasing, from

$$a(t)w''(t) \leqslant a(t_0)w''(t_0),$$
$$\Rightarrow \quad w''(t) \leqslant \frac{a(t_0)}{a(t)} w''(t_0),$$

we integrate over $[t_0, t]$ (two times) to get:

$$w'(t) \leqslant w'(t_0) + a(t_0)w''(t_0) \int_{t_0}^{t} \frac{1}{a(s)}\, ds \quad \text{(first integration)},$$

$$\Rightarrow \quad w(t) \leqslant w(t_0) + w'(t_0)[t - t_0] + a(t_0)w''(t_0) \int_{t_0}^{t}\int_{t_0}^{z} \frac{1}{a(s)}\, ds\, dz$$

(second integration),

$$\Rightarrow \quad w(t) \leqslant \big[w(t_0) - w'(t_0)t_0\big] + w'(t_0)t + a(t_0)w''(t_0)\left(\left[z\int_{t_0}^{z} \frac{1}{a(s)}\, ds\right]_{t_0}^{t} - \int_{t_0}^{t} \frac{s}{a(s)}\, ds\right)$$

(the integration by parts formula is used to compute double integral),

$$\Rightarrow \quad w(t) \leqslant \big[w(t_0) - w'(t_0)t_0\big] + w'(t_0)t + a(t_0)w''(t_0) \int_{t_0}^{t} \frac{t-s}{a(s)}\, ds.$$

This inequality leads easily to (2) by a suitable choice of positive values c and $\bar{t} \geqslant t_0$.

For the sake of simplicity, we assume here that A is monotone non-increasing, with respect to the first variable. In view of Lemma 2.1 and (2), we note that an (eventually) positive solution to (1), namely w, is such that

$$m \leqslant w(t) \leqslant M_c(t) := c \int_{t_0}^{t} \frac{t-s}{a(s)}\, ds \quad \text{for some } m \geqslant 0,\ c > 0 \text{ and all } t \geqslant \bar{t} \text{ large enough.}$$

Adopting the terminology of [11], we can say that those solutions such that $w(t)$ is asymptotic to $M_c(t)$, as $t \to +\infty$, are the "maximal solutions" of (1). We can formalize this asymptotic behavior of an (eventually) positive solution to (1), by the following property:

(L) $\lim_{t\to+\infty} \frac{w(t)}{M_1(t)} = \gamma > 0$ (constant).

We also refer the reader to Kusano–Akio–Hiroyuki [9], for some similar general considerations over a class of second-order differential equations.

Now, we are ready to introduce the precise hypotheses on the data of (1):

(H_2) $\int_{t_0}^{\infty}(A(M_c^2(t), t)M_1(t))\,dt < +\infty$ for some $c > 0$.

(H_3) One of the following conditions holds:

a. $\int_z^{\infty} A(\kappa, s)\,ds = +\infty$,
b. $\int_v^{\infty} \frac{1}{a(z)} \int_z^{\infty} A(\kappa, s)\,ds\,dz = +\infty$,
c. $\int_{t_0}^{\infty} \int_v^{\infty} \frac{1}{a(z)} \int_z^{\infty} A(\kappa, s)\,ds\,dz\,dv = +\infty$,

for all $\kappa > 0$.

(H_4) There exists a function $\eta \in C([t_0, \infty), \mathbb{R}_+)$ such that, for all $t_1 \geqslant t_0$ (large enough) and some $t_* \geqslant t_1$, we have that the first-order retarded differential equation

$$u'(t) + A\big(\eta(t), t\big)\left(\int_{t_*}^{\delta(t)} \int_{t_1}^{v} a^{-1}(s)\,ds\,dv\right)u\big(\delta(t)\big) = 0$$

is oscillatory, where $\delta \in C([t_0, \infty), \mathbb{R}_+)$ is such that $\delta(t) < t$ and $\delta(t)$ goes to infinity as t goes to infinity.

(H_5) There exists a function $\widetilde{\eta} \in C([t_0, \infty), \mathbb{R}_+)$ such that, for all $t_1 \geqslant t_0$ (large enough) and some $t_* \geqslant t_1$, we have that the inequality

$$u'(t) + A\big(\widetilde{\eta}(t), t\big)\left(\int_{t_*}^{\delta(t)} \int_{t_1}^{v} a^{-1}(s)\,ds\,dv\right)u\big(\delta(t)\big) \leqslant 0,$$
$$\delta \in C\big([t_0, \infty), \mathbb{R}_+\big) \text{ as given in } (H_4),$$

has no positive solutions.

Remark 2.3. Consider the following type version of the first-order general retarded differential equation in hypothesis (H_4):

$$u'(t) + \widetilde{q}(t)u\big(\delta(t)\big) = 0, \quad \widetilde{q} \in C\big([t_0, \infty), \mathbb{R}_+\big), \widetilde{q}(t) > 0, t \geqslant t_0 > 0. \tag{3}$$

By Ladde–Lakshmikantham–Zhang [12] (Theorem 2.1.1 (iii), p. 16) we know that (3) is oscillatory, provided that

$$\liminf_{t \to +\infty} \int_{\delta(t)}^{t} \widetilde{q}(s)\,ds > e^{-1}.$$

In the particular case $\widetilde{q}(t) = q_0$ (constant case) and $\delta(t) = t - \delta_0$ with $\delta_0 > 0$, then

$$q_0\delta_0 > e^{-1}$$

is a necessary and sufficient condition for oscillations of solutions to (3) (see [12, Corollary 2.1.1, p. 18]).

According to Remark 2.3, the first-order retarded differential equation

$$u'(t) + (2 - \sin t)u(t - \pi) = 0, \quad t \geqslant 2\pi > 0,$$

is oscillatory, since

$$\liminf_{t\to+\infty} \int_{t-\pi}^{t} (2 - \sin s)\, ds > \pi > e^{-1}.$$

The last result of this section is a key-proposition establishing the asymptotic behavior of a (eventually) positive solution of equation (1) provided that (H_3) holds true. Precisely, the next result deals with the following asymptotic property:

$(L)'$ $\lim_{t\to+\infty} w(t) = 0$.

Proposition 2.4. *If (H_3) holds and $w \in C^2([t_w, \infty), \mathbb{R}_+)$ is a (S_2)-type solution of (1), then $(L)'$ holds true.*

Proof. Since (S_2) holds, we know that $w(t) > 0$ and $w'(t) < 0$ definitively. So, there exists $\ell \geqslant 0$ such that $w(t) \to \ell$ as $t \to +\infty$. If we assume $\ell > 0$, then there exist $c > 1$ and $\widetilde{t} \geqslant t_0$ such that $\ell < w(t) \leqslant c\ell$ for all $t \geqslant \widetilde{t}$. We construct the proof in three steps.

Step 1. Assume that $(H_3)_a$ holds, that is

$$\int_z^\infty A(\kappa, s)\, ds = +\infty, \quad \text{for all } \kappa > 0.$$

Then, from (1) and from the fact that A is non-increasing with respect to the first variable, we deduce that

$$\begin{aligned}
&\big(a(t)w''(t)\big)' + \ell A\big(c^2\ell^2, t\big) \leqslant 0, \quad t \geqslant \widetilde{t},\\
&\quad \Rightarrow \quad \int_z^y \big[\big(a(s)w''(s)\big)' + \ell A\big(c^2\ell^2, s\big)\big]\, ds \leqslant 0, \quad y > z \geqslant \widetilde{t},\\
&\quad \Rightarrow \quad \ell \int_z^y A\big(c^2\ell^2, s\big)\, ds \leqslant a(z)w''(z) - a(y)w''(y) \leqslant a(z)w''(z),\\
&\quad \Rightarrow \quad a(z)w''(z) \geqslant \ell \int_z^\infty A\big(c^2\ell^2, s\big)\, ds,
\end{aligned} \tag{4}$$

which leads to contradiction, by $(H_3)_a$.

Step 2. Assuming that $\int_z^\infty A(\kappa, s)\, ds < +\infty$ for some $\kappa > 0$ (that is $(H_3)_a$ does not hold), we consider the situation where $(H_3)_b$ is true. Fixing $c > 1$ such that $c^2\ell^2 > \kappa$, and satisfying also the assumption of step 1, we deduce that the right hand side of (4) is finite. After dividing each side of (4) by $a(z)$, we integrate over $[v, t]$ to obtain

$$w'(t) - w'(v) \geqslant \int_v^t \frac{\ell}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz,$$

$$\Rightarrow \quad -w'(v) \geqslant \ell \int_v^\infty \frac{1}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz, \tag{5}$$

which leads to contradiction, by $(H_3)_b$.

Step 3. Assuming that $(H_3)_a$ and $(H_3)_b$ do not hold, we have that

$$\int_z^\infty A(\kappa_a, s)\, ds < +\infty \quad \text{for some } \kappa_a > 0$$

and

$$\int_v^\infty \frac{1}{a(z)} \int_z^\infty A(\kappa_b, s)\, ds\, dz < +\infty \quad \text{for some } \kappa_b > 0.$$

Take $c > 1$ such that $c^2\ell^2 > \kappa := \max\{\kappa_a, \kappa_b\}$. Then, due to the monotonicity of A (non-increasing in its first variable), we have

$$\int_z^\infty A\big(c^2\ell^2, s\big)\, ds \leqslant \int_z^\infty A(\kappa, s)\, ds \leqslant \int_z^\infty A(\kappa_a, s)\, ds < +\infty.$$

Then we can divide (4) by $a(z)$ and integrate over $[v, t]$ to obtain

$$-w'(v) \geqslant \ell \int_v^\infty \frac{1}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz.$$

Due to $a(z) > 0$ and again using monotonicity of A, we get

$$\begin{aligned}\int_v^\infty \frac{1}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz &\leqslant \int_v^\infty \frac{1}{a(z)} \int_z^\infty A(\kappa, s)\, ds\, dz \\ &\leqslant \int_v^\infty \frac{1}{a(z)} \int_z^\infty A(\kappa_b, s)\, ds\, dz < +\infty.\end{aligned}$$

Therefore the right hand side of (5) is finite. Then, we integrate each side of (5) over $[t_*, t]$ to obtain

$$\begin{aligned}&-w(t) + w(t_*) \geqslant \int_{t_*}^t \ell \int_v^\infty \frac{1}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz\, dv, \\ &\Rightarrow \quad w(t_*) \geqslant \ell \int_{t_*}^\infty \int_v^\infty \frac{1}{a(z)} \int_z^\infty A\big(c^2\ell^2, s\big)\, ds\, dz\, dv,\end{aligned}$$

which leads to contradiction, by $(H_3)_c$.

We conclude that $\ell = 0$, that is $w(t)$ goes to zero as t goes to infinity, and hence $(L)'$ holds true. □

A particular case of $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ satisfying immediately the hypothesis $(H_3)_a$ is as follows:

$$A(\kappa, t) = \frac{|\sin t|}{2 + \sin t} \quad \text{for all } t \geqslant t_0 > 0,$$

where, for the sake of simplicity, we drop the κ-dependence. We mention it, because the sign-changing version of this function (that is, without the absolute value above) is used by Travis [16] to construct and study a second-order retarded differential equation, whose solution is neither oscillatory nor satisfying the asymptotic property $(L)'$.

Finally, we point out that, for all $t \geqslant t_0 > 0$, the couple of functions

$$\begin{cases} A(\kappa, t) = \frac{\kappa+2}{\kappa+1} t^{-(1+\alpha)} & 0 < \alpha < 1, \\ a(t) = t^{\beta} & 0 < \beta \leqslant 1 - \alpha, \end{cases}$$

satisfies $(H_3)_b$. On the other hand, for all $t \geqslant t_0 > 0$, the couple of functions

$$\begin{cases} A(\kappa, t) = \frac{\kappa+2}{\kappa+1} t^{-(1+\alpha)} & 0 < \alpha < 1, \\ a(t) = t^{\beta} & 1 - \alpha < \beta < 2 - \alpha, \end{cases}$$

satisfies $(H_3)_c$.

3. Main results

In this section, we present both the non-oscillatory and oscillatory criteria.

The first theorem establishes the existence of a (eventually) positive solution to (1) with the property (L).

Theorem 3.1. *If (H_0)–(H_2) hold, then there exists a (eventually) positive solution $w \in C^2([t_w, \infty), \mathbb{R}_+)$ of (1) with the property (L).*

Proof. By (H_2) we can find $T > 0$ (large enough) satisfying

$$\int_T^{\infty} M_1(s) A\big(M_c^2(s), s\big)\, ds < \frac{1}{4}.$$

We introduce an integral equation of the form

$$w(t) = (\Phi w)(t) \tag{6}$$

by setting the integral operator

$$(\Phi w)(t) := M_c(t) + M_1(t) \int_t^{\infty} w(s) A\big(w^2(s), s\big)\, ds + \int_T^t w(s) M_1(s) A\big(w^2(s), s\big)\, ds \\ + \int_T^t \left(\int_{t_0}^s \frac{1}{a(v)}\, dv \right) (t - s) w(s) A\big(w^2(s), s\big)\, ds.$$

This means that we put the problem of existence of solutions to (1) in an equivalent fixed-point problem of equation (6) (that is, the solutions of (6) are solutions to (1)).

Now, we consider the linear space $C([T,\infty),\mathbb{R})$ of all continuous functions $w:[T,\infty)\to\mathbb{R}$ with

$$\|w\| = \sup\{M_1^{-2}(t)|w(t)| : t \geqslant T\} < +\infty.$$

Clearly, $(C([T,\infty),\mathbb{R}), \|\cdot\|)$ is a Banach space. To conclude the proof we have to establish the existence of a fixed point of Φ by an application of Schauder's theorem. Let c be the constant of hypothesis (H_2), we look at the set $W \subseteq (C([T,\infty),\mathbb{R}), \|\cdot\|)$ given as:

$$W = \{w \in C([T,\infty),\mathbb{R}) : M_c(t) \leqslant w(t) \leqslant M_{2c}(t) \text{ for } t \geqslant T\}.$$

We show in some steps that W is bounded, convex and closed.
Step 1. Φ maps W into W. Indeed, let $w \in W$, then by the definition of Φ we obtain:

$$\begin{aligned}
M_c(t) \leqslant (\Phi w)(t) &\leqslant M_c(t) + M_1(t)\int_t^\infty w(s)A(w^2(s),s)\,ds + 2M_1(t)\int_T^t w(s)A(w^2(s),s)\,ds \\
&\leqslant M_c(t) + 2M_1(t)\int_T^\infty w(s)A(w^2(s),s)\,ds \\
&\leqslant M_c(t) + 2M_1(t)\int_T^\infty 2M_c(t)A(w^2(s),s)\,ds \\
&\leqslant M_c(t) + 4M_c(t)\int_T^\infty M_1(s)A(M_c^2(s),s)\,ds \leqslant 2M_c(t) \quad \text{for } t \geqslant T.
\end{aligned}$$

Step 2. Φ is continuous. Let $\{w_n\} \subseteq W$ satisfying $\|w_n - w\| \to 0$ as $n \to +\infty$. Since W is closed, $w \in W$ and

$$\begin{aligned}
|(\Phi w_n)(t) - (\Phi w)(t)| &= \bigg| M_1(t)\int_t^\infty \big(w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)\big)\,ds \\
&\quad + M_1(t)\int_T^t \big(w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)\big)\,ds \\
&\quad + \int_T^t \left(\int_{t_0}^s \frac{dv}{a(v)}\right)(t-s)\big(w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)\big)\,ds \bigg| \\
&\leqslant M_1(t)\int_t^\infty |w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)|\,ds \\
&\quad + M_1(t)\int_T^t |w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)|\,ds \\
&\quad + M_1(t)\int_T^t |w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)|\,ds \\
&\leqslant 2M_1(t)\int_T^\infty |w_n(s)A(w_n^2(s),s) - w(s)A(w^2(s),s)|\,ds.
\end{aligned}$$

We note that $\|w_n - w\| \to 0$ as $n \to +\infty$, and so $\sup_{t \geqslant T} |(w_n(t) - w(t))M_1^{-2}(t)| \to 0$ too. Since $M_1^{-2}(t) > 0$ for all $t \geqslant T$, then we have $w_n(t) \to w(t)$ for all $t \geqslant T$. We deduce that

$$w_n(t)A\big(w_n^2(t), t\big) \to w(t)A\big(w^2(t), t\big) \quad \text{as } n \to +\infty, \text{ for all } t \geqslant T.$$

From

$$\begin{aligned}
&\big|w_n(s)A\big(w_n^2(s), s\big) - w(s)A\big(w^2(s), s\big)\big| \\
&\quad \leqslant \big|w_n(s)A\big(w_n^2(s), s\big)\big| + \big|w(s)A\big(w^2(s), s\big)\big| \\
&\quad \leqslant 4M_c(s)A\big(M_c^2(s), s\big) \quad \text{for } s \geqslant T, \\
&\quad \Rightarrow \quad \int_T^\infty \big|w_n(s)A\big(w_n^2(s), s\big) - w(s)A\big(w^2(s), s\big)\big|\, ds \to 0.
\end{aligned}$$

Now, for all $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for all $n \geqslant n(\varepsilon)$ we have

$$\begin{aligned}
&\int_T^\infty \big|w_n(s)A\big(w_n^2(s), s\big) - w(s)A\big(w^2(s), s\big)\big|\, ds < \frac{\varepsilon}{2} M_1(T) \\
&\quad \Rightarrow \quad \sup_{n \geqslant n(\varepsilon)} \big|(\Phi w_n)(t) - (\Phi w)(t)\big| M_1^{-2}(t) \\
&\qquad\qquad \leqslant 2M_1(T)^{-1} \sup_{n \geqslant n(\varepsilon)} \int_T^\infty \big|w_n(s)A\big(w_n^2(s), s\big) - w(s)A\big(w^2(s), s\big)\big|\, ds \\
&\qquad\qquad \leqslant 2M_1(T)^{-1} M_1(T) \frac{\varepsilon}{2} = \varepsilon, \\
&\quad \Rightarrow \quad \big\|(\Phi w_n)(t) - (\Phi w)(t)\big\| \to 0 \quad \text{as } n \to +\infty \text{ (as } \varepsilon \text{ is arbitrary).}
\end{aligned}$$

It follows that Φ is continuous.

Step 3. We show that ΦW is compact. To this end, we follow the arguments in the proof of [11, Theorem 1]. Therefore, if $w \in W$, then for $t_2 > t_1 \geqslant T$ we get

$$\begin{aligned}
&\big|\big(M_1^{-2}\Phi w\big)(t_2) - \big(M_1^{-2}\Phi w\big)(t_1)\big| \\
&\quad \leqslant \big|M_c^{-1}(t_2) - M_c^{-1}(t_1)\big| \\
&\qquad + \left|M_1^{-1}(t_2) \int_{t_2}^\infty w(s)A\big(w^2(s), s\big)\, ds - M_1^{-1}(t_1) \int_{t_1}^\infty w(s)A\big(w^2(s), s\big)\, ds\right| \\
&\qquad + \left|M_1^{-2}(t_2) \int_T^{t_2} M_1(s)w(s)A\big(w^2(s), s\big)\, ds - M_1^{-2}(t_1) \int_T^{t_1} M_1(s)w(s)A\big(w^2(s), s\big)\, ds\right| \\
&\qquad + \left|M_1^{-2}(t_2) \int_T^{t_2} \left(\int_{t_0}^s \frac{1}{a(v)}\, dv\right)(t_2 - s)w(s)A\big(w^2(s), s\big)\, ds \right. \\
&\qquad \left. - M_1^{-2}(t_1) \int_T^{t_1} \left(\int_{t_0}^s \frac{1}{a(v)}\, dv\right)(t_1 - s)w(s)A\big(w^2(s), s\big)\, ds\right|
\end{aligned}$$

$$\begin{aligned}
&\leqslant 2cM_1^{-1}(t_1) + 2M_1^{-1}(t_1)\int_T^\infty w(s)A\bigl(w^2(s),s\bigr)\,ds + 2M_1^{-1}(t_1)\int_T^\infty w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&= 2cM_1^{-1}(t_1) + 4M_1^{-1}(t_1)\int_T^\infty w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\leqslant 3cM_1^{-1}(t_1)\\
&\to 0 \quad \text{as } t_1 \to +\infty.
\end{aligned}$$

Now, there exists $T^* > T$ such that

$$\bigl|\bigl(M_1^{-2}\Phi w\bigr)(t_2) - \bigl(M_1^{-2}\Phi w\bigr)(t_1)\bigr| < \varepsilon \quad \text{for all } t_2 > t_1 \geqslant T^*.$$

Also, $T^* \geqslant t_2 > t_1 \geqslant T$ imply

$$\begin{aligned}
&\bigl|\bigl(M_1^{-2}\Phi w\bigr)(t_2) - \bigl(M_1^{-2}\Phi w\bigr)(t_1)\bigr|\\
&\quad\leqslant \bigl|M_c^{-1}(t_2) - M_c^{-1}(t_1)\bigr|\\
&\qquad + \bigl|M_1^{-1}(t_2) - M_1^{-1}(t_1)\bigr|\int_{t_2}^\infty w(s)A\bigl(w^2(s),s\bigr)\,ds + M_1^{-1}(t_2)\int_{t_1}^{t_2} w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + \bigl|M_1^{-1}(t_2) - M_1^{-1}(t_1)\bigr|\int_T^{t_2} M_1(s)w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + M_1^{-1}(t_1)\int_{t_1}^{t_2} M_1(s)w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + \bigl|t_2M_1^{-2}(t_2) - t_1M_1^{-2}(t_1)\bigr|\int_T^{t_2}\biggl(\int_{t_0}^s \frac{1}{a(v)}\,dv\biggr)w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + t_1M_1^{-2}(t_1)\int_{t_1}^{t_2}\biggl(\int_{t_0}^s \frac{1}{a(v)}\,dv\biggr)w(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + \bigl|M_1^{-1}(t_2) - M_1^{-1}(t_1)\bigr|\int_T^{t_2}\biggl(\int_{t_0}^s \frac{d\sigma}{a(\sigma)}\biggr)sw(s)A\bigl(w^2(s),s\bigr)\,ds\\
&\qquad + M_1^{-2}(t_1)\int_{t_1}^{t_2}\biggl(\int_{t_0}^s \frac{d\sigma}{a(\sigma)}\biggr)sw(s)A\bigl(w^2(s),s\bigr)\,ds.
\end{aligned}$$

We know that $w(t)A(w^2(t),t) \leqslant M_{2c}(t)A(M_c^2(t),t)$, for $t \geqslant T$.
We deduce easily that there exists $\delta > 0$ such that for all $w \in W$ we have

$$\bigl|\bigl(M_1^{-2}\Phi w\bigr)(t_2) - \bigl(M_1^{-2}\Phi w\bigr)(t_1)\bigr| < \varepsilon \quad \text{if } |t_2 - t_1| < \delta.$$

The above calculations assure that the interval $[T,\infty)$ can be decomposed into a finite number of subintervals with the following property:

- Each function of the form $M_1^{-2}\Phi w$, $w \in W$, has oscillations less than ε, on each of the above subintervals.

This means that the family $\{M_1^{-2}\Phi w : w \in W\}$ is equicontinuous on $[T, \infty)$. On the other hand, the family $\{M_1^{-2}\Phi w : w \in W\}$ is uniformly bounded too. Therefore, the compactness of ΦW is established.

The above steps authorize the use of Schauder's fixed point theorem so that we can find a fixed point of Φ in W, namely $\overline{w} \in W$. Such a point $\overline{w} = \overline{w}(t)$ solves the fixed point equation $w(t) = (\Phi w)(t)$ on the interval $[T, \infty)$, We can use the L'Hopital rule to deduce that

$$\lim_{t\to+\infty} \frac{\overline{w}(t)}{M_1(t)} = \lim_{t\to+\infty} \frac{\overline{w}'(t)}{M_1'(t)} = \lim_{t\to+\infty} \frac{\overline{w}''(t)}{M_1''(t)} = \lim_{t\to+\infty} a(t)\overline{w}''(t).$$

The last limit exists since the function $a\overline{w}''$ is decreasing and positive. Moreover, $\overline{w} \in W$ implies that

$$c \leqslant \lim_{t\to+\infty} \frac{\overline{w}(t)}{M_1(t)} \leqslant 2c.$$

Thus $\overline{w}$ is a solution of (1) with the property (L). $\square$

Remark 3.2. Let $w(t) > 0$ such that the property (L) holds. We show that (H_2) necessarily occurs in such a situation. Indeed, we can find some $c_1, c_2 > 0$ and $t_1 \geqslant t_0 > 0$ such that

$$M_{c_1}(t) \leqslant w(t) \leqslant M_{c_2}(t) \quad \text{for } t \geqslant t_1. \tag{7}$$

We integrate (1) over $[t_1, t]$ to have

$$\begin{aligned}
&a(t)w''(t) - a(t_1)w''(t_1) + \int_{t_1}^{t} w(s)A\big(w^2(s), s\big)\, ds = 0,\\
&\quad\Rightarrow \int_{t_1}^{\infty} w(s)A\big(w^2(s), s\big)\, ds < +\infty\\
&\qquad\quad \big(\text{recall that } a(t)w''(t) > 0 \text{ for all } t \geqslant t_1, \text{ see Lemma 2.1}\big)\\
&\quad\Rightarrow \int_{t_1}^{\infty} M_1(s)A\big(M_{c_2}^2(s), s\big)\, ds < +\infty \quad (\text{by (7), recall } A \text{ is non-increasing}),
\end{aligned}$$

and so (H_2) holds.

Example 3.3. Consider the third-order differential equation

$$\big(tw''(t)\big)' + e^{-t}w(t) = 0, \quad t \geqslant t_0 > 0.$$

Applying Theorem 3.1 to this equation, then we deduce that it admits a positive solution $w \in C^2([t_w, \infty), \mathbb{R})$ with the property (L).

Here, using the set of hypotheses (H_0), (H_1), (H_3) and (H_4), we establish the following oscillatory criteria of (1).

Theorem 3.4. *If (H_0), (H_1), (H_3) and (H_4) hold, then every solution $w \in C^2([t_w, \infty), \mathbb{R})$ of (1) is either oscillatory or satisfies $(L)'$.*

Proof. If case (S_2) holds, then the proof follows easily by Proposition 2.4. So, we focalize on case (S_1). Since the function $a(t)w''(t)$ is decreasing, then we deduce that

$$\begin{aligned} w'(t) &= w'(t_1) + \int_{t_1}^{t} w''(s)\,ds \\ &= w'(t_1) + \int_{t_1}^{t} \frac{a(s)w''(s)}{a(s)}\,ds \geqslant a(t)w''(t)\int_{t_1}^{t} a^{-1}(s)\,ds, \\ \Rightarrow \quad & w(t) \geqslant a(t)w''(t)\int_{t_*}^{t}\int_{t_1}^{v} a^{-1}(s)\,ds\,dv \quad \big(\text{we integrate over } [t_*, t]\big). \end{aligned}$$

Let $\delta \in C([t_0, \infty), \mathbb{R}_+)$ be the function given in (H_4). We observe that

$$\begin{aligned} &\big(a(t)w''(t)\big)' + w(t)A\big(w^2(t), t\big) = 0 \quad \big(\text{by (1); recall that } w''(t) > 0 \text{ for } (S_1)\big), \\ \Rightarrow \quad & \big(a(t)w''(t)\big)' + w\big(\delta(t)\big)A\big(w^2(t), t\big) \leqslant 0 \quad \big(\text{by } \delta(t) < t\text{; recall that } w'(t) > 0 \text{ for } (S_1)\big), \\ \Rightarrow \quad & \big(a(t)w''(t)\big)' + A\big(w^2(t), t\big)a\big(\delta(t)\big)w''\big(\delta(t)\big)\int_{t_*}^{\delta(t)}\int_{t_1}^{v} a^{-1}(s)\,ds\,dv \leqslant 0. \end{aligned}$$

Comparing the last inequality with the retarded differential equation in hypothesis (H_4) with $\eta = w^2$, we deduce that $u = aw''$ is a positive solution of a first-order oscillatory retarded differential inequality, related to that equation. Now, Corollary 1 of Philos [15] (see also [15, Theorem 1] for the complete proof) gives us that there exists $0 < u_* \leqslant u$ such that

$$u_*'(t) + u_*\big(\delta(t)\big)A\big(w^2(t), t\big)\int_{t_*}^{\delta(t)}\int_{t_1}^{v} a^{-1}(s)\,ds\,dv = 0, \tag{8}$$

which means that u_* is a positive solution to (8), a contradiction to (H_4). $\square$

Example 3.5. The third-order differential equation

$$\big(tw''(t)\big)' + \frac{c_0}{t^\alpha}w(t) = 0, \quad c_0 > 0, \alpha \in (0, 1), t \geqslant t_0 > 0,$$

where $A(w^2(t), t) = \frac{c_0}{t^\alpha}$ and $a(t) = t$, satisfies the hypotheses of Theorem 3.4, where in (H_4) we assume $\delta(t) = t - \delta_0$, with $\delta_0 > 0$.

In view of the conclusive part of the proof of Theorem 3.4, we state the following result (more precisely, we substitute hypothesis (H_4) by (H_5)), without proof.

Theorem 3.6. *If (H_0), (H_1), (H_3) and (H_5) hold, then every solution $w \in C^2([t_w, \infty), \mathbb{R})$ of (1) is either oscillatory or satisfies $(L)'$.*

4. Complementary results

In this section, we focus on the asymptotic behavior of solutions to the following modification of our main equation (1):

$$\begin{cases} (a(t)w'(t))'' + w(t)A(w^2(t), t) = 0, \\ a(t) > 0, \qquad a'(t) \geqslant 0, \qquad A(z,t) > 0, \qquad t \geqslant t_0 > 0, \qquad z > 0, \end{cases} \tag{9}$$

imposing the following regularities over (9): $a \in C^1([t_0, \infty), \mathbb{R}_+)$, $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ is monotone with respect to its first variable. By a solution of (9) we mean a function $w \in C^1([t_w, \infty), \mathbb{R})$, $t_w \geqslant t_0$, which has the property $aw' \in C^2([t_w, \infty), \mathbb{R})$, and satisfies (9) on $[t_w, \infty)$. Our aim here is to understand how the technical hypotheses (H_2) and (H_3) change.

About the classification of positive solutions to (9), we adapt Lemma 2.1 as follows.

Lemma 4.1. *Let $aw' \in C^2([t_0, \infty), \mathbb{R})$ be a (eventually) positive solution of (1). Then, we have the following situations:*

(S_1) $w(t) > 0$, $w'(t) > 0$, $(a(t)w'(t))' > 0$, $(a(t)w'(t))'' \leqslant 0$,
(S_2) $w(t) > 0$, $w'(t) < 0$, $(a(t)w'(t))' > 0$, $(a(t)w'(t))'' \leqslant 0$,

for $t \geqslant t_1$, where $t_1 \geqslant t_0$ is large enough.

Proof. Let w be a (eventually) positive solution to (1). By (9) and $A(z,t) > 0$ for all $t \geqslant t_0$, $z > 0$, we have:

$$\begin{aligned} &\big(a(t)w'(t)\big)'' = -w(t)A\big(w^2, t\big) < 0, \quad \text{for all } t \geqslant t_0, \\ &\quad \Rightarrow \quad \big(aw'\big)' \text{ is decreasing and does not change sign definitively.} \end{aligned}$$

So, we distinguish two cases: $(aw')'$ is negative definitively, and $(aw')'$ is positive definitively.

If we assume $(a(t)w'(t))' < 0$ for $t \geqslant t_1 \geqslant t_0$, then we have:

$$\begin{aligned} &\big(a(t)w'(t)\big)' < 0, \\ &\quad \Rightarrow \quad aw' \text{ is decreasing and does not change sign definitively,} \\ &\quad \Rightarrow \quad w' \text{ does not change sign definitively too.} \end{aligned}$$

So, we distinguish two cases: w' is negative definitively, and w' is positive definitively.

If we assume $w'(t) < 0$ for $t \geqslant t_1 \geqslant t_0$, then we can find a positive real number K with $a(t)w'(t) \leqslant -K < 0$. This leads to

$$\begin{aligned} &w'(t) \leqslant -\frac{K}{a(t)}, \\ &\quad \Rightarrow \quad \int_{t_1}^{t} w'(s)\,ds \leqslant -K \int_{t_1}^{t} \frac{1}{a(s)}\,ds \quad \big(\text{we integrate over } [t_1, t]\big), \end{aligned}$$

$$\Rightarrow \quad w(t) \leqslant w(t_1) - K \int_{t_1}^{t} \frac{1}{a(s)}\, ds$$

$$\Rightarrow \quad w(t) \to -\infty \quad \bigl(\text{since the last integral goes to } \infty \text{ as } t \text{ goes to } \infty, \text{ by } (H_0)\bigr).$$

So, we have a contradiction to the fact that w is positive. We conclude that w' must be positive definitively.

On the other side, we observe that

$$\bigl(a(t)w'(t)\bigr)'' < 0 \quad \text{and} \quad \bigl(a(t)w'(t)\bigr)' < 0 \quad \text{implies} \quad \lim_{t\to+\infty} \bigl(a(t)w'(t)\bigr) = -\infty.$$

It follows that $w'(t) < 0$ for sufficiently large t, which leads to contradiction. So it remains to examine the case where $(aw')'$ is positive definitively.

If we assume $(a(t)w'(t))' > 0$ for $t \geqslant t_1 \geqslant t_0$, then we have:

$$\bigl(a(t)w'(t)\bigr)' > 0,$$

$\Rightarrow$ aw' is increasing and does not change sign definitively,

$\Rightarrow$ w' does not change sign definitively too.

Thus, we conclude easily that only the situations (S_1) and (S_2) may occur. □

Remark 4.2. In the case (S_2), since aw' is increasing and does not change sign definitively, then there exists $\ell \leqslant 0$ such that $a(t)w'(t) \to \ell$ as $t \to +\infty$. Now $\ell = 0$ implies $w'(t) \to 0$ as $t \to +\infty$, and $w(t)$ is asymptotic to a finite constant.

