SUPERALGEBRAS: POLYNOMIAL IDENTITIES AND ASYMPTOTICS

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ABSTRACT. To any superalgebra A is attached a numerical sequence $c_n^{sup}(A)$, $n \geq 1$, called the sequence of supercodimensions of A. In characteristic zero its asymptotics are an invariant of the superidentities satisfied by A. It is well-known that for a PI-superalgebra such sequence is exponentially bounded and $exp^{sup}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}$ is an integer that can be explicitly computed.

Here we introduce a notion of fundamental superalgebra over a field of characteristic zero. We prove that if A is such an algebra, then

$$C_1 n^t exp^{sup}(A)^n \leq c_n^{sup}(A) \leq C_2 n^t exp^{sup}(A)^n, \tag{1}$$

where $C_1 > 0, C_2, t$ are constants and t is a half integer that can be explicitly written as a linear function of the dimension of the even part of A and the nilpotency index of the Jacobson radical. We also give a characterization of fundamental superalgebras through the representation theory of the symmetric group.

As a consequence we prove that if A is a finitely generated PI-superalgebra, then the inequalities in (1) still hold and t is a half integer. It follows that $\lim_{n\to\infty} \log_n \frac{c_n^{sup}(A)}{exp^{sup}(A)^n}$ exists and is a half integer.

1. INTRODUCTION

In the study of the polynomial identities satisfied by an algebra, an important role is played by finite dimensional superalgebras. In fact from the theory of varieties developed by Kemer ([27]), it follows that any algebra over a field of characteristic zero has the same identities as a suitable graded tensor product, called Grassmann envelope, involving a finite dimensional superalgebra and the Grassmann algebra.

Now, the growth of the polynomial identities satisfied by an algebra A is measured through a numerical sequence called the sequence of codimensions. On the other hand, one considers the growth of the superpolynomial identities of the finite dimensional superalgebra related to A as mentioned above through the Grassmann algebra and there seems to be a connection between the two growths.

To be more specific, let A be an algebra over a field of characteristic zero and suppose that A is a PI-algebra, i.e., it satisfies a non-trivial polynomial identity. The sequence of codimensions of A is $c_n(A)$, $n = 1, 2, \ldots$, where $n! - c_n(A)$ is the dimension of the space of multilinear polynomial identities in n fixed variables satisfied by A.

Such sequence has been extensively studied ([13], [16], [19], [28], [29]), and Regev in [32] proved that if A is a PI-algebra, the corresponding sequence of codimensions is exponentially bounded. Moreover in [33] he proved that if $M_k(F)$ is the algebra of $k \times k$ matrices over F, then the asymptotic equality $c_n(M_k(F)) \simeq Cn^t(k^2)^n$ holds, where C is an explicitly computed constant and $t = -\frac{1}{2}(k^2 - 1)$.

Later in [21] and [22] the authors proved that for any PI-algebra A, there exist constants $C_1 > 0, C_2, t, s, d$ such that

$$C_1 n^t d^n \leq c_n(A) \leq C_2 n^s d^n,$$

for all $n \geq 1$, and d is an integer called the exponent $exp(A)$ of A. Moreover in [9] and [8] (see also [25]) it was proved that in the above inequalities $t = s \in \frac{1}{2}\mathbb{Z}$ and, if $1 \in A$, then also $C_1 = C_2$.

Now, the exponent $exp(A)$ and

$$t = \lim_{n\to\infty} \log_n \frac{c_n(A)}{exp(A)^n}$$

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are two invariants of the identities of the algebra A. The exponent has an interpretation as the dimension of a superalgebra related to the identities of A, whereas in [5] the authors computed explicitly the invariant t for any so-called fundamental algebra.

Next we turn our attention to superalgebras. Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra over a field F of characteristic zero. As for the general setting, one considers the free superalgebra $F\langle Y, Z\rangle$ on countable sets Y and Z of even and odd variables. Then one constructs the sequence of supercodimensions of A by setting $c_n^{sup}(A) = \dim P_n^{sup}(A)$, $n = 1, 2, \ldots$, where $P_n^{sup}(A)$ is the space of multilinear superpolynomials in n fixed variables of degree 0 or 1, modulo the superidentities of A.

Now, in [20] it was shown that for any PI-algebra A, $c_n(A) \le c_n^{sup}(A) \le 2^n c_n(A)$, holds for all $n \ge 1$. On the other hand the explicit computation of the supercodimensions has been carried out in very few examples ([15], [31], [30]) and, as in the ordinary case, the attention focused on computing their asymptotics.

The exponential growth of the sequence of supercodimensions was computed in [7, 1, 2, 14]. It turns out that if A is a PI-superalgebra there exist constants $C_1 > 0, C_2, t_1, t_2, d$ such that

$$C_1 n^{t_1} d^n \le c_n^{sup}(A) \le C_2 n^{t_2} d^n, \tag{2}$$

for all $n \ge 1$, and d is an integer called the superexponent $exp^{sup}(A)$ of A.

Next step is to ask if the polynomial factor in (2) is uniquely determined, i.e., $t_1 = t_2$, giving in this way a second invariant of the superindenties of A after the superexponent. Moreover, can one explicitly compute such polynomial factor for a certain class of superalgebras relating it to the structure of the algebra itself? Now, if A is a finite dimensional simple superalgebra, in [26] it was proved that $t_1 = t_2 = -\frac{1}{2}(\dim A^{(0)} - 1)$.

In this paper we are able to give a positive answer to this question for the class of fundamental superalgebras that here we call φ-fundamental algebras, where φ is the automorphism giving the superstructure of A. A basic property of these algebras is that any finite dimensional superalgebra has the same superidentities as a finite direct sum of φ-fundamental algebras.

We are able to compute the polynomial factor of the supercodimensions of any φ-fundamental algebra in terms of some fixed parameters. More precisely we prove the following: let A be a φ-fundamental algebra over an algebraically closed field of characteristic zero and let $A = \bar{A} + J$ be its Wedderburn-Malcev decomposition, where $\bar{A}$ is a semisimple superalgebra and J is the Jacobson radical of A. Let $s \ge 0$ be the least integer such that $J^{s+1} = 0$. Then

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A}^{(0)})-q)+s} (\dim \bar{A})^n \le c_n^{sup}(A) \le C_2 n^{-\frac{1}{2}(\dim(\bar{A}^{(0)})-q)+s} (\dim \bar{A})^n,$$

for some constants $C_1 > 0, C_2$ where q is the number of simple superalgebras appearing in the decomposition of $\bar{A}$.

As a corollary, from [3] we get that if A is any finitely generated PI-superalgebra over a field of characteristic 0, then in (2) $t_1 = t_2 \in \frac{1}{2}\mathbb{Z}$.

Hence $\lim_{n\to\infty} \log_n \frac{c_n^{sup}(A)}{exp^{sup}(A)^n}$ exists and is a half integer.

The reader should be aware that the definition of φ-fundamental algebra given here is paired with the notion of $*$-fundamental algebra given in [17, 18] and differs from the one given by Kemer ([27] see also [3]) or by Procesi ([4]).

2. The general setting

Throughout this paper we shall denote by F a field of characteristic zero and by $A = A^{(0)} \oplus A^{(1)}$ an associative superalgebra over F ($\mathbb{Z}_2$-graded algebra). Recall that the map $\varphi : A \to A$ such that $\varphi(a^{(0)} + a^{(1)}) = a^{(0)} - a^{(1)}$, for $a^{(0)} \in A^{(0)}, a^{(1)} \in A^{(1)}$ is an automorphism of order ≤ 2 of A. Hence A can be viewed as an algebra with φ-action.

Conversely, if A is any algebra with an automorphism φ of order ≤ 2, then A is a superalgebra if we define $A^{(0)} = \{a \in A \mid \varphi(a) = a\}$ and $A^{(1)} = \{a \in A \mid \varphi(a) = -a\}$.

Recall that the elements of $A^{(0)}$ and $A^{(1)}$ are called homogeneous of degree zero (or even elements) and degree one (or odd elements), respectively.

As in [17] we shall write the homogeneous elements in terms of the automorphism φ. Hence the free superalgebra on a countable set will be viewed as $F\langle\mathcal{X},\varphi\rangle = F\langle x_1, x_1^{\varphi}, x_2, x_2^{\varphi}, \ldots\rangle$, the free associative algebra with an automorphism φ of order ≤ 2 on the set $\mathcal{X} = \{x_1, x_2, \ldots\}$.

Its elements are called φ-polynomials (or superpolynomials) and $f(x_1, \ldots, x_n) \in F\langle\mathcal{X},\varphi\rangle$ is a φ-identity (or superidentity) of A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$. Here we simply write $f(x_1, \ldots, x_n)$ to indicate a φ-polynomial in which the variables $x_1, \ldots, x_n$ or their image by φ appear.

