



Hamiltonians Generated by Parseval Frames

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Abstract It is known that self-adjoint Hamiltonians with purely discrete eigenvalues can be written as (infinite) linear combination of mutually orthogonal projectors with eigenvalues as coefficients of the expansion. The projectors are defined by the eigenvectors of the Hamiltonians. In some recent papers, this expansion has been extended to the case in which these eigenvectors form a Riesz basis or, more recently, a \mathcal{D} -quasi basis (Bagarello and Bellomonte in J. Phys. A 50:145203, 2017, Bagarello et al. in J. Math. Phys. 59:033506, 2018), rather than an orthonormal basis. Here we discuss what can be done when these sets are replaced by Parseval frames. This interest is motivated by physical reasons, and in particular by the fact that the *mathematical* Hilbert space where the physical system is originally defined, contains sometimes also states which cannot really be occupied by the *physical* system itself. In particular, we show what changes in the spectrum of the observables, when going from orthonormal bases to Parseval frames. In this perspective we propose the notion of E -connection for observables. Several examples are discussed.

Keywords Frames · Hamiltonian operators · Orthonormal bases

1 Introduction

As it is well known, in quantum mechanics the dynamics of closed systems is quite often deduced out of a self-adjoint operator, the Hamiltonian H of the system \mathcal{S} , which is the main ingredient to write down the Schrödinger equation for the wave function of \mathcal{S} , $i\dot{\Psi}(t) = H\Psi(t)$. An important aspect of H , useful for the analysis of \mathcal{S} , is the set of its eigenvalues and eigenvectors. Assuming that H has only discrete spectrum, each eigenvalue E_j of H is real, and the eigenvectors corresponding to different eigenvalues are orthogonal. In particular, if each E_j has multiplicity one, the set of related normalized eigenvectors,

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$\mathcal{F}_e = \{e_j\}$, is an orthonormal basis (ONB) of the Hilbert space \mathcal{H} , which can be thought as the closure of the linear span of the e_j 's. The various linear combinations of e_j represent different physical states of the system \mathcal{S} , e.g., different energy configurations. Of course, the same point of view can be extended to other self-adjoint operators having the same properties required to H , or even to other operators with not only discrete spectrum. For instance, if \hat{p} is the momentum operator defined on $\mathcal{L}^2(\mathbb{R})$, its spectrum coincides with \mathbb{R} , and each $p \in \mathbb{R}$ corresponds to a generalized eigenvectors of \hat{p} , the plane wave $\frac{1}{\sqrt{2\pi}} e^{-ipx}$, which does not belong to $\mathcal{L}^2(\mathbb{R})$ but can still be normalized in a distributional sense, [20, 21].

In the analysis of a concrete physical situation it may happen that *not all* vectors of \mathcal{H} are *relevant* in the analysis of \mathcal{S} . This is the case, for instance, when the energy of \mathcal{S} cannot *really* increase too much, or when \mathcal{S} is localized in a bounded region, or still when the value of the momentum of \mathcal{S} cannot be too large. In all these cases, but not only, it is reasonable to consider a *physical vector space*, \mathcal{H}_{ph} , as the subset of the *mathematical* Hilbert space \mathcal{H} on which \mathcal{S} is originally defined. This is exactly the point of view in [16], just to cite one, where \mathcal{H}_{ph} contains those functions of $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ which are zero outside a certain compact subset \mathcal{D} of \mathbb{R} . Depending on the interpretation of \mathcal{H} , this approach can be used to describe particles localized in \mathcal{D} , or particles with a bounded momentum. In [16] the main idea is to consider \mathcal{H}_{ph} as a subspace of $\mathcal{L}^2(\mathbb{R})$ for quantization purposes.

In general, the physical Hilbert space \mathcal{H}_{ph} can be constructed as the projection of \mathcal{H} , via some suitable orthogonal projector operator P i.e., $\mathcal{H}_{ph} = P\mathcal{H}$. Then, if H is any (bounded, for the moment) operator on \mathcal{H} , and $f, g \in \mathcal{H}_{ph}$, its matrix elements $\langle f, Hg \rangle$ can be rewritten as follows

$$\langle f, Hg \rangle = \langle Pf, HPg \rangle = \langle Pf, (PHP)Pg \rangle = \langle Pf, H_{ph}Pg \rangle,$$

where, $H_{ph} = PHP$ is the *physical part* of H acting in \mathcal{H}_{ph} . Of course, if $H = H^\dagger$, then $H_{ph} = H_{ph}^\dagger$ as well.

Let the set $\mathcal{F}_e = \{e_n, n = 1, 2, \dots, N\}$ of normalized eigenvectors of H be an ONB of \mathcal{H} and let $\{E_n, n = 1, 2, \dots, N\}$ be the corresponding set of real eigenvalues (here $N = \dim \mathcal{H} \leq \infty$). Then the action of the corresponding Hamiltonian H can be presented as

$$Hg = \sum_{n=1}^N E_n \langle e_n, g \rangle e_n \quad (1.1)$$

and its maximal domain of definition is $D(H) = \{g \in \mathcal{H} : \sum_{n=1}^N E_n \langle e_n, g \rangle e_n \in \mathcal{H}\} = \{g \in \mathcal{H} : \sum_{n=1}^N E_n^2 |\langle e_n, g \rangle|^2 < \infty\}$. In view of (1.1), the physical part of H has the form

$$PHPg = \sum_{n=1}^N E_n \langle e_n, Pg \rangle \varphi_n = \sum_{n=1}^N E_n \langle Pe_n, Pg \rangle \varphi_n = \sum_{n=1}^N E_n \langle \varphi_n, f \rangle \varphi_n,$$

where $f = Pg$ and $\varphi_n = Pe_n$. Therefore, the operator $H_{ph} = PHP$ acts in a subspace \mathcal{H}_{ph} of \mathcal{H} and

$$H_{ph}f = \sum_{n=1}^N E_n \langle \varphi_n, f \rangle \varphi_n, \quad f \in D(H_{ph}) = PD(H). \quad (1.2)$$

In general, the set of vectors $\mathcal{F}_\varphi = \{\varphi_n = Pe_n, n = 1, 2, \dots, N\}$ loses the property of being an ONB of \mathcal{H}_{ph} and, instead, it turns out to be a Parseval frame¹ of \mathcal{H}_{ph} . For this

¹See Sect. 2.1 for definition and elementary properties of frames.

reason, the operator H_{ph} in (1.2) can be considered as a Hamiltonian $H_\varphi (= H_{ph})$ generated by a Parseval frame \mathcal{F}_φ , in analogy with what discussed in [2, 5]. The operators defined by (1.2) are particular case of more general concept of *multipliers*, which was introduced and studied in [7, 22]. Furthermore, for finite Parseval frames ($N < \infty$), the operators H_{ph} can be regarded as quantum observables that are obtained by using a finite version of the Klauder-Berezin-Toeplitz-type coherent state quantization [14] of a real-valued function f defined on a set of data $\{a_1, \dots, a_N\}$ related to a physical system. The eigenvalues of H_{ph} form the ‘quantum spectrum’ of a classical observable f whereas its ‘classical spectrum’ coincides with the set of values $\{E_n = f(a_n)\}_{n=1}^N$ [12, 13]. These results can be generalized to the case of infinite Parseval frames with the use of POVM quantization developed in [15].

The main objective of this paper is the investigation of Hamiltonians generated by Parseval frames. As we will see, an interesting aspect is that, when we project from \mathcal{H} to \mathcal{H}_{ph} , the eigenvalues of the observables are not preserved in general. This could have relevant consequences in concrete situations, where H is just a formal *simple* operator, while H_{ph} is its really useful physical counterpart. In other words, while each E_n is an eigenvalue for H , see (1.1), E_n is not an eigenvalue for H_{ph} if $\|\varphi_n\| \neq 1$, despite of the fact that H and H_{ph} share similar expansions (1.1) and (1.2), but in terms of families of vectors which have different properties. In particular, \mathcal{F}_φ is not an ONB, and therefore $\langle \varphi_j, \varphi_k \rangle \neq \delta_{j,k}$, in general. We construct five explicit examples in Sect. 2.2 showing, among other features, how eigenvalues change.

