EXISTENCE OF NON-ZERO SOLUTIONS FOR A DIRICHLET PROBLEM DRIVEN BY $(p(x), q(x))$-LAPLACIAN

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Abstract. The paper focuses on a Dirichlet problem driven by the $(p(x), q(x))$-Laplacian. The existence of at least two non-zero solutions under suitable conditions on the nonlinear term is established. The approach is based on variational methods.

1. Introduction

The aim of this paper is to study the following nonlinear elliptic boundary value problem with variable exponent

$$(P_\lambda) \qquad \begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and $\Delta_{q(x)}u = \operatorname{div}(|\nabla u|^{q(x)-2}\nabla u)$ are the $p(x)-$Laplacian operator and the $q(x)-$Laplacian operator respectively, λ is a positive real paremeter, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and p and $q \in C(\overline{\Omega})$ satisfying following condition

$$1 < q(x) < p(x) < N, \text{ for all } x \in \overline{\Omega}. \tag{1.1}$$

The nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function i.e. $f(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f(x, \cdot)$ is continuous for almost every $x \in \Omega$.

Problems with $p(x)-$Laplacian appear in different models of physical problems. Indeed there are particular innovative substances based on nanostructured materials, the so–called electrorheological fluids, which change their viscosity dramatically, under the action of an electric field and in proportion to the electric field itself. This their physical characteristic produces the anisotropy of differential equations describing the behavior of the electrorheological fluid, translates with the study of differential equations with non–standard growth conditions.
For more details on Lebesgue and Sobolev spaces with variable exponent we refer to [8], [11], [13] and references therein.

Mathematical models of electroreological fluids appear in [17] and [18], in which authors present the following equation of generalized mean curvature

$$\begin{cases} \operatorname{div} u = 0, \\ -\operatorname{div}\left(\left(1 + |\epsilon(u)|^2\right)^{\frac{p(x)-2}{2}} \epsilon(u)\right) + D\pi = f\left(x, u(x), \nabla u(x)\right), \end{cases} \tag{1.2}$$

with $p(x)$ is the variable exponent such that $1 < p(x) \leq 2$, $u : \Omega \to \mathbb{R}$ is a real function defined in an bounded domain Ω of $\mathbb{R}^3$ and $\epsilon(u)$ is the symmetric part of the gradient $\nabla u(x)$.
Regularity results for weak solutions to problems like (1.2) are proved in [1].

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About others physical applications, in [21] the author presents a model so called thermistor, describing the electric current which varies in a conductor according to the temperature $p = p(x)$. The main purpose is to find an electric potential $u = u(x)$ which satisfies the following differential equation

$$\Delta_{p(x)}u(x) + \mathcal{F}\left(x, u(x)\right) = 0 \text{ in } \Omega,$$

where $p(x) > 1$ for all $x \in \overline{\Omega}$. It is worth noticing that the study of materials which have particular inhomogeneity, as the electrorheological fluids mentioned above, is conducted using the Lebesgue spaces $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{m,p(x)}(\Omega)$ with variable exponent, contrary to what occurs for most of the materials that can be patterned after with sufficient accuracy using the classical Lebesgue spaces $L^{p}(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$. The $p(x)-$Laplacian operator is a generalization of the well know $p-$Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$. Note that, according to the anisotropy of these kind of problems, the $p(x)-$Laplacian operator is in general not homogeneous.
For other details about these applications we refer to [5], [7], [12] and references therein.

Our aim is point out existence results of two non–zero solutions for a Dirichlet problem expressed by a differential operator $A(u) := \Delta_{p(x)}u + \Delta_{q(x)}u$ where p and q satisfy condition (1.1).
The case $p(\cdot) = p$ and $q(\cdot) = q$ has been intensively studied as highlighted by authors in the survey [14], in which some important application fields of these results are listed. See also [19] in which variational methos are used together with critical points theorems.

Among differential problems with $(p(x), q(x))-$Laplacian containing a parameter, we cite [2], [20], [15]. In [2] the authors study problem (P_λ) in $\mathbb{R}^N$ and for $f(x,t) = m(x)|t|^{r(x)}$ with $1 < q(x) < r^- = \inf_{x\in\mathbb{R}^N} r(x) \le r^+ = \sup_{x\in\mathbb{R}^N} r(x) < p(x) < N$ and m an opportune function.
In [20], with a similar nonlinear term used in [2], the author prove the existence of infinitely many non–trivial weak solutions for $\lambda = 1$. In [15] the authors study Dirichlet problem (P_λ) when the nonlinearity is $f(x,u) = \pm(\lambda|u|^{m(x)-2} + |u|^{r(x)-2})$ and prove the existence of infinitely many weak solutions for each positive real parameter λ.