Let $\mathrm{Id}_2(A) = \{f \in F\langle\mathcal{X},\varphi\rangle \mid f \equiv 0 \text{ on } A\}$ the set of φ-identities of A. Clearly $\mathrm{Id}_2(A)$ is a T_2-ideal of $F\langle\mathcal{X},\varphi\rangle$, i.e., an ideal invariant by endomorphisms of the free algebra commuting with the φ-action. It is well known that in characteristic zero $\mathrm{Id}_2(A)$ is completely determined by its multilinear polynomials and we denote by

$$P_n^{sup} = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} |\ \sigma \in S_n, w_i = x_i \text{ or } w_i = x_i^{\varphi}, 1 \leq i \leq n\}$$

the space of multilinear φ-polynomials of degree n in $x_1, \ldots, x_n$.

We act with the symmetric group S_n on the left on P_n^{sup} : if $\sigma \in S_n$ and $f = f(x_1, \ldots, x_n) \in P_n^{sup}$, then $\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The outcome is that

$$P_n^{sup}(A) = \frac{P_n^{sup}}{P_n^{sup} \cap \mathrm{Id}_2(A)}$$

has a structure of S_n-module and its dimension $c_n^{sup}(A)$ is called the nth supercodimension of A.

Recall that in [2, 14] the authors proved that for any associative superalgebra A satisfying an ordinary identity, there exist constants $C_1 > 0, C_2, t_1, t_2, d$ such that

$$C_1 n^{t_1} d^n \leq c_n^{sup}(A) \leq C_2 n^{t_2} d^n, \tag{3}$$

for all $n \geq 1$, and $d = \exp^{sup}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}$ is an integer called the superexponent of A.

Now the character $\chi_n^{sup}(A)$ of $P_n^{sup}(A)$, is called the nth supercocharacter of A, and by complete reducibility, we can write

$$\chi_n^{sup}(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{4}$$

where χ_λ is the irreducible S_n-character associated to the partition λ and $m_\lambda \geq 0$ is the corresponding multiplicity.

We recall that a φ-polynomial $f(x_1, \ldots, x_n, Y)$ linear in the variables $x_1, \ldots, x_n$ (and in some other set of variables Y) is alternating in $x_1, \ldots, x_n$ if f vanishes whenever we identify any two of these variables. This is equivalent to say that the polynomial changes sign whenever we exchange any two of these variables (here we exchange the indices of the two variables).

We also recall that any irreducible left S_n-module $M \subseteq P_n^{sup}$ with character χ_λ can be generated as an S_n-module by an element of the form $e_{T_\lambda} f$, for some $f \in M$ and some Young tableau T_λ of shape λ. Here $e_{T_\lambda} = R_{T_\lambda}^{+} C_{T_\lambda}^{-}$ is an essential idempotent, where $R_{T_\lambda}^{+} = \sum_{\sigma \in R_{T_\lambda}} \sigma$, $C_{T_\lambda}^{-} = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sign}\tau)\tau$, and R_{T_λ} and C_{T_λ} are the row and column stabilizers of T_λ, respectively. Notice that the φ-polynomial $f_{T_\lambda} = C_{T_\lambda}^{-}(e_{T_\lambda} f)$ generates also M, as an S_n-module, and it is alternating in each set of variables indexed by the columns of T_λ.

Now assume that A is a finite dimensional superalgebra over an algebraically closed field F of characteristic zero.

By the Wedderburn-Malcev theorem [24, Theorem 3.4.4] for superalgebras we can write

$$A = \bar{A} \oplus J$$

where $\bar{A}$ is a semisimple subalgebra of A, $J = J(A)$ is the Jacobson radical and both $\bar{A}$ and J are stable under the φ-action. Moreover

$$\bar{A} = A_1 \oplus \cdots \oplus A_q, \tag{5}$$

where $A_1, \ldots, A_q$ are simple superalgebras.

Recall that a finite dimensional simple superalgebra over an algebraically closed field is isomorphic to one of the following:

1. $A = M_{k,l}(F) = \{\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, k \geq l \geq 0,$ where P, Q, R, S are $k\times k, k\times l, l\times k, l\times l$ matrices, respectively$\}$ and $\varphi(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}) = \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}$;
2. $A = M_n(F) \oplus M_n(F)$ where φ is the automorphism such that $\varphi(a, b) = (b, a)$.

Hence in (5) we shall assume that $A_i \equiv M_{d_i}(F) \oplus M_{d_i}(F)$ or $A_i \equiv M_{k_i,l_i}(F)$.

If we denote by e_i the unit element of A_i, then $1_{\bar{A}} = \sum_{i=1}^{q} e_i$ is the unit element of $\bar{A}$. In case A has a unit element $1 = 1_{\bar{A}}$, we have the decomposition

$$A = \oplus_{i,k=1}^{q} e_i A e_k = (\oplus_{i=1}^{q} e_i A_i e_i) \oplus (\oplus_{i,k=1}^{q} e_i J e_k).$$

If A has not a unit element, we consider the algebra $A' = F \oplus A$ obtained from A by adjoining 1 and we extend φ to A' by defining $\varphi(\alpha + a) = \alpha + \varphi(a)$. Then we define $e_0 = 1_{A'} - \sum_{i=1}^{q} e_i \in A'$, where $1_{A'}$ is the unit element of A', and we have the decomposition

$$A' = F \oplus (\oplus_{i=1}^{q} e_i A_i e_i) \oplus (\oplus_{i,k=0}^{q} e_i J e_k).$$

The relation between the supercodimensions of A and A' is $c_n^{sup}(A) \leq c_n^{sup}(A')$.

We choose a basis of the finite dimensional superalgebra $A = \bar{A} + J$ as the union of a basis of J and a basis of $\bar{A}$, which is the union of bases of the simple components.

Now, since $A_i = M_{k_i,l_i}(F)$ or $A_i = M_{d_i}(F) \oplus M_{d_i}(F)$, we can decompose $e_i = \sum_{j=1}^{d_i} e_{j,j}^{(i)}$, where $d_i = k_i + l_i$ or $e_i = \sum_{j=1}^{d_i} (e_{j,j}^{(i)}, 0) + \sum_{j=1}^{d_i} (0, e_{j,j}^{(i)})$, where the $e_{j,j}^{(i)}$'s are the matrix units of $M_{d_i}(F)$. By abuse of notation, in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)$, $e_{j,j}^{(i)}$ will denote also $(e_{j,j}^{(i)}, 0)$ or $(0, e_{j,j}^{(i)})$. Hence we can write the elements of A as a linear combination of elements of the spaces

$$e_{j,j}^{(i)} A_i e_{k,k}^{(i)}, \quad e_{j,j}^{(i)} J e_{m,m}^{(l)}, \quad 1 \leq j, k \leq d_i,\ 1 \leq m \leq d_l,\ 1 \leq i, l \leq q$$

and, when A does not have a unit element, we have to add the spaces

$$e_{1,1}^{(0)} J e_{k,k}^{(i)}, \quad e_{j,j}^{(i)} J e_{1,1}^{(0)}, \quad e_{1,1}^{(0)} J e_{1,1}^{(0)},$$

where $e_{1,1}^{(0)} = e_0$.

Definition 2.1. *Let f be a φ-polynomial. A substitution of all the variables of f with elements of one of the spaces $e_{j,j}^{(i)} A_i e_{k,k}^{(i)}$, $e_{j,j}^{(i)} J e_{k,k}^{(l)}$ is called an elementary substitution.*

3. The Kemer index of a superalgebra

Let

$$Cap_n(X, Y) = Cap_n(x_1, \ldots, x_n, y_1, \ldots, y_{n+1}) = \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma) y_1 x_{\sigma(1)} y_2 x_{\sigma(2)} \cdots y_n x_{\sigma(n)} y_{n+1}$$

be the nth Capelli polynomial, where S_n is the symmetric group, which is alternating in the variables $x_1, \ldots, x_n$. It is well known that $Cap_{d^2}(X, Y)$ is not an identity of $d \times d$ matrices over F (see for instance [24, Proposition 1.7.1]). Hence $Cap_{d^2}(X, Y)$ is not a φ-identity of A, where either $A = M_d(F) \oplus M_d(F)$ or $A = M_{k,l}(F)$, with $d = k + l$. Moreover, any element $e_{h,k}$ of A can be obtained as an evaluation of $Cap_{d^2}(X, Y)$, where in case $A = M_d(F) \oplus M_d(F)$, $e_{h,k}$ denotes $(e_{h,k}, 0)$ or $(0, e_{h,k})$.

The following proposition holds.