In the present paper, we concentrate on the case of bounded operators. Unbounded Hamiltonians generated by Parseval frames have a lot of delicate properties and they will be considered in a forthcoming paper.

The paper is structured as follows: after short preliminaries about frames in Sect. 2.1, we begin our analysis of the bounded Hamiltonians H_φ . General methods of the calculation of eigenvalues and eigenvectors of H_φ are presented in Theorem 6 and Corollary 7. In Sect. 3, we introduce and study a physically motivated relation (*E*-connection) between Parseval frames and ONBs (Definition 8). It should be mentioned that this relation, when it exists, is not related to the possibility of getting an ONB in a larger space (see the Naimark dilation theorem 2, [18]) since it works in the same Hilbert space. In fact, it is more connected with what is stated in [19, Corollary 8.33] or [11, Theorem 5.5.5]. Conclusions are given in Sect. 4.

2 Bounded Hamiltonians Defined by Parseval Frames

2.1 Frames and Parseval Frames

Here all necessary information about frame theory are presented in a form convenient for our exposition. More details on frames can be found in monographs [11, 19]. The papers [6, 8, 10, 18] are recommended as complementary reading on the subject.

Let \mathcal{K} be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ linear in the second argument. Let \mathbb{J} denote a generic countable (or finite) index set such as \mathbb{Z} , \mathbb{N} , $\mathbb{N} \cup \{0\}$, etc. By $|\mathbb{J}|$ we denote the cardinality of \mathbb{J} .

Definition 1 A set of vectors $\mathcal{F}_\varphi = \{\varphi_j, j \in \mathbb{J}\}$ is called a frame in \mathcal{K} if there are constants A and B , $0 < A \leq B < \infty$, such that, for all $f \in \mathcal{K}$,

$$A\|f\|^2 \leq \sum_{j \in \mathbb{J}} |\langle \varphi_j, f \rangle|^2 \leq B\|f\|^2. \quad (2.1)$$

The optimal constants in (2.1) (maximal for A and minimal for B) are called the frame bounds. A frame \mathcal{F}_φ is called a *tight frame* if $A = B$, and is called a *Parseval frame* (PF in the following) if $A = B = 1$.

The potential of a frame \mathcal{F}_φ is defined by $\mathbf{FP}[\mathcal{F}_\varphi] = \sum_{j,i \in \mathbb{J}} |(\varphi_i, \varphi_j)|^2$. Following [6], we recall that the excess $\mathbf{e}[\mathcal{F}_\varphi]$ of \mathcal{F}_φ is the greatest integer n such that n elements can be deleted from the frame \mathcal{F}_φ and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed.

Formula (2.1) with $A = B = 1$ is quite similar to the Parseval equality for ONBs and in fact it is well known that a PF is indeed an ONB if each vector in \mathcal{F}_φ is normalized: $\|\varphi_j\| = 1$ for all $j \in \mathbb{J}$, and vice-versa. On the other hand, some of the vectors in a PF may be the zero vector, while this is not allowed for ONBs. Similarly to ONBs, PFs have the maximality property: they cannot be enlarged to a PF by adding non-zero vectors (a unique way to enlarge is to add zero vectors).

It is well known that, given an ONB $\mathcal{F}_e = \{e_j, j \in \mathbb{J}\}$ in a Hilbert space \mathcal{H} containing \mathcal{K} as a subspace, and an orthogonal projection $P : \mathcal{H} \rightarrow \mathcal{K}$, the set $\mathcal{F}_\varphi = \{\varphi_j = P e_j, j \in \mathbb{J}\}$ is a PF for \mathcal{K} . The inverse of this statement is the so-called Naimark theorem (see, for instance, [18, Propositions 1.1, 1.4]), which is crucial for our investigations. More precisely:

Theorem 2 *Let $\mathcal{F}_\varphi = \{\varphi_j, j \in \mathbb{J}\}$ be a PF in a Hilbert space \mathcal{K} . Then there exists a Hilbert space \mathcal{M} and a PF $\mathcal{F}_\psi = \{\psi_j, j \in \mathbb{J}\}$ in \mathcal{M} such that*

$$\mathcal{F}_e = \{e_j = \varphi_j \oplus \psi_j : j \in \mathbb{J}\} \quad (2.2)$$

is an ONB for² $\mathcal{H} = \mathcal{K} \oplus \mathcal{M}$.

Remark 3 It follows from [6, Lemma 4.1] and the proof of [18, Proposition 1.1] that the dimension of spaces \mathcal{K} and \mathcal{M} in Theorem 2 coincide, respectively, with the potential $\mathbf{FP}[\mathcal{F}_\varphi]$ and with the excess $\mathbf{e}[\mathcal{F}_\varphi]$ of \mathcal{F}_φ . Moreover, $\dim \mathcal{H} = |\mathbb{J}| = \mathbf{FP}[\mathcal{F}_\varphi] + \mathbf{e}[\mathcal{F}_\varphi]$.

Each Parseval frame \mathcal{F}_φ determines an analysis operator $\theta_\varphi : \mathcal{K} \rightarrow \ell_2(\mathbb{J})$:

$$\theta_\varphi(f) = \{\langle \varphi_j, f \rangle\}_{j \in \mathbb{J}}, \quad f \in \mathcal{K}, \quad (2.3)$$

which is an isometry $\theta_\varphi : \mathcal{K} \rightarrow \ell_2(\mathbb{J})$. The image set $\mathcal{R}(\theta_\varphi)$ is a closed subspace of $\ell_2(\mathbb{J})$ and, because of the Theorem 2,

$$\ell_2(\mathbb{J}) = \mathcal{R}(\theta_\varphi) \oplus \mathcal{R}(\theta_\psi), \quad (2.4)$$

where the isometry $\theta_\psi : \mathcal{M} \rightarrow \ell_2(\mathbb{J})$ is defined by the PF \mathcal{F}_ψ introduced in Theorem 2.

The next statement is well known in the frame theory. For convenience of the reader we give its simple proof.

Lemma 4 *Let $\{c_j\}_{j \in \mathbb{J}} \in \ell_2(\mathbb{J})$. Then $\sum_{j \in \mathbb{J}} c_j \varphi_j = 0$ if and only if $\{c_j\}_{j \in \mathbb{J}} \in \mathcal{R}(\theta_\psi)$.*

Proof If $\sum c_j \varphi_j = 0$, then $f = \sum c_j e_j = \sum c_j \varphi_j + \sum c_j \psi_j = \sum c_j \psi_j$ due to (2.2). This means that $f \in \mathcal{M}$ and $c_j = \langle e_j, f \rangle = \langle \psi_j, f \rangle$. Hence, $\theta_\psi(f) = \{c_j\}$.

Conversely, if $\{c_j\} \in \mathcal{R}(\theta_\psi)$, then there exists $f \in \mathcal{M}$ such that $c_j = \langle \psi_j, f \rangle = \langle e_j, f \rangle$. Hence, $f = \sum c_j e_j = \sum c_j \varphi_j + \sum c_j \psi_j$ that gives $\sum c_j \varphi_j = 0$, since $f \in \mathcal{M}$. \square

²If \mathcal{F}_φ is an ONB, the understanding is that \mathcal{M} will be the zero space and each ψ_j will be zero vector.