In all mentioned papers the nonlinearity f is a specific function. Vice versa, in the present paper, the nonlinearity is a generic Carathéodory function, satisfying a subcritical growth condition and the usual Ambrosetti–Rabinowithz condition. In our paper, using variational methods we prove the existence of at least two non–zero weak solutions of problem (P_λ).

The paper is organized as follows. In Section 2 some definitions and preliminary results are collected. Section 3 deals with the existence and non-negativity of solutions, together with some remarks and an example.

2. Preliminaries and basic notations

In this section, we recall definitions and tools used in the paper.
Let $(X, \|\cdot\|)$ be a Banach space, its dual space is X^* and the corresponding duality pairing is denoted by $\langle\cdot,\cdot\rangle$. Let $I : X \to \mathbb{R}$ be a Gâteaux differentiable functional. We say that I satisfies the Palais–Smale condition, (in short $(PS)-$condition), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

(β_1) $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
(β_2) $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in X^*,

admits a convergent subsequence in X.
Now we recall the following definition, that we will use to prove the $(PS)-$ condition (see [6]).

Definition 2.1. *Let $A : X \to X^*$ be a functional. We say that A has S_+-property iff every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $u_n \rightharpoonup u$ in X and $\limsup\limits_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0$ implies that $u_n \to u$ in X.*

Remark 2.1. We observe that operator $-\Delta_{p(x)}$ has the S_+-property (see [9, Theorem 3.1]). Furthermore, by applying proposition 2.70 in [16], operator $-\Delta_{p(x)} - \Delta_{q(x)}$ has the S_+-property in turn.

Let $p \in C(\overline{\Omega})$ and denote by

$$p^- = \min_{\overline{\Omega}} p(x), \quad p^+ = \max_{\overline{\Omega}} p(x).$$

For any $p \in C(\overline{\Omega})$, we introduce the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \quad \text{such that } u \text{ is measurable and } \int_\Omega |u|^{p(x)}\, dx < +\infty \right\},$$

and

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

in which the following norms are considered

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \delta > 0 : \int_\Omega \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\},$$

and

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

By $W_0^{1,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{W_0^{1,p(x)}(\Omega)} := \||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

It is well known (see [11]) that, in view of (1.1), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

Since $q(x) < p(x)$ for all $x \in \Omega$, theorem 2.8 in [13] ensures that $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ with

$$\|u\|_{L^{q(x)}(\Omega)} \leq (1 + |\Omega|)\|u\|_{L^{p(x)}(\Omega)},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbb{R}^N$. From this fact it results that $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega)$ and the following relation between the norms holds

$$\|u\|_{W_0^{1,q(x)}(\Omega)} := \|\nabla u\|_{L^{q(x)}(\Omega)} \leq (1 + |\Omega|)\|\nabla u\|_{L^{p(x)}(\Omega)} := (1 + |\Omega|)\|u\|_{W_0^{1,p(x)}(\Omega)}.$$

Denoted by $p^*(x) = \dfrac{p(x)N}{N - p(x)}$ the critical exponent of $p(x)$, we have that for $s \in C(\overline{\Omega})$ with $s(x) < p^*(x)$ the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ (see [8, Corollary 8.3.2]) is continuous and compact. In the sequel we denote by k_s the best constant for which one has

$$\|u\|_{L^{s(x)}(\Omega)} \leq k_s \|u\|_{W_0^{1,p(x)}(\Omega)}. \tag{2.1}$$

Put

$$R := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega), \tag{2.2}$$

simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. In the sequel we denote by

$$m := |B(x_0, 1)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}, \tag{2.3}$$

the measure of unit ball in $\mathbb{R}^N$ where Γ is the Gamma function.

