

THE ACM PROPERTY FOR UNIONS OF LINES IN $\mathbb{P}^1 \times \mathbb{P}^2$

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ABSTRACT. This paper examines the Arithmetically Cohen-Macaulay (ACM) property for certain codimension 2 varieties in $\mathbb{P}^1 \times \mathbb{P}^2$ called *sets of lines* in $\mathbb{P}^1 \times \mathbb{P}^2$ (not necessarily reduced). We discuss some obstacles to finding a general characterization. We then consider certain classes of such curves, and we address two questions. First, when are they themselves ACM? Second, in a non-ACM reduced configuration, is it possible to replace one component of a primary (prime) decomposition by a suitable power (i.e. to “fatten” one line) to make the resulting scheme ACM? Finally, for our classes of such curves, we characterize the locally Cohen-Macaulay property in combinatorial terms by introducing the definition of a *fully v -connected* configuration. We apply some of our results to give analogous ACM results for sets of lines in \mathbb{P}^3 .

1. INTRODUCTION

It is still an open problem to determine a geometric characterization of the arithmetically Cohen-Macaulay (ACM) property for varieties in a multiprojective space. While this problem has strong connections to the analogous problem for varieties in a projective space, there are also striking differences. To give just two simple illustrations, any finite set of points in a projective space is automatically ACM, while this is not true in any multiprojective space. Indeed, a whole book has been devoted to this topic [14] just for the case of $\mathbb{P}^1 \times \mathbb{P}^1$. Secondly, the structure of a multihomogeneous ideal differs from that of a homogeneous ideal in important ways, and this contributes to the difficulty.

In both the projective and the multiprojective settings, the problem can have a strong combinatorial component in addition to geometric and algebraic ones. One specific problem that has been studied in the projective setting in many different ways is the question of when a union of lines is ACM. This is still wide open in projective spaces, in general. In this paper we begin the study of this problem in a specific multiprojective space, namely $\mathbb{P}^1 \times \mathbb{P}^2$.

Included in this problem (in either setting) is the question of what can be added to a non-ACM variety to make it become ACM, hopefully adding as little as possible. We propose a new variation of this question (and solve it in our special setting): given a reduced configuration of lines that is not ACM, can one “fatten” one of the components and make the new scheme ACM? Here, by “fatten” we mean that in a primary decomposition we replace a prime ideal \mathfrak{p} , corresponding to one of the lines, by a power \mathfrak{p}^k for suitable k . (Notice that \mathfrak{p} is a complete intersection, so \mathfrak{p}^k is unmixed.) After solving this problem in the situation of this paper, we give as a corollary the analogous result for unions of lines in \mathbb{P}^3 . (See Theorem 4.15 for $\mathbb{P}^1 \times \mathbb{P}^2$ and Corollary 4.16 for \mathbb{P}^3 .) We also explore the property of being locally Cohen-Macaulay, giving a characterization for our curves in $\mathbb{P}^1 \times \mathbb{P}^2$.

Many papers in the literature have studied the ACM property for different kinds of subvarieties of multiprojective spaces, especially for sets of points. Despite the fact that in $\mathbb{P}^1 \times \mathbb{P}^1$ there are

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several characterizations of the ACM property (again, see [14] for a detailed discussion of the topic) only a few other results are known in general. In particular, a characterization was given in [7] of the ACM property for sets of points in $(\mathbb{P}^1)^r = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, and in [9] the authors described, under certain conditions, the ACM property for sets of points in $\mathbb{P}^m \times \mathbb{P}^n$ and, with more details, in $\mathbb{P}^1 \times \mathbb{P}^n$. See also [15] for some other results in this direction.

A crucial difference in the study of the ACM property of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and a set of points in any other multiprojective space is given by the codimension. A set of points has codimension two in $\mathbb{P}^1 \times \mathbb{P}^1$, and strictly larger than two in a different multiprojective space. So, it is interesting to approach the study of the ACM property by looking at the codimension two varieties more generally. The paper [8] investigates the ACM property for 2-codimensional varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Such varieties have a different nature and meaning from sets of points in $(\mathbb{P}^1)^r$, $r \geq 2$, but the characterization of the ACM property deeply uses a common fact. All of them are defined by ideals generated by particular products of linear forms. In this case, this peculiarity makes the description of the ACM property merely combinatorial.

The vast nature of the problem draws in many standard tools and techniques from the homogeneous setting. These include, just to cite some of them, hyperplane sections, basic double G-linkage, liaison addition, and liaison. On the other hand, the study of varieties in multiprojective spaces plays an important role in several branches of mathematics, and it finds an application in different contexts; these include the study of monomial ideals (see for instance [2, 20]), scrolls ([6]), symbolic powers ([5, 12, 13]), tensor analysis ([4]) and virtual resolutions ([3, 10]), just to give a partial list.

In this paper we call *sets of lines* of $\mathbb{P}^1 \times \mathbb{P}^2$ certain codimension two varieties in $\mathbb{P}^1 \times \mathbb{P}^2$. We will also look at these varieties as unions of planes in \mathbb{P}^4 . This is possible since a bihomogeneous linear form of $k[\mathbb{P}^1 \times \mathbb{P}^2]$ also defines a hyperplane in \mathbb{P}^4 . Thus, a line in $\mathbb{P}^1 \times \mathbb{P}^2$ can be viewed as a plane (the intersection of two hyperplanes) of \mathbb{P}^4 . This correspondence allows us to move the study of several properties, including the Cohen-Macaulay property, from $\mathbb{P}^1 \times \mathbb{P}^2$ to \mathbb{P}^4 . The converse is not always possible for two reasons due to the nature of the problem. First, a linear form in $k[\mathbb{P}^4]$ is not necessarily an element of $k[\mathbb{P}^1 \times \mathbb{P}^2]_{(1,0)}$ or $k[\mathbb{P}^1 \times \mathbb{P}^2]_{(0,1)}$. Moreover, a plane in \mathbb{P}^4 could be defined by an ideal that is, even if bihomogeneous, not saturated in $\mathbb{P}^1 \times \mathbb{P}^2$. See Remark 3.1 for more detail about this connection.

From what has been observed above, we only have lines of two different types in $\mathbb{P}^1 \times \mathbb{P}^2$. The *horizontal lines*, given by the intersection of a hyperplane of degree $(1, 0)$ with one of degree $(0, 1)$, and the *vertical lines*, that are intersections of two hyperplanes of degree $(0, 1)$. The latter gives an immediate relation with \mathbb{P}^2 since a set of vertical lines in $\mathbb{P}^1 \times \mathbb{P}^2$ corresponds, in a way made precise in Remark 3.3, to a cone over a set of points in \mathbb{P}^2 .

The main result in Section 2, Proposition 2.1, concerns sets of points in \mathbb{P}^2 , which will be used later in the paper. Given a set of (fat) points Y and a curve $\mathbb{V}(F)$ passing through some of them, we give a characterization, in terms of the h -vector of a certain subscheme of Y , to guarantee that $I_Y + (F)$ is a saturated ideal. In Section 3 we introduce the notation and examine some obstacles to the characterization of the ACM property; see Remark 3.5 and Example 3.6. In Section 4 we apply Proposition 2.1 to study the ACM property for sets of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ satisfying a particular condition introduced in Notation 4.1. Precisely, we assume that in the sets under consideration the horizontal lines are reduced and two different horizontal lines of Z are not contained in a hyperplane defined by a form of degree $(0, 1)$.