Proposition 3.1. *Let A be a simple superalgebra. For every $\mu \geq 1$ there exists a multilinear φ-polynomial*

$$f(X_1, \ldots, X_\mu, Y) \notin Id_2(A) \tag{6}$$

alternating on each of the disjoint sets $X_1, \ldots, X_\mu$, where $|X_1| = \cdots = |X_\mu| = \dim A$ and $|Y| < \infty$.

Such a polynomial has the property that it can take any value of the type $e_{i,i}$, $1 \leq i \leq d$, when evaluated in A.

Proof. For every $\mu \geq 1$, define

$$Cap_{\mu,n}(X_1,\dots,X_\mu,Y) := \prod_{i=1}^{\mu} Cap_n(X_i,Y_i),$$

where $X_i = \{x_{i,1},\dots,x_{i,n}\}$, $Y_i = \{y_{i,1},\dots,y_{i,n+1}\}$, $i = 1,\dots,\mu$, are distinct sets of variables and $Y = \cup Y_i$. It is clear that if $A = M_{k,l}(F)$, Cap_{μ,d^2}, with $d = k+l$, is the required polynomial. Hence we may assume that $A = M_d(F) \oplus M_d(F)$.

Given $X = \{x_1,\dots,x_{2n}\}$ and $Y = \{y_1,\dots,y_{2n+2}\}$ define

$$Cap_n(X,Y,\varphi) := Cap_n(x_1,\dots,x_n,y_1,\dots,y_{n+1})Cap_n(x_{n+1}^\varphi,\dots,x_{2n}^\varphi,y_{n+2}^\varphi,\dots,y_{2n+2}^\varphi).$$

Notice that $Cap_{d^2}(X,Y,\varphi) \notin \mathrm{Id}_2(A)$, and any value of the type $e_{i,i} \in A$ can be obtained by evaluation.

Now let Alt_X be the operator of alternation on the set X. Then

$$G_{2n}(X,Y,\varphi) = Alt_X Cap_n(X,Y,\varphi)$$

is a multilinear φ-polynomial alternating in $X = \{x_1,\dots,x_{2n}\}$.

For every $\mu \geq 1$, let $X_i = \{x_{i,1},\dots,x_{i,2n}\}$, $Y_i = \{y_{i,1},\dots,y_{i,2n+2}\}$, $i = 1,\dots,\mu$, be distinct sets of variables. Then if $Y = \cup Y_i$, the polynomial

$$G_{\mu,2n}(X_1,\dots,X_\mu,Y,\varphi) := \prod_{i=1}^{\mu} G_{2n}(X_i,Y_i,\varphi)$$

is the required one. □

From now on A will be a finite dimensional superalgebra over an algebraically closed field F of characteristic zero. We write $A = \bar{A} \oplus J$, where $\bar{A} = A_1 \oplus \cdots \oplus A_q$ with $A_1,\dots,A_q$ simple superalgebras and $s \geq 0$ is the smallest integer such that $J^{s+1} = 0$. Recall that A is reduced if up to a rearrangement of the simple components we have that $A_1 J A_2 J \cdots J A_q \neq 0$

Let $f(X_1,\dots,X_r,Y)$ be a multilinear φ-polynomial in disjoint sets of variables $X_1,\dots,X_r,Y$. Then f is said to be r-fold t-alternating if f is alternating on each set X_i and $|X_i| = t$, $1 \leq i \leq r$.

Lemma 3.1. *Let $A = \bar{A} + J$ be a finite dimensional reduced superalgebra with $\dim \bar{A} = \dim(A_1 \oplus \cdots \oplus A_q) = t$. Then, for any $\mu \geq 1$, there exists a multilinear φ-polynomial*

$$f(X_1,\dots,X_\mu,Z_1,\dots,Z_{q-1},Y) \notin Id_2(A)$$

with $|X_j| = t$, $1 \leq j \leq \mu$, $|Z_i| = t+1$, $1 \leq i \leq q-1$ such that f is μ-fold t-alternating and $(q-1)$-fold $(t+1)$-alternating.

Proof. We assume as we may that $A_1 J A_2 J \cdots J A_q \neq 0$. Now let $\mu \geq 1$ and for every $1 \leq i \leq q$, let $f_i(X_{1,i},\dots,X_{\mu+q-1,i},Y_i) \notin \mathrm{Id}_2(A_i)$ be the multilinear φ-polynomial constructed in Proposition 3.1. Hence f_i is alternating on the disjoint sets $X_{1,i},\dots,X_{\mu+q-1,i}$, with $|X_{1,i}| = \cdots = |X_{\mu+q-1,i}| = \dim A_i$. Also such φ-polynomial has the property that it can take any value of the type $e_{j_i \cdot j_i} \in A_i$.

Let $f = f_1 z_1 f_2 z_2 \cdots z_{q-1} f_q$, where $z_1,\dots,z_{q-1}$ are new variables, and define

$$\tilde{f} = Alt_{X,Z} f$$

where $Alt_{X,Z}$ is the operator of alternation on each set

$$X_j = X_{j,1} \cup \cdots \cup X_{j,q},\ q \leq j \leq \mu+q-1$$

and on each set

$$Z_j = X_{j,1} \cup \cdots \cup X_{j,q} \cup \{z_j\},\ 1 \leq j \leq q-1.$$

Since $\dim A_1 + \cdots + \dim A_q = t$, $\tilde{f}$ is $(q-1)$-fold $(t+1)$-alternating, and μ-fold t-alternating.

Now it is not hard to prove (see for instance [18, Lemma 1]) that $\tilde{f}$ is not a φ-identity of A and the proof is complete. □

Next we define the Kemer φ-index of the superalgebra A. Let $\Gamma = \mathrm{Id}_2(A)$.

Definition 3.1. *Let $\beta(\Gamma)$ be the greatest integer t such that for every $\mu \geq 1$, there exists a multilinear φ-polynomial $f(X_1, \ldots, X_\mu, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = t$. Also, let $\gamma(\Gamma)$ be the greatest integer s for which there exists for all $\mu \geq 1$, a multilinear φ-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_s, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = \beta(\Gamma)$ and in the s sets Z_j with $|Z_j| = \beta(\Gamma)+1$. $Ind_K^\varphi(\Gamma) = (\beta(\Gamma), \gamma(\Gamma))$ is called the Kemer φ-index of Γ.*

We also say that $(\beta(\Gamma), \gamma(\Gamma)) = (\beta(A), \gamma(A)) = Ind_K^\varphi(A)$ is the Kemer φ-index of A.

As an example we consider $A = M_d(F) \oplus M_d(F)$. For every $\mu \geq 1$ the polynomial $G_{\mu,2d^2}$ is alternating in the μ sets X_i with $|X_i| = 2d^2$ and is not a φ-identity of A. Moreover since $\dim A = 2d^2$ any φ-polynomial alternating in $2d^2 + 1$ elements is a φ-identity of A. Hence $\beta(A) = 2d^2$ and $\gamma(A) = 0$. Notice that in case $A = M_{k,l}(F)$, $k + l = d$, $Ind_K^\varphi(A) = (d^2, 0)$. In fact $Cap_{\mu,d^2} \notin \mathrm{Id}_2(A)$ and it has the prescribed properties.

We remark that by the definition of $\gamma(\Gamma)$ there exists a smallest integer μ_0 such that every multilinear φ-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{\gamma(\Gamma)+1}, Y)$, alternating in $\mu \geq \mu_0$ sets X_i with $\beta(\Gamma)$ elements and in $\gamma(\Gamma)+1$ sets Z_j with $\beta(\Gamma) + 1$ elements lies in Γ.

Definition 3.2. *A multilinear φ-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{\gamma(\Gamma)}, Y) \notin \Gamma$ which is alternating in $\mu > \mu_0$ sets X_i with $|X_i| = \beta(\Gamma)$ and in $\gamma(\Gamma)$ sets Z_i with $|Z_i| = \beta(\Gamma) + 1$ is called a Kemer φ-polynomial related to Γ.*

Definition 3.3. *The (t, s)-index of A is $Ind_{t,s}(A) = (\dim \bar{A}, s_A)$ where $s_A \geq 0$ is the smallest integer such that $J^{s_A+1} = 0$.*

Notice that this is the same as the (t, s)-index of A with trivial grading. Also if A is a finite dimensional superalgebra, then $Ind_K^\varphi(A) \leq Ind_{t,s}(A)$ in the left lexicographic order.

Recall (see [7]) that for a finite dimensional superalgebra A, the superexponent $exp^{sup}(A)$ is the largest dimension of a semisimple sub-superalgebra of $\bar{A}$, say, $A_1 \oplus \cdots \oplus A_n$, such that

$$A_1 J A_2 J \cdots J A_n \neq 0.$$

The connection between the superexponent and the Kemer φ-index is given in the following.

