Quasilinear Dirichlet Problems with Degenerated $p$-Laplacian and Convection Term

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Abstract: The paper develops a sub-supersolution approach for quasilinear elliptic equations driven by degenerated $p$-Laplacian and containing a convection term. The presence of the degenerated operator forces a substantial change to the functional setting of previous works. The existence and location of solutions through a sub-supersolution is established. The abstract result is applied to find nontrivial, nonnegative and bounded solutions.

Keywords: quasilinear elliptic problem; degenerated $p$-Laplacian; convection term; sub-supersolution; nonnegative solution

1. Introduction

In this paper, we study the following quasilinear elliptic problem

$$\begin{cases}
-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases} \quad (P)$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $p \in (1,N)$. We assume that the boundary $\partial\Omega$ of $\Omega$ is locally Lipschitzian, i.e., each point of $\partial\Omega$ has a neighborhood whose intersection with $\partial\Omega$ is the graph of a Lipschitz continuous function. Throughout the text we denote by $|\cdot|$ and $\cdot \cdot$ the standard Euclidean norm and scalar product on $\mathbb{R}^N$, respectively. A main feature of the present work is that the leading part of the equation in $(P)$ is the differential operator in divergence form $\text{div}(a(x)|\nabla u|^{p-2}\nabla u)$ known as the degenerated $p$-Laplacian with the weight $a \in L^1_{\text{loc}}(\Omega)$. It is supposed that the function $a$ be positive almost everywhere in $\Omega$ and that the following condition holds

$$a^{-s} \in L^1(\Omega) \text{ for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right). \quad (1)$$

In the case where $a(x) \equiv 1$ we recover the ordinary $p$-Laplacian. Various examples of useful weights meeting the requirement $(1)$ are given in [1]. For instance, it is obvious that defining $a(x) = \text{dist}(x,S)$ for $x \in \Omega$, with a nonempty closed subset $S$ of $\partial\Omega$, one obtains a function $a$ on $\Omega$ for which $(1)$ holds true with any listed $s$.

The natural space associated with problem $(P)$ is $W^{1,p}_0(a,\Omega)$ that is the closure of $C_0^\infty(\Omega)$ in the weighted Sobolev space $W^{1,p}(a,\Omega)$. In Section 2 we briefly survey the spaces $W^{1,p}(a,\Omega)$ and $W^{1,p}_0(a,\Omega)$. The (negative) degenerated $p$-Laplacian with the weight $a \in L^1_{\text{loc}}(\Omega)$ under condition $(1)$ is defined on $W^{1,p}_0(a,\Omega)$ and takes values in the dual space $(W^{1,p}_0(a,\Omega))^*$. 

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Corresponding to the constant $s$ in (1) we set
\[ p_s = \frac{ps}{s+1} \]
and the Sobolev critical exponent $p^*_s = \frac{Np_s}{N-mp_s}$ (we note that $1 \leq p_s < N$). There is a continuous embedding $W^{1,p}(a,u) \hookrightarrow L^{p_s'}(\Omega)$, so a continuous embedding $L^{p_s'}(\Omega) \hookrightarrow (W^{1,p}_0(a,\Omega))^*$, where $(p^*_s)'$ stands for the Hölder conjugate of $p^*_s$, i.e., $(p^*_s)' = \frac{p^*_s}{p^*_s - 1}$. In order to handle problem (P) the idea is to arrange that the right-hand side $f(x,u,\nabla u)$ become an element of $L^{(p^*_s)'}(\Omega)$, which basically will be achieved through an adequate growth condition (see assumption $(H)$). We emphasize that the nonlinearity $f(x,u,\nabla u)$ depends on the solution $u$ and on its gradient $\nabla u$, which generally makes the variational methods be ineffective. Such a term $f(x,u,\nabla u)$ is often called convection. It is expressed by means of a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that is Carathéodory, i.e., $f(\cdot,t,\xi)$ is measurable for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$.

The goal of our work is to build a systematical approach to problem (P) via the method of sub-supersolution. It is for the first time when the method of sub-supersolution is implemented for problem (P) involving the degenerated $p$-Laplacian and related convection. In this respect, the functional setting is adapted to the novel situation of degenerated operators relying in an essential way on the associated exponent $p_s$. For results on the method of sub-supersolution applied to problems exhibiting convection terms but not driven by degenerated differential operators we refer to [2–6].

By a (weak) solution to problem (P) we mean a function $u \in W^{1,p}_0(a,\Omega)$ such that $f(x,u,\nabla u) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx, \forall v \in W^{1,p}_0(a,\Omega). \]

A function $\underline{u} \in W^{1,p}_0(a,\Omega)$ is called a subsolution for problem (P) if $\underline{u} \leq 0$ on $\partial \Omega$ (in the sense of traces), $f(\cdot,\underline{u}(\cdot),\nabla \underline{u}(\cdot)) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_\Omega a(x)|\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x) \cdot \nabla v(x)dx \leq \int_\Omega f(x,\underline{u}(x),\nabla \underline{u}(x))v(x)dx \]
for all $v \in W^{1,p}_0(a,\Omega)$, $v \geq 0$ a.e. in $\Omega$. Symmetrically, a function $\overline{u} \in W^{1,p}_0(a,\Omega)$ is called a supersolution for problem (P) if $\overline{u} \geq 0$ on $\partial \Omega$ (in the sense of traces), $f(\cdot,\overline{u}(\cdot),\nabla \overline{u}(\cdot)) \in L^{(p^*_s)'}(\Omega)$ and
\[ \int_\Omega a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx \geq \int_\Omega f(x,\overline{u}(x),\nabla \overline{u}(x))v(x)dx \]
for all $v \in W^{1,p}_0(a,\Omega)$, $v \geq 0$ a.e. in $\Omega$. Corresponding to a subsolution $\underline{u}$ and a supersolution $\overline{u}$ with $\underline{u} \leq \overline{u}$ a.e. in $\Omega$ we can consider the ordered interval
\[ [\underline{u},\overline{u}] = \{ w \in W^{1,p}_0(a,\Omega) : \underline{u} \leq w \leq \overline{u} \}. \]

The following hypothesis for $f(x,s,\xi)$ is adapted to an ordered sub-supersolution $\underline{u} \leq \overline{u}$.