We begin with an additional assumption (see Theorem 4.5 and Remark 4.6) and then show that this can be extended (Proposition 4.7, Proposition 4.8 and Corollary 4.9). We then apply these results to a class of configurations of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ where only one is non-reduced. In this situation we give a necessary and sufficient condition for ACMness in terms of the multiplicity

of the non-reduced component. As a corollary we give a result for \mathbb{P}^3 , as mentioned above. Our most general result (still with some assumptions) gives a characterization for when the configuration is locally Cohen-Macaulay (see Theorem 4.18).

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2. A PRELIMINARY RESULT ABOUT SATURATION IN \mathbb{P}^2

In this section we give a criterion, in terms of the h -vector, to establish if an ideal of fat points plus a form is saturated or not. In the next sections, we will relate it to the ACM property for some sets of lines in $\mathbb{P}^1 \times \mathbb{P}^2$; see Remark 4.6. There is an immediate connection with points in \mathbb{P}^2 since a set of vertical lines in $\mathbb{P}^1 \times \mathbb{P}^2$ is the cone over a set of points in \mathbb{P}^2 .

We start the section by recalling some standard notation. Given a finite set of n distinct points $W = \{P_1, \dots, P_n\}$ in \mathbb{P}^N , and m_1, \dots, m_n positive integers, we write $Y = m_1P_1 + \dots + m_nP_n$ for the set of fat points defined by the saturated homogeneous ideal

$$I_Y = \bigcap_{P_i \in W} (I_{P_i})^{m_i} \subseteq S = k[\mathbb{P}^N].$$

The degree of Y is $\deg(Y) = \sum \binom{m_i + N - 1}{N - 1}$. Recall that for a zero-dimensional scheme $Y \subset \mathbb{P}^N$ the Hilbert function of Y is defined as the numerical function $H_Y : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$H_Y(i) = \dim_k(S/I_Y)_i = \dim_k S_i - \dim_k (I_Y)_i.$$

Since $H_Y(\tau) = \deg(Y)$ for τ large enough, the first difference of the Hilbert function $\Delta H_Y(i) = H_Y(i) - H_Y(i - 1)$ is eventually zero. The h -vector of Y is

$$h_Y = h = (1, h_1, \dots, h_d)$$

where $h_i = \Delta H_Y(i)$ and d is the last index such that $\Delta H_Y(i) > 0$.

Proposition 2.1. *Let $Y = m_1P_1 + m_2P_2 + \dots + m_nP_n$ be a set of (fat) points in \mathbb{P}^2 . Let $\mathcal{C} = \mathbb{V}(F) \subseteq \mathbb{P}^2$ be a curve defined by a form F of degree $\deg(F) = d$. For any i , assume that if F vanishes at $P_i \in Y$, then the vanishing order is at least m_i , so $F \in [I_{m_iP_i}]_d$. Let Y_1 and Y_2 be the two zero-dimensional schemes such that $I_{Y_1} = (I_Y + F)^{\text{sat}}$ and $I_{Y_2} = I_Y : F$. Then $Y = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = \emptyset$, and $I_Y + (F) \subseteq k[\mathbb{P}^2]$ is saturated if and only if, for each $\tau \geq 0$, we have*

$$h_Y(\tau) = h_{Y_1}(\tau) + h_{Y_2}(\tau - d).$$

Proof. With these hypotheses we have $Y_1 = Y \cap \mathcal{C}$ and $Y_2 = Y \setminus \mathcal{C}$, and $Y_1 \cap Y_2 = \emptyset$ (the latter is because the multiplicity of F at each point is big enough). Note that $I_{Y_2} = I_Y : F$ is automatically saturated since I_Y is saturated. On the other hand, $I_{Y_1} = (I_Y + (F))^{\text{sat}}$ but $I_Y + (F)$ is not necessarily already saturated. Consider the short exact sequence

$$0 \rightarrow \frac{R}{I_Y : F}(-d) \xrightarrow{\cdot F} \frac{R}{I_Y} \rightarrow \frac{R}{I_Y + (F)} \rightarrow 0.$$

In any degree τ , the first vector space in the sequence has dimension $h_{Y_2}(\tau - d)$ and the second has dimension $h_Y(\tau)$ since the corresponding ideals are saturated. Then the third ideal, $I_Y + (F)$ is saturated if and only if the third vector space has dimension $h_{Y_1}(\tau)$, since the latter Hilbert function uses the saturated ideal of Y_1 , which is $(I_Y + (F))^{\text{sat}}$. \square

Remark 2.2. In Proposition 2.1, if Y is a reduced set of points then the condition about the vanishing order of F is automatically satisfied.

In the next example we show different ways to add points, in a “non-saturated ideal case”, in order to construct a saturated ideal.

Example 2.3. Denote by ℓ_i the line of \mathbb{P}^2 defined by the linear form $x + iy$, $1 \leq i \leq 4$. Let \mathcal{C} be the curve given by the union of the lines $\ell_1, \ell_2, \ell_3, \ell_4$, and let F be a form defining \mathcal{C} . Let Y be a set of five distinct points (no three on a line) such that only four of them belong to \mathcal{C} . Say for instance $P_1 = (1, -1, 1)$, $P_2 = (2, -1, 1)$, $P_3 = (6, -2, 1)$, $P_4 = (8, -2, 1)$ and $P_5 = (1, 1, 1)$.

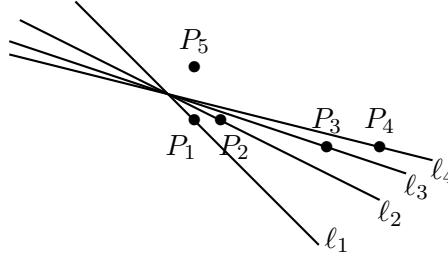


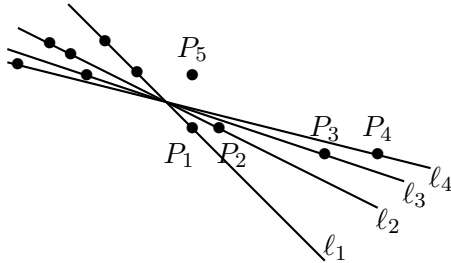
FIGURE 1. Five points, four of them on a quartic.

Using the notation in Proposition 2.1, we set $Y_1 = \{P_1, P_2, P_3, P_4\}$ and $Y_2 = \{P_5\}$. So we have

h_Y	1	2	2	0	0
h_{Y_1}	1	2	1	0	0
h_{Y_2}	0	0	0	0	1

where the third line in the table is the h -vector of Y_2 shifted by 4 (the degree of F). By Proposition 2.1, we note that $(F) + I_Y$ is not a saturated ideal. Now, by adding points to Y_1 , we show three different ways to make the resulting ideal $(F) + I_Y$ saturated.

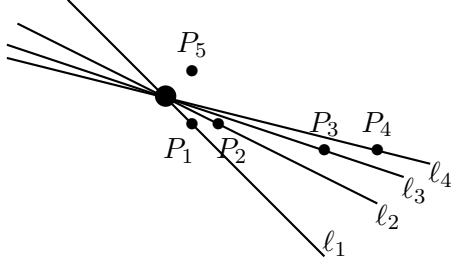
- i) We add to Y six points lying on \mathcal{C} , consisting of two general points in ℓ_1 and ℓ_2 and one general point in ℓ_3 and ℓ_4 . (Note that we have a similar situation occurs by adding a total of 7, 8, 9 or 10 general points on ℓ_1, \dots, ℓ_4 .)