Lemma 5 *The Gram matrix $G_\varphi = [\langle \varphi_i, \varphi_j \rangle]_{i,j \in \mathbb{J}}$ of a PF \mathcal{F}_φ determines a bounded self-adjoint operator in $\ell_2(\mathbb{J})$ which is the orthogonal projector in $\ell_2(\mathbb{J})$ onto $\mathcal{R}(\theta_\varphi)$.*

Proof In view of [19, Theorem 7.5], the Gram matrix G_φ determines a bounded self-adjoint operator in $\ell_2(\mathbb{J})$, which acts as follows $G_\varphi : \{c_j\}_{j \in \mathbb{J}} \rightarrow \{\sum_{j \in \mathbb{J}} \langle \varphi_i, \varphi_j \rangle c_j\}_{i \in \mathbb{J}}$. This operator coincides with the orthogonal projection in $\ell_2(\mathbb{J})$ onto $\mathcal{R}(\theta_\varphi)$ constructed in [11, Proposition 5.3.6] (here, we should take into account that the frame operator S in the formula (5.12) of [11] is the identity operator, since \mathcal{F}_φ is a PF and the scalar product in [11] is linear in the first argument). \square

2.2 Working with Bounded Operators

Assume that $\{E_j\}_{j \in \mathbb{J}}$ is a bounded sequence of real numbers. Hence, there exist $E_{\min} = \inf_{j \in \mathbb{J}} E_j$ and $E_{\max} = \sup_{j \in \mathbb{J}} E_j$ such that $-\infty < E_{\min} \leq E_j \leq E_{\max} < \infty$. The Hamiltonian generated by a PF \mathcal{F}_φ and a collection of numbers $\{E_j\}$:

$$H_\varphi f = \sum_{j \in \mathbb{J}} E_j \langle \varphi_j, f \rangle \varphi_j, \quad f \in \mathcal{K} \quad (2.5)$$

is a bounded everywhere defined self-adjoint operator in \mathcal{K} . These properties of H_φ can be established as follows: due to Theorem 2 there exists an ONB $\{e_j\}$ of \mathcal{H} such that $P e_j = \varphi_j$, where P is the orthogonal projection in \mathcal{H} on \mathcal{K} . Therefore, $H_\varphi = P H_e P$, where

$$H_e g = \sum_{j \in \mathbb{J}} E_j \langle e_j, g \rangle e_j, \quad g \in \mathcal{H} \quad (2.6)$$

is a bounded self-adjoint operator in \mathcal{H} with eigenvalues $\{E_j\}$ and the corresponding eigenvectors $\{e_j\}$.

The spectrum of H_e coincides with the closure of the set of its eigenvalues: $\sigma(H_e) = \overline{\{E_j\}}$. Since $\langle H_e f, f \rangle = \langle H_\varphi f, f \rangle$ for $f \in \mathcal{K}$, the spectrum $\sigma(H_\varphi)$ is contained in $[E_{\min}, E_{\max}]$. In general, as already noticed, we cannot claim that the quantities $\{E_j\}$ in (2.5) are also eigenvalues of H_φ . This will become evident in the examples below.

Let us assume that $H_\varphi f = \mu f$ for nonzero $f \in \mathcal{K}$. Since \mathcal{F}_φ is a PF, $f = \sum_{j \in \mathbb{J}} \langle \varphi_j, f \rangle \varphi_j$ and the eigenvalue equation takes the form:

$$\sum_{j \in \mathbb{J}} (E_j - \mu) \langle \varphi_j, f \rangle \varphi_j = 0. \quad (2.7)$$

In view of Lemma 4, the relation (2.7) is equivalent to the condition $\{(E_j - \mu)c_j\} \in \mathcal{R}(\theta_\psi)$, where $\{c_j = \langle \varphi_j, f \rangle\} \in \mathcal{R}(\theta_\varphi)$. Summing up, this proves:

Theorem 6 *The operator H_φ has an eigenvalue μ if and only if there exists a sequence $\{c_j\} \in \mathcal{R}(\theta_\varphi)$ such that $\{(E_j - \mu)c_j\} \in \mathcal{R}(\theta_\psi)$, where $\mathcal{R}(\theta_\varphi)$ and $\mathcal{R}(\theta_\psi)$ are the orthogonal subspaces of $\ell_2(\mathbb{J})$, see (2.4). In this case, the corresponding eigenvector is $f = \sum_{j \in \mathbb{J}} c_j \varphi_j$.*

Lemma 5 allows one to carry out modifications of Theorem 6 which, sometimes, are more convenient for the calculation of eigenvalues. Simultaneously with the Gram matrices $G_\varphi = [\langle \varphi_i, \varphi_j \rangle]_{i,j \in \mathbb{J}}$ and $G_\psi = [\langle \psi_i, \psi_j \rangle]_{i,j \in \mathbb{J}}$, we will use the matrix-valued functions $B_\varphi(\mu) = [b_{ij}(\mu)]_{i,j \in \mathbb{J}}$ and $D_\varphi(\mu) = [d_{ij}(\mu)]_{i,j \in \mathbb{J}}$, where

$$b_{ij}(\mu) = \sum_{k \in \mathbb{J}} (E_k - \mu) \langle \varphi_i, \varphi_k \rangle \langle \varphi_k, \varphi_j \rangle, \quad d_{ij}(\mu) = (E_i - \mu) \langle \varphi_i, \varphi_j \rangle. \quad (2.8)$$

Corollary 7 Let H_φ be a Hamiltonian generated by a PF \mathcal{F}_φ , see (2.5). The following are equivalent:

- (i) $\mu \in \mathbb{R}$ is an eigenvalue of H_φ ;
- (ii) there exists $\{c_j\} \in \ell_2(\mathbb{J})$ such that $G_\varphi\{c_j\} \neq 0$ and $B_\varphi(\mu)\{c_j\} = 0$;
- (iii) there exist $\{c_j\}, \{\tilde{c}_j\} \in \ell_2(\mathbb{J})$ such that $G_\varphi\{c_j\} \neq 0$ and $D_\varphi(\mu)\{c_j\} = G_\psi\{\tilde{c}_j\}$.

Proof Let $\mathcal{F}_e = \{e_j = \varphi_j \oplus \psi_j : j \in \mathbb{J}\}$ be an ONB of \mathcal{H} determined in Theorem 2 and let

$$\theta_e(g) = \{\langle e_j, g \rangle\}_{j \in \mathbb{J}}, \quad g \in \mathcal{H} \quad (2.9)$$

be the corresponding analysis operator that isometrically maps \mathcal{H} onto $\ell_2(\mathbb{J})$. In view of (2.6), $\theta_e(H_e g) = \mathcal{E}\theta_e(g)$, where \mathcal{E} is a bounded multiplication operator in $\ell^2(\mathbb{J})$:

$$\mathcal{E}\{c_j\} = \{E_j c_j\}, \quad D(\mathcal{E}) = \ell^2(\mathbb{J}). \quad (2.10)$$

Since, for all $f \in \mathcal{K}$, $\theta_e(f) = \{\langle e_j, f \rangle\} = \langle P e_j, f \rangle = \langle \varphi_j, f \rangle = \theta_\varphi(f)$, where P is the orthogonal projection in \mathcal{H} on \mathcal{K} and θ_e is defined by (2.9), we arrive at the conclusion that θ_e maps \mathcal{K} onto $\mathcal{R}(\theta_\varphi)$ (see the decomposition (2.4)). Taking into account that $H_\varphi = P H_e P$, we obtain that H_φ is unitary equivalent to the operator \mathcal{E}_φ acting in the subspace $\mathcal{R}(\theta_\varphi)$ of $\ell^2(\mathbb{J})$:

$$\mathcal{E}_\varphi = \mathcal{P} \mathcal{E} \mathcal{P}, \quad D(\mathcal{E}_\varphi) = \mathcal{R}(\theta_\varphi) = \mathcal{P} \ell^2(\mathbb{J}), \quad (2.11)$$

where $\mathcal{P} = \theta_e P \theta_e^{-1}$ is the orthogonal projection operator in $\ell^2(\mathbb{J})$ on $\mathcal{R}(\theta_\varphi)$.

