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GROWTH OF CENTRAL POLYNOMIALS OF ALGEBRAS WITH INVOLUTION

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Abstract. Let A be an associative algebra with involution $*$ over a field of characteristic zero. A central $*$-polynomial of A is a polynomial in non-commutative variables that takes central values in A.
Here we prove the existence of two limits called the central $*$-exponent and the proper central $*$-exponent that give a measure of the growth of the central $*$-polynomials and proper central $*$-polynomials, respectively. Moreover, we compare them with the PI-$*$-exponent of the algebra.

1. Introduction

Let A be an associative algebra with involution $*$ over a field F of characteristic zero and let $F\langle X, *\rangle$ be the free associative algebra with involution freely generated over F by the countable set X of non-commutative variables.

A $*$-polynomial $f(x_1^+, \dots, x_r^+, x_1^-, \dots, x_s^-) \in F\langle X, *\rangle$ is a central $*$-polynomial of A if for all $a_1^+, \dots, a_r^+ \in A^+$ and for all $a_1^-, \dots, a_s^- \in A^-$, $f(a_1^+, \dots, a_r^+, a_1^-, \dots, a_s^-) \in Z(A)$, the center of A. If f takes only the zero value, then f is called $*$-polynomial identity of A whereas if f takes a non-zero value in $Z(A)$, then it is said to be a proper central $*$-polynomial.

The study of central polynomials was motivated by a famous conjecture of Kaplansky (see [18]) that later on was proved independently by Formanek and Razmyslov ([6], [27]), asserting that the algebra $M_n(F)$ of $n \times n$ matrices without any additional structure has proper central polynomials. This result is highly non-trivial since even if an algebra has a non-zero center, the existence of proper central polynomials is not granted. For instance, it is well-known (see [15, Lemma 1]) that the algebra of block upper triangular matrices has no proper central polynomials.

Here we are interested in a comparison among the growth of the spaces of central $*$-polynomials, proper central $*$-polynomials and $*$-polynomial identities. To this end, we use an idea introduced by Regev in [28] in the setting of ordinary central polynomials. We will consider the space P_n^* of multilinear $*$-polynomials of degree n and we shall attach to it three numerical sequences: $c_n^*(A)$, the dimension of P_n^* modulo the $*$-polynomial identities of A; $c_n^{*,z}(A)$, the dimension of P_n^* modulo the central $*$-polynomials of A; $\delta_n^*(A)$, the dimension of the central $*$-polynomials modulo the $*$-identities of A. It is easily seen that for all $n \geq 1$,

$$c_n^*(A) = c_n^{*,z}(A) + \delta_n^*(A). \tag{1}$$

We analize the growth of $c_n^{*,z}(A)$ and $\delta_n^*(A)$ by means of their asymptotic behavior.

In particular, in [10] it was proved that the limit

$$\exp^*(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^*(A)}$$

exists and is a non-negative integer called the $*$-exponent of A. Because of equality (1), it is worth asking if the limits

$$\exp_*^z(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^{*,z}(A)}, \quad \exp_*^\delta(A) = \lim_{n\to+\infty} \sqrt[n]{\delta_n^*(A)}$$

exist.

In this paper, we firstly prove that for any $*$-algebra A, $\exp_*^\delta(A)$ exists and is a non-negative integer, secondly that $\exp_*^z(A) = \exp^*(A)$ provided $\exp^*(A) \geq 2$. Moreover $\exp_*^z(A) = 0$ or 1 if $\exp^*(A) \leq 1$. A similar result was obtained for ordinary algebras in [15] and [16].

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2. Background on $*$-algebras

Throughout the paper F will denote a field of characteristic zero. Let A be an associative F-algebra with involution $*$, i.e., a linear map $*: A \to A$ of order two such that $(ab)^* = b^*a^*$, for all $a, b \in A$. We write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively.