Remark 3.1. *If $A = \bar{A} + J$ is a finite dimensional superalgebra and $Ind_K^\varphi(A) = (\beta, \gamma)$, then $\beta = exp^{sup}(A)$.*

Proof. Let $exp^{sup}(A) = \dim A_1 \oplus \cdots \oplus A_n$ with $A_1 J A_2 J \cdots J A_n \neq 0$. Then the superalgebra $B = A_1 \oplus \cdots \oplus A_n + J$ is reduced and by Lemma 3.1 for every $\mu \geq 1$ there exists a multilinear φ-polynomial $f \notin \mathrm{Id}_2(B)$ which is μ-fold t-alternating, where $t = \dim(A_1 \oplus \cdots \oplus A_n) = exp^{sup}(A)$. Since $f \notin \mathrm{Id}_2(A)$, then for the first Kemer φ-index we have that $\beta \geq exp^{sup}(A)$.

Next let $J^k = 0$ and let g be a multilinear φ-polynomial k-fold $(exp^{sup}(A) + 1)$-alternating.

Suppose that there is a non-zero evaluation ψ of g in A.

If we evaluate all variables of g in $\bar{A}$, $\psi(g) \neq 0$ implies that all variables must be evaluated in one simple component, say A_i. Since g is alternating on $exp^{sup}(A)+1 > \dim A_i$ variables we get $\psi(g) = 0$, a contradiction. It follows that some variables of g must be evaluated in J.

Suppose now that we evaluate all variables of an alternating set X_u in $\bar{A}$, say in $A_{i_1}, \ldots, A_{i_s}$. But then $0 \neq \psi(g) \in B = A_{i_1} + \cdots + A_{i_s} + J$ and, by definition of superexponent, $\dim(A_{i_1} + \cdots + A_{i_s}) \leq exp^{sup}(A)$. Since $|X_u| = exp^{sup}(A) + 1$, we get that $\psi(g) = 0$, a contradiction.

It follows that at least one variable of each alternating set must be evaluated under ψ in J. Since there are k alternating sets, we get that $\psi(g) \in J^k = 0$.

Since for any $l \geq k$, any multilinear φ-polynomial l-fold $(exp^{sup}(A) + 1)$-alternating is a φ-identity of A, it follows that $\beta = exp^{sup}(A)$. $\square$

4. Asymptotics for finite dimensional superalgebras

We start by recalling that two functions $f(x), g(x)$ of a real variable are asymptotically equal and we write $f(x) \simeq g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

We recall the following result proved in [26]

Theorem 4.1. *If $A = A^{(0)} \oplus A^{(1)}$ is a finite dimensional simple superalgebra over an algebraically closed field F of characteristic zero then*

$$c_n^{sup}(A) \simeq Cn^{-\frac{1}{2}(\dim A^{(0)}-1)}(\dim A)^n,$$

for some constant C.

We can also prove the following result on the asymptotics of product of supercodimensions.

Lemma 4.1. *Let $\bar{A} = A_1 \oplus \cdots \oplus A_q$ be a finite dimensional semisimple superalgebra over an algebraically closed field of characteristic zero, where the A_i's are simple superalgebras. Then*

$$\sum_{m_1+\cdots+m_q=m} \binom{m}{m_1,\ldots,m_q} c_{m_1}^{sup}(A_1)\cdots c_{m_q}^{sup}(A_q) \simeq Cm^{-\frac{1}{2}(\dim \bar{A}^{(0)}-q)}(\dim \bar{A})^m,$$

where C is a constant.

Proof. The proof consists in an application of a result of Beckner and Regev in [6]. They proved that if $(p_1,\ldots,p_q) \in \mathbb{Q}^q$ is such that $\sum_i p_i = 1$ and $F(x_1,\ldots,x_q)$ is a continuous function homogeneous of degree r with $0 < F(p_1,\ldots,p_q) < \infty$, then

$$\sum_{m_1+\cdots+m_q=m} \binom{m}{m_1,\ldots,m_q} p_1^{m_1}\cdots p_q^{m_q} F(m_1,\ldots,m_q) \simeq m^r F(p_1,\ldots,p_q). \tag{7}$$

Now, set $p_j = \frac{d_j}{d}$, $1 \leq j \leq q$, where $d_j = \dim A_j$ and $d = \dim \bar{A} = \sum_{j=1}^q d_j$, and consider the polynomial function

$$F(x_1,\ldots,x_q) = \prod_{j=1}^q x_j^{-\frac{1}{2}(\dim A_j^{(0)}-1)}$$

which is of degree $-\frac{1}{2}\sum_{j=1}^q(\dim A_j^{(0)} - 1) = -\frac{1}{2}(\dim \bar{A}^{(0)} - q)$.

Then by applying the formula (7) we get

$$\sum_{m_1+\cdots+m_q=m} \binom{m}{m_1,\ldots,m_q} \prod_{j=1}^q (\frac{d_j}{d})^{m_j} \prod_{j=1}^q m_j^{-\frac{1}{2}(dimA_j^{(0)}-1)} \simeq Cm^{-\frac{1}{2}(\dim \bar{A}^{(0)}-q)}, \tag{8}$$

where $C = F(\frac{d_1}{d},\ldots,\frac{d_q}{d})$ is a constant. Thus recalling that by Theorem 4.1, $c_{m_j}^{sup}(A_j) \simeq C_j m_j^{-\frac{1}{2}(\dim A_j^{(0)}-1)}(\dim A_j)^{m_j}$,

$$\sum_{m_1+\cdots+m_q=m} \binom{m}{m_1,\ldots,m_q} c_{m_1}^{sup}(A_1)\cdots c_{m_q}^{sup}(A_q)$$

$$\simeq C_1 \sum_{m_1+\cdots+m_q=m} \binom{m}{m_1,\ldots,m_q} \prod_{j=1}^q m_j^{-\frac{1}{2}(\dim A_j^{(0)}-1)}(\dim A_j)^{m_j}$$

$$\simeq C_2 m^{-\frac{1}{2}(\dim \bar{A}^{(0)}-q)}(\dim \bar{A})^m.$$

□

From now on $A = \bar{A} + J$ will be a superalgebra over an algebraically closed field F of characteristic zero. Let $\bar{A} = A_1 \oplus \cdots \oplus A_q$, $J^s \neq 0$, $J^{s+1} = 0$ and the A_i's are simple superalgebras with $A_i \equiv M_{d_i}(F) \oplus M_{d_i}(F)$ or $A_i \equiv M_{k_i,l_i}(F)$, $1 \leq i \leq q$. In the following we shall denote the algebra $M_{k_i,l_i}(F)$ with $M_{d_i}(F)$ if $k_i + l_i = d_i$.

Fix a basis of A which is the union of the standard bases $\{a_1^{(l)},\ldots,a_{m_l}^{(l)}\}$ of the simple superalgebras A_l, $1 \leq l \leq q$. If $A_l = M_{d_l}(F) = M_{k_l,l_l}(F)$, then we take a basis consisting of the matrix units $e_{i,j}^{(l)}$, and if $A_l = M_{d_l}(F) \oplus M_{d_l}(F)$, we take the elements $(e_{i,j}^{(l)}, 0)$ and $(0, e_{i,j}^{(l)})$.

Let also $\{w_1,\ldots,w_p\}$ be a basis of J. If $1 \in A$, $w_t \in e_{j,j}^{(i)} J e_{k,k}^{(l)}$, for some $1 \leq j \leq d_i$, $1 \leq k \leq d_l$, $1 \leq i,l \leq q$, where in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)$, $e_{j,l}^{(i)}$ stands for $(e_{j,l}^{(i)}, 0)$ or $(0, e_{l,j}^{(i)})$. When A does not have a unit element, w_t can belong also to the spaces $e_{1,1}^{(0)} J e_{k,k}^{(l)}$, $e_{j,j}^{(i)} J e_{1,1}^{(0)}$, $e_{1,1}^{(0)} J e_{1,1}^{(0)}$, where $e_{1,1}^{(0)} = e_0$.

Let $F[\xi_{i,j}^{(l)}, \eta_{k,j} \mid 1 \leq j \leq n, 1 \leq i \leq m_l, 1 \leq k \leq p, 1 \leq l \leq q]$ be the polynomial ring in the variables $\xi_{i,j}^{(l)}, \eta_{k,j}$. Then for any $j \geq 1$ we define

$$U_j^{(l)} = \sum_{i=1}^{m_l} \xi_{i,j}^{(l)} a_i^{(l)}, \quad (U_j^{(l)})^\varphi = \sum_{i=1}^{m_l} \xi_{i,j}^{(l)} (a_i^{(l)})^\varphi, \quad W_j = \sum_{i=1}^{p} \eta_{i,j} w_i, \quad W_j^\varphi = \sum_{i=1}^{p} \eta_{i,j} w_i^\varphi$$

that are elements of $A \otimes F[\xi_{i,j}^{(l)}, \eta_{k,j} \mid 1 \leq j \leq n, 1 \leq i \leq m_l, 1 \leq k \leq p, 1 \leq l \leq q]$.