(i) \iff (ii). In view of (2.11), $\mu \in \sigma_p(H_\varphi)$ if and only if there exists $\{c_j\} \in \ell_2(\mathbb{J})$ such that $\mathcal{P}\{c_j\} \neq 0$ and $\mathcal{P}(\mathcal{E} - \mu I)\mathcal{P}\{c_j\} = 0$. By Lemma 5, the operator \mathcal{P} coincides with the Gram operator G_φ . Therefore, the first condition takes the form $G_\varphi\{c_j\} \neq 0$, whereas the second one: $G_\varphi(\mathcal{E} - \mu I)G_\varphi\{c_j\} = B_\varphi(\mu)\{c_j\} = 0$.

To prove the equivalence of (i) and (iii) it suffices to repeat the previous arguments and rewrite the condition $\mathcal{P}(\mathcal{E} - \mu I)\mathcal{P}\{c_j\} = 0$ as $(\mathcal{E} - \mu I)\mathcal{P}\{c_j\} \in \mathcal{R}(\theta_\psi)$. Taking into account that G_ψ is an orthogonal projection on $\mathcal{R}(\theta_\psi)$ and $(\mathcal{E} - \mu I)\mathcal{P}\{c_j\} = D_\varphi(\mu)\{c_j\}$ we complete the proof. \square

2.3 Examples

In this section we propose some examples of H and their related H_{ph} , starting with purely mathematical examples, and then considering applications arising in concrete applications discussed in the literature. In particular, in Examples 1, 2, and 3, we will show how a physical Hamiltonian $H_\varphi = \sum_{j=1}^N E_j \langle \varphi_j, \cdot \rangle \varphi_j$, given in terms of some particular PF, can be rewritten in terms of an ONB of its eigenvectors, and how its related eigenvalues are different from the E_j in the expansion above. Of course, in view of what we have discussed in the Introduction, H_φ should be understood as the physical part of another, *larger*, Hamiltonian, H , which produces H_{ph} after a suitable projection. This means that we have a first ONB, given by the eigenvectors of H , which, when projected to \mathcal{H}_{ph} , defines a PF. Hence, to find the physical content of H_{ph} , we need to diagonalize H_{ph} , getting a second ONB, different from the first one, which can really be considered as the *physical eigenvectors* of the system. In Examples 4 and 5 these steps will be particularly emphasized.

Example 1 (Mercedes frame) In the Hilbert space $\mathcal{H} = \mathbb{C}^3$, we consider the ONB $\mathcal{F}_e = \{e_j, j \in \mathbb{J}\}$, where $\mathbb{J} = \{1, 2, 3\}$ and

$$e_1 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad e_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad e_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The orthogonal projection of $\mathcal{F}_e = \{e_1, e_2, e_3\}$ onto the subspace $\mathcal{K} = \mathbb{C} \oplus \mathbb{C} \oplus 0$ gives rise to a PF $\mathcal{F}_\varphi = \{\varphi_j, j \in \mathbb{J}\}$, where

$$\varphi_1 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \quad \varphi_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}.$$

The PF \mathcal{F}_φ is called the Mercedes frame. Its dual PF $\mathcal{F}_\psi = \{\psi_j, j \in \mathbb{J}\}$ in (2.1) consists of vectors of the subspace $\mathcal{M} = 0 \oplus 0 \oplus \mathbb{C}$ (see formula (2.2))

$$\psi_1 = \psi_2 = \psi_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

The image sets of the analysis operators θ_φ and θ_ψ are subspaces of $\ell_2(\mathbb{J}) = \mathbb{C}^3$:

$$\mathcal{R}(\theta_\varphi) = \left\{ \begin{bmatrix} 2x_1 \\ -x_1 + \sqrt{3}x_2 \\ -x_1 + \sqrt{3}x_2 \end{bmatrix}, x_1, x_2 \in \mathbb{C} \right\}, \quad \mathcal{R}(\theta_\psi) = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}, x_3 \in \mathbb{C} \right\}.$$

Let E_1, E_2 , and E_3 be real quantities. Then the Hamiltonian³ $H_\varphi = \sum_{j=1}^3 E_j \langle \varphi_j, \cdot \rangle \varphi_j$ generated by the Mercedes frame acts in \mathbb{C}^2 (we identify \mathcal{K} with \mathbb{C}^2) and, due to Theorem 6, $\mu \in \mathbb{R}$ is an eigenvalue of H_φ if and only if the linear system

$$\begin{cases} 2(E_1 - \mu)x_1 = x_3 \\ (E_2 - \mu)(-x_1 + \sqrt{3}x_2) = x_3 \\ (E_3 - \mu)(-x_1 - \sqrt{3}x_2) = x_3 \end{cases}$$

has a nonzero solution x_1, x_2, x_3 . An elementary calculation shows that $\mu \in \sigma(H_\varphi)$ if and only if

$$(E_1 - \mu)(E_2 - \mu) + (E_2 - \mu)(E_3 - \mu) + (E_1 - \mu)(E_3 - \mu) = 0. \quad (2.12)$$

Another way to obtain (2.12) is, of course, to present H_φ (acting in \mathbb{C}^2) in the matrix form

$$H_\varphi = \frac{1}{6} \begin{bmatrix} 4E_1 + E_2 + E_3 & \sqrt{3}(E_3 - E_2) \\ \sqrt{3}(E_3 - E_2) & 3(E_2 + E_3) \end{bmatrix} \quad (2.13)$$

³We are assuming here that H_φ arises from a different operator H , acting on a larger Hilbert space \mathbb{C}^3 , after taking its projection $H_\varphi = PHP$ on the physical Hilbert space $\mathcal{H}_{ph} = \mathbb{C}^2$. Notice that we are not giving here the explicit form of H , since it is not really relevant. We will do this in Example 4 and Example 5, since in those cases H has a physical meaning.

and, then, to solve the characteristic equation $\det[H_\varphi - \mu I] = 0$. More detailed calculations show that the eigenvalues are

$$\tilde{E}_\pm = \frac{1}{3} \left(E_1 + E_2 + E_3 \pm \sqrt{E_1^2 + E_2^2 + E_3^2 - E_1 E_2 - E_2 E_3 - E_1 E_3} \right),$$

with eigenvectors (if $E_2 \neq E_3$)

$$\tilde{e}_\pm = N_\pm \begin{bmatrix} -2E_1 + E_2 + E_3 \mp 2\sqrt{E_1^2 + E_2^2 + E_3^2 - E_1 E_2 - E_2 E_3 - E_1 E_3} \\ \sqrt{3}(E_2 - E_3) \end{bmatrix}.$$

Here N_\pm are normalization constants fixed to have $\|\tilde{e}_\pm\| = 1$. This means that, using a bracket notation, the operator H_φ can be rewritten in term of an ONB of its eigenstates:

$$H_\varphi = \sum_{j=1}^3 E_j |\varphi_j\rangle \langle \varphi_j| = \sum_{i=\pm} \tilde{E}_i |\tilde{e}_i\rangle \langle \tilde{e}_i|.$$

This equality clarifies once more that, while \tilde{E}_i is an eigenvalue of H_φ , E_j is not.