**Hypthesis 1.** Given an ordered sub-supersolution $\underline{u} \leq \overline{u}$ for problem (P), the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition
\[ |f(x,t,\xi)| \leq \sigma(x) + b|\xi|^r \text{ for a.e. } x \in \Omega, \text{ for all } t \in [\underline{u}(x),\overline{u}(x)], \xi \in \mathbb{R}^N, \]
with a function $\sigma \in L^{\frac{p}{r}'}(\Omega)$ and constants $b > 0$ and $r \in (0,\frac{p}{p^*_s})$. 
According to Hypothesis 1 we have
\[ f(x, u, \nabla u) \in L^{(p_s')} (\Omega), \forall u \in [u, \overline{u}], \]
thus the integrals in the definitions above exist since
\[ f(x, u, \nabla u)v \in L^1(\Omega), \forall u \in [u, \overline{u}], v \in W_0^{1,p}(a, \Omega). \]

Under Hypothesis 1, our main result establishes the existence of a weak solution to problem \((P)\) with the additional location property \(u \in [u, \overline{u}]\). We stress that this location property represents a significant qualitative information for the solution giving actually a priori estimates for it. As an application we prove the existence of a nontrivial nonnegative solution for a class of problems of type \((P)\). The applicability of the stated result is demonstrated by an example.

2. Preliminary Material

The notation \(|\Omega|\) stands for the Lebesgue measure of the bounded domain \(\Omega\) in \(\mathbb{R}^N\). In this section we discuss a few facts about the degenerated \(p\)-Laplacian entering problem \((P)\). More details can be found in [1].

We note that (1) implies
\[ a^{-\frac{1}{p_s}} \in L^1(\Omega). \]

Indeed, it is seen that
\[
\int_{\Omega} a(x)^{-\frac{1}{p_s}} dx = \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p_s}} dx + \int_{\{a(x)\geq 1\}} a(x)^{-\frac{1}{p_s}} dx 
\leq \int_{\{a(x)<1\}} a(x)^{-s} dx + |\Omega| < \infty
\]
since according to (1) one has \(s \geq \frac{1}{p_s} - 1\) and \(a^{-s} \in L^1(\Omega)\).

The weighted Sobolev space \(W^{1,p}(a, \Omega)\) consists of all the functions \(u \in L^p(\Omega)\) for which \(a^\frac{1}{p} |\nabla u| \in L^p(\Omega)\). It is endowed with the norm
\[
\|u\|_{1,p,a} = \left( \int_{\Omega} |u|^p dx + \int_{\Omega} a(x)|\nabla u|^p dx \right)^{\frac{1}{p}}
\]
becoming a uniformly convex Banach space (due to the preceding property of the weight \(a(x)\), see ([1], [Theorem 1.3])), thus reflexive, that contains \(C_0^\infty(\Omega)\). The space \(W_0^{1,p}(a, \Omega)\) is the closure of \(C_0^\infty(\Omega)\) with respect to the norm \(\| \cdot \|_{1,p,a}\).

There is an extensive literature devoted to the weighted Sobolev spaces including embeddings and traces related to different boundary value problems (see, e.g., [1,7,8]). The results depend strongly on what type of weight is used, generally attempting reduction to nonweighted spaces. As described below, under assumption (1), we can embed the space \(W^{1,p}(a, \Omega)\) into the ordinary Sobolev space \(W^{1,p_s}(\Omega)\), hence automatically having the trace (note the boundary \(\partial\Omega\) is Lipschitz). This fact is needed in the definition of the sub-supersolution.

From (1) it is known that \(s \geq \frac{1}{p_s^*} - 1\), so one has \(p_s \geq 1\) and the continuous embedding
\[
W^{1,p}(a, \Omega) \hookrightarrow W^{1,p_s}(\Omega),
\]
which is relation (1.22) in [1]. More precisely, observing that \(p > p_s\), through Holder’s inequality and (1) we get
\[
\int_{\Omega} |\nabla u|^{p_s} dx = \int_{\Omega} a^{-\frac{p_s}{p}} a^{\frac{p_s}{p}} |\nabla u|^{p_s} dx \leq \left( \int_{\Omega} a^{\frac{1}{p_s}} dx \right)^{\frac{p_s}{p}} \left( \int_{\Omega} a |\nabla u|^p dx \right)^{\frac{p}{p}}
\]
for all $u \in W^{1,p}(a, \Omega)$. As a consequence of the above inequality, we can endow $W^{1,p}_0(a, \Omega)$ with an equivalent norm

$$||u|| = \left(\int_\Omega a(x)|\nabla u|^pdx\right)^{\frac{1}{p}}$$

for which it holds

$$||u||_{W^{1,p}_0(\Omega)} \leq ||a^{-\frac{1}{2}}||_{L^1(\Omega)} ||u||.$$  \hfill (6)

The Sobolev embedding theorem ensures the continuous embedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*_s}(\Omega)$, with the critical exponent $p^*_s = \frac{Np_s}{N-p_s}$ (note that $1 \leq p_s < N$). Hence there exists a constant $T_0 > 0$ such that

$$||u||_{L^{p^*_s}(\Omega)} \leq T_0 ||u||_{W^{1,p}_0(\Omega)} \quad \forall u \in W^{1,p}_0(\Omega).$$  \hfill (7)