Then

$$U_j = \sum_{l=1}^{q} U_j^{(l)} + W_j, \quad U_j^\varphi = \sum_{l=1}^{q} (U_j^{(l)})^\varphi + W_j^\varphi, \tag{9}$$

are called generic elements of A. Their basic property is that a multilinear φ-polynomial is a φ-identity of A if and only if it vanishes on the generic elements U_j (see [24, Theorem 1.4.4]).

Hence if we define

$$V_n = \operatorname{span}\{U_{\sigma(1)}^{\varepsilon_1} \cdots U_{\sigma(n)}^{\varepsilon_n} \mid \sigma \in S_n, \ \varepsilon_i = 1 \text{ or } \varphi\},$$

we have that

$$c_n^{sup}(A) = \dim V_n. \tag{10}$$

Also, keeping in mind the equalities in (9) by linearity we can write $c_n^{sup}(A) = \dim V_n \leq \dim Y_n$ where

$$Y_n = \operatorname{span}\{T_{\sigma(1)} \cdots T_{\sigma(n)} \mid T_j = U_j^{(l)} \text{ or } (U_j^{(l)})^\varphi \text{ or } \eta_{i,j} w_i \text{ or } \eta_{i,j} w_i^\varphi, \text{ for some } 1 \leq l \leq q, 1 \leq i \leq p\}.$$

Notice that, since $J^{s+1} = 0$, in every non-zero monomial of Y_n at most s elements w_i or w_i^φ can appear. Hence for $u \leq s$, let $(w_{t_1}^{\varepsilon_1}, \ldots, w_{t_u}^{\varepsilon_u}) = \vec{w}$ denote a fixed sequence of elements of the basis of J and their images, and consider the subspace

$$Y_n^{\vec{w}} = \operatorname{span}\{M_0 \eta_{t_1,j_1} w_{t_1}^{\varepsilon_1} M_1 \eta_{t_2,j_2} w_{t_2}^{\varepsilon_2} \cdots M_{u-1} \eta_{t_u,j_u} w_{t_u}^{\varepsilon_u} M_u \mid \{j_1, \ldots j_u\} \subseteq \{1, \ldots, n\},$$
$$M_0, M_1, \ldots, M_u \ \text{ multilinear monomials in } U_k^{(l)} \text{ or } (U_k^{(l)})^\varphi, \ k \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_u\}, \ 1 \leq l \leq q\}.$$

Thus

$$c_n^{sup}(A) \leq \dim Y_n \leq \sum_{\vec{w}} \dim Y_n^{\vec{w}} \tag{11}$$

where the sum runs over all sequences $\vec{w} = (w_{t_1}^{\varepsilon_1}, \ldots, w_{t_u}^{\varepsilon_u})$ with $0 \leq u \leq s$.

Since

$$M_0 \eta_{t_1,j_1} w_{t_1}^{\varepsilon_1} M_1 \eta_{t_2,j_2} w_{t_2}^{\varepsilon_2} \cdots M_{u-1} \eta_{t_u,j_u} w_{t_u}^{\varepsilon_u} M_u = \prod_{i=1}^{u} \eta_{t_i,j_i} M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u,$$

if we define

$$\bar{Y}_n^{\vec{w}} = \operatorname{span}\{M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u \mid M_0, M_1, \ldots, M_u \ \text{ monomials in } U_k^{(l)} \text{ or } (U_k^{(l)})^\varphi,$$
$$k \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_u\}, \ 1 \leq l \leq q\}. \tag{12}$$

we have that

$$\dim Y_n^{\vec{w}} \leq u! \binom{n}{u} \dim \bar{Y}_n^{\vec{w}}, \tag{13}$$

where $\binom{n}{u}$ counts the distinct subsets $\{j_1, \ldots, j_u\}$ of $\{1, \ldots, , n\}$ and $u!$ counts the number of permutations of $j_1, \ldots, j_u$.

Putting together (11) and (13) we get for $n >> s$,

$$c_n^{sup}(A) \leq Cs! \binom{n}{s} \dim \bar{Y}_n^{\vec{w}}, \tag{14}$$

where C counts the number of sequences (of length $\leq s$) of the basis of J and their star, and $\vec{w}$ is a sequence such that $\dim \bar{Y}_n^{\vec{w}}$ is maximal.

For any $\vec{w} = (w_{t_1}^{\varepsilon_1}, \dots, w_{t_u}^{\varepsilon_u})$ and fixed $n_1, \dots, n_q$ such that $n_1 + \cdots + n_q = n - u$, define

$$Z_{n_1,\dots,n_q} = \mathrm{span}\{M = M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u \in \bar{Y}_n^{\vec{w}} \mid n_i = \text{number of } U_j^{(i)} \text{ or } (U_j^{(i)})^{\varphi} \text{ appearing in } M,\ 1 \le i \le q\}.$$

Clearly

$$\dim \bar{Y}_n^{\vec{w}} \le \sum_{n_1+\cdots+n_q=n-u} \binom{n-u}{n_1,\dots,n_q} \dim Z_{n_1,\dots,n_q} \tag{15}$$

and an upper bound of $\dim Z_{n_1,\dots,n_q}$. is given in the following.

Lemma 4.2. ([17, Lemma 8.1]) *There are constants $b_1, \dots, b_q, \sum_{i=1}^q b_i = u + 1 \le s + 1$, such that*

$$\dim Z_{n_1,\dots,n_q} \le C c_{k_1}^{sup}(A_1) \cdots c_{k_q}^{sup}(A_q),$$

for some constant C, where $k_i = n_i + b_i + 1$, for $i = 1, \dots, q$.

An upper bound of $c_n^{sup}(A)$ is given in the following theorem.

Theorem 4.2. *Let $A = \bar{A} + J$ be a finite dimensional superalgebra over an algebraically closed field F of characteristic zero. Let $\bar{A} = A_1 \oplus \cdots \oplus A_q$ be a direct sum of simple superalgebras and let $J^s \neq 0, J^{s+1} = 0$. Then*

$$c_n^{sup}(A) \le C n^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q) + s} (\dim \bar{A})^n.$$

for some constant C.

Proof. By (14) and (15) we have that

$$c_n^{sup}(A) \le C' \binom{n}{s} \sum_{n_1+\cdots+n_q=n-u} \binom{n-u}{n_1,\dots,n_q} \dim Z_{n_1,\dots,n_q},$$

for some constant C'. Then by Lemma 4.2 and Lemma 4.1 we get that

$$c_n^{sup}(A) \le C' \binom{n}{s} \sum_{n_1+\cdots+n_q=n-u} \binom{n-u}{n_1,\dots,n_q} c_{k_1}^{sup}(A_1) \cdots c_{k_q}^{sup}(A_q),$$

where $k_i = n_i + b_i + 1$, $\sum_{i=1}^q b_i = u + 1$. Hence

$$c_n^{sup}(A) \le C' \binom{n}{s} \sum_{k_1+\cdots+k_q=k} \binom{k}{k_1,\dots,k_q} c_{k_1}^{sup}(A_1) \cdots c_{k_q}^{sup}(A_q) \le C' \binom{n}{s} k^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q)} (\dim \bar{A})^k$$

$$\le C' \binom{n}{s} (n+K)^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q)} (\dim \bar{A})^{n+K} \le C'' n^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q) + s} (\dim \bar{A})^n,$$

for some constant C'', K. □

5. φ-fundamental algebras

We start with the following construction. Let $A = \bar{A} + J$ be a finite dimensional superalgebra over an algebraically closed field F, $J^s \neq 0$, $J^{s+1} = 0$ and let $n = \dim J$. Then define

$$A' = \bar{A} * F\langle x_1, \dots, x_n, \varphi\rangle,$$

the free product of $\bar{A}$ and the free algebra $F\langle x_1, \dots, x_n, \varphi\rangle$. If I_1 is the φ-ideal generated by $\{f(A') \mid f \in \mathrm{Id}_2(A)\}$, then since $f(\bar{A}) = 0$, for $f \in \mathrm{Id}_2(A)$, we have that $I_1 \subseteq I$, the φ-ideal of A' generated by $x_1, \dots, x_n$.