Let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ be the free associative algebra with involution on a countable set $X = \{x_1, x_2, \dots\}$ over F. It is useful to regard $F\langle X, *\rangle$ as generated by the symmetric variables and by the skew variables , i.e., $F\langle X, *\rangle = F\langle x_1^+, x_1^-, x_2^+, x_2^-, \dots\rangle$, where $x_i^+ = x_i + x_i^*$, $x_i^- = x_i - x_i^*$, $i \geq 1$. Recall that a polynomial $f(x_1^+, \dots, x_r^+, x_1^-, \dots, x_s^-) \in F\langle X, *\rangle$ is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if $f(a_1^+, \dots, a_r^+, a_1^-, \dots, a_s^-) = 0$ for all $a_1^+, \dots, a_r^+ \in A^+$, $a_1^-, \dots, a_s^- \in A^-$. The set $\mathrm{Id}^*(A)$ of all $*$-identities of A is a T_*-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra commuting with the involution $*$.

Let us denote by P_n^* the space of multilinear polynomials of degree n in $x_1^+, x_1^-, \dots, x_n^+, x_n^-$, i.e.,

$$P_n^* = \mathrm{span}_F\Big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i \in \left\{x_i^+, x_i^-\right\},\ i = 1, \dots, n\Big\};$$

hence for every $i = 1, 2, \dots, n$ either x_i^+ or x_i^- appears in every monomial of P_n^* at degree 1 (but not both), for any $i = 1, \dots, n$. Since in characteristic zero every $*$-identity is equivalent to a system of multilinear $*$-identities, the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. Thus we construct the quotient space

$$P_n^*(A) = \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)},$$

and we call its dimension, $c_n^*(A) = \dim P_n^*(A)$, $n \geq 1$ the nth $*$-codimension of A.

We can define an action of the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$ on the space P_n^* by the rule: for $h = (\varepsilon_1, \dots, \varepsilon_n; \sigma) \in \mathbb{Z}_2 \wr S_n$, $hx_i^+ = x_{\sigma(i)}^+$ and $hx_i^- = (x_{\sigma(i)}^-)^{\varepsilon_{\sigma(i)}} = x_{\sigma(i)}^-$ or $-x_{\sigma(i)}^-$ according as $\varepsilon_{\sigma(i)} = 1$ or -1, respectively (see [11]). Thus, since $P_n^* \cap Id^*(A)$ is invariant under the $\mathbb{Z}_2 \wr S_n$ action, the space $P_n^*(A)$ inherits the structure of left $\mathbb{Z}_2 \wr S_n$-module. The structure of a $\mathbb{Z}_2 \wr S_n$-submodule M can be described as follows (see [5] and [11]): let $0 \leq r \leq n$ and let T_λ and T_μ be Young tableaux of shape λ and μ filled with the integers $1, \dots, r$ and $r+1, \dots, n$, respectively. Moreover, let e_{T_λ} and e_{T_μ} be the corresponding essential idempotents of FS_r and $FS_{(r+1,\dots,n)} \equiv FS_{n-r}$, respectively. Then, if Γ is a set of left coset representatives of $S_r \times S_{n-r}$ in S_n, we have that

$$M \cong \Big(\bigoplus_{\gamma \in \Gamma} \gamma FS_r e_{T_\lambda} \otimes FS_{n-r} e_{T_\mu}\Big)(x_1^+ \cdots x_r^+ x_{r+1}^- \cdots x_n^-),$$

where S_r and S_{n-r} act on the set of variables $\{x_1^+, \dots, x_r^+\}$ and $\{x_{r+1}^-, \dots, x_n^-\}$, respectively.

The $\mathbb{Z}_2 \wr S_n$-character of $P_n^*(A)$, denoted by $\chi_n^*(A)$, is called the nth $*$-cocharacter of A and by complete reducibility we can write

$$\chi_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu} \chi_{\lambda,\mu},$$

where $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to the multipartition (λ, μ), $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity and $|\lambda| + |\mu| = n$ means that $\lambda \vdash r$ and $\mu \vdash n - r$.

For fixed $r = 0, \dots, n$, $P_{r,n-r}^*$ denotes the space of multilinear polynomials in the variables $x_1^+, \dots, x_r^+$, $x_1^-, \dots, x_{n-r}^-$. It is clear that $P_n^*(A)$ can be determined by studying $P_{r,n-r}^*(A) = \frac{P_{r,n-r}^*}{P_{r,n-r}^* \cap \mathrm{Id}^*(A)}$ for all $0 \leq r \leq n$. To this end, we define an action of $S_r \times S_{n-r}$ on $P_{r,n-r}^*$ in a natural way: S_r and S_{n-r} act by permutation on the variables $x_1^+, \dots, x_r^+$ and $x_1^-, \dots, x_{n-r}^-$, respectively. We get that $P_{r,n-r}^*(A)$ has an induced structure of $S_r \times S_{n-r}-$module and its character arises. If we denote it by $\chi_{r,n-r}^*(A)$, then by complete reducibility, we can write

$$\chi_{r,n-r}^*(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} \overline{m}_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu)$$

and, according to [5, Theorem 1.3], we have that

$$m_{\lambda,\mu} = \overline{m}_{\lambda,\mu}, \tag{2}$$

for all $\lambda \vdash r$, $\mu \vdash n-r$, $r = 0, \dots, n$. One of the most challenging problem is to determine these multiplicities (see for instance [24] and [29]).

Next we shall introduce a powerful tool that one can use in order to reduce the problem of computing the T^*-ideal of $*$-identities of any $*$-algebra to that of a suitable finite dimensional superalgebra. Recall that an algebra A is a superalgebra if A is graded by $\mathbb{Z}_2$, the cyclic group of order 2, i.e. $A = A_0 \oplus A_1$, as vector space, such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. The elements of A_0 and A_1 are called homogeneous elements of degree 0 and 1, respectively.

Let G denote the infinite Grassmann algebra generated by the elements $1, e_1, e_2, \dots$ subject to the relations $e_ie_j = -e_je_i$, for all i, j. Let $G = G_0 \oplus G_1$ be its natural $\mathbb{Z}_2$-grading, where

$$G_0 = \operatorname{span}\{e_{i_1} \dots e_{i_{2k}} | 1 \leq i_1 < \dots < i_{2k}, k \geq 0\}$$

and

$$G_1 = \operatorname{span}\{e_{i_1} \dots e_{i_{2k+1}} | 1 \leq i_1 < \dots < i_{2k+1}, k \geq 0\}.$$

If $A = A_0 \oplus A_1$ is a superalgebra, then the algebra $G(A) = G_0 \otimes A_0 \oplus G_1 \otimes A_1$ is called the Grassmann envelope of A.

Now recall that a superinvolution on a superalgebra A is a graded linear map $\sharp : A \to A$ such that $(a^\sharp)^\sharp = a$ for all $a \in A$ and $(ab)^\sharp = (-1)^{|a||b|} b^\sharp a^\sharp$, for every $a, b \in A_0 \cup A_1$ of homogeneous degree $|a|$ and $|b|$, respectively. Since $\operatorname{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i | a^\sharp = a\}$ and $A_i^- = \{a \in A_i | a^\sharp = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively. In recent years, superalgebras with superinvolution have played a prominent role in the theory of polynomial identities and they have been extensively studied by many authors (see for examples [7, 8, 9, 21, 22, 23]).

Notice that if A is a superalgebra with superinvolution $\sharp$, the Grassmann envelope $G(A)$ can be regarded as an algebra with involution $* : G(A) \to G(A)$ such that $(a \otimes g)^* = a^\sharp \otimes g^\star$, where $\star : G \to G$ is the superinvolution defined by $e_i^\star = -e_i$, for $i \geq 1$. Furthermore, since $G(A)$ has a natural structure of superalgebra defined by $G(A)_0 = A_0 \otimes G_0$ and $G(A)_1 = A_1 \otimes G_1$, the involution $*$ can be regarded also as a graded involution, i.e. an involution preserving the $\mathbb{Z}_2$-grading.

In what follows, in order to simplify the notation, we shall always use $\sharp$ to denote the superinvolution of the superalgebra A and $*$ to denote the involution on $G(A)$.

In the theory of algebras with polynomial identity without any additional structure, a celebrated theorem of Kemer states that an arbitrary algebra satisfying a (ordinary) polynomial identity over a field of characteristic zero has the same identities of the Grassmann envelope $G(A)$ of a finite dimensional superalgebra A (see [20]). In [1] Aljadeff, Giambruno and Karasik proved the analogous result in case of algebras with involution.

Theorem 1 ([1],Theorem 4)**.** *If A is an algebra with involution satisfying a non-trivial $*$-identity, then there exists a finite dimensional superalgebra with superinvolution B such that $Id^*(A) = Id^*(G(B))$.*

As in the case of algebras with involution, one can define a superinvolution on the free algebra $F\langle X\rangle$ in a natural way. We write the set X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1 respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements respectively. The free algebra with superinvolution is denoted by $F\langle Y \cup Z, \sharp\rangle$ and it is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y \cup Z, \sharp\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

A $\sharp$-polynomial $f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_t^+, z_1^-, \dots, z_s^-) \in F\langle Y \cup Z, \sharp\rangle$ is a $\sharp$-polynomial identity of A (or simply a $\sharp$-identity), and we write $f \equiv 0$, if

$$f(u_1^+, \dots, u_n^+, u_1^-, \dots, u_m^-, v_1^+, \dots, v_t^+, v_1^-, \dots, v_s^-) = 0$$

for all $u_1^+, \dots, u_n^+ \in A_0^+$, $u_1^-, \dots, u_m^- \in A_0^-$, $v_1^+, \dots, v_t^+ \in A_1^+$ and $v_1^-, \dots, v_s^- \in A_1^-$.

We denote by $\operatorname{Id}^\sharp(A) = \{f \in F\langle Y \cup Z, \sharp\rangle \mid f \equiv 0 \text{ on } A\}$ the $T_2^\sharp$-ideal of $\sharp$-identities of A, i.e., $\operatorname{Id}^\sharp(A)$ is an ideal of $F\langle Y \cup Z, \sharp\rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra commuting with the superinvolution $\sharp$. It is well known that in characteristic zero, every $\sharp$-identity is equivalent to a system of multilinear $\sharp$-identities. Hence if we denote by

$$P_n^\sharp = \mathrm{span}_F\Big\{w_{\sigma(1)}\cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \ldots, n\Big\}$$

the space of multilinear polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \ldots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1) the study of $\mathrm{Id}^\sharp(A)$ is equivalent to the study of $P_n^\sharp \cap \mathrm{Id}^\sharp(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^\sharp(A) = \dim_F \frac{P_n^\sharp}{P_n^\sharp \cap \mathrm{Id}^\sharp(A)},\ n \geq 1$$

is called the n-th $\sharp$-codimension of A.

Let now $N \geq 1$ and write $N = m + n + p + q$ as a sum of four non-negative integers. We denote by $P^\sharp_{m,n,p,q} \subseteq P_N^\sharp$ the vector space of multilinear $\sharp$-polynomials in which m variables are symmetric of even degree, n variables are skew of even degree, p variables are symmetric of odd degree and q variables are skew of odd degree. As already done previously, one can define a natural action of $S_m \times S_n \times S_p \times S_q$ on $P^\sharp_{m,n,p,q}$ by permuting the variables of the same homogeneous degree and the same symmetry with respect to $\sharp$. Thus the $S_m \times S_n \times S_p \times S_q$-character arises and we call it the (m, n, p, q)-th cocharacter of the superalgebra A.

We now define a map whose properties will be used very often (see [1]).

If $w \in P^\sharp_{m,n,p,q}$, we write

$$w = w_1 z_{\sigma(1)}^{\varepsilon_{\sigma(1)}} \ldots z_{\sigma(i_1)}^{\varepsilon_{\sigma(i_1)}} w_2 z_{\sigma(i_1+1)}^{\varepsilon_{\sigma(i_1+1)}} \ldots z_{\sigma(i_2)}^{\varepsilon_{\sigma(i_2)}} w_3 \ldots w_{r+1},$$

where $\sigma \in S_{p+q}$, $\varepsilon_{i_j} \in \{+, -\}$ and the w_i's are (possibly empty) monomials in even variables.

Then we consider the linear map

$$\tilde{}: P^\sharp_{m,n,p,q} \to P^\sharp_{m,n,q,p}$$

defined on the monomials by

$$\tilde{w} = (\mathrm{sgn}\,\sigma) w_1 z_{\sigma(1)}^{\eta_{\sigma(1)}} \ldots z_{\sigma(i_1)}^{\eta_{\sigma(i_1)}} w_2 z_{\sigma(i_1+1)}^{\eta_{\sigma(i_1+1)}} \ldots z_