For any $a \in \mathbb{R}$, $a > 0$ and $h \in C(\overline{\Omega})$ with $0 < h^-$, we put

$$[a]^h := \max\left\{a^{h^-}, a^{h^+}\right\}, \tag{2.4}$$

and

$$[a]_h := \min\left\{a^{h^-}, a^{h^+}\right\}. \tag{2.5}$$

For each $a, b \in \mathbb{R}_+$ and $x \in \overline{\Omega}$ it is easy to verify that,

$$\begin{gathered} [a]_h \le a^{h(x)} \le [a]^h, \\ [ab]^h \le [a]^h [b]^h, \\ [a]^{\frac{1}{h}} = \max\left\{a^{\frac{1}{h^-}}, a^{\frac{1}{h^+}}\right\}, \end{gathered} \tag{2.6}$$

and

$$[a]_{\frac{1}{h}} = \min\left\{a^{\frac{1}{h^-}}, a^{\frac{1}{h^+}}\right\}.$$

Throughout the paper, we assume that the nonlinearity f satisfies the following subcritical growth condition and the usual Ambrosetti–Rabinowitz condition (in short $(AR)-$condition).

(H) There exist two nonnegative constants a_1, a_2 and a function $\alpha \in C(\overline{\Omega})$ with $p^+ \le \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ such that

$$|f(x,t)| \le a_1 + a_2|t|^{\alpha(x)-1} \quad \text{for all}\,(x,t) \in \Omega \times \mathbb{R}.$$

(AR) There exist constants $\mu > p^+$ and $M > 0$ such that, $0 < \mu F(x,t) \le t f(x,t)$, for all $x \in \Omega$ and for all $|t| \ge M$, where

$$F(x,t) = \int_0^t f(x,\xi)d\xi,$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

A weak solution of problem (P_λ) is any $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |\nabla u(x)|^{q(x)-2}\nabla u(x)\cdot\nabla v(x)dx = \lambda\int_\Omega f(x,u(x))v(x)dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Notice that, under hypothesis (H), the integrals in definition of weak solutions exist.

In order to study problem (P_λ), we introduce the functionals Φ, $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) := \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla u|^{q(x)}\right)dx, \tag{2.7}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.8}$$

Clearly Φ and Ψ are continuously Gâteaux differentiable functionals, whose Gâteaux derivatives at point $u \in W_0^{1,p(x)}(\Omega)$ are given by

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |\nabla u(x)|^{q(x)-2}\nabla u(x)\cdot\nabla v(x)dx,$$

and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in W_0^{1,p(x)}(\Omega)$.
Put $I_\lambda = \Phi - \lambda\Psi$, we observe that critical points of I_λ are weak solutions of (P_λ). We have the following Lemma.

Lemma 2.1. *Assume that (H) and $(AR)-$condition hold. Then, for all $\lambda > 0$, I_λ satisfies the $(PS)-$condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a sequence such that (β_1) and (β_2) hold. Then

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle &= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right)|\nabla u_n|^{p(x)}\,dx + \int_\Omega \left(\frac{1}{q(x)} - \frac{1}{\mu}\right)|\nabla u_n|^{q(x)}\,dx \\ &- \lambda \int_\Omega \left(F(x,u_n) - \frac{1}{\mu} f(x,u_n(x))u_n(x)\right)dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \left(\frac{1}{q^+} - \frac{1}{\mu}\right)\|u_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)} + C. \end{aligned}$$

Where $C = C(M)$ is a constant. If $\{u_n\}_{n\in\mathbb{N}}$ is unbounded, noting that $p^+ < \mu$, $p^- > 1$ and taking n goes to infinity in the inequatity above, we get a contradiction. Therefore $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Without loss of generality we assume that $u_n \rightharpoonup u$.

Hypothesis (H) and the embedding properties of spaces $W^{1,p(x)}(\Omega)$ imply that

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)(u_n - u)dx = 0. \tag{2.9}$$

From (β_2) and (2.9) follows

$$\langle I'_\lambda(u_n), u_n - u\rangle = \langle -\Delta_{p(x)}u_n - \Delta_{q(x)}u_n, u_n - u\rangle - \lambda \int_\Omega f(x,u_n)(u_n-u)dx \to 0$$

for $n \to +\infty$. Then, one has

$$\limsup_{n\to\infty}\langle -\Delta_{p(x)}u_n - \Delta_{q(x)}u_n, u_n - u\rangle \leq 0.$$

By the S_+-property of $-\Delta_{p(x)} - \Delta_{q(x)}$ in $W_0^{1,p(x)}(\Omega)$ (see Remark 2.1) we have $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Hence I_λ fulfills the $(PS)-$condition. □

Our main tool is a two non–zero critical points theorem established in [4]. We recall it here for convenience of the reader.

Theorem 2.1. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{2.10}$$

and, for each

$$\lambda \in \Lambda = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non–zero critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

3. Main Results

In this section, we present our main results. To be precise, we establish the existence results of at least two non–zero weak solutions of problem (P_λ).