Example 2 Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_K\}$, $K \geq 2$ be an ONB of a Hilbert space \mathcal{K} . The set $\mathcal{F}_\varphi = \{\varphi_j, j \in \mathbb{J}\}$ where $\mathbb{J} = \{1, 2, \dots, K+1\}$ and

$$\varphi_j = \mathbf{e}_j - \frac{1}{K} \sum_{i=1}^K \mathbf{e}_i, \quad 1 \leq j \leq K, \quad \varphi_{K+1} = \frac{1}{\sqrt{K}} \sum_{i=1}^K \mathbf{e}_i, \quad (2.14)$$

is a PF for \mathcal{K} [9, Lemma 2.5]. Representing each vector $f \in \mathcal{K}$ as $f = \sum_{i=1}^K x_i \mathbf{e}_i$ and using (2.9), we define the subspaces $\mathcal{R}(\theta_\varphi)$, $\mathcal{R}(\theta_\psi)$ of $\ell_2(\mathbb{J}) = \mathbb{C}^{K+1}$:

$$\mathcal{R}(\theta_\varphi) = \left\{ \begin{bmatrix} x_1 - \frac{1}{K} \sum_{i=1}^K x_i \\ \vdots \\ x_K - \frac{1}{K} \sum_{i=1}^K x_i \\ \frac{1}{\sqrt{K}} \sum_{i=1}^K x_i \end{bmatrix}, x_i \in \mathbb{C} \right\}, \quad \mathcal{R}(\theta_\psi) = \mathbb{C}^{K+1} \ominus \mathcal{R}(\theta_\varphi) = \left\{ \begin{bmatrix} x \\ \vdots \\ x \\ 0 \end{bmatrix}, x \in \mathbb{C} \right\}.$$

Let $E_1 < E_2 < \dots < E_{K+1}$ be real quantities. Then the Hamiltonian $H_\varphi = \sum_{j=1}^{K+1} E_j |\varphi_j\rangle \langle \varphi_j|$ generated by the frame \mathcal{F}_φ acts in \mathcal{K} and, due to Theorem 6, its eigenvalues \tilde{E}_j coincide with $\mu \in \mathbb{R}$ for which the linear system

$$\begin{cases} (E_1 - \mu)(x_1 - \frac{1}{K} \sum_{i=1}^K x_i) = x \\ \vdots \\ (E_K - \mu)(x_K - \frac{1}{K} \sum_{i=1}^K x_i) = x \\ (E_{K+1} - \mu)(\frac{1}{\sqrt{K}} \sum_{i=1}^K x_i) = 0 \end{cases}$$

has a nonzero solution x_1, x_2, \dots, x_K, x . An elementary analysis shows that the largest eigenvalue \tilde{E}_K of H_φ coincides with E_{K+1} , while the other eigenvalues \tilde{E}_j , $j = 1, \dots, K-1$ are the roots of the equation

$$\frac{1}{E_1 - \mu} + \frac{1}{E_2 - \mu} + \cdots + \frac{1}{E_K - \mu} = 0. \quad (2.15)$$

The corresponding eigenfunctions are

$$f_K = \frac{1}{K} \sum_{i=1}^K \mathbf{e}_i \quad (\text{for } \tilde{E}_K = E_{K+1}) \quad \text{and} \quad f_j = \sum_{i=1}^K \frac{1}{E_i - \mu_j} \mathbf{e}_i,$$

where μ_j ($j = 1, \dots, K-1$) is a solution of (2.15). After the normalization of f_j we obtain $H_\varphi = \sum_{j=1}^K \tilde{E}_j |f_j\rangle\langle f_j|$, which is a different way to write $H_\varphi = \sum_{j=1}^{K+1} E_j |\varphi_j\rangle\langle \varphi_j|$.

Example 3 The finite PF considered above is a key counterpart of an infinite PF which contains no subset that is a Riesz basis [9]. We investigate spectral properties of Hamiltonians generated by this curious PF.

Let \mathcal{K} be a separable Hilbert space. Index an ONB for \mathcal{K} as $\{\mathbf{e}_j^K\}_{K \in \mathbb{N}, j=1, \dots, K}$. Set $\mathcal{K}_K = \text{span}\{\mathbf{e}_1^K, \mathbf{e}_2^K, \dots, \mathbf{e}_K^K\}$. The vectors $\varphi_j^K \equiv \varphi_j$ defined by (2.14) form a PF $\mathcal{F}_{\varphi^K} = \{\varphi_j^K, j = 1, 2, \dots, K+1\}$ of the space \mathcal{K}_K . Since $\mathcal{K} = \sum_{K=1}^\infty \oplus \mathcal{K}_K$, the collection of vectors $\mathcal{F}_\varphi = \sum_{K=1}^\infty \oplus \mathcal{F}_{\varphi^K}$ is a PF for \mathcal{K} (and it was constructed in [9]).

Assume that $\{E_j^K\}_{K \in \mathbb{N}, j=1, \dots, K+1}$ is a bounded strictly increasing sequence, i.e. $E_1^1 < E_2^1 < E_1^2 < E_2^2 < E_3^2 \dots < E_1^K < E_2^K \dots < E_K^K < E_{K+1}^K \dots$. The Hamiltonian generated by \mathcal{F}_φ is

$$H_\varphi = \sum_{K=1}^\infty \sum_{j=1}^{K+1} E_j^K |\varphi_j^K\rangle\langle \varphi_j^K|$$

It is easy to verify (using the previous example) that the point spectrum of H_φ involves the subset $\{E_{K+1}^K\}_{K=1}^\infty$ of the original quantities and the solutions of the equations, cf. (2.15):

$$\frac{1}{E_1^K - \mu} + \frac{1}{E_2^K - \mu} + \cdots + \frac{1}{E_K^K - \mu} = 0, \quad K \geq 2.$$

Let us now discuss a fourth, physically motivated, example, based on the anti-commutation relations (CAR) for two fermionic modes.

Example 4 Let a_1 and a_2 be the operators satisfying the CAR $\{a_j, a_k^\dagger\} = \delta_{j,k} \mathbb{I}$, with $\{a_j, a_k\} = 0$, $j, k = 1, 2$. Here \mathbb{I} is the identity operator in the Hilbert space of the system, which is $\mathcal{H} = \mathbb{C}^4$. Calling $\eta_{0,0}$ the vacuum of a_j , $a_j \eta_{0,0} = 0$, $j = 1, 2$, we can construct three more vectors acting on it with a_j^\dagger : $\eta_{1,0} = a_1^\dagger \eta_{0,0}$, $\eta_{0,1} = a_2^\dagger \eta_{0,0}$ and $\eta_{1,1} = a_1^\dagger a_2^\dagger \eta_{0,0}$. An explicit realization of these vectors and operators is the following: $\eta_{0,0} = \delta_1$, $\eta_{1,0} = \delta_2$, $\eta_{0,1} = \delta_3$, $\eta_{1,1} = \delta_4$, where $\{\delta_j\}$ is the canonical ONB in \mathbb{C}^4 , and

$$a_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In [3] these operators have been used to construct a dynamical system describing two populations moving in a two-dimensional lattice, and interacting adopting a sort of predator-prey mechanism. The dynamics has been produced by the sum of several copies of a single-cell

term, H_α (α labels the lattice cells), plus a global contribution responsible for the migration of the species. Here we only consider the single-cell term, which we rewrite as follows:

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda(a_1^\dagger a_2 + a_2^\dagger a_1),$$

where ω_1 , ω_2 and λ are parameters whose values can be fixed in different way, according to which aspect of the system we want to put in evidence, see [3]. We also refer to [3] for the meaning of this Hamiltonian, for the rationale for its introduction, and for the dynamics connected to it. The matrix expression for H is