So, we have

h_Y	1	2	3	4	1
h_{Y_1}	1	2	3	4	0
h_{Y_2}	0	0	0	0	1

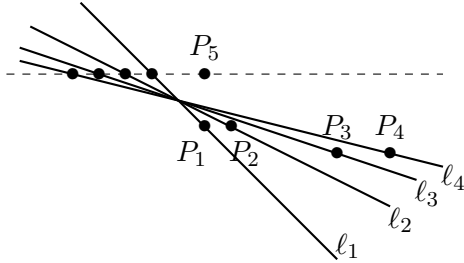
- ii) We add to Y the triple point defined by the ideal $(x, y)^3$. The vanishing order of F at this point is 4.



So, we get

h_Y	1	2	3	4	1
h_{Y_1}	1	2	3	4	0
h_{Y_2}	0	0	0	0	1

- iii) Let ℓ be a line containing P_5 , say for instance ℓ defined by the linear form $y - z$, and add to Y the four intersection points of ℓ and \mathcal{C} i.e. $P_6 = (-1, 1, 1)$, $P_7 = (-2, 1, 1)$, $P_8 = (-3, 1, 1)$, $P_9 = (-4, 1, 1)$.



So, we have

h_Y	1	2	3	2	1
h_{Y_1}	1	2	3	2	0
h_{Y_2}	0	0	0	0	1

- Remark 2.4.**
- If $I_Y + F$ is saturated then F is a “separator” for Y_2 , i.e. the form F vanishes at $Y \setminus Y_2$ and it doesn't vanish at any point of Y_2 ; see for instance the paper [21] where it first appears. So, the ideal defining Y_1 is obtained just adding F to I_Y .
 - Let Y be a reduced complete intersection of two planar curves of degrees $d_1 \geq 1$ and $d_2 \geq 1$. Assume that Y_2 is a single point, and that F vanishes on Y_1 but not Y_2 . If $I_Y + (F)$ is saturated then F cannot have degree $d_1 + d_2 - 3$, because by the Cayley-Bacharach theorem, any F vanishing on Y_1 must also vanish on Y_2 .
 - Using liaison, this kind of remark can be strengthened. For example, let Y be a reduced complete intersection in linear general position and Y_2 consists of three points, if $I_Y + (F)$ is saturated then F cannot have degree $d_1 + d_2 - 4$, by a similar analysis.

3. NOTATION, TERMINOLOGY AND EXAMPLES

We work over a field of characteristic zero. For a product of two projective spaces we define $V = \mathbb{P}^{a_1} \times \mathbb{P}^{a_2}$ and

$$\pi_i : V \rightarrow \mathbb{P}^{a_i}$$

to be the projection to the i -th component ($i = 1, 2$).

Let $\{\underline{e}_1, \underline{e}_2\}$ be the standard basis of \mathbb{N}^2 . Let $x_{i,j}$, with $1 \leq i \leq 2$ and $0 \leq j \leq a_i$ for all i, j , be the variables for \mathbb{P}^{a_1} and \mathbb{P}^{a_2} . Let

$$R = k[x_{1,0}, \dots, x_{1,a_1}, x_{2,0}, \dots, x_{2,a_2}] = k[V],$$

where the degree of $x_{i,j}$ is \underline{e}_i .

Remark 3.1. A subscheme, X , of V is defined by a bihomogeneous ideal I_X that is saturated with respect to $(x_{1,0}, \dots, x_{1,a_1})$ and to $(x_{2,0}, \dots, x_{2,a_2})$. The ideal I_X is generated by a system of multihomogeneous polynomials in R .

Given a scheme $X \subseteq V$, its defining ideal, I_X , is also saturated with respect to the ideal $(x_{1,0}, \dots, x_{1,a_1}, x_{2,0}, \dots, x_{2,a_2})$. Thus, the ideal I_X defines a different scheme X' in $\mathbb{P}^{a_1+a_2+1}$.

The converse is, of course, not true: consider for instance the ideal $(x_{1,0}, \dots, x_{1,a_1})$, which defines a subscheme of $\mathbb{P}^{a_1+a_2-1}$ but not of $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2}$.

Moreover, the maps in a minimal free resolution of R/I_X can be seen as those of a minimal free resolution of $k[x_{1,0}, \dots, x_{1,a_1}, x_{2,0}, \dots, x_{2,a_n}]/I_{X'}$. In fact, the only difference consists in the degree of the forms appearing as entries of the matrices and not in the forms themselves – they have bidegree (a, b) in the first case and degree $a + b$ in the second. So, many homological invariants of X and X' are strictly connected and some of them, such as the projective dimension, agree. This idea has often been exploited in the literature, for example in [5, 7, 9, 12]. See also [14], Remark 3.3, where they say (in the context of $\mathbb{P}^1 \times \mathbb{P}^1$) that “our study of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be seen as an investigation of these special unions of lines in \mathbb{P}^3 .”

We say that X is *arithmetically Cohen-Macaulay (ACM)* if R/I_X is a Cohen-Macaulay ring.

Let $N = a_1 + a_2 + 2$. Given a subscheme X of V together with its homogeneous ideal I_X , we can also consider the subscheme \bar{X} of \mathbb{P}^{N-1} defined by I_X . Notice that if X is a zero-dimensional subscheme of V , then I_X almost never defines a zero-dimensional subscheme of \mathbb{P}^{N-1} .

In the following, to shorten the notation, we write $R = k[s, t, x, y, z] = k[\mathbb{P}^1 \times \mathbb{P}^2]$. We will use the letters “ A, A', A_i, \dots ” to denote the elements in $R_{(1,0)}$, and the letters “ B, B', B_j, \dots ” for elements in $R_{(0,1)}$. These bihomogeneous linear forms define hyperplanes in $\mathbb{P}^1 \times \mathbb{P}^2$.

Definition 3.2. We call a *line* in $\mathbb{P}^1 \times \mathbb{P}^2$ the intersection of two hyperplanes defined by a saturated ideal. More precisely, we only have two different kind of lines, depending on the bidegree of the hyperplanes defining it: we call a *horizontal line* a line of type $\mathbb{V}(A, B)$ and a *vertical line* a line of type $\mathbb{V}(B, B')$.

Remark 3.3. Note that two different vertical lines are disjoint in $\mathbb{P}^1 \times \mathbb{P}^2$. However, there is a useful geometric connection to \mathbb{P}^4 . A vertical line is defined by two linear forms in x, y, z . Thus the ideal defines, in \mathbb{P}^4 , a 2-dimensional linear space that contains the line λ defined by $x = y = z = 0$. Let X be a union of vertical lines in $\mathbb{P}^1 \times \mathbb{P}^2$. The projection of X to \mathbb{P}^2 (i.e. to the second component) is a finite set of points in \mathbb{P}^2 . In \mathbb{P}^4 the same ideal I_X (viewed only with the standard grading) defines a finite union of 2-planes whose intersection with the 2-plane defined by $s = t = 0$ is also a finite set of points, a copy of the projection of X . In this sense we can view I_X as defining a cone in \mathbb{P}^4 over a finite set of points in \mathbb{P}^2 , whose vertex is the line λ (as opposed to being a point) and whose components are 2-planes (as opposed to lines). So the cone notion is not useful in $\mathbb{P}^1 \times \mathbb{P}^2$, but is useful if we are viewing the corresponding subschemes of \mathbb{P}^4 . As a result we sometimes refer to an ideal like I_X as a “cone ideal.”