The best embedding constant $T_0$ has been estimated by Talenti [9] as follows

$$T_0 \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{2p}} \left( \frac{p_s-1}{N-p_s} \right)^{\frac{1}{p}} \left( \frac{\Gamma\left(1+\frac{N}{p}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p_s}\right)\Gamma\left(1+N-\frac{N}{p_s}\right)} \right)^{\frac{1}{N}},$$

where $\Gamma$ is the Euler function

$$\Gamma(t) = \int_0^{+\infty} z^{t-1}e^{-z}dz, \quad \forall t > 0.$$

Moreover, by the Rellich–Kondrachov compact embedding theorem, if $1 \leq r < p^*_s$ then the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

By (7) and Hölder’s inequality we infer that

$$||u||_{L^r(\Omega)} \leq T_0 ||u||_{W^{1,p}_0(\Omega)}$$  \hfill (8)

for every $u \in W^{1,p}_0(\Omega)$ and $r \in [1, p^*_s]$. Combining (6) and (8) we arrive at

$$||u||_{L^r(\Omega)} \leq \kappa_r ||u||$$  \hfill (9)

for all $u \in W^{1,p}_0(a, \Omega)$ and $r \in [1, p^*_s]$, with the constant

$$\kappa_r = T_0 ||a^{-\frac{1}{2}}||_{L^1(\Omega)}^{\frac{p}{p^*_s}} ||a^{-\frac{1}{2}}||_{L^1(\Omega)}^{\frac{1}{p^*_s}}.$$

The (negative) degenerated $p$-Laplacian with the weight $a \in L^1_{\text{loc}}(\Omega)$ satisfying condition (1) is the operator $A : W^{1,p}_0(a, \Omega) \to (W^{1,p}_0(a, \Omega))^*$ defined by

$$\langle A(u), v \rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \quad \forall u, v \in W^{1,p}_0(a, \Omega).$$  \hfill (10)

We readily check that the operator $A$ in (10) is well defined noticing by means of Hölder’s inequality that for all $u, v \in W^{1,p}_0(a, \Omega)$ it holds

$$\left| \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx \right| \leq \int_\Omega a(x)^\frac{p-1}{p} |\nabla u|^{p-1}a(x)^\frac{1}{p} |\nabla v|dx$$

$$\leq \left( \int_\Omega a(x)|\nabla u|^{p}dx \right)^{\frac{p-1}{p}} \left( \int_\Omega a(x)|\nabla v|^{p}dx \right)^{\frac{1}{p}} < \infty.$$  \hfill (11)
Important properties of the operator $A$ introduced in (10) are listed in the statement below.

**Proposition 1.** Assume that the measurable function $a : \Omega \rightarrow \mathbb{R}$ satisfies condition (1). Then the (negative) degenerated $p$-Laplacian $A : W^{1,p}_0(a, \Omega) \rightarrow (W^{1,p}_0(a, \Omega))^*$ defined by (10) has the following properties:

(i) $A$ is a bounded operator in the sense that it maps bounded sets to bounded sets;

(ii) $A$ is a coercive operator, i.e.,
$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty;$$

(iii) $A$ is a strictly monotone operator, i.e.,
$$\langle Au - Av, u - v \rangle > 0, \quad u \neq v;$$

(iv) $A$ has the $S_+$ property meaning that any sequence $\{u_n\} \subset W^{1,p}_0(a, \Omega)$ that satisfies $u_n \rightharpoonup u$ in $W^{1,p}_0(a, \Omega)$ and
$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0$$

is strongly convergent.

**Proof.** (i) From (10) and (11) we infer that
$$|\langle Au, v \rangle| = \left| \int_{\Omega} a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla vdx \right| \leq \|u\|^{p-1}\|v\|, \quad \forall u, v \in W^{1,p}_0(a, \Omega).$$

We obtain
$$\|Au\|_{(W^{1,p}_0(a, \Omega))^*} = \sup_{v \in W^{1,p}_0(a, \Omega), \|v\| \leq 1} |\langle Au, v \rangle| \leq \|u\|^{p-1}, \quad \forall u \in W^{1,p}_0(a, \Omega),$$
whence $A$ is bounded.

(ii) By (10) we have that
$$\langle Au, u \rangle = \int_{\Omega} a(x)|\nabla u|^{p}dx = \|u\|^{p}, \quad \forall u \in W^{1,p}_0(a, \Omega).$$

Taking into account that $p > 1$, it follows that the operator $A$ is coercive.

(iii) In view of the strict monotonicity of the mapping $\xi \mapsto |\xi|^{p-2}(\xi)$ on $\mathbb{R}^N$, it turns out
$$\langle Au - Av, u - v \rangle = \int_{\Omega} a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot (\nabla u - \nabla v)dx > 0, \quad u \neq v,$$
so $A$ is a strictly monotone operator.