We define

$$\mathcal{A}_s = A'/(I^{s+1} + I_1),$$

a finite dimensional algebra with $\mathrm{Id}_2(\mathcal{A}_s) = \mathrm{Id}_2(A)$. Also, if $I' = I/(I^{s+1} + I_1)$, then $\mathcal{A}_s \cong \bar{A} + I'$, $(I')^s \neq 0$, $(I')^{s+1} = 0$ and $Ind_{t,s}(\mathcal{A}_s) = (\dim \bar{A}, s) = Ind_{t,s}(A)$.

Then we define

$$\mathcal{B}_0 = \mathcal{A}_s/(I')^s.$$

Hence $\mathrm{Id}_2(A) = \mathrm{Id}_2(\mathcal{A}_s) \subseteq \mathrm{Id}_2(\mathcal{B}_0)$, and $Ind_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s-1)$.

Let $\bar{A} = A_1 \oplus \cdots \oplus A_q$, where the A_i's are simple superalgebras and, for every $1 \le i \le q$, define

$$B_i = A_1 \oplus \cdots \hat{A}_i \cdots \oplus A_q + J,$$

where the symbol $\hat{A}_i$ means that the algebra A_i is omitted in the direct sum.

Definition 5.1. *The superalgebra A is φ-fundamental if either A is a simple superalgebra or $s > 0$ and*

$$Id_2(A) \subsetneq \cap_{i=1}^{q} Id_2(B_i) \cap Id_2(\mathcal{B}_0).$$

In this case a multilinear φ-polynomial f is φ-fundamental if $f \in \cap_{i=1}^{q} Id_2(B_i) \cap Id_2(\mathcal{B}_0)$ and $f \notin Id_2(A)$.

We remark that in case $s > 0$, the algebras B_i have a lower (t,s)-index. In fact the first index is lower. Also the algebra $\mathcal{B}_0$ has a lower (t,s)-index, since $Ind_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s-1)$.

The following proposition is proved by induction on the (t,s)-index of A as in [17, Proposition 6.1].

Proposition 5.1. *Every finite dimensional superalgebra satisfies the same φ-identities as a finite direct sum of φ-fundamental algebras.*

Lemma 5.1. *Let f be a multilinear φ-polynomial. Then f is φ-fundamental for A if and only if in every non-zero elementary evaluation all the simple components participate and also precisely s variables are evaluated in J.*

Proof. Suppose first that f is φ-fundamental and let $f(a_1, \ldots, a_m) \neq 0$ be an elementary evaluation in A.

If no variable is evaluated in a simple component, say A_j, then f is actually evaluated in B_j. Hence, since f is φ-fundamental, $f \in \mathrm{Id}_2(B_j)$ and $f(a_1, \ldots, a_m) = 0$, a contradiction.

Also since $J^{s+1} = 0$, at most s among the a_i's belong to J and let $a_1, \ldots, a_k \in J$, $a_{k+1}, \ldots, a_m \in \bar{A}$.

By the universal property of $\mathcal{A}_s$, there is a φ-homomorphism $\psi : \mathcal{A}_s \to A$ which is the identity on $\bar{A}$ and maps x_i to a_i, $1 \le i \le k$. Since $0 \neq f(a_1, \ldots, a_m) = \psi(f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_m))$ if $k \le s-1$ we get that $f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_m)$ is still non-zero in the projection $\mathcal{A}_s \to \mathcal{A}_s/(I')^s = \mathcal{B}_0$. Since by hypothesis f is a φ-identity of $\mathcal{B}_0$, we get a contradiction.

Conversely, suppose that in every non-zero elementary evaluation of f all the simple components participate and also precisely s variables are evaluated in J.

Then since for $1 \le i \le q$, the algebra B_i is missing the simple component A_i, we clearly get that $f \in \cap_{i=1}^{q} \mathrm{Id}_2(B_i)$. Moreover in the above φ-homomorphism $\psi : \mathcal{A}_s \to A$, $f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_m)$ maps to $f(a_1, \ldots, a_m) \neq 0$ and it becomes zero in the projection $\mathcal{A}_s \to \mathcal{A}_s/(I')^s = \mathcal{B}_0$ since s variables are evaluated in the radical. Hence $f \in \mathrm{Id}_2(B_0)$. ☐

The proof of the following theorem can be read from [17, Theorem 6.1] recalling that φ is an automorphism versus an antiautomorphism.

Theorem 5.1. *Let A be a finite dimensional superalgebra. Then A is φ-fundamental if and only if $Ind_K^{\varphi}(A) = Ind_{t,s}(A)$.*

Write $A = \bar{A} + J$, $\bar{A} = A_1 \oplus \cdots \oplus A_q$ and let $d = \dim \bar{A}$. Notice that as a consequence of the above theorem we get that for any $\mu \ge 1$, there exists a φ-fundamental polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_s, Y) \notin \mathrm{Id}_2(A)$ with $|X_j| = d$, $1 \le j \le \mu$, $|Z_i| = d+1$, $1 \le i \le s$, such that f is μ-fold d-alternating and s-fold $(d+1)$-alternating.

In the following theorem we give a characterization of φ-fundamental algebras through the representation theory of the symmetric group.

Theorem 5.2. *Let A be a finite dimensional superalgebra with $Ind_{t,s}(A) = (t,s)$ and let $\chi_n^{sup}(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be its nth supercocharacter. Then A is φ-fundamental if and only if for all n large enough, there is a partition λ of n having exactly s boxes under the first t rows such that $m_\lambda \neq 0$.*

Proof. Suppose first that for all n large enough there is a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_{t+1} + \cdots + \lambda_r = s$ such that $m_\lambda \neq 0$.

If $s = 0$, A is semisimple and by hypothesis there is an irreducible S_n-module $M \not\subseteq \mathrm{Id}_2(A)$ generated by a φ-polynomial $f_{T_\lambda} \in P_n^{sup}$ alternating in t variables. Since $f_{T_\lambda} \notin \mathrm{Id}_2(A)$ is multilinear, there is a non-zero evaluation of the variables on a basis of A and being $t = \dim A$, it is clear that A must be simple, hence φ-fundamental.

If $s > 0$, there exists an irreducible S_n-module $M \not\subseteq \mathrm{Id}_2(A)$ generated by a φ-polynomial $f_{T_\lambda}(x_1, \dots, x_n) \in P_n^{sup}$, alternating on λ_{t+1} sets of variables each of size greater or equal than $t+1$.

We shall show that f_{T_λ} is φ-fundamental, i.e., $f_{T_\lambda} \in \cap_{i=1}^q \mathrm{Id}_2(B_i) \cap \mathrm{Id}_2(\mathcal{B}_0)$. Now, since $Ind_{t,s}(\mathcal{B}_0) = (t, s-1)$, then $f_{T_\lambda} \in \mathrm{Id}_2(\mathcal{B}_0)$. In fact, f_{T_λ} is alternating on λ_{t+1} sets of variables each of size greater or equal than $t+1$. Hence if we want to get a non-zero evaluation, in each of the alternating sets we have to substitute at most t linearly independent elements from the maximal semisimple subalgebra of $\mathcal{B}_0$. Hence we need to evaluate the remaining variables in the Jacobson radical J' of $\mathcal{B}_0$. Now, since $\lambda_{t+1} + \cdots + \lambda_r = s$ and $J'^s = 0$ we get that any evaluation of f_{T_λ} into $\mathcal{B}_0$ gives zero.

Since for any $i = 1, \dots, q$, we have that $\dim \bar{B}_i < t$, if we evaluate f_{T_λ} in B_i, in order to get a non-zero value we have to evaluate more than s variables in J. Thus $f_{T_\lambda} \in \mathrm{Id}_2(B_i)$. We have proved that $f_{T_\lambda} \in \cap_{i=1}^q \mathrm{Id}_2(B_i) \cap \mathrm{Id}_2(\mathcal{B}_0)$ and, so, A is φ-fundamental.

Conversely, suppose that A is φ-fundamental.

If $s = 0$ A is simple and by Proposition 3.1 there exists a multilinear φ-polynomial $f(x_1, \dots, x_t, Y) \notin \mathrm{Id}_2(A)$ alternating in $x_1, \dots, x_t$, $|Y| < \infty$.

Now, for any $n > \deg f$, by multiplying on the right by new variables we may assume that $f \in P_n^{sup}$. By the permutation action of the symmetric group S_t on $x_1, \dots, x_t$, we can regard P_n^{sup} and, so, $P_n^{sup}(A)$ as an S_t-module. Since $f \notin \mathrm{Id}_2(A)$, for any tableau T of shape (1^t) we have that $f_T = e_T f \notin \mathrm{Id}_2(A)$. Thus the character $\chi_{(1^t)}$ appears with non-zero multiplicity in the decomposition of the S_t-character of $P_n^{sup}(A)$. Now, by the branching rule ([24, Theorem 2.3.1]) the induced character $\chi_{(1^t)} \uparrow S_n$ has an irreducible component $\chi_{\lambda'}$ with non-zero multiplicity and $ht(\lambda') = t$ since $\dim A = t$.