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega_1 & \lambda & 0 \\ 0 & \lambda & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_1 + \omega_2 \end{bmatrix},$$

whose eigenvalues are

$$E_1 = 0, \quad E_2 = \omega_1 + \omega_2 \quad E_3 = \frac{1}{2} \left(\omega_1 + \omega_2 - \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2} \right)$$

and

$$E_4 = \frac{1}{2} \left(\omega_1 + \omega_2 + \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2} \right)$$

with (normalized) eigenvectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad e_3 = \frac{f_3}{\|f_3\|}, \quad e_4 = \frac{f_4}{\|f_4\|},$$

where

$$f_3 = \begin{bmatrix} 0 \\ \frac{1}{2\lambda} \left(\omega_1 - \omega_2 - \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2} \right) \\ 1 \\ 0 \end{bmatrix},$$

$$f_4 = \begin{bmatrix} 0 \\ \frac{1}{2\lambda} \left(\omega_1 - \omega_2 + \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2} \right) \\ 1 \\ 0 \end{bmatrix}.$$

For concreteness, if we take $\lambda = 2$, $\omega_1 = \frac{1}{2}$ and $\omega_2 = \frac{7}{2}$, then

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 2 & 0 \\ 0 & 2 & 7/2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad e_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \sqrt{\frac{4}{5}} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix},$$

while e_1 and e_2 are those previously introduced. Also, $E_1 = 0$, $E_2 = 4$, $E_3 = -\frac{1}{2}$, $E_4 = \frac{9}{2}$, and we can rewrite $H = \sum_{j=1}^4 E_j |e_j\rangle\langle e_j|$.

Due to the physical interpretation of the model, [3], if we consider the orthogonal projector $P = \mathbb{I} - P_{1,0}$, $P_{1,0}f = \langle \eta_{1,0}, f \rangle \eta_{1,0}$, P project the system on a space in which it is impossible to find the system in a state with high density of the first species, and low density of the other species. In other words, this state is forbidden for us. The (biological) reason for requiring this is that, for instance, we want to avoid a dominance of the first species on the second one. The computation of H_{ph} is easy: as for the effect of P on the eigenvectors $\{e_j\}$ of H , we get four vectors $\varphi_j = Pe_j$ in $\mathcal{H} = \mathbb{C}^4$. Removing the zero second component from each one of these vectors⁴ we recover the following PF $\mathcal{F}_\varphi = \{\varphi_j, 1 \leq j \leq 4\}$ in $\mathcal{K} = \mathbb{C}^3$,

$$\varphi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \varphi_3 = \cos \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \varphi_4 = \sin \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad (2.16)$$

where

$$\cos \beta = \frac{1}{\|f_3\|} = \frac{2|\lambda|}{\sqrt{[\omega_1 - \omega_2 - \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2}]^2 + 4\lambda^2}},$$

$$\sin \beta = \frac{1}{\|f_4\|} = \frac{2|\lambda|}{\sqrt{[\omega_1 - \omega_2 + \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2}]^2 + 4\lambda^2}}, \quad \beta \in (0, \pi/2).$$

The operator H_{ph} acting in $\mathcal{K} = \mathbb{C}^3$ coincides with the operator $H_\varphi = \sum_{j=1}^4 E_j |\varphi_j\rangle\langle \varphi_j|$ generated by \mathcal{F}_φ . In view of (2.8) and (2.16), the Gram matrix G_φ and $B_\varphi(\mu)$ are:

$$G_\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos^2 \beta & \cos \beta \sin \beta \\ 0 & 0 & \cos \beta \sin \beta & \sin^2 \beta \end{bmatrix}, \quad B_\varphi(\mu) = \begin{bmatrix} E_1 - \mu & 0 & 0 & 0 \\ 0 & E_2 - \mu & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix},$$

where $b_{33} = (E_3 - \mu) \cos^4 \beta + (E_4 - \mu) \cos^2 \beta \sin^2 \beta$, $b_{44} = (E_3 - \mu) \cos^2 \beta \sin^2 \beta + (E_4 - \mu) \sin^4 \beta$, $b_{34} = b_{43} = (E_3 - \mu) \cos^3 \beta \sin \beta + (E_4 - \mu) \cos \beta \sin^3 \beta$.

These matrices and Corollary 7 allow one to describe eigenvalues \tilde{E}_1 , \tilde{E}_2 , and \tilde{E}_3 of $H_{ph} = H_\varphi$. Precisely, $\tilde{E}_1 = E_1 = 0$, $\tilde{E}_2 = E_2 = \omega_1 + \omega_2$, while \tilde{E}_3 coincides with the solution μ of the equation $(E_3 - \mu) \cos^2 \beta + (E_4 - \mu) \sin^2 \beta = 0$. So we see how the eigenvalues of H_{ph} are different from those of the original Hamiltonian H . Of course, these differences have consequences on the dynamics of the system, but this aspect will not be discussed in this paper. However, we want to stress that differences in the eigenvalues imply, among other differences, different stationary states. This might have serious, and very interesting consequences: a system which, in principle, should not evolve in time (since it is in a supposed stationary state), does indeed evolve. Or vice-versa. In both cases this can be related to the fact that the *real* Hamiltonian for the system is not H , but H_{ph} . This is because of some constraint on the system.

⁴This component must be zero due to the action of P on e_j .

Fig. 1 A schematic view to the single-garbage ecosystem

The last example we want to discuss here is again built in terms of anti-commutation relations (CAR) and fermionic modes, and is based on an ecological system considered first in [4] and then in [1].

Example 5 The system we want to describe is made by two levels of organisms (L_1 and L_2), one compartment for the nutrients and another compartment for the garbage, see Fig. 1 for a schematic view. The nutrients feed the organisms of L_1 , which feed those of L_2 . Moreover, when dying, the organisms of both levels contribute to increase the density of the garbage which, after some time, turns into nutrients. This is a simple example of closed ecosystem. More complicated systems (with more levels and with different kind of garbages) are considered in [4].

The dynamics of our system is described by the following Hamiltonian:

$$\left\{ \begin{array}{l} H = H_0 + H_I, \quad \text{with} \\ H_0 = \sum_{j=0}^3 \omega_j a_j^\dagger a_j, \\ H_I = \sum_{j=0}^2 \lambda_j (a_j a_3^\dagger + a_3 a_j^\dagger) + \sum_{j=0}^1 v_j (a_j a_{j+1}^\dagger + a_{j+1} a_j^\dagger), \end{array} \right.$$

where $\{a_j, a_k^\dagger\} = \delta_{j,k} \mathbb{I}$, and $a_j^2 = 0$, for all $j, k = 0, 1, 2, 3$, and where ω_j, v_j and λ_j are real constants, whose meaning is explained in [1, 4]. The zero-th fermionic mode ($j = k = 0$) is related to the nutrients, the 3-th mode to the garbage, while the two remaining modes describe the organisms of the various trophic levels. We again refer to [4] for the biological meaning of the various terms in H . Here, we just comment that, for instance, the term $\lambda_j a_j a_3^\dagger$, describes the fact that the garbage is recycled by decomposers and transformed into nutrients: this is due to the fact that the density of the garbage decrease (because of the presence of the lowering operator a_3), and simultaneously the density of the nutrients increases (because of the raising operator a_0^\dagger).