(iv) Let a sequence $\{u_n\} \subset W^{1,p}_0(a, \Omega)$ satisfy $u_n \rightharpoonup u$ in $W^{1,p}_0(a, \Omega)$ and (12). Using the monotonicity of the operator $A$ and (12) we have
$$\lim_{n \to \infty} \langle A(u_n) - A(u), u_n - u \rangle = 0.$$
\begin{equation}
\langle A(u_n) - A(u), u_n - u \rangle = \int_{\Omega} a(x) \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot (\nabla u_n - \nabla u) dx
\end{equation}

\begin{align*}
\leq & \int_{\Omega} a(x)|\nabla u_n|^p dx - \int_{\Omega} a(x)|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u dx - \int_{\Omega} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla u dx + \int_{\Omega} a(x)|\nabla u|^p dx \\
\geq & \int_{\Omega} a(x)|\nabla u_n|^p dx - \left( \int_{\Omega} a(x)|\nabla u_n|^p dx \right) \frac{1}{p} \left( \int_{\Omega} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla u dx \right)^{\frac{1}{p}} \\
& - \left( \int_{\Omega} a(x)|\nabla u|^p dx \right) \frac{1}{p} \left( \int_{\Omega} a(x)|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u dx \right)^{\frac{1}{p}} + \int_{\Omega} a(x)|\nabla u|^p dx \\
= & (\|u_n\| - \|u\|) (\|u_n\|^{p-1} - \|u\|^{p-1}) \geq 0,
\end{align*}

from which we find that \( \lim_{n \to +\infty} \|u_n\| = \|u\| \). Due to the uniform convexity of \( W_0^{1,p}(a,\Omega) \) it follows that \( u_n \to u \) in \( W_0^{1,p}(a,\Omega) \), thus completing the proof. \( \square \)

We also need the first eigenvalue \( \lambda_1 \) of the operator \( A : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^* \) in (10). Precisely, \( \lambda_1 > 0 \) is the least (positive) number for which the equation

\begin{align*}
\begin{cases}
-\text{div}(a(x)|\nabla u|^{p-2} \nabla u) = \lambda_1 |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\end{align*}

(13)

admits a nontrivial solution called eigenfunction corresponding to the first eigenvalue \( \lambda_1 \). A solution to (13) is understood in the weak sense, i.e., \( u \in W_0^{1,p}(a,\Omega) \) satisfying

\begin{equation}
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx = \lambda_1 \int_{\Omega} |u(x)|^{p-2} u(x)v(x) dx, \forall v \in W_0^{1,p}(a,\Omega).
\end{equation}

It is known that there exists an eigenfunction \( u_1 \in W_0^{1,p}(a,\Omega) \) corresponding to the first eigenvalue \( \lambda_1 \) such that \( u_1(x) \geq 0 \) for a.e. \( x \in \Omega \), \( u_1 \neq 0 \), and \( u_1 \in L^\infty(\Omega) \). For the proofs of these properties we refer to ([1], Chapter 3).

3. Main Results

Our main abstract result provides the existence of a solution to problem (P) and its location within the ordered interval determined by a sub-supersolution.

**Theorem 1.** Let the weight \( a \in L^1_{\text{loc}}(\Omega) \) fulfill the requirement (1) and assume that the condition (H) for a subsolution \( \underline{u} \) and a supersolution \( \overline{u} \) with \( \underline{u} \leq \overline{u} \) a.e. is satisfied. Then problem (P) possesses at least a solution \( u \in W_0^{1,p}(a,\Omega) \) with the location property \( \underline{u} \leq u \leq \overline{u} \) for a.e. \( x \in \Omega \).

**Proof.** By means of the given sub-supersolution \( \underline{u} \leq \overline{u} \) for problem (P), we introduce some related mappings. The cut-off function \( \pi : \Omega \times \mathbb{R} \to \mathbb{R} \) is defined by

\begin{equation}
\pi(x,t) = \begin{cases}
-(\underline{u}(x) - t)^{s/r} & \text{if } t < \underline{u}(x) \\
0 & \text{if } \underline{u}(x) \leq t \leq \overline{u}(x) \\
(t - \overline{u}(x))^{s/r} & \text{if } t > \overline{u}(x),
\end{cases}
\end{equation}

(14)

where \( s \) and \( r \) are the constants given in (1) and Hypothesis 1. Using (14) in conjunction with \( \underline{u}, \overline{u} \in L^p(\Omega) \) enables us to find that

\begin{equation}
|\pi(x,t)| \leq c|t|^{s/r} + q(x) \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R},
\end{equation}

(15)
with a constant $c > 0$ and a function $q \in L^{\frac{p}{p(r)}}(\Omega)$. Moreover, proceeding as in [4], we can establish that

$$
\int_\Omega \pi(x, u(x))u(x) \, dx \geq b_1 \|u\|_{L^{\frac{p}{p(r)}}(\Omega)}^p - b_2 \quad \text{for all } u \in W_0^{1,p}(a, \Omega),
$$

(16)

with positive constants $b_1$ and $b_2$.

In view of (15), the Nemytskij operator $u \mapsto \pi(\cdot, u(\cdot))$ generated by $\pi$ maps continuously $L^{p_r}(\Omega)$ to $L^{\frac{p_r(p_r-1)}{p_r(p(r)-1)}}(\Omega)$. Therefore, the mapping $\Pi : W_0^{1,p}(a, \Omega) \rightarrow (W_0^{1,p}(a, \Omega))^*$ defined by

$$
\langle \Pi(u), v \rangle = \int_\Omega \pi(x, u)vdx, \quad \forall u, v \in W_0^{1,p}(a, \Omega)
$$

is completely continuous. This is true because the inclusion $L^{\frac{p_r(p_r-1)}{p_r(p(r)-1)}}(\Omega) \subset (W_0^{1,p}(a, \Omega))^*$ is compact being the adjoint of the compact inclusion $W_0^{1,p}(a, \Omega) \subset L^{p_r}(\Omega)$ (note that $\frac{p_r(p_r-1)}{p_r(p(r)-1)} < p^*_r$ owing to the assumption $r \in (0, \frac{p_r}{p(r)-1})$ in (H)).