Theorem 3.1. *Assume that (H) and $(AR)-$condition hold. Moreover, assume that there are two positive constants r and d, with*

$$m\left(\frac{R}{2}\right)^N (2^N-1)\left(\frac{1}{p^-}\left[\frac{2d}{R}\right]^p + \frac{1}{q^-}\left[\frac{2d}{R}\right]^q\right) < r, \tag{3.1}$$

such that

$$F(x,t) \geq 0, \quad \text{for all} \quad (x,t) \in \Omega \times [0,d], \tag{3.2}$$

and

$$\frac{1}{r}\left(a_1 k_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\frac{\alpha^+}{p^-}} \left[[r]^{\frac{1}{p}}\right]^\alpha\right) < \frac{\inf\limits_{x\in\Omega} F(x,d)}{(2^N-1)\left(\frac{1}{p^-}\left[\frac{2d}{R}\right]^p + \frac{1}{q^-}\left[\frac{2d}{R}\right]^q\right)}, \tag{3.3}$$

where m, k_1, k_α, a_1, a_2 and R are given by (2.1), (2.2), (2.3) and by hypothesis (H).

Then, for each

$$\lambda \in \Lambda := \left]\frac{(2^N-1)\left(\frac{1}{p^-}\left[\frac{2d}{R}\right]^p + \frac{1}{q^-}\left[\frac{2d}{R}\right]^q\right)}{\inf\limits_{x\in\Omega} F(x,d)}, \frac{r}{a_1 k_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\frac{\alpha^+}{p^-}} \left[[r]^{\frac{1}{p}}\right]^\alpha}\right[,$$

problem (P_λ) has at least two non–zero weak solutions.

Proof. Put Φ and Ψ as in (2.7) and (2.8). It is well known that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.1, moreover $\inf\limits_{u\in W_0^{1,p}(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Our aim is to verify condition (2.10) of Theorem 2.1.
Fix $\lambda \in \Lambda$ we observe that from (3.3), one has $\Lambda \neq \emptyset$. Let $u \in W_0^{1,p(x)}(\Omega)$ be such that $\Phi(u) \leq r$. From (2.6) and Theorem 1.3 in [11] , we have

$$\|u\|_{W_0^{1,p(x)}(\Omega)} \leq \left[p^+ r\right]^{\frac{1}{p}} \leq \left[p^+\right]^{\frac{1}{p}} [r]^{\frac{1}{p}} = \left(p^+\right)^{\frac{1}{p^-}} [r]^{\frac{1}{p}}, \tag{3.4}$$

from (H), (2.1) and (3.4) one has

$$
\begin{aligned}
\frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &\leq \frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{\alpha^-}\left[\|u\|_{L^{\alpha(x)}(\Omega)}\right]^{\alpha}\right)}{r}\\
&\leq \frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(a_1k_1\|u\|_{W_0^{1,p(x)}(\Omega)}+\frac{a_2}{\alpha^-}[k_\alpha]^{\alpha}\left[\|u\|_{W_0^{1,p(x)}(\Omega)}\right]^{\alpha}\right)}{r}\\
&\leq \frac{a_1k_1\left(p^+\right)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}+\frac{a_2}{\alpha^-}[k_\alpha]^{\alpha}\left[\left(p^+\right)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}\right]^{\alpha}}{r}\\
&\leq \frac{a_1k_1\left(p^+\right)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}+\frac{a_2}{\alpha^-}[k_\alpha]^{\alpha}\left(p^+\right)^{\frac{\alpha^+}{p^-}}\left[[r]^{\frac{1}{p}}\right]^{\alpha}}{r}.
\end{aligned}
\tag{3.5}
$$

Now, define

$$
\tilde{u}(x)=\begin{cases} 0 & \text{if } x\in\Omega\setminus B(x_0,R),\\ \dfrac{2d}{R}\left(R-|x-x_0|\right) & \text{if } x\in B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right),\\ d & \text{if } x\in B\left(x_0,\frac{R}{2}\right).\end{cases}
$$

Clearly, $\tilde{u}\in W_0^{1,p(x)}(\Omega)$, and we have

$$
\begin{aligned}
\Phi(\tilde{u}) &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)}\left\{\frac{1}{p(x)}\left(\frac{2d}{R}\right)^{p(x)}+\frac{1}{q(x)}\left(\frac{2d}{R}\right)^{q(x)}\right\}dx\\
&\leq \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)}\left\{\frac{1}{p^-}\left(\frac{2d}{R}\right)^{p(x)}+\frac{1}{q^-}\left(\frac{2d}{R}\right)^{q(x)}\right\}dx\\
&\leq m\left(\frac{R}{2}\right)^{N}\left(2^N-1\right)\left(\frac{1}{p^-}\left[\frac{2d}{R}\right]^{p}+\frac{1}{q^-}\left[\frac{2d}{R}\right]^{q}\right).
\end{aligned}
$$