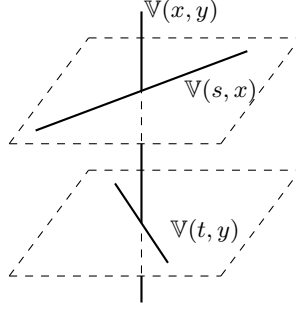
Note that the ideal generated by two different forms A, A' is not saturated in $\mathbb{P}^1 \times \mathbb{P}^2$. (For instance, $\mathbb{V}(s, t) = \emptyset$.) Whenever it is not necessary to make the type of line explicit, we just use the letter L . A *set of lines* X is a finite collection of lines. We will write either $X = L_1 + L_2 + \dots + L_n \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ or $X = \{L_1, L_2, \dots, L_n\} \subseteq \mathbb{P}^1 \times \mathbb{P}^2$.

There is a natural partition on a set of lines X : we will write $X = X_1 \cup X_2$ to denote the partition of X into horizontal and vertical lines respectively.

Example 3.4. Let X be the set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$

$$X = \mathbb{V}(s, x) \cup \mathbb{V}(t, y) \cup \mathbb{V}(x, y).$$

Then X consists of two horizontal lines, that are $\mathbb{V}(s, x)$ and $\mathbb{V}(t, y)$, and one vertical line $\mathbb{V}(x, y)$. In Figure 2 a representation of such set.

FIGURE 2. The above set X .

Given a set of n lines X and positive integers m_1, \dots, m_n , we let Z denote the subscheme of $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the saturated bihomogeneous ideal

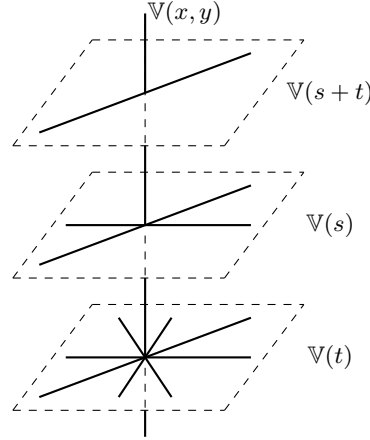
$$I_Z = \bigcap_{L_i \in X} (I_{L_i})^{m_i} \subseteq R.$$

We call Z a set of *fat lines* in $\mathbb{P}^1 \times \mathbb{P}^2$ whose support is $X = L_1 + L_2 + \dots + L_n \subseteq \mathbb{P}^1 \times \mathbb{P}^2$. The scheme Z will be denoted by $Z = m_1 L_1 + m_2 L_2 + \dots + m_n L_n$.

Remark 3.5. The aim of this paper is to investigate which properties make a set of (fat) lines ACM. In [7] and [9] the ACM property (or its failure) was studied for sets of points. Although the ambient spaces in these two papers are different, and the descriptions of the ACM sets of points are certainly different, they have a property in common. Given a set of points in a multiprojective space, there is a complete intersection of points containing it. So its residual is again a set of points. An ACM set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ is a codimension 2 scheme, hence viewed in \mathbb{P}^4 it is in the same liaison class as a complete intersection. One can hope that liaison tricks will continue to work here. One powerful trick is the result of Gaeta that says that if $X \subset \mathbb{P}^N$ is ACM of codimension two then one can link in a finite number of steps, always using complete intersections that are generated by minimal generators of the ideal, in such a way that at each step the number of minimal generators drops by one, and the end result is a complete intersection. Conversely, if such a sequence of links exists then X is ACM. If $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ is viewed in \mathbb{P}^4 , this can still be done using homogeneous complete intersections. However, if we want to insist that our complete intersections are generated by bihomogeneous polynomials (so all varieties in the sequence of links lie in $\mathbb{P}^1 \times \mathbb{P}^2$), it might not be possible. (See for instance [18, 19].) This is illustrated in the following example.

Example 3.6. Let

$$X = \{\mathbb{V}(x, y), \mathbb{V}(s, x), \mathbb{V}(s, y), \mathbb{V}(t, x), \mathbb{V}(t, y), \mathbb{V}(t, z), \mathbb{V}(t, x + y + z), \mathbb{V}(s + t, y)\}.$$

FIGURE 3. A representation of the above set X .

Then one can check using Macaulay2 or CoCoA that X is ACM, with four minimal generators:

$$txy, sty, x^2yz + xy^2z + xyz^2, \text{ and } s^2tx + st^2x.$$

These have bidegree, respectively, $(1,2)$, $(2,1)$, $(0,4)$ and $(3,1)$. In \mathbb{P}^4 it is clear that one can link X using a complete intersection consisting of homogeneous polynomials of degrees 3 and 4 that are part of a minimal generating set. However, if we want to link using bihomogeneous polynomials that are minimal generators, we claim that no such link exists. Indeed, note first that the given generators have pairwise common factors. A bihomogeneous minimal generator of degree 3 has to be one of the two given generators, since any linear combination is no longer bihomogeneous. To get a bihomogeneous minimal generator of degree 4, the only possibility is a polynomial $L \cdot sty + (s^2tx + st^2x)$ for some linear form L of type $(1,0)$. Note that st would be a factor of any such bihomogeneous form. Thus any such bihomogeneous polynomial has a factor in common with both txy and sty , so no such link is possible.

One could envision an approach wherein we begin with a set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ and we somehow link in \mathbb{P}^4 without regard to having the residual be viewable as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^2$, looking only to whether or not we can arrive at a complete intersection (hence X is ACM). We have not seen how to make such an approach work.

4. ACM SETS OF LINES IN $\mathbb{P}^1 \times \mathbb{P}^2$: THE STARTING CASE

In this section we focus on the ACM property for a set of fat lines Z in $\mathbb{P}^1 \times \mathbb{P}^2$ satisfying extra conditions. From now on we will work under the following hypothesis.

Notation 4.1. Let $Z = Z_1 \cup Z_2$ be the partition of a set Z of fat lines in $\mathbb{P}^1 \times \mathbb{P}^2$ into horizontal and vertical lines respectively. Throughout this section we will assume that

- (a) Z_1 is a non-empty set of horizontal reduced lines;
- (b) two different horizontal lines of Z are not contained in a hyperplane defined by a form of degree $(0,1)$, i.e., if $\mathbb{V}(A, B), \mathbb{V}(A', B) \in Z$, then $(A) = (A')$.

Remark 4.2. Note that a set of minimal generators of I_{Z_2} is only in the variables x, y, z so I_{Z_2} is the cone ideal of a set of (fat) points Y_2 in \mathbb{P}^2 .

Remark 4.3. In this section we will use the connection between sheafification of a graded module and the direct sum over all twists of the cohomology of the sheaf. This has a version from local cohomology, but we will use the sheaf version. We refer to Hartshorne [16] for sheafification, and to [18, Section 1.1] for the rest, but we recall the basic facts here that we will use in this section.

If M is a graded R -module then \tilde{M} denotes its sheafification. If \mathcal{F} is a sheaf on \mathbb{P}^n and $i \geq 0$ is an integer then

$$H_*^i(\mathcal{F}) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t)).$$

This is a graded R -module. In particular, if I is a homogeneous ideal and $\mathcal{I} = \tilde{I}$ then $I = H_*^0(\mathcal{I})$ if and only if I is already saturated; in general, the module on the right is the saturation of I . If

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is a short exact sequence of graded modules then

$$0 \rightarrow \tilde{M}_1 \rightarrow \tilde{M}_2 \rightarrow \tilde{M}_3 \rightarrow 0$$

is a short exact sequence of sheaves, but this is only left exact on global sections in general, and we get a long exact on cohomology.

Remark 4.4. Let Z be as in Notation 4.1. If $|\pi_1(Z_1)| = 1$ then all the horizontal lines of Z are contained in the same plane, say $\mathbb{V}(A)$. Hence Z_1 is a complete intersection of codimension 2. Indeed, say $Z_1 = \mathbb{V}(A, B_1) \cup \dots \cup \mathbb{V}(A, B_N)$, we note that the ideal defining Z_1 is $I_{Z_1} = (A, F)$, where $F = B_1 B_2 \dots B_N$. Since $\deg A = (1, 0)$ and $\deg F = (0, N)$ the ideal (A, F) is generated by a regular sequence.