Hypothesis (H) and (5) imply that the Nemytskij operator $u \mapsto f(\cdot, u(\cdot), \nabla u(\cdot))$ maps continuously $[u, \varpi] \subset W^{1,p}(a, \Omega)$ to $L^p(\Omega)$ with $r \in (0, \frac{p_r}{p(r)-1})$. Composing the preceding Nemytskij operator with the inclusion $L^p(\Omega) \subset (W_0^{1,p}(a, \Omega))^*$, which is compact because it is the adjoint operator of the compact inclusion $W_0^{1,p}(a, \Omega) \subset L^p(\Omega)$ (note that $\frac{p_r}{p(r)-1} < p^*_r$ since $r \in (0, \frac{p_r}{p(r)-1})$ in (H)), we obtain a completely continuous mapping $N_f : [u, \varpi] \rightarrow (W_0^{1,p}(a, \Omega))^*$ given by

$$
\langle N_f(u), v \rangle = \int_\Omega f(x, u(x), \nabla v(x))dx
$$

for all $u \in [u, \varpi]$ and $v \in W_0^{1,p}(a, \Omega)$.

We also make use of the truncation operator $T : W_0^{1,p}(a, \Omega) \rightarrow W^{1,p}(a, \Omega)$ given by

$$
(Tu)(x) = \begin{cases} u(x) & \text{if } u(x) < \underline{u}(x) \\ \underline{u}(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \overline{u}(x) & \text{if } u(x) > \overline{u}(x) \end{cases}
$$

(17)

for all $u \in W_0^{1,p}(a, \Omega)$ and a.e. $x \in \Omega$. It is a continuous and bounded mapping (in the sense that it maps bounded sets to bounded sets). Notice that its range lies in $[\underline{u}, \overline{u}]$, so $T$ can be composed with the operator $N_f$.

Now we consider for every $\lambda > 0$ the operator $A_\lambda : W_0^{1,p}(a, \Omega) \rightarrow (W_0^{1,p}(a, \Omega))^*$ defined by

$$
A_\lambda = A + \lambda \Pi - N_f \circ T.
$$

(18)

Explicitly, it reads as

$$
\langle A_\lambda(u), v \rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \lambda \int_\Omega \pi(x, u)v \, dx
$$

$$
- \int_\Omega f(x, Tu, \nabla(Tu))v \, dx \quad \text{for all } u, v \in W_0^{1,p}(a, \Omega).
$$

(19)

From Proposition 1(i) it is known that the operator $A : W_0^{1,p}(a, \Omega) \rightarrow (W_0^{1,p}(a, \Omega))^*$ is bounded, while the above comments demonstrate that the operators $\Pi$, $N_f$ and $T$ are all of them bounded. Therefore from (18) we infer that the operator $A_\lambda : W_0^{1,p}(a, \Omega) \rightarrow (W_0^{1,p}(a, \Omega))^*$ is bounded.
We claim that $A_\lambda : W_0^{1,p}(a, \Omega) \to (W_0^{1,p}(a, \Omega))^\ast$ is a pseudomonotone operator. In this respect, let a sequence $\{u_n\} \subset W_0^{1,p}(a, \Omega)$ satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(a, \Omega)$ and
\[
\limsup_{n \to \infty} \langle A_\lambda(u_n), u_n - u \rangle \leq 0. \tag{20}
\]

The sequence $\{\Pi(u_n)\}$ is bounded in $L^{p/(p-1)}(\Omega)$, while $u_n \to u$ in $L^{p/(p-1)}(\Omega)$ by the compact embedding $W_0^{1,p}(a, \Omega) \subset L^{p/(p-1)}(\Omega)$, thus
\[
\lim_{n \to \infty} \langle \Pi(u_n), u_n - u \rangle = 0.
\]

The sequence $\{N_f \circ T(u_n)\}$ is bounded in $L^p(\Omega)$, while $u_n \to u$ in $L^p(\Omega)$ by the compact embedding $W_0^{1,p}(a, \Omega) \subset L^p(\Omega)$, producing
\[
\lim_{n \to \infty} (N_f \circ T(u_n), u_n - u) = 0.
\]

Consequently, complying with (18), we see that (20) reduces to (12). This, in conjunction with the weak convergence $u_n \rightharpoonup u$, enables us to apply Proposition 1(iv) ensuring that the strong convergence $u_n \to u$ in $W_0^{1,p}(a, \Omega)$ holds.

From the strong convergence $a(\cdot)^{p/(p-1)} \nabla u_n(\cdot) \to a(\cdot)^{p/(p-1)} \nabla u(\cdot)$ in $(L^p(\Omega))^N$ it follows the strong convergence $a(\cdot)^{(p-1)/p-1} |\nabla u_n(\cdot)|^{p-2} \nabla u_n(\cdot) \to a(\cdot)^{(p-1)/p-1} |\nabla u(\cdot)|^{p-2} \nabla u(\cdot)$ in $(L^{p/(p-1)}(\Omega))^N$. This amounts to saying that $Au_n \to Au$ in $(W_0^{1,p}(a, \Omega))^\ast$ since
\[
\langle Au_n, v \rangle = \int_{\Omega} a(x)|\nabla u_n|^p \nabla u_n \cdot \nabla vdx \to \int_{\Omega} a(x)|\nabla u|^p \nabla u \cdot \nabla vdx = \langle Au, v \rangle, \forall v \in W_0^{1,p}(a, \Omega).
\]

Again, from the strong convergence $a(\cdot)^{1/p} \nabla u_n(\cdot) \to a(\cdot)^{1/p} \nabla u(\cdot)$ in $(L^p(\Omega))^N$ we infer that
\[
\langle Au_n, u_n \rangle = \int_{\Omega} a(x)|\nabla u_n|^p dx \to \int_{\Omega} a(x)|\nabla u|^p dx = \langle Au, u \rangle
\]
as $n \to \infty$. Taking into account the continuity of the mappings $\Pi$ and $N_f \circ T$, we have
\[
\langle A_\lambda u_n, v \rangle \to \langle A_\lambda u, v \rangle, \forall v \in W_0^{1,p}(a, \Omega),
\]
and
\[
\langle A_\lambda u_n, u_n \rangle \to \langle A_\lambda u, u \rangle
\]
as $n \to \infty$, for every $\lambda > 0$. We can conclude that $A_\lambda : W_0^{1,p}(a, \Omega) \to (W_0^{1,p}(a, \Omega))^\ast$ is a pseudomonotone operator (see, e.g., (2), Definition 2.97)).