The equations of motion can be deduced using the Heisenberg rule $\dot{X} = i[H, X]$, and the solution (which is not essential here) can be found in [4]. Here we want to see how the idea introduced in this paper works for this particular example. For that, let $\mathbf{e}_{0,0,0,0}$ be the vacuum of the system: $a_j \mathbf{e}_{0,0,0,0} = 0$ for $j = 0, 1, 2, 3$. This corresponds to an *essentially empty* system: very low densities in all the levels (including garbage and nutrients). Analogously, $\mathbf{e}_{0,0,1,0}$ is a vector describing a situation where only L_2 is filled, while all the other levels are essentially empty, and so on. We now fix, for concreteness, $\omega_0 = \omega_3 = 2$, $\omega_1 = 3$, $\omega_2 = 4$, $\lambda_0 = 1$, $\lambda_1 = 2$, $\lambda_2 = 3$, $v_0 = 1$ and $v_1 = 3$. The criterium for choosing some particular values of the parameters of the Hamiltonian is widely discussed in [1]. Here this choice is relevant just to allow a simple computation of the eigenvalues and the eigenvectors of H , which acts

in $\mathcal{H} = \mathbb{C}^{16}$ and turns out to be the Hermitian matrix:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & -3 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -3 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 4 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 6 & -1 & 0 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 7 & 0 & 0 & 0 & -3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & -3 & 0 & 2 & -1 & 0 \\ 0 & -1 & -2 & 0 & -3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -3 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & -1 & 5 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 7 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -3 & 0 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 \end{bmatrix}.$$

Using any mathematical software one easily deduces the following list of eigenvalues $\{E_j\}_{j=1}^{16}$, ordered in decreasing absolute value:

$$\begin{aligned} &13.6645, \quad 11.4925, \quad 11, \quad 9.17202, \quad 8.82798, \quad 7, \quad 6.66453, \quad 6.50749, \\ &4.49251, \quad 4.33547, \quad 4, \quad -2.66453, \quad 2.17202, \quad 1.82798, \quad -0.492505, \quad 0. \end{aligned}$$

Denote by e_j the corresponding normalized eigenvectors for H . Then $\mathcal{F}_e = \{e_j\}_{j=1}^{16}$ is an ONB of \mathcal{H} and $H = \sum_{j=1}^{16} E_j |e_j\rangle\langle e_j|$. H is the analogous of the Hamiltonian H_e in (2.6).

What we are interested in here is to see the mathematical consequences if we assume that the second level, L_2 , cannot be filled because of, say, some particular reason: for instance, the organisms in L_2 receive nutrients from L_1 , see Fig. 1, but the external conditions can be not sufficiently good to allow these organisms to survive. From a mathematical point of view, this situation can be described by introducing the following orthogonal projector

$$P_{L_2} = \sum_{i,j,k=0,1} P_{i,j,0,k},$$

where $P_{i,j,0,k} f = \langle \mathbf{e}_{i,j,0,k}, f \rangle \mathbf{e}_{i,j,0,k}$, for all vector f in \mathcal{H} . The physical Hamiltonian $H_{ph} = P_{L_2} H P_{L_2}$ is an operator acting in the subspace $\mathcal{K} = \text{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}\}$ of \mathcal{H} ,

where $\{\delta_j\}$ is the canonical ONB of \mathbb{C}^{16} :

$$H_{ph} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

whose eigenvalues are the following:

$$7.38849, \quad 7, \quad 4.57577, \quad 4.18728, \quad 2.81272, \quad 2.42423, \quad -0.38849, \quad 0.$$

We see that the eigenvalues of H and H_{ph} are different, and that the spectrum $\sigma(H_{ph})$ is contained in $[E_{\min}, E_{\max}]$, as stated before.

Finally, the set $\mathcal{F}_\varphi = \{\varphi_j = P_{L_2} e_j\}_{j=1}^{16}$ is a PF in the Hilbert space \mathcal{K} and

$$H_{ph} = \sum_{j=1}^{16} E_j |\varphi_j\rangle \langle \varphi_j| = \sum_{j=1}^8 \tilde{E}_j |\tilde{\varphi}_j\rangle \langle \tilde{\varphi}_j|,$$

where \tilde{E}_j are the eigenvalues above and $\tilde{\varphi}_j$ are the related eigenvectors, whose set is an ONB in $\mathcal{H}_{ph} = \mathbb{C}^8$. As in the previous example, the difference between E_j and \tilde{E}_j may have consequences on the dynamics of the system, but this aspect will not be considered in this paper.

3 E-Connection Between PFs and ONBs

The examples above suggests to introduce the following, physically motivated, relation between PFs and ONBs in a given Hilbert space \mathcal{K} :

Definition 8 Given a PF $\mathcal{F}_\varphi = \{\varphi_j \in \mathcal{K}, j \in \mathbb{J}\}$ and a bounded set of real numbers $E = \{E_j, j \in \mathbb{J}\}$, we say that the pair (\mathcal{F}_φ, E) is E -connected to an ONB $\mathcal{F}_{\tilde{e}} = \{\tilde{e}_k \in \mathcal{K}, k \in \mathbb{J}'\}$ if a set of real numbers exists, $\tilde{E} = \{\tilde{E}_k, k \in \mathbb{J}'\}$, such that

$$H_\varphi = \sum_{j \in \mathbb{J}} E_j |\varphi_j\rangle \langle \varphi_j| = \sum_{k \in \mathbb{J}'} \tilde{E}_k |\tilde{e}_k\rangle \langle \tilde{e}_k|. \quad (3.1)$$

This is exactly what we have seen in the examples considered in Sect. 2.3. In other words, (3.1) can be used to give two different representations of the same operator H_φ (the physical Hamiltonian, in our examples), one in terms of a PF, \mathcal{F}_φ , and one, possibly more relevant for its physical interpretation, in terms of $\mathcal{F}_{\tilde{e}}$.

If a pair (\mathcal{F}_φ, E) is E -connected to an ONB $\mathcal{F}_{\tilde{e}}$, then the real numbers \tilde{E}_k in (3.1) are eigenvalues of the self-adjoint operator H_φ , whereas \tilde{e}_k are the corresponding normalized eigenvectors. The eigenvalues \tilde{E}_k can be defined with the use of Theorem 6 and obviously, they are defined uniquely by the pair (\mathcal{F}_φ, E) . Moreover, due to Remark 3, the cardinality $|\mathbb{J}'|$ coincides with the frame potential $\mathbf{FP}[\mathcal{F}_\varphi]$ and it cannot exceed the cardinality $|\mathbb{J}| = \mathbf{FP}[\mathcal{F}_\varphi] + \mathbf{e}[\mathcal{F}_\varphi]$ of \mathbb{J} .

If \mathcal{F}_φ is an ONB in \mathcal{K} , then the pair (\mathcal{F}_φ, E) is E -connected to the same ONB $\mathcal{F}_\varphi = \mathcal{F}_{\tilde{e}}$ and $E = \tilde{E}$. A slightly more interesting example is the following: if $\varphi_{2k-1} = \varphi_{2k} = \frac{1}{\sqrt{2}}\mathbf{e}_k$, where $\mathcal{F}_{\mathbf{e}} = \{\mathbf{e}_k, k \in \mathbb{N}\}$ is an ONB of \mathcal{K} , then the set $\mathcal{F}_\varphi = \{\varphi_j, j \in \mathbb{N}\}$ is a PF in \mathcal{K} and the pair (\mathcal{F}_φ, E) is E -connected to $\mathcal{F}_{\mathbf{e}} = \mathcal{F}_{\tilde{e}}$. In this case, $\tilde{E} = \{\tilde{E}_k = \frac{1}{2}(E_{2k-1} + E_{2k}), k \in \mathbb{N}\}$.