The next result make evident what connection there is between the saturation problem studied in Section 2 and the ACM property of Z .

Theorem 4.5. *Let Z be as in Notation 4.1 and assume $|\pi_1(Z_1)| = 1$. Let F be of the form in Remark 4.4. Then Z is ACM if and only if $I_{Z_2} + (F)$ is saturated in $k[x, y, z]$.*

Proof. Notice that if $I_{Z_2} + (F)$ is artinian then it is not saturated, so the last condition includes the statement that $I_{Z_2} + (F)$ has height 2 in $k[x, y, z]$.

We look at Z , Z_1 and Z_2 as unions of planes in \mathbb{P}^4 . Consider the short exact sequence

$$(4.1) \quad 0 \rightarrow I_Z \rightarrow I_{Z_1} \oplus I_{Z_2} \rightarrow I_{Z_1} + I_{Z_2} \rightarrow 0.$$

Since $|\pi_1(Z_1)| = 1$, we have $I_{Z_1} = (A, F)$, a complete intersection (see Remark 4.4). In particular, note that Z_1 and Z_2 are ACM unions of planes in \mathbb{P}^4 (see Remark 4.2).

Sheafifying the exact sequence (4.1) and taking cohomology, we obtain the following exact diagrams:

$$(4.2) \quad \begin{array}{ccccccc} 0 \rightarrow I_Z \rightarrow I_{Z_1} \oplus I_{Z_2} & \longrightarrow & [I_{Z_1} + I_{Z_2}]^{sat} & \rightarrow & H_*^1(\mathcal{I}_Z) & \rightarrow & 0 \\ & \searrow & \nearrow & & & & \\ & I_{Z_1} + I_{Z_2} & & & & & \\ & \nearrow & \searrow & & & & \\ & 0 & & & 0 & & \end{array}$$

and

$$(4.3) \quad 0 \rightarrow H_*^1(\mathcal{I}_{Z_1} + \mathcal{I}_{Z_2}) \rightarrow H_*^2(\mathcal{I}_Z) \rightarrow 0.$$

Recall that $F \in k[x, y, z]$. Recall also that Z is ACM if and only if $H_*^1(\mathcal{I}_Z) = H_*^2(\mathcal{I}_Z) = 0$. Note that the form A is a regular element in $\frac{R}{I_{Z_2} + (F)}$ and recall that

$$I_{Z_1} + I_{Z_2} = (A, F) + I_{Z_2}.$$

Suppose first that F does not vanish on any component of Z_2 . Then $F + I_{Z_2}$ is an unmixed (in particular saturated) height 3 ideal in R , and $(A, F) + I_{Z_2} = I_{Z_1} + I_{Z_2}$ is a saturated ideal of height 4 in R . From (4.3) we know that

$$H_*^2(\mathcal{I}_Z) = 0 \text{ if and only if } H_*^1(\mathcal{I}_{Z_1} + \mathcal{I}_{Z_2}) = 0.$$

Since in the current situation $I_{Z_1} + I_{Z_2}$ defines a zero-dimensional scheme, the first cohomology module $H_*^1(\mathcal{I}_{Z_1} + \mathcal{I}_{Z_2})$ does not vanish. Thus Z has no hope of being ACM.

What we have just shown is that if $H_*^2(\mathcal{I}_Z) = 0$ then $I_{Z_2} + (F)$ has height 2 (either in R or in $k[x, y, z]$). We claim that the converse is also true. Indeed, if $I_{Z_2} + (F)$ has height 2 in $k[x, y, z]$, its saturation defines a zero-dimensional scheme in \mathbb{P}^2 , which is automatically ACM. Thus viewed in R , this saturation defines an ACM surface in \mathbb{P}^4 , so the (not necessarily saturated) ideal $I_{Z_1} + I_{Z_2}$ defines the hyperplane section, which is also ACM. But the first cohomology of the ideal sheaf does not depend on whether the original ideal was saturated or not, so it vanishes and hence $H_*^2(\mathcal{I}_Z) = 0$ by (4.3).

Thus we can assume without loss of generality that $I_{Z_2} + (F)$ has height 2 (either in R or in $k[x, y, z]$) and that $H_*^2(\mathcal{I}_Z) = 0$, and we focus on $H_*^1(\mathcal{I}_Z)$. From (4.2) we obtain

$$\begin{aligned} H_*^1(\mathcal{I}_Z) = 0 & \text{ if and only if } I_{Z_1} + I_{Z_2} \text{ is saturated in } R \\ & \text{ if and only if } I_{Z_2} + (F) \text{ is saturated in } k[x, y, z]. \end{aligned}$$

(As before, an ideal in $k[x, y, z]$ of height 2 is Cohen-Macaulay if and only if it is saturated if and only if it is unmixed.) \square

Remark 4.6. Theorem 4.5 shows that the ACM property of a set of (fat) lines $Z = Z_1 \cup Z_2$ where $|\pi_1(Z_1)| = 1$ only depends on the saturation of the ideal $(F) + I_{Z_2} \subseteq k[\mathbb{P}^2]$. In \mathbb{P}^2 , the set Z_2 can be viewed as a set of points and the form F defines a curve, \mathcal{C} , that is a union of lines. So, if the hypotheses of Proposition 2.1 are satisfied, the ACM property only depends on the h -vectors of I_{Z_2} , $I_{Y_1} = (F) + I_{Z_2}$ and $I_{Y_2} = I_{Z_2} : I_{\mathcal{C}}$. More precisely, Z is ACM if and only if $h_{Z_2}(\tau) = h_{Y_1}(\tau) + h_{Y_2}(\tau - d)$ where d is the degree of $F \in k[\mathbb{P}^2]$.

We introduce the following definition. Recall that s and t are the variables of degree $(1, 0)$ in the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^2$, $R = k[s, t, x, y, z]$.

Definition 4.7. Let Z be a set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ as in Notation 4.1. We define

$$\hat{Z} = \{\mathbb{V}(s, B) \mid \mathbb{V}(A, B) \in Z_1 \text{ for some } A\} \cup Z_2.$$

The set \hat{Z} consists of all the vertical lines in Z , together with the projections of all the horizontal lines in Z down to the hyperplane defined by $\mathbb{V}(s)$. See Figure 4 and Figure 5 for two examples.

Note that in Definition 4.7 the set \hat{Z} satisfies the hypotheses of Theorem 4.5; indeed we have $|\pi_1(\hat{Z}_1)| = 1$.

Proposition 4.8. *Let Z be as in Notation 4.1. If Z is ACM then \hat{Z} is ACM, and Z and \hat{Z} share the same multigraded homological invariants.*

Proof. Without loss of generality we can assume that the linear form t (one of the indeterminates) is a regular element in R/I_Z . If we look at Z and \hat{Z} as unions of planes in \mathbb{P}^4 , the hyperplane defined by t does not contain any component either of Z or of \hat{Z} , and so meets each such component in a line.

Let us examine the effect on both kinds of components of Z . A “horizontal” component is defined by an ideal of the form (ℓ_i, m_i) with $\ell_i \in k[s, t]$ and $m_i \in k[x, y, z]$, so the hyperplane section defined by the linear form t is an ideal of the form $(\ell_i, m_i, t) = (s, t, m_i)$. A “vertical” component is defined by an ideal of the form (L_j, M_j) , with $L_j, M_j \in k[x, y, z]$, so the hyperplane section by t is defined by (L_j, M_j, t) .