The next step in the proof is to show that the operator $A_\lambda : W_0^{1,p}(a, \Omega) \to (W_0^{1,p}(a, \Omega))^\ast$ is coercive provided $\lambda > 0$ is large enough. Taking advantage of the fact that $Tu \in [\underline{u}, \overline{u}]$ whenever $u \in W_0^{1,p}(a, \Omega)$, let us note by (16), (19) and Hypothesis 1 that
\[
\langle A_\lambda(u), u \rangle = \langle A(u), u \rangle + \lambda \int_{\Omega} \pi(x, u) u dx - \int_{\Omega} f(x, Tu, \nabla(Tu)) u dx \geq ||u||^p + \lambda(b_1||u||_{L^{p/(p-1)}(\Omega)} - b_2) - ||\sigma||_{L^{p/(p-1)}(\Omega)} ||u||_{L^{p/(p-1)}(\Omega)} - b \int_{\Omega} \nabla(Tu)^t ||u|| dx \tag{21}
\]
for all \( u \in W_0^{1,p}(a,\Omega) \). Now we estimate the last term in (21) based on the fact that by (5) we know that \( \nabla u \in (L^p(\Omega))^N \), and so \( \nabla (Tu) \in (L^p(\Omega))^N \). Using the definition of \( Tu \) in (17), Hölder’s inequality and the continuous embedding in (9) it turns out that
\[
\int_{\Omega} |\nabla (Tu)|^p |u| dx = \int_{\{z \leq u \leq \pi\}} |\nabla u|^p |u| dx + \int_{\{u \leq z\}} |\nabla u|^p |u| dx + \int_{\{u > \pi\}} |\nabla u|^p |u| dx \\
\leq \int_{\Omega} |\nabla u|^p |u| dx + c_1 \|u\|_p, \quad \forall u \in W_0^{1,p}(a,\Omega),
\]
with a constant \( c_1 > 0 \). We can insert the preceding inequality in (21) to derive
\[
\langle A_\lambda(u), u \rangle \geq \|u\|^p + \lambda(b_1\|u\|_{L^{p_s}(\Omega)}^{p_s} - b_2) - c_2 \|u\| - b \int_{\Omega} |\nabla u|^p |u| dx,
\]
with a constant \( c_2 > 0 \). The Hölder’s and Young’s inequalities in conjunction with embedding (5) imply
\[
\int_{\Omega} |\nabla u|^p |u| dx \leq \|\nabla u\|_{L^{p_s}(\Omega)} \|u\|_{L^{p_s}(\Omega)} \leq c_3 \|u\|^{p_s} + c_4 \|u\|^{p_s_{\lambda}}_{L^{p_s}(\Omega)},
\]
with constants \( c_3 > 0 \) and \( c_4 > 0 \). Then (22) entails
\[
\langle A_\lambda(u), u \rangle \geq \|u\|^p + \lambda(b_1\|u\|_{L^{p_s}(\Omega)}^{p_s} - b_2) - c_2 \|u\| - b(c_3 \|u\|^{p_s} + c_4 \|u\|^{p_s_{\lambda}}_{L^{p_s}(\Omega)}) \quad (23)
\]
for all \( u \in W_0^{1,p}(a,\Omega) \). Recalling from (16) that \( b_1 > 0 \), we can choose \( \lambda > 0 \) so large to have \( \lambda b_1 > bc_4 \). Hence due to \( p > p_s \geq 1 \) (see (1)), (23) yields the coercivity of \( A_\lambda \), i.e.,
\[
\lim_{\|u\| \to +\infty} \frac{\langle A_\lambda(u), u \rangle}{\|u\|} = +\infty.
\]

We have shown that the nonlinear operator \( A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^* \) is bounded, pseudomonotone and coercive provided \( \lambda > 0 \) is sufficiently large. Therefore, for such an \( A_\lambda \) we can apply the main theorem of pseudomonotone operators (see, e.g., ([2], Theorem 2.99)) ensuring that there exists a solution \( u \in W_0^{1,p}(a,\Omega) \) to the equation
\[
A_\lambda(u) = 0.
\]

Fix an admissible \( \lambda > 0 \) as pointed out above. We are going to prove that \( u \in W_0^{1,p}(a,\Omega) \) resolving (24) is a weak solution of the original problem \((P)\), which means that (2) is satisfied. To this end, notice that (19) and (24) yield
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx + \lambda \int_{\Omega} \pi(x,u)v dx = \int_{\Omega} f(x,Tu,n(Tu))v dx \quad \text{for all } v \in W_0^{1,p}(a,\Omega).
\]