Proposition 9 *If a PF $\mathcal{F}_\varphi = \{\varphi_j, j = 1, 2, \dots, N\}$ has the finite potential $M = \mathbf{FP}[\mathcal{F}_\varphi]$, then the pair (\mathcal{F}_φ, E) is E -connected to an ONB $\mathcal{F}_{\tilde{e}} = \{\tilde{e}_k \in \mathcal{K}, k = 1, \dots, M\}$. If, additionally, \mathcal{F}_φ has the finite excess $\mathbf{e}[\mathcal{F}_\varphi]$ and $E = \{E_j, j = 1, \dots, N\}$ is a sequence of positive quantities, then the quantities $\tilde{E}_k, k = 1, \dots, M$ in (3.1) satisfy the relations:*

$$\sum_{k=1}^M \tilde{E}_k = \sum_{j=1}^N E_j \|\varphi_j\|^2, \quad \sum_{k=1}^M \tilde{E}_k^2 = \sum_{j=1}^N \sum_{i=1}^N E_j E_i |\langle \varphi_i, \varphi_j \rangle|^2, \quad \sum_{k=1}^M \tilde{E}_k^2 \geq \frac{1}{M} \left(\sum_{j=1}^N E_j \|\varphi_j\|^2 \right)^2.$$

If all $\varphi_j \neq 0$ and $0 < \tilde{E}_1 \leq \tilde{E}_2 \leq \dots \leq \tilde{E}_M$, then for all $1 \leq n \leq M$

$$\sum_{k=1}^n \tilde{E}_k \geq \sum_{j=1}^n E_j \|\varphi_j\|^2. \quad (3.2)$$

Proof The self-adjoint operator $H_\varphi = \sum_{j=1}^N E_j |\varphi_j\rangle\langle\varphi_j|$ acts in a finite dimensional space \mathcal{K} with $\dim \mathcal{K} = M$. Hence its spectrum is discrete and the corresponding normalized eigenfunctions $\{\tilde{e}_k \in \mathcal{K}, k = 1, \dots, M\}$ form an ONB of \mathcal{K} .

If all E_j are positive, then the set $\mathcal{F}_{\hat{\varphi}} = \{\hat{\varphi}_j = \sqrt{E_j}\varphi_j, j = 1, \dots, N\}$ is a frame in \mathcal{K} and the operator H_φ turns out to be a frame operator $S = \sum_{j=1}^N |\hat{\varphi}_j\rangle\langle\hat{\varphi}_j|$ of $\mathcal{F}_{\hat{\varphi}}$. In this case, the first three relations in Proposition 9 follow from [10, Sect. 5] and [8, Corollary 2.3]. The inequality (3.2) follows from [10, Theorem 6.3]. \square

Another simple result is given by the following Lemma, where the validity of the resolution of the identity is used both for the ONB and for the PF.

Lemma 10 *Let \mathcal{F}_φ be a PF and let the real numbers E_j be the same for all $j \in \mathbb{J}$ (i.e., $E_j = \lambda$ for all $j \in \mathbb{J}$). Then the pair (\mathcal{F}_φ, E) is E -connected to every ONB $\mathcal{F}_{\tilde{e}}$ of \mathcal{K} and $E = \tilde{E} = \{\lambda\}$.*

Proof Due to the characteristic properties of PFs and the condition $E_j = \lambda$ we obtain $H_\varphi = \sum_{j \in \mathbb{J}} E_j |\varphi_j\rangle\langle\varphi_j| = \lambda \sum_{j \in \mathbb{J}} |\varphi_j\rangle\langle\varphi_j| = \lambda I$. Hence, formula (3.1) holds for any arbitrary choice of ONB $\mathcal{F}_{\tilde{e}}$, since $\sum_{k \in \mathbb{J}'} \tilde{E}_k |\tilde{e}_k\rangle\langle\tilde{e}_k| = \lambda \sum_{k \in \mathbb{J}'} |\tilde{e}_k\rangle\langle\tilde{e}_k| = \lambda I$. \square

The following result is surely more interesting.

Proposition 11 *Let \mathcal{F}_φ be a PF and let the set of real numbers $E = \{E_j, j \in \mathbb{J}\}$ have only one accumulation point λ . Then there exist an ONB $\mathcal{F}_{\tilde{e}}$ and a set of numbers \tilde{E}_k such that the pair (\mathcal{F}_φ, E) is E -connected to $\mathcal{F}_{\tilde{e}}$.*

Proof Rewrite the left-hand side of (3.1) as follows:

$$H_\varphi = \sum_{j \in \mathbb{J}} E_j |\varphi_j\rangle \langle \varphi_j| = \sum_{j \in \mathbb{J}} (E_j - \lambda) |\varphi_j\rangle \langle \varphi_j| + \lambda \sum_{j \in \mathbb{J}} |\varphi_j\rangle \langle \varphi_j| \quad (3.3)$$

Due to Lemma 10, $\lambda \sum_{j \in \mathbb{J}} |\varphi_j\rangle \langle \varphi_j| = \sum_{k \in \mathbb{J}'} \lambda |\tilde{e}_k\rangle \langle \tilde{e}_k|$, where $\{\tilde{e}_k\}_{k \in \mathbb{J}'}$ is an arbitrary ONB of \mathcal{K} . On the other hand, since $\lim_{j \rightarrow \infty} (E_j - \lambda) = 0$, the operator (see (2.10))

$$(\mathcal{E} - \lambda I)\{c_j\} = \{(E_j - \lambda)c_j\}, \quad D(\mathcal{E}) = \ell^2(\mathbb{J})$$

is compact in $\ell^2(\mathbb{J})$, [17, problem 132]. Therefore, the operator (see (2.11))

$$(\mathcal{E}_\varphi - \lambda I)\{c_j\} = \mathcal{P}(\mathcal{E} - \lambda I)\{c_j\}, \quad D(\mathcal{E}_\varphi) = \mathcal{R}(\theta_\varphi)$$

acting in the subspace $\mathcal{R}(\theta_\varphi)$ of $\ell^2(\mathbb{J})$ is compact. This means that

$$H_\varphi - \lambda I = \sum_{j \in \mathbb{J}} (E_j - \lambda) |\varphi_j\rangle \langle \varphi_j|$$

is self-adjoint and compact in \mathcal{K} (since $H_\varphi - \lambda I$ is unitary equivalent to $\mathcal{E}_\varphi - \lambda I$). Therefore, there exists an ONB $\{\tilde{e}_k\}_{k \in \mathbb{J}'}$ of \mathcal{K} formed by normalized eigenvectors \tilde{e}_k :

$$(H_\varphi - \lambda I)\tilde{e}_k = \mu_k \tilde{e}_k, \quad k \in \mathbb{J}' \quad (3.4)$$

and $H_\varphi - \lambda I = \sum_{j \in \mathbb{J}} (E_j - \lambda) |\varphi_j\rangle \langle \varphi_j| = \sum_{k \in \mathbb{J}'} \mu_k |\tilde{e}_k\rangle \langle \tilde{e}_k|$. Substituting the obtained formulas into (3.3), we obtain

$$\sum_{j \in \mathbb{J}} E_j |\varphi_j\rangle \langle \varphi_j| = \sum_{k \in \mathbb{J}'} (\lambda + \mu_k) |\tilde{e}_k\rangle \langle \tilde{e}_k|,$$

where the choice of ONB $\{\tilde{e}_k\}_{k \in \mathbb{J}'}$ is determined by (3.4). \square

Remark 12 The proof of Proposition 11 cannot be modified for the case of two or more accumulation points of $E = \{E_j, j \in \mathbb{J}\}$ and the problem of checking E -connection in this case is still open.

4 Conclusions

This paper is a continuation of a series of contributions on possible definitions of Hamiltonian-like operators in terms of sets of vectors which are not necessarily ONB, as in [2, 5]. In particular, this is relevant when the physical part of the Hilbert space of the system under analysis produces, in a quite natural way, a PF in terms of which the physical part of the Hamiltonian, H_{ph} , can be expanded. With this in mind we have studied some properties of H_{ph} , with particular focus on its eigenvalues. In this perspective we have introduced the

notion of E -connection, and we have analyzed it for bounded operators. Unbounded Hamiltonians generated by PF's have a lot of curious properties that require a deeper analysis and they will be considered in a forthcoming paper.

We have also discussed few examples of our general framework, two of which directly connected to biological models considered in the literature in recent years. These are both given in terms of fermionic operators. In this paper we have considered mainly the mathematical consequences of our settings. We plan to consider what changes in the dynamics (and, more in general, in the physics) of the systems in a future paper. In this perspective, a possible application is the use of the idea of localized systems as described in [16]. This is particularly relevant for quantization procedures.

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