Since Z is ACM (viewed in \mathbb{P}^4), we have that $I_Z + (t)$ is the saturated ideal of the hyperplane section (a union of lines), which is ACM. The entire hyperplane section by t then has the saturated ideal

$$I_Z + (t) = \bigcap_i (s, m_i, t) \cap \bigcap_j (L_j, M_j, t),$$

since Z is ACM, and defines an ACM curve in \mathbb{P}^4 .

On the other hand, up to saturation the latter is also the hyperplane section by t of \hat{Z} . Since the curve is ACM, the union of planes \hat{Z} must also be ACM by [17, Proposition 2.1]. Thus

$$I_{\hat{Z}} + (t) = \left(\bigcap_i (s, m_i) \cap \bigcap_j (L_j, M_j) \right) + (t).$$

Finally, since Z and \hat{Z} are both ACM with the same hyperplane section, they must in fact have the same homological invariants.

Up to this point we have only shown that Z and \hat{Z} have the same *graded* homological invariants, by viewing Z and \hat{Z} as subschemes of \mathbb{P}^4 . But in fact they began in $\mathbb{P}^1 \times \mathbb{P}^2$, so I_Z and $I_{\hat{Z}}$ have multigraded Betti numbers and in particular multigraded minimal generators. When we reduce by the non-zerodivisor t , this preserves the multigrading. Hence we have the result. \square

The following is an immediate consequence of Proposition 4.8. It gives us a necessary condition for the Cohen-Macaulayness of Z . We denote by $\beta_{0,(a,b)}(I_Z)$ the minimal number of generators of I_Z of degree (a, b) .

Corollary 4.9. *Let Z be a set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ as in Notation 4.1. If Z is ACM, then $\beta_{0,(a,b)}(I_Z) = 0$ for each $a > 1$.*

Proof. Indeed this is true for \hat{Z} . \square

Remark 4.10. Corollary 4.9 seems very surprising, at first glance. But it assumes from the beginning that Z is ACM, and then that it satisfies condition (b) of Notation 4.1. Neither of these is particularly restrictive by itself, but the point is that the combination is restrictive. Still, having both conditions does not by any means imply that $Z = \hat{Z}$. For instance, consider Example 3.4. One might think at first that there is a minimal generator of type $(2, 1)$ (e.g. stx), but in fact one can check that the ideal has two minimal generators of bidegree $(1, 1)$ and one of bidegree $(0, 2)$.

Furthermore, starting with a Z that is ACM and satisfies Notation 4.1, it is easy to use basic double linkage to produce a new Z' that is ACM and does not satisfy $\beta_{0,(a,b)}(I_Z) = 0$ for each $a > 1$, but we lose the property given in Notation 4.1 (b). For instance, returning to Example 3.4, one could form $(s + t) \cdot I_Z + (xy)$, which is ACM and has a minimal generator of bidegree $(2, 1)$; but now it does not satisfy Notation 4.1.

We do not know if the converse of Corollary 4.9 is true. So we pose the following question:

Let Z be as in Notation 4.1. Assume \hat{Z} is ACM and $\beta_{0,(a,b)}(I_Z) = 0$ for each $a > 1$. Is Z ACM?

However, the next example shows that, without the condition on $\beta_{0,(a,b)}(I_Z)$, \hat{Z} ACM does not imply Z ACM.

Example 4.11. Let Z the set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the ideal $I_Z = (t, x) \cap (s, y)$. The set Z is clearly not ACM. However, \hat{Z} , defined by $I_{\hat{Z}} = (s, x) \cap (s, y)$, is ACM.

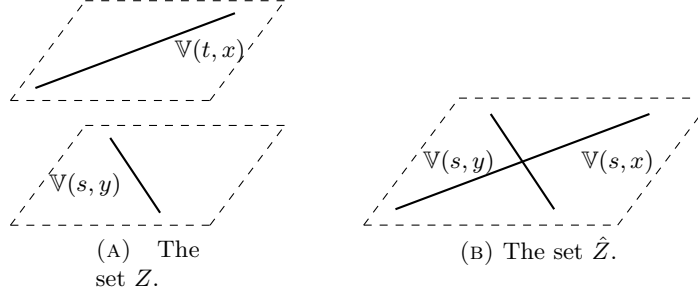


FIGURE 4. Example 4.11.

From Proposition 4.8 we know that the ACM property of \hat{Z} is a necessary condition for the ACM property of Z . Example 4.11 shows that it is not sufficient. The following lemma will give us another necessary condition.

Lemma 4.12. *Let Z be an ACM set of lines as in Notation 4.1. Let $\mathbb{V}(A, B), \mathbb{V}(A', B') \in Z$ be two horizontal lines and $A \neq A'$. Then $\mathbb{V}(B, B') \in Z$.*

Proof. Look at Z as a set of planes in \mathbb{P}^4 . If $\mathbb{V}(B, B') \notin Z$ then Z is not locally Cohen-Macaulay at the point defined by the ideal $\mathfrak{p} = (A, A', B, B')$, hence Z is not ACM. Contradiction. \square

Then it is natural to give the next definition.

Definition 4.13. Let Z be as in Notation 4.1. We say that Z is *v-connected* if, for each $\mathbb{V}(A, B), \mathbb{V}(A', B') \in Z_1$ where $A \neq A'$ and $B \neq B'$, we have $\mathbb{V}(B, B') \in Z_2$.

In the next example we show that this property is still not enough to ensure the ACM property.

Example 4.14. Let X be the set of lines in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the ideal

$$I_X = (s, x) \cap (s, y) \cap (t, x + y) \cap (x, y).$$

Note that X is v-connected. Moreover, from Theorem 4.5, the set \hat{X} defined by

$$I_{\hat{X}} = (s, x) \cap (s, y) \cap (s, x + y) \cap (x, y)$$

is ACM.

According to CoCoA, X is not ACM. A computer experiment shows that the set of fat lines, Z , whose ideal is

$$I_Z = (s, x) \cap (s, y) \cap (t, x + y) \cap (x, y)^2$$

is ACM. Thus that Z is a (non-reduced) ACM set of lines whose support X is not ACM. The next result explores this idea further.

Theorem 4.15. *Let $Z = \mathbb{V}(A_1, B_1) + \dots + \mathbb{V}(A_n, B_n) + m \cdot \mathbb{V}(B_1, B_2)$ be a set of lines of $\mathbb{P}^1 \times \mathbb{P}^2$ such that*

- (1) $\mathbb{V}(A_{n-1}) \neq \mathbb{V}(A_n)$ (i.e. not all the horizontal lines are in the same plane);
- (2) $\mathbb{V}(B_i) \neq \mathbb{V}(B_j)$ for $i \neq j$, (i.e. B_1, \dots, B_n define different planes);
- (3) $\mathbb{V}(B_1, \dots, B_n) = \mathbb{V}(B_1, B_2)$, (i.e. B_1, \dots, B_n define planes in the pencil containing $\mathbb{V}(B_1, B_2)$, therefore $\mathbb{V}(B_1, B_2) = \mathbb{V}(B_i, B_j)$ for all $i \neq j$).