We proceed by comparing \( u \) with the subsolution \( \underline{u} \) and supersolution \( \overline{u} \) postulated in assumption (H). We claim that \( u \leq \overline{u} \) a.e. in \( \Omega \). Towards this, it can be readily checked that \( (u - \overline{u})^+ = \max\{u - \overline{u}, 0\} \in W_0^{1,p}(a,\Omega) \), where the condition \( \overline{u} \geq 0 \) on \( \partial \Omega \) in the sense of traces is essentially used. Thus, we can insert \( v = (u - \overline{u})^+ \) in (25) and (4) which gives
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (u - \overline{u})^+(x) dx + \lambda \int_{\Omega} \pi(x,u(x))(u - \overline{u})^+(x) dx = \int_{\Omega} f(x,Tu(x),n(Tu(x)))(u - \overline{u})^+(x) dx
\]
(26)
and
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla (u - \bar{u})^+(x) dx \geq \int_{\Omega} f(x, u(x), \nabla u(x)) (u - \bar{u})^+(x) dx. \quad (27)
\]

From (26) and (27), by subtraction we are led to
\[
\int_{\Omega} a(x) \left( |\nabla u(x)|^{p-2} \nabla u(x) - |\nabla \bar{u}(x)|^{p-2} \nabla \bar{u}(x) \right) \cdot \nabla (u - \bar{u})^+(x) dx + \lambda \int_{\Omega} \pi(x, u(x))(u - \bar{u})^+(x) dx
\leq \int_{\Omega} \left( f(x, T(x), \nabla (T(x))) - f(x, \pi(x), \nabla (\pi(x))) \right) (u - \bar{u})^+(x) dx.
\]

By (14), (17), and the preceding inequality we get
\[
\int_{\{u > \bar{u}\}} a(x) \left( |\nabla u(x)|^{p-2} \nabla u(x) - |\nabla u(x)|^{p-2} \nabla \bar{u}(x) \right) \cdot \nabla (u - \bar{u})^+(x) dx + \lambda \int_{\{u > \bar{u}\}} (u(x) - \bar{u}(x)) \frac{|\nabla u(x)|^p}{\bar{u}(x)} dx
\leq \int_{\{u > \bar{u}\}} \left( f(x, T(x), \nabla (T(x))) - f(x, \pi(x), \nabla (\pi(x))) \right) (u - \bar{u})^+(x) dx = 0.
\]

Since the function \(a(x)\) is positive almost everywhere in \(\Omega\) and the mapping \(\xi \mapsto |\xi|^{p-2} \xi\) on \(\mathbb{R}^N\) is monotone, we arrive at
\[
\int_{\{u > \bar{u}\}} (u(x) - \bar{u}(x)) \frac{|\nabla u(x)|^p}{\bar{u}(x)} dx \leq 0.
\]

Therefore, the Lebesgue measure of the set \(\{u > \bar{u}\}\) is zero, i.e., \(u \leq \bar{u}\) a.e. in \(\Omega\).

Similarly, we can prove that \(u \leq \bar{u}\) a.e. in \(\Omega\). Specifically, relying on the condition \(\bar{u} \leq 0\) on \(\partial \Omega\) (in the sense of traces), it holds \((\bar{u} - u)^+ = \max\{\bar{u} - u, 0\} \in W_0^{1,p}(\Omega, \mathbb{R})\), which allows us to test (25) and (3) with \(v = (\bar{u} - u)^+ \in W_0^{1,p}(\Omega, \mathbb{R})\). This results in
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla (\bar{u} - u)^+(x) dx + \lambda \int_{\Omega} \pi(x, u(x))(\bar{u} - u)^+(x) dx \]
\[
= \int_{\Omega} f(x, Tu(x), \nabla (Tu(x)))(\bar{u} - u)^+(x) dx \quad \text{(28)}
\]

and
\[
\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla (u - \bar{u})^+(x) dx \leq \int_{\Omega} f(x, u(x), \nabla u(x))(u - \bar{u})^+(x) dx. \quad \text{(29)}
\]

Arguing as before, we deduce from (28), (29), (14), and (17) the following estimate
\[
\int_{\{u > \bar{u}\}} a(x) \left( |\nabla u(x)|^{p-2} \nabla u(x) - |\nabla \bar{u}(x)|^{p-2} \nabla \bar{u}(x) \right) \cdot \nabla (u - \bar{u})^+(x) dx + \lambda \int_{\{u > \bar{u}\}} (u(x) - \bar{u}(x)) \frac{|\nabla u(x)|^p}{\bar{u}(x)} dx
\leq \int_{\Omega} \left( f(x, T(x, \nabla (T(x))) - f(x, \pi(x), \nabla (\pi(x))) \right)(\bar{u} - u)^+ dx
\]
\[
= \int_{\{u > \bar{u}\}} \left( f(x, T(x),\nabla (T(x))) - f(x, \pi(x), \nabla (\pi(x))) \right)(\bar{u} - u)^+ dx = 0.
\]

At this point, the positivity of the function \(a(x)\) on \(\Omega\) and the monotonicity of the mapping \(\xi \mapsto |\xi|^{p-2} \xi\) on \(\mathbb{R}^N\) confirm that
\[
\int_{\{u > \bar{u}\}} (u(x) - \bar{u}(x)) \frac{|\nabla u(x)|^p}{\bar{u}(x)} dx \leq 0,
\]
from which we can readily derive that \(u \leq \bar{u}\) a.e. in \(\Omega\).