Therefore, from (3.1) one has $0<\Phi(\tilde{u})<r$.
On the other hand, from (3.2) one has

$$
\Psi(\tilde{u})\geq\int_{B\left(x_0,\frac{R}{2}\right)}F\left(x,d\right)dx\geq m\left(\frac{R}{2}\right)^{N}\inf_{x\in\Omega}F(x,d),
$$

and hence, we obtain

$$
\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}>\frac{\inf_{x\in\Omega}F(x,d)}{\left(2^N-1\right)\left(\frac{1}{p^-}\left[\frac{2d}{R}\right]^{p}+\frac{1}{q^-}\left[\frac{2d}{R}\right]^{q}\right)}.
\tag{3.6}
$$

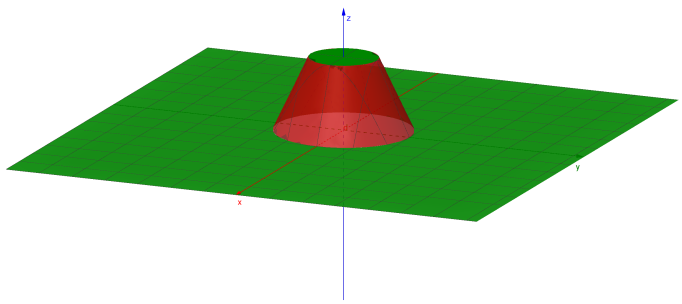
Therefore, from (3.3), (3.5) and (3.6), condition (2.10) of Theorem 2.1 is verified.

By virtue of Lemma 2.1, for all $\lambda>0$ the functional I_λ satisfies the $(PS)-$condition. Using $(AR)-$condition, it is easy to prove that the functional I_λ is unbounded from below. Therefore, all assumptions of Theorem 2.1 are satisfied. So, for all $\lambda\in\Lambda$ problem (P_λ) admits at least two non–zero weak solutions. □

Remark 3.1. Similar calculations to those carried out in [3] allow us to estimate the constants k_1 and k_α when $\alpha^+<(p^*)^-$.

Remark 3.2. If $p(x) = p$, $q(x) = q$ and $\alpha(x) = \alpha$ Theorem 3.1 gets back a similar result to that obtained in [19] in which the embedding constants have been explicit.

Remark 3.3. Test function $\tilde{u}$ used in the proof of Theorem 3.1 has the following graph and it seems that this type of funcion allows to obtain the best estimate for $\Phi(\tilde{u})$.



Remark 3.4. If in Theorem 3.1, we assume $f(x, 0) = 0$ for a.e. $x \in \Omega$, then weak solutions obtained are nonnegative and non–zero.
Indeed, define

$$f_+(x,t) = \begin{cases} f(x,t) & \text{if } t \geq 0, \\ \\ 0 & \text{if } t < 0, \end{cases}$$

and consider the following problem

$$(P_{+,\lambda}) \qquad \begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u = \lambda f_+(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If u_0 is a weak solution of $(P_{+,\lambda})$ then one has

$$\int_\Omega |\nabla u_0(x)|^{p(x)-2} \nabla u_0(x) \nabla v(x) dx + \int_\Omega |\nabla u_0(x)|^{q(x)-2} \nabla u_0(x) \nabla v(x) dx = \lambda \int_\Omega f_+(x, u_0(x)) v(x) dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Choosing $v = u_0^- := \min\{u_0, 0\}$, it results

$$\int_\Omega \left|\nabla u_0^-(x)\right|^{p(x)} dx + \int_\Omega \left|\nabla u_0^-(x)\right|^{q(x)} dx = 0,$$

that, by simple calculations, implies $\|u_0^-\|_{W_0^{1,p(x)}(\Omega)} = 0$ i.e. $u_0 \geq 0$ a.e. $x \in \Omega$. For this reason u_0 is solution of (P_λ). By applying Theorem 3.1 to the function f_+, we obtain the existence of two nonnegative and non-zero weak solutions of problem $(P_{+,\lambda})$ which, as proven above, are solutions of (P_λ).
We also observe that, when $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, the same conclusion holds (see [10, Theorem 1.1]).