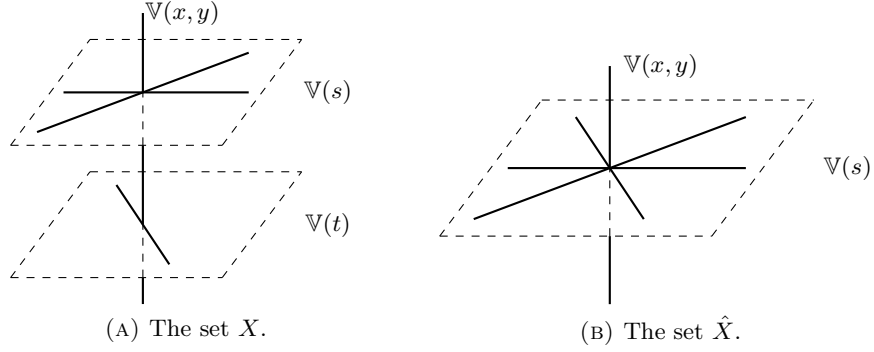


FIGURE 5. Example 4.14.

Then Z is ACM if and only if $m \geq n - 1$.

Proof. First assume $n = 2$, i.e., $Z = \mathbb{V}(A_1, B_1) + \mathbb{V}(A_2, B_2) + m \cdot \mathbb{V}(B_1, B_2)$.

(\Rightarrow) By Lemma 4.12, Z ACM implies $m \geq 1$.

(\Leftarrow) On the other hand, take $m \geq 1$ and consider the following short exact sequences

$$(4.4) \quad 0 \rightarrow I_Z \rightarrow (A_1, B_1) \oplus [(A_2, B_2) \cap (B_1, B_2)^m] \rightarrow (A_1, B_1) + [(A_2, B_2) \cap (B_1, B_2)^m] \rightarrow 0,$$

and

$$(4.5) \quad 0 \rightarrow (A_2, B_2) \cap (B_1, B_2)^m \rightarrow (A_2, B_2) \oplus (B_1, B_2)^m \rightarrow (A_2, B_2) + (B_1, B_2)^m \rightarrow 0.$$

Look at (4.5) and note that $(A_2, B_2) + (B_1, B_2)^m = (A_2, B_2, B_1^m)$. Thus, the ideal $(A_2, B_2) + (B_1, B_2)^m$ defines an ACM scheme of height 3. Since the schemes defined by (A_2, B_2) and $(B_1, B_2)^m$ are ACM of height 2, then a mapping cone argument gives that the ideal $(A_2, B_2) \cap (B_1, B_2)^m$ has projective dimension 2. Now consider (4.4), and note that

$$(A_1, B_1) + [(A_2, B_2) \cap (B_1, B_2)^m] = (A_1, B_1, B_2^m),$$

so it defines an ACM scheme of height 3. Moreover, the module in the middle of the sequence has projective dimension 2, hence Z is ACM.

Now we proceed by induction, the base case having just been proven. Assume that the statement is true for $n - 1 \geq 1$ and assume that $n \geq 3$. That is, assume that if Z has $n - 1$ reduced components, then Z is ACM if and only if $m \geq n - 2$. Set

$$Z' = \mathbb{V}(A_2, B_2) + \cdots + \mathbb{V}(A_n, B_n) + m\mathbb{V}(B_1, B_2)$$

and consider the following short exact sequence:

$$(4.6) \quad 0 \rightarrow I_Z \rightarrow (A_1, B_1) \oplus I_{Z'} \rightarrow (A_1, B_1) + I_{Z'} \rightarrow 0.$$

(\Leftarrow) If $m \geq n - 1$ then, by the inductive hypothesis, Z' is ACM. Thus, in order to prove that Z is ACM, by the sequence (4.6), it is enough to show that $(A_1, B_1) + I_{Z'}$ defines an ACM variety of height 3. We show that $(A_1, B_1) + I_{Z'} = (A_1, B_1, B_2^m)$.

- $(A_1, B_1) + I_{Z'} \subseteq (A_1, B_1, B_2^m)$. Indeed, if $F \in I_{Z'}$ then $F \in (B_1, B_2)^m \subseteq (B_1, B_2^m)$;
- $(A_1, B_1) + I_{Z'} \supseteq (A_1, B_1, B_2^m)$. We first claim that the form $F = B_2^{m-n+2} \cdot B_3 \cdots B_n$ belongs to $I_{Z'}$. By hypotheses 2) and 3), for $j > 3$ we have $B_j = \lambda_j B_1 + \mu_j B_2$, where $\lambda_j, \mu_j \neq 0$. Thus, $F = F' B_1 + a B_2^m \in I_{Z'}$ and therefore $B_2^m \in (A_1, B_1) + I_{Z'}$.

(\Rightarrow) Assume that Z is ACM. View Z as a set of planes in \mathbb{P}^4 . By the sequence (4.6), the ideal $(A_1, B_1) + I_{Z'}$ is saturated in $k[\mathbb{P}^4]$. Without loss of generality we can assume the linear form z corresponds to a general form of degree one in $k[x, y, z]$, so that the intersection of Z with $\mathbb{V}(z)$ is a proper hyperplane section of Z in \mathbb{P}^4 , all but one component of $Z \cap \mathbb{V}(z)$ are reduced, and $(\bar{B}_1), \dots, (\bar{B}_n)$ are all different in $R/(z)$. Denoting by J the ideal defining $Z \cap \mathbb{V}(z)$ in \mathbb{P}^3 , we have that $J \cong (I_Z + (z))/(z)$ and it is Cohen-Macaulay in $k[\mathbb{P}^3] \cong R/(z)$. Let us denote by Y the scheme defined by J in \mathbb{P}^3 . Hence Y is an ACM set of lines (one non-reduced) in \mathbb{P}^3 ; precisely

$$J = (A_1, \bar{B}_1) \cap \dots \cap (A_n, \bar{B}_n) \cap (\bar{B}_1, \bar{B}_2)^m$$

where $\mathbb{V}(\bar{B}_j) = \mathbb{V}(B_j) \cap \mathbb{V}(z)$. (Recall that $A_i \in k[s, t]$ while $B_j \in k[x, y, z]$, so abusing notation we will view $A_i \in k[\mathbb{P}^3]$.) We set

$$(4.7) \quad J' = (A_2, \bar{B}_2) \cap \dots \cap (A_n, \bar{B}_n) \cap (\bar{B}_1, \bar{B}_2)^m$$

and denote by Y' the scheme defined by J' , and by λ the line defined by (A_1, \bar{B}_1) . Consider the short exact sequence

$$(4.8) \quad 0 \rightarrow J \rightarrow (A_1, \bar{B}_1) \oplus J' \rightarrow (A_1, \bar{B}_1) + J' \rightarrow 0.$$

Since J defines an ACM curve Y , sheaffying and taking the long exact sequence in cohomology gives us that the ideal $(A_1, \bar{B}_1) + J'$ is saturated in $k[\mathbb{P}^3]$; in particular it has height 3, and

$$(4.9) \quad (A_1, \bar{B}_1) + J' = (A_1, \bar{B}_1, \bar{B}_2^u).$$

What is u ? We have by (4.9) that u is the minimum such that $[(A_1, \bar{B}_1) + J']_{(0,u)} \neq [(A_1, \bar{B}_1)]_{(0,u)}$. Since $\bar{B}_2 \bar{B}_3 \dots \bar{B}_n$ is the only minimal generator of $(A_2, \bar{B}_2) \cap \dots \cap (A_n, \bar{B}_n)$ with bidegree $(0, v)$, the equation (4.7) implies that $u = \max(n-1, m)$. In particular, $u \geq n-1$.