Suppose now that $s > 0$ and consider a Kemer φ-polynomial $f \in P_n^{sup}$. Since $Ind_K^\varphi(A) = Ind_{t,s}(A)$ f can be written as

$$f = f(X_1, \dots, X_s, z_1, \dots, z_s, Y) \notin \mathrm{Id}_2(A),$$

where $|X_1| = \cdots = |X_s| = t$ and f is alternating on each set $\{X_i, z_i\}$, for $i = 1, \dots, s$.

We regard P_n^{sup} as an S_k-module, where $k = (t+1)s$, by letting S_k act on the set $\{X_1, \dots, X_s, z_1, \dots, z_s\}$. Then the quotient space $P_n^{sup}(A)$ has also a structure of S_k-module.

Since $f \notin \mathrm{Id}_2(A)$, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash k$ and a tableau T_λ such that $f_{T_\lambda} = C^-_{T_\lambda} e_{T_\lambda} f \notin \mathrm{Id}_2(A)$. Let χ be the S_k-character of $P_n^{sup}(A)$. Then, if we decompose $\chi = \sum_{\mu \vdash k} m'_\mu \chi'_\mu$, we have that $m'_\lambda \neq 0$. Since f_{T_λ} is not a φ-identity for A, we get that also $f' = R^+_{T_\lambda} f_{T_\lambda} \notin \mathrm{Id}_2(A)$. Notice that f' is symmetric in λ_1 variables and f is alternating on s disjoint sets of variables; hence $m'_\lambda \neq 0$ implies that $\lambda_1 \leq s$ and, also, $\lambda_1 + \cdots + \lambda_t \leq ts$.

If $\lambda_1 + \cdots + \lambda_t < ts$, then $\lambda_{t+1} + \cdots + \lambda_r \geq s+1$ and as above we get $m'_\lambda = 0$, a contradiction. Thus $\lambda_1 + \cdots + \lambda_t = ts$ and $\lambda_{t+1} + \cdots + \lambda_r = s$. Now consider the induced character $\chi'_\lambda \uparrow S_n$. By the branching rule we have $\chi'_\lambda \uparrow S_n = \sum_{\mu \vdash n} m_\mu \chi_\mu$, a decomposition into irreducible S_n-characters. Since $m'_\lambda \neq 0$, there exists at least one χ_μ which appears with non-zero multiplicity. Then as in the first part of the proof we get that $\mu_{t+1} + \cdots + \mu_r + \cdots = s$ and we are done. $\square$

Next we recall the definition of minimal superalgebra introduced in [23]. Let $A = \bar{A} + J$ be a finite dimensional superalgebra over an algebraically closed field F, $\bar{A} = A_1 \oplus \cdots \oplus A_q$. Then A is minimal if either A is simple or there exist homogeneous elements $w_{12}, \dots, w_{q-1,q} \in J$ and minimal homogeneous idempotents $e_1 \in A_1, \dots, e_q \in A_q$ such that $e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}$ for $1 \leq i \leq q-1$, and $w_{12} \cdots w_{q-1,q} \neq 0$. Moreover $w_{12}, \dots, w_{q-1,q}$ generate J as a two-sided ideal.

The minimal superalgebras are strictly related to the minimal varieties of given exponent. Recall the if $\mathcal{V}$ is a variety of algebras, then we say that $\mathcal{V}$ is minimal of exponent $d \geq 2$ if $exp(\mathcal{V}) = d$ and $exp(\mathcal{U}) < d$ for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$. Here the exponent of a variety is defined as the exponent of a generating algebra.

In [23] it was proved that a variety $\mathcal{V}$ is minimal of exponent d if and only if $\mathcal{V}$ is generated by the Grassmann envelope of a minimal superalgebra. As a consequence, if $\mathcal{V}$ is of finite basic rank i.e., it is generated by a finite dimensional algebra, then it can be generated by an algebra of upper-block triangular matrices $UT(A_1, \dots, A_q)$ with $\dim(A_1 \oplus \cdots \oplus A_q) = d$.

The problem of classifying the minimal supervarieties of given superexponent is still open in general, but it has been solved in [11] (see also [10], [12]) in the setting of finite dimensional superalgebras. In particular they proved that a variety of superalgebras of finite basic rank is minimal of given superexponent d if and only if it is generated by a suitable minimal superalgebra.

Theorem 5.3. *Every minimal superalgebra is φ-fundamental.*

Proof. Let $A = \bar{A} + J$ be a minimal superalgebra with $\bar{A} = A_1 \oplus \cdots \oplus A_q$ and J is generated by $w_{12}, \dots, w_{q-1,q}$. Since $w_{12} \cdots w_{q-1,q} \neq 0$, then $J^{q-1} \neq 0$ and $J^q = 0$. Also, since $A_1 w_{12} A_2 \cdots A_{q-1} w_{q-1,q} A_q \neq 0$, the superalgebra A is reduced. Hence $Ind_{t,s}(A) = (\dim(A_1 + \cdots + A_q), q-1) = (exp^{sup}(A), q-1)$. Since $Ind_K^{\varphi}(A) \leq Ind_{t,s}(A)$, by Lemma 3.1 and Remark 3.1 we get that $Ind_K^{\varphi}(A) = Ind_{t,s}(A)$ and by Theorem 5.1 A is φ-fundamental. $\square$

It is worth mentioning that if $A = \bar{A} + J$ is a minimal superalgebra and $\bar{A}$ contains a simple superalgebra of the type $M_d(F) \oplus M_d(F)$, then A is φ-fundamental but not fundamental. This is easily seen by noticing that the algebra A is not reduced in the ordinary (non super) sense.

6. Asymptotics for φ-fundamental algebras

Next we shall find a lower bound of $c_n^{sup}(A)$ in case A is a φ-fundamental algebra. The final result is the following.

Theorem 6.1. *Let $A = \bar{A} + J$ be a φ-fundamental algebra over an algebraically closed field F of characteristic zero and let $\bar{A} = A_1 \oplus \cdots \oplus A_q$ be a direct sum of simple superalgebras, $J^{s+1} = 0, s \geq 0$. Then*

$$C_1 n^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q) + s} (\dim \bar{A})^n \leq c_n^{sup}(A) \leq C_2 n^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q) + s} (\dim \bar{A})^n,$$

for some constants $C_1 > 0, C_2$. It follows that

$$\lim_{n \to \infty} \log_n \frac{c_n^{sup}(A)}{exp^{sup}(A)^n} = -\frac{1}{2}(\dim \bar{A})^{(0)} - q) + s.$$

Proof. Since the upper bound was computed in Theorem 4.2 we need only to compute the lower bound.

If A is a simple superalgebra the result follows from Theorem 4.1. Hence we may assume that $s > 0$ and let (d, s) be the (t, s)-index of A.

Since A is φ-fundamental, by Theorem 5.1 there exists a φ-fundamental polynomial $f(Z_1, \dots, Z_s, y_1, \dots, y_q, X) \notin \mathrm{Id}_2(A)$ multilinear and alternating in s sets of variables $Z_i = \{x_{i,1}, \dots, x_{i,d+1}\}$ of order $d+1$, where $d = \dim \bar{A}$, linear in the variables $y_1, \dots, y_q$ and eventually depending on some extra variables X.

By Lemma 5.1 we may assume that in a non-zero elementary evaluation η the variable y_i is evaluated in A_i, $1 \leq i \leq q$, and the variable $z_i := x_{i,1} \in Z_i$, $1 \leq i \leq s$, is evaluated in J.

Let y_i be evaluated in $e^{(i)}_{h_i,k_i}$, $1 \leq i \leq q$. Recall that if $A_i = M_{d_i}(F) \oplus M_{d_i}(F)$, $e^{(i)}_{h_i,k_i}$ stands for $(e^{(i)}_{h_i,k_i}, 0)$ or $(0, e^{(i)}_{k_i,h_i})$.

Let $\tilde{f}$ be the polynomial obtained from f by replacing y_i by $u_i v_i w_i y_i$, where u_i, v_i, w_i are new variables different from the ones appearing in f. Extend the evaluation η to an evaluation η' of $\tilde{f}$ by evaluating the new variables u_i, v_i, w_i in $e^{(i)}_{h_i,h_i}$, $1 \leq i \leq q$. Clearly $\eta'(\tilde{f}) \neq 0$.

We partially evaluate $\tilde{f}$ by substituting the variables $u_i, w_i, y_i, x_{j,k}$, $k > 1$, $1 \leq j \leq s$, as in η' and let $g(v_1, \dots, v_q, z_1, \dots, z_s)$ be the generalized φ-polynomial so obtained. Recall that if we evaluate some z_j in $\bar{A}$ or some v_i in J then g vanishes.