Theorem 3.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative Carathéodory function that satisfies (H) and $(AR)-$condition. Moreover, assume that*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{q^-}} = +\infty. \tag{3.7}$$

Put $\lambda^* = \sup_{r>0} \dfrac{r}{a_1 k_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \dfrac{a_2}{\alpha^-} [k_\alpha]^\alpha (p^+)^{\frac{\alpha^+}{p^-}} \left[[r]^{\frac{1}{p}}\right]^\alpha}$. *Then, for each $\lambda \in]0, \lambda^*[$, problem (P_λ) admits at least two nonnegative and non–zero weak solutions.*

Proof. First of all we observe that condition (3.7) implies

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{\dfrac{1}{p^-}\left[\dfrac{2t}{R}\right]^p + \dfrac{1}{q^-}\left[\dfrac{2t}{R}\right]^q} = +\infty, \tag{3.8}$$

Fix $\lambda \in]0, \lambda^*[$. Then, there is $r > 0$ such that $\lambda < \dfrac{r}{a_1 k_1 (p^+)^{\frac{1}{p^-}} [r]^{\frac{1}{p}} + \dfrac{a_2}{\alpha^-} [k_\alpha]^\alpha (p^+)^{\frac{\alpha^+}{p^-}} \left[[r]^{\frac{1}{p}}\right]^\alpha}$.

From (3.8) there is $d > 0$ small enough such that (3.1) is verified and $\dfrac{\inf_{x\in\Omega} F(x,d)}{2^{N-1}\left(\dfrac{1}{p^-}\left[\dfrac{2d}{R}\right]^p + \dfrac{1}{q^-}\left[\dfrac{2d}{R}\right]^q\right)} > \dfrac{1}{\lambda}$. Hence, Theorem 3.1 ensures the conclusion. □

We illustrate the applicability of Theorem 3.2 by an example.

Example 3.1. Consider four positive real constants α, p, q and R such that

$$1 < q < p < \sqrt{N}, \quad N - p^2 < \alpha^2 < 1,$$

and

$$R^2 < \min\left\{p^2\left(\frac{1}{\alpha^2} - 1\right),\ N - p^2\right\}.$$

Given the domain $\Omega = \{x \in \mathbb{R}^N \ : \ |x|^2 < R^2\}$, we consider functions defined as follows

$$p(x) = |x|^2 + p^2,\ q(x) = |x|^2 + q^2,\ \alpha(x) = \frac{p(x)}{\alpha^2 - |x|^2} \quad \forall x \in \overline{\Omega}.$$

$$f(x,t) = \begin{cases} a + b|t|^{\alpha(x)-1} & \forall x \in \Omega,\ t \in \mathbb{R},\ t \neq 0, \\ a & \forall x \in \Omega,\ t = 0, \end{cases}$$

where a, b are positive constants. It is easy to verify that

$$1 < q(x) < p(x) < N,\ p^+ < \alpha(x) < p^*(x) \quad \forall x \in \overline{\Omega}.$$

Function f satisfies hypothesis (H) with $a_1 = a$ and $b = a_2$, moreover we have

$$F(x,t) = at + \frac{b}{\alpha(x)}|t|^{\alpha(x)} \quad \forall x \in \Omega,\ t \in \mathbb{R}.$$

Hence $F(x,t) \geq 0$ for all $(x,t) \in \Omega \times [0, +\infty[$.

Put $h(\cdot) = \frac{1}{\alpha(x)-1}$ and fix $p^+ < \mu < \alpha^-$, we can choose $M > \max\left\{\left[\frac{(\mu-1)a\alpha^+}{(\alpha^- - \mu)b}\right]^h, \left[\frac{a}{b}\alpha^+\right]^h\right\}$. We infer that f satisfies the $(AR)-$condition. Therefore, taking into account that $\inf\limits_{x\in\Omega} F(x,t) \geq at$ for all $t \in \mathbb{R}$, we have

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{q^-}} = +\infty.$$

Then, thanks to Theorem 3.2, for each $\lambda \in]0, \lambda^*[$, where $\lambda^* = \sup\limits_{r>0} \dfrac{r}{ak_1(p^+)^{\frac{1}{p^2}}[r]^{\frac{1}{p}} + \dfrac{b}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\frac{\alpha^+}{p^2}}\left[[r]^{\frac{1}{p}}\right]^\alpha}$, problem (P_λ) with the chosen function f admits at least two nonnegative non–zero weak solutions.

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