Now we assume by contradiction that $m < n-1$. Consider the two forms

$$\begin{aligned} F_1 &= \bar{B}_2 \cdots \bar{B}_{n-2} \bar{B}_{n-1} A_n \text{ and} \\ F_2 &= \bar{B}_2 \cdots \bar{B}_{n-2} A_{n-1} \bar{B}_n. \end{aligned}$$

Both F_1 and F_2 have degree $(1, n-2)$ and both belong to J' , and so,

$$F_1, F_2 \in (A_1, \bar{B}_1) + J' = (A_1, \bar{B}_1, \bar{B}_2^u).$$

By hypothesis (2), $F_1, F_2 \notin (\bar{B}_1)$. By hypothesis (1), at least one among F_1 and F_2 does not belong to (A_1) , say F_1 . This means that $F_1 \in (\bar{B}_1, \bar{B}_2^u)$ and hence $\bar{B}_2 \cdots \bar{B}_{n-1} \in (\bar{B}_1, \bar{B}_2^u)$, so $u \leq n-2$ which contradicts what we have shown above. Therefore we have shown $m \geq n-1$ as desired. \square

As a corollary, Theorem 4.15 can be translated to a result about lines in \mathbb{P}^3 .

Corollary 4.16. *Let λ_1 and λ_2 be two skew lines in \mathbb{P}^3 . Let $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n be planes containing λ_1 and λ_2 respectively, such that*

- $\beta_i \neq \beta_j$ for any $i \neq j$;
- $\alpha_{n-1} \neq \alpha_n$.

Set $\ell_i = \alpha_i \cap \beta_i$. Then, the set of lines $Y = \ell_1 + \dots + \ell_n + m \cdot \lambda_2$ is ACM if and only if $m \geq n-1$.

Proof. We have $k[\mathbb{P}^4] = k[s, t, x, y, z]$ as usual. In Theorem 4.15, viewing the configuration as a union of planes in \mathbb{P}^4 (one not reduced), the A_i are hyperplanes vanishing on the 2-plane defined by $s = t = 0$ and (without loss of generality, after possibly a change of variables) the B_i can be taken to be hyperplanes vanishing on the 2-plane defined by $x = y = 0$. In our current setting, without loss of generality (after possibly a change of variables) we can take $k[\mathbb{P}^3] = k[s, t, x, y]$, and set λ_1 to be the line defined by $s = t = 0$ and set λ_2 to be the line defined by $x = y = 0$. Note that z is a non-zero-divisor for R/I_Z in Theorem 4.15. Then Y is the hyperplane section of Z cut out by the hyperplane defined by z . By [17, Proposition 2.1], Y is ACM if and only if Z is ACM. The result follows immediately. \square

Theorem 4.15 leads us to introduce a new definition.

Definition 4.17. Let Z be as in Notation 4.1. Assume that whenever $\mathbb{V}(A, B), \mathbb{V}(A', B') \in Z_1$ with $A \neq A'$ and $B \neq B'$, the vertical line $V(B, B')$ meets n horizontal lines of Z (where the integer $n \geq 2$ depends on the choice of B and B'). We say that Z is *fully v-connected* if $(n-1)\mathbb{V}(B, B') \subseteq Z$, i.e., for each such B, B' the line $\mathbb{V}(B, B')$ is contained in Z with multiplicity at least $n - 1$.

It was shown in [5, Lemma 4.1] that any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is locally a complete intersection, and hence locally Cohen-Macaulay. In the next theorem we characterize the local Cohen-Macaulay property of Z in terms of the fully v-connected property.

Theorem 4.18. *Let $Z = Z_1 \cup Z_2$ be a set of lines of $\mathbb{P}^1 \times \mathbb{P}^2$ such that the assumptions of Notation 4.1 hold, i.e.*

- (a) Z_1 is a non-empty set of reduced horizontal lines;
- (b) two different horizontal lines of Z are not contained in a hyperplane defined by a form of degree $(0, 1)$, i.e. if $(A, B), (A', B) \in Z_1$ then $(A) = (A')$.

Then, Z is locally Cohen-Macaulay if and only if Z is fully v-connected.

Proof. First assume that Z is locally Cohen-Macaulay. Let $(B, B')^m$ define a vertical fat line mL in Z . It is not restrictive to assume that $(B, B') = (x, y)$, and also that z is a regular element in R/I_Z . Let $\mathbb{V}(A_1, B_1), \dots, \mathbb{V}(A_n, B_n) \in Z_1$ be the horizontal lines meeting L . Let $P \in \mathbb{P}^4$ be the point defined by $I_P = (s, t, x, y)$. In particular, $(x, y) = (B_1, \dots, B_n)$, where B_1, \dots, B_n define different hyperplanes. Let $W = \mathbb{V}(A_1, B_1) + \dots + \mathbb{V}(A_n, B_n) + mL$ be the corresponding subscheme of Z . Note that W satisfies the hypotheses of Theorem 4.15, so W is ACM if and only if $m \geq n - 1$.

Notice that all components of W (viewed in \mathbb{P}^4) contain the point P , and no other component of Z does. Thus since the components of W are linear, W is the cone in \mathbb{P}^4 with vertex P over a curve, C , in the hyperplane \mathbb{P}^3 defined by $z = 0$, and C is precisely of the form given in Corollary 4.16. Now, since Z is locally Cohen-Macaulay, in particular at P , we conclude that C is ACM, and hence by Corollary 4.16 we have that $m \geq n - 1$. Since this holds for each vertical line (noting that m and n will change for different vertical lines), Z is fully v-connected.

Conversely, assume that Z is fully v-connected. Certainly at any smooth point of Z (viewed in \mathbb{P}^4), Z is locally Cohen-Macaulay. Similarly, at the intersection of two or more horizontal lines that do not have a vertical line through the point of intersection, locally it is a complete intersection, hence locally Cohen-Macaulay.

Although the vertical (possibly fat) lines are disjoint in $\mathbb{P}^1 \times \mathbb{P}^2$, the corresponding (fat) planes do meet in \mathbb{P}^4 . The intersection of any finite number of such planes is one-dimensional. Still, this is a cone over a set of (fat) points in \mathbb{P}^2 , so it is ACM and hence is locally Cohen-Macaulay along this locus.

Consider the intersection in $\mathbb{P}^1 \times \mathbb{P}^2$ of a (fat) vertical line of multiplicity m and a collection of n horizontal lines. Without loss of generality assume that the vertical line has support given by (x, y) . Let $J = (s, t, x, y)$. Note that the ideal J defines a point P in \mathbb{P}^4 . Let $W = \mathbb{V}(A_1, B_1) + \cdots + \mathbb{V}(A_n, B_n) + mL$ be the subscheme of Z corresponding to the components, viewed in \mathbb{P}^4 , containing P . Note that W satisfies the hypotheses of Theorem 4.15, so W is ACM if and only if $m \geq n - 1$.

Since the components of W are linear, W is the cone in \mathbb{P}^4 with vertex P over a curve, C , in the hyperplane \mathbb{P}^3 defined by $z = 0$ (which is a non-zerodivisor), and C is precisely of the form given in Corollary 4.16. Since Z is fully v-connected, we know that $m \geq n - 1$. Thus, by Corollary 4.16, C is ACM. Hence Z is locally Cohen-Macaulay at P . \square

Remark 4.19. Let $Z = Z_1 \cup Z_2$ be a set of lines of $\mathbb{P}^1 \times \mathbb{P}^2$ such that the assumptions of Notation 4.1 hold. We wonder which hypotheses ensure Z to be ACM. Note that the fully v-connected condition is not enough. Take for instance $Z = \mathbb{V}(s, x) \cup \mathbb{V}(y, z)$; it trivially satisfies Definition 4.17 and it is not ACM. So, as shown in Proposition 4.8, we at least need to ask \hat{Z} to be ACM. It would be interesting to show if these two conditions, Z fully v-connected together with \hat{Z} ACM, are sufficient to guarantee the ACM property for Z .

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