Based on the enclosure property \(\bar{u} \leq u \leq \bar{u}\) a.e. in \(\Omega\), it follows through (17) that \(T(u) = u\) and through (14) that \(\Pi(u) = 0\). As a result, (25) takes the form of (2), thus the proof is complete. \(\square\)

Now we present an application of Theorem 1 describing how the existence of a nontrivial nonnegative solution can be established by effectively determining a sub-supersolution. In the sequel, by \(\lambda_1\) we denote the first eigenvalue of problem (13) (see Section 2).
Theorem 2. Let the weight $a \in L^1_{\text{loc}}(\Omega)$ fulfill the requirement (1). Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions:

(j) there is a constant $\mu > 0$ such that

$$\lambda_1 t^{p-1} \leq f(x, t, \xi) \text{ for a.e. } x \in \Omega, \text{ all } t \in [0, p], \xi \in \mathbb{R}^N;$$

(jj) there is a constant $C > 0$ such that

$$f(x, C, 0) \leq 0 \text{ for a.e. } x \in \Omega;$$

(jjj) there is a function $\sigma \in L^p_u(\Omega)$ and constants $b > 0$ and $r \in (0, \frac{p}{p+1})$ such that

$$|f(x, t, \xi)| \leq \sigma(x) + b|\xi|^r \text{ for a.e. } x \in \Omega, \text{ all } t \in [0, C], \xi \in \mathbb{R}^N.$$

Then problem $(P)$ has a nondegenerate, nonnegative and bounded weak solution $u \in W^{1,p}_0(a, \Omega)$ satisfying the estimate $u \leq C$.

Proof. Our goal is to apply Theorem 1 by constructing an appropriate sub-supersolution. In order to determine a subsolution, we use an eigenfunction $u_1 \in W^{1,p}_0(a, \Omega)$ corresponding to the first eigenvalue $\lambda_1$ of problem (13) with the properties $u_1(x) \geq 0$ for a.e. $x \in \Omega$, $u_1 \neq 0$, and $u_1 \in L^\infty(\Omega)$ as mentioned in Section 2. Then we choose an $\varepsilon > 0$ sufficiently small to verify

$$\varepsilon u_1(x) \leq \mu \text{ for a.e. } x \in \Omega,$$  \hfill (30)

where $\mu$ is the positive constant postulated in assumption (j). Then assumption (j) implies

$$\lambda_1 (\varepsilon u_1)^{p-1} \leq f(x, \varepsilon u_1, \nabla (\varepsilon u_1)) \text{ for a.e. } x \in \Omega.$$  \hfill (31)

For a possibly smaller $\varepsilon > 0$ we can suppose

$$\varepsilon u_1(x) \leq C \text{ for a.e. } x \in \Omega,$$  \hfill (32)

with $C > 0$ in assumption (jj).

Let us fix an $\varepsilon > 0$ for which (30) and (32) are fulfilled. We claim that $\underline{u} = \varepsilon u_1$ is a subsolution to problem $(P)$. Indeed, by (13) with $u_1$ in place of $u$ and (31) we note that

$$\int_\Omega a(x) |\nabla \underline{u}(x)|^{p-2} \nabla \underline{u}(x) \cdot \nabla v(x) dx = \varepsilon^{p-1} \int_\Omega u_1(x)^{p-1} v(x) dx \leq \int_\Omega f(x, \varepsilon u_1(x), \nabla (\varepsilon u_1)) v(x) dx = \int_\Omega f(x, \underline{u}(x), \nabla \underline{u}(x)) v(x) dx$$

for all $v \in W^{1,p}_0(a, \Omega)$, $v \geq 0$ a.e. in $\Omega$, thereby proving the claim.

Next we claim that the constant function $\overline{u} = C$, with $C > 0$ in assumption (jj), is a supersolution to problem $(P)$. Accordingly, from assumption (jj) we find that

$$\int_\Omega a(x) |\nabla \overline{u}(x)|^{p-2} \nabla \overline{u}(x) \cdot \nabla v(x) dx = 0 \geq \int_\Omega f(x, C, 0) v(x) dx = \int_\Omega f(x, \overline{u}(x), \nabla \overline{u}(x)) v(x) dx$$

for all $v \in W^{1,p}_0(a, \Omega)$, $v \geq 0$ a.e. in $\Omega$, which proves the claim.

It is clear from (32) that $\underline{u}(x) \leq \overline{u}(x)$ for a.e. in $\Omega$. Assumption (jj) ensures that the growth condition required in Hypothesis 1 of Theorem 1 holds true. Therefore, all the hypotheses of Theorem 1 are verified, which permits the conclusion that there exists a solution $u \in W^{1,p}_0(a, \Omega)$ of problem $(P)$ within the ordered interval $[\underline{u}, \overline{u}]$. Since the function $\overline{u} = \varepsilon u_1$ is nontrivial and nonnegative, and $u \geq \underline{u}$, we have that $u$ is nontrivial and
nonnegative, whereas \( u \in [u, \bar{u}] \) renders the boundedness of \( u \) and the a priori estimate \( u \leq C \). The proof is complete. \( \square \)

We end the paper with a simple example for which Theorem 2 applies.

**Example 1.** Fix a positive weight \( a \in L^1_{\text{loc}}(\Omega) \) with the property (1). Let the function \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) be defined by

\[
f(x,t,\xi) = \begin{cases} 0 & \text{if } t < 0 \\ t^{p-1} \left( \rho(x) + |\xi|^r \right) & \text{if } 0 \leq t \leq 1 \\ (2-t) \left( \rho(x) + |\xi|^r \right) & \text{if } t > 1, \end{cases}
\]

with some \( r \in [1, \frac{p}{p^*}] \) and \( \rho \in L^\infty(\Omega) \) satisfying \( \rho(x) \geq \lambda_1 \) for a.e. \( x \in \Omega \). It follows that \( f \) is a Carathéodory function for which conditions (j) – (jjj) in Theorem 2 are verified. Precisely, condition (j) holds with \( \mu = 1 \) because \( \rho(x) \geq \lambda_1 \), condition (jj) holds with \( C = 2 \), and condition (jjj) is fulfilled with the given \( r \). Hence Theorem 2 applies to problem (P) whose equation has the right-hand side expressed with the function \( f(x,t,\xi) \) given above.

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