Remark 4.3. In both cases (S_1) and (S_2), we can find $c > 0$ and $\bar{t} \geqslant t_0$ such that

$$w(t) \leqslant c \int_{t_0}^{t} \frac{s}{a(s)}\, ds, \qquad t \geqslant \bar{t}. \tag{10}$$

Keeping in mind Remark 4.2, we consider only the case (S_1), where $(aw')'$ is decreasing in $[t_0, \infty)$. So, from

$$\bigl(a(t)w'(t)\bigr)' \leqslant \bigl(a(t_0)w'(t_0)\bigr)',$$

we integrate over $[t_0, t]$ (two times) to get:

$$a(t)w'(t) \leqslant a(t_0)w'(t_0) + \bigl(a(t_0)w'(t_0)\bigr)'[t - t_0] \quad \text{(first integration)},$$

$$\Rightarrow \quad w'(t) \leqslant \frac{a(t_0)w'(t_0) - (a(t_0)w'(t_0))'t_0}{a(t)} + \frac{(a(t_0)w'(t_0))'t}{a(t)},$$

$$\Rightarrow \quad w(t) \leqslant w(t_0) + \bigl[a(t_0)w'(t_0) - \bigl(a(t_0)w'(t_0)\bigr)'t_0\bigr] \int_{t_0}^{t} \frac{ds}{a(s)} + \bigl(a(t_0)w'(t_0)\bigr)' \int_{t_0}^{t} \frac{s}{a(s)}\, ds$$

(second integration).

This inequality leads easily to (10) by a suitable choice of positive values c and $\bar{t} \geqslant t_0$.

In this section, we assume again that A is monotone non-increasing, with respect to the first variable. In view of Lemma 4.1 and (10), we note that a (eventually) positive solution to (9), namely again w, is such that

$$r \leqslant w(t) \leqslant R_c(t) := c \int_{t_0}^{t} \frac{s}{a(s)}\, ds \quad \text{for some } r \geqslant 0,\ c > 0 \text{ and all } t \geqslant \bar{t} \text{ large enough.}$$

Reasoning as in Section 2 we can study the positive solutions to (9) with the asymptotic property:

$(L)_r$ $\lim_{t\to+\infty} \frac{w(t)}{R_1(t)} = \gamma > 0$ (constant).

Now, we are ready to introduce the precise hypotheses on the data of (9):

$(H_2)'$ $\int_{t_0}^{\infty} (A(R_c^2(t), t) R_1(t))\, dt < +\infty$ for some $c > 0$.
$(H_3)'$ For all $\kappa > 0$, one of the following conditions holds:

a. $\int_z^{\infty} A(\kappa, s)\, ds = +\infty$.
b. $\int_v^{\infty} \int_z^{\infty} A(\kappa, s)\, ds\, dz = +\infty$.
c. $\int_{t_0}^{\infty} \frac{1}{a(v)} \int_v^{\infty} \int_z^{\infty} A(\kappa, s)\, ds\, dz\, dv = +\infty$.

Consequently we have the result:

Theorem 4.4. *If (H_0), (H_1), $(H_2)'$ hold, then there exists a (eventually) positive solution $w \in C^1([t_w, \infty), \mathbb{R}_+)$ of (9) with the property $(L)_r$.*

Proof. The proof of Theorem 4.4 can be easily obtained, following and adapting the proof of Theorem 3.1. This time, we will use the integral operator

$$(\Phi w)(t) := R_c(t) + R_1(t) \int_t^{\infty} w(s) A\big(w^2(s), s\big)\, ds + \int_T^t w(s) R_1(s) A\big(w^2(s), s\big)\, ds + \int_T^t \frac{1}{a(v)} \int_{t_0}^{v} s w(s) A\big(w^2(s), s\big)\, ds\, dv,$$

to solve (again) the integral equation of the form

$$w(t) = (\Phi w)(t).$$

Clearly, we have to consider $R_1(t)$ and $R_c(t)$, respectively, instead of $M_1(t)$ and $M_c(t)$ in whole the proof. Consequently the steps 1, 2, and 3 remain the same.

This means that we can apply the Schauder's fixed point theorem to get a fixed point of Φ in W, namely (again) $\overline{w} \in W$, where this time we have

$$W = \big\{ w \in C\big([T, \infty), \mathbb{R}\big) : R_c(t) \leqslant w(t) \leqslant R_{2c}(t) \text{ for } t \geqslant T \big\}.$$

Such a point $\overline{w}$ solves the fixed point equation $w(t) = (\Phi w)(t)$ on the interval $[T, \infty)$. By L'Hopital rule, we deduce that

$$\lim_{t\to+\infty} \frac{\overline{w}(t)}{R_1(t)} = \lim_{t\to+\infty} \frac{\overline{w}'(t)}{R_1'(t)} = \lim_{t\to+\infty} \frac{a(t)\overline{w}'(t)}{t} = \lim_{t\to+\infty} \big(a(t)\overline{w}'(t)\big)'.$$

The last limit exists since the function $(a\overline{w}')'$ is decreasing and positive. Moreover, $\overline{w} \in W$ implies that

$$c \leqslant \lim_{t\to+\infty} \frac{\overline{w}(t)}{R_1(t)} \leqslant 2c.$$

Thus $\overline{w}$ is a solution of (1) with the property $(L)_r$. $\square$

Now, we establish the analogous of Proposition 2.4 in Section 2.

Proposition 4.5. *If $(H_3)'$ holds and $w \in C^1([t_w,\infty),\mathbb{R}_+)$ is a (S_2)-type solution of (9), then $(L)'$ holds true.*

Proof. Since (S_2) holds, we know that $w(t) > 0$ and $w'(t) < 0$ for $t \geqslant t_1$, where $t_1 \geqslant t_0$ is large enough. So, there exists $\ell \geqslant 0$ such that $w(t) \to \ell$ as $t \to +\infty$. If we assume $\ell > 0$, then there exist $c > 1$ and $\widetilde{t} \geqslant t_0$ such that $\ell < w(t) \leqslant c\ell$ for all $t \geqslant \widetilde{t}$. We construct the proof in three steps.

Step 1. Assume that $(H_3)'_a$ holds, that is

$$\int_z^\infty A(\kappa,s)\,ds = +\infty, \quad \text{for all } \kappa > 0.$$

Then, from (9) and from the fact that A is non-increasing with respect to the first variable, we deduce that

$$\begin{aligned}
&\big(a(t)w'(t)\big)'' + \ell A\big(c^2\ell^2,t\big) \leqslant 0, \quad t \geqslant \widetilde{t},\\
&\quad\Rightarrow\quad \int_z^y \big[\big(a(s)w'(s)\big)'' + \ell A\big(c^2\ell^2,s\big)\big]\,ds \leqslant 0, \quad y > z \geqslant \widetilde{t},\\
&\quad\Rightarrow\quad \ell\int_z^y A\big(c^2\ell^2,s\big)\,ds \leqslant \big(a(z)w'(z)\big)' - \big(a(y)w'(y)\big)' \leqslant \big(a(z)w'(z)\big)',\\
&\quad\Rightarrow\quad \big(a(z)w'(z)\big)' \geqslant \ell\int_z^\infty A\big(c^2\ell^2,s\big)\,ds,
\end{aligned} \tag{11}$$

which leads to contradiction, by $(H_3)'_a$.

Step 2. Assuming that $\int_z^\infty A(\kappa,s)\,ds < +\infty$ for some $\kappa > 0$ (that is $(H_3)'_a$ does not hold), we consider the situation where $(H_3)'_b$ is true. Fixing $c > 1$ such that $c^2\ell^2 > \kappa$, and satisfying also the assumption of step 1, we deduce that the right hand side of (11) is finite. Thus we integrate each side of (11) over $[v,t]$ to obtain

$$\begin{aligned}
&\int_v^t \big(a(z)w'(z)\big)'\,dz \geqslant \ell\int_v^t\int_z^\infty A\big(c^2\ell^2,s\big)\,ds\,dz,\\
&\quad\Rightarrow\quad a(t)w'(t) - a(v)w'(v) \geqslant \ell\int_v^t\int_z^\infty A\big(c^2\ell^2,s\big)\,ds\,dz,\\
&\quad\Rightarrow\quad -a(v)w'(v) \geqslant \ell\int_v^\infty\int_z^\infty A\big(c^2\ell^2,s\big)\,ds\,dz,
\end{aligned} \tag{12}$$

which leads to contradiction, by $(H_3)'_b$.

Step 3. Assuming that $(H_3)'_a$ and $(H_3)'_b$ do not hold, we have that

$$\int_z^{\infty} A(\kappa_a, s)\, ds < +\infty \quad \text{for some } \kappa_a > 0$$

and

$$\int_v^{\infty}\int_z^{\infty} A(\kappa_b, s)\, ds\, dz < +\infty \quad \text{for some } \kappa_b > 0.$$

Take $c > 1$ such that $c^2\ell^2 > \kappa := \max\{\kappa_a, \kappa_b\}$. Then, due to the monotonicity of A (nonincreasing in its first variable), we have

$$\int_z^{\infty} A\big(c^2\ell^2, s\big)\, ds \leqslant \int_z^{\infty} A(\kappa, s)\, ds \leqslant \int_z^{\infty} A(\kappa_a, s)\, ds < +\infty.$$

Then we can integrate (11) over $[v, t]$ to obtain

$$-a(v)w'(v) \geqslant \ell \int_v^{\infty}\int_z^{\infty} A\big(c^2\ell^2, s\big)\, ds\, dz.$$

Again using monotonicity of A, we get

$$\begin{aligned}\int_v^{\infty}\int_z^{\infty} A\big(c^2\ell^2, s\big)\, ds\, dz &\leqslant \int_v^{\infty}\int_z^{\infty} A(\kappa, s)\, ds\, dz \\ &\leqslant \int_v^{\infty}\int_z^{\infty} A(\kappa_b, s)\, ds\, dz < +\infty.\end{aligned}$$

Therefore the right hand side of (12) is finite. After dividing each side of (12) by $a(v) > 0$, we integrate over $[t_*, t]$ to obtain

$$\begin{aligned}&-w(t) + w(t_*) \geqslant \int_{t_*}^{t} \frac{\ell}{a(v)} \int_v^{\infty}\int_z^{\infty} A\big(c^2\ell^2, s\big)\, ds\, dz\, dv, \\ &\quad\Rightarrow \quad w(t_*) \geqslant \ell \int_{t_*}^{\infty} \frac{1}{a(v)} \int_v^{\infty}\int_z^{\infty} A\big(c^2\ell^2, s\big)\, ds\, dz\, dv,\end{aligned}$$

which leads to contradiction, by $(H_3)'_c$.

We conclude that $\ell = 0$, that is $w(t)$ goes to zero as t goes to infinity, and hence $(L)'$ holds true. □

5. Conclusions

Our work here starts from a characterization into two classes of (eventually) positive solutions (hence non-oscillatory solutions) to certain third order nonlinear differential equations. For both the classes, we provide informations about the asymptotic behavior of solutions. This study leads to establish sufficient criteria for the existence of (non-)oscillatory solutions to (1), complementing the existing literature on the topic. Here, the coefficient function $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ is of one sign, but it will be interesting to know how an oscillatory coefficient function affects the analysis of the problem (1) (that is, $A \in C(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R})$ may change sign as its second variable t goes to infinity). In addition, Koplatadze–Čanturija [8] and Fukagai–Kusano [6] pointed out the existence of a sort of "duality" between retarded and advanced differential equations, with related positive, negative and sign changing coefficient functions. So, a similar duality can be investigated in respect to the equation (1).

Competing interests

The authors declare that they have no competing interests.

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