Write $n = \sum_{i=1}^q n_i$, and set $m = n + n_{q+1}$, where $n_{q+1} = s$. Then the set $\{1, \dots, m\}$ is partitioned in $q+1$ subsets $B_1, \dots, B_{q+1}$, with

$$B_i = \{b_{i-1} + 1, b_{i-1} + 2, \dots, b_i\}, \ |B_i| = n_i,$$

where $b_0 = 0, b_i := n_1 + \ldots + n_i, \ 1 \leq i \leq q+1$, and we set $\mathcal{P} = \{B_1, \dots, B_{q+1}\}$.

Fix a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_i = 1$ or φ, and consider the set of variables $T = \{x'_1, \dots, x'_n, x_{n+1}, \dots, x_m\}$, where $x'_i = x_i^{\varepsilon_i}$, $1 \le i \le n$. Let $M_1^\varepsilon, \dots, M_q^\varepsilon$ be the monomials in some of the variables of T defined by

$$M_i^\varepsilon = M_i^\varepsilon(x'_{b_{i-1}+1}, x'_{b_{i-1}+2}, \dots, x'_{b_i}) = x'_{b_{i-1}+1} x'_{b_{i-1}+2} \cdots x'_{b_i}.$$

Now if in g we replace v_i by the monomial M_i^ε, $1 \le i \le q$, and z_j by the variable $x_{n+j}, 1 \le j \le s$, then we get a generalized φ-polynomial:

$$g_{\mathcal{P}}^\varepsilon(x'_1, \dots, x'_n, x_{n+1}, \dots, x_m) := g(M_1^\varepsilon, \dots, M_q^\varepsilon, x_{n+1}, \dots, x_m).$$

Consider m generic elements $U_j = \sum_{i=1}^q U_j^{(i)} + W_j,\ 1 \le j \le m$, as in (9).

For $\sigma \in S_m$, set

$$M_i^{\varepsilon,\sigma} = M_i^\varepsilon(U^{(i)}_{\sigma(b_{i-1}+1)}, U^{(i)}_{\sigma(b_{i-1}+2)}, \dots, U^{(i)}_{\sigma(b_i)}) = (U^{(i)}_{\sigma(b_{i-1}+1)})^{\varepsilon_{b_{i-1}+1}} (U^{(i)}_{\sigma(b_{i-1}+2)})^{\varepsilon_{b_{i-1}+2}} \cdots (U^{(i)}_{\sigma(b_i)})^{\varepsilon_{b_i}}.$$

We have

$$g_{\mathcal{P}}^\varepsilon(U_{\sigma(1)}, \dots, U_{\sigma(m)}) = \prod_{i=1}^q (M_i^{\varepsilon,\sigma})_{h_i,h_i} G(W_{\sigma(n+1)}, \dots, W_{\sigma(m)}), \tag{16}$$

where

$$G(W_{\sigma(n+1)}, \dots, W_{\sigma(m)}) = g(e^{(1)}_{h_1,h_1}, \dots, e^{(q)}_{h_q,h_q}, W_{\sigma(n+1)} \dots, W_{\sigma(m)}).$$

Now by [17, Lemma 9.2] if $\Sigma = \{\sigma \in S_m \mid \sigma(B_i) = B_i, 1 \le i \le q \text{ and } \sigma(n+j) = n+j, 1 \le j \le s\}$ and

$$G_{\mathcal{P}} = span\{g_{\mathcal{P}}^\varepsilon(U_{\sigma(1)}, \dots, U_{\sigma(m)}),\ \sigma \in \Sigma,\ \varepsilon \in \{1, \varphi\}^n\},$$

we have that

$$\dim G_{\mathcal{P}} = c_{n_1}^{sup}(A_1) \cdots c_{n_q}^{sup}(A_q).$$

For $\mathcal{C} = \{C_1, \dots, C_{q+1}\}$ a multipartition of m such that each C_j has n_j elements, define

$$\mathcal{M}_{\mathcal{C}} = span\{g_{\mathcal{P}}^\varepsilon(U_{\sigma(1)}, \dots, U_{\sigma(m)}) \mid \varepsilon \in \{1, \varphi\}^n,\ \sigma \in S_m \text{ and } \sigma(B_i) = C_i, 1 \le i \le q\}.$$

Let $\mathcal{C}_{\mathbf{n}}$ be the set of such multipartitions of m. Clearly $|\mathcal{C}_{\mathbf{n}}| = \frac{m!}{\prod_{i=1}^{q+1} n_i!} = \binom{m}{n_1, \dots, n_{q+1}}$.

The sum $\sum_{\mathcal{C} \in \mathcal{C}_{\mathbf{n}}} \mathcal{M}_{\mathcal{C}}$ is direct since the spaces $\mathcal{M}_{\mathcal{C}}$ are made of homogeneous elements of different multidegree. Such a sum is contained in the space obtained by specializations of the span of multilinear products of $n+t$ generic elements of A, where t is independent of n and is equal to s plus the number of variables of $\tilde{f}$ which are evaluated in A in the partial evaluation. Hence

$$\sum_{\substack{\mathbf{n}=(n_1,\dots,n_q) \\ \sum_i n_i = n}} \sum_{\mathcal{C} \in \mathcal{C}_{\mathbf{n}}} \dim(\mathcal{M}_{\mathcal{C}}) \le c_{n+t}^{sup}(A) \tag{17}$$

with t some fixed number. By symmetry we have $\dim \mathcal{M}_{\mathcal{C}} = \dim \mathcal{M}_{\mathcal{P}}$ for all $\mathcal{C} \in \mathcal{C}_{\mathbf{n}}$. Now, since

$$\dim \mathcal{M}_{\mathcal{P}} \ge \dim G_{\mathcal{P}} = c_{n_1}^{sup}(A_1) \cdots c_{n_q}^{sup}(A_q)$$

and $n_{q+1} = s$, we have

$$\sum_{\mathcal{C} \in \mathcal{C}_{\mathbf{n}}} \dim \mathcal{M}_{\mathcal{C}} = \binom{m}{n_1, \dots, n_{q+1}} \dim \mathcal{M}_{\mathcal{P}} = \binom{m}{s} \binom{n}{n_1, \dots, n_q} \dim \mathcal{M}_{\mathcal{P}}$$

$$\ge C_1 n^s \binom{n}{n_1, \dots, n_q} c_{n_1}^{sup}(A_1) \cdots c_{n_q}^{sup}(A_q),$$

for some constant $C_1 > 0$. Hence, by (17) and Lemma 4.1 we get

$$c_{n+t}^{sup}(A) \ge C_1 n^s \sum_{\substack{\mathbf{n}=(n_1,\dots,n_q) \\ \sum_i n_i = n}} \binom{n}{n_1, \dots, n_q} c_{n_1}^{sup}(A_1) \cdots c_{n_q}^{sup}(A_q) \tag{18}$$

$$\ge C_2 n^{-\frac{1}{2}(\dim \bar{A}^{(0)} - q) + s} (\dim \bar{A})^n,$$

for some constant $C_2 > 0$. Hence formula (18) yields the conclusion

$$c_n^{sup}(A) \geq C_2 n^{-\frac{1}{2}(\dim \bar{A}^{(0)}-q)+s}(\dim \bar{A})^n, \tag{19}$$

for some constant C_2.

Since the superalgebra A is φ-fundamental, it has a φ-fundamental polynomial f. By Lemma 5.1 in every non-zero elementary evaluation of f all the simple components participate and this says that the superalgebra A is reduced. Hence $exp^{sup}(A) = \dim \bar{A}$ and the proof of the theorem is completed. □

Now let A be a finitely generated superalgebra satisfying a polynomial identity. By [27] A has the same superidentities as a finite dimensional superalgebra B. By Proposition 5.1, B has the same superidentities as a finite direct sum of φ-fundamental algebras $D_1 \oplus \cdots \oplus D_m$. Since $c_n^{sup}(B) \simeq c_n^{sup}(D_l)$ for a suitable D_l, we get the following.

Theorem 6.2. *Let A be a finitely generated superalgebra over a field F of characteristic zero. If A satisfies a polynomial identity then*

$$C_1 n^t exp^{sup}(A)^n \leq c_n^{sup}(A) \leq C_2 n^t exp^{sup}(A)^n,$$

where $t \in \frac{1}{2}\mathbb{Z}$, for some constants $C_1 > 0, C_2$. Hence $\lim_{n\to\infty} \log_n \frac{c_n^{sup}(A)}{exp^{sup}(A)^n}$ exists and is a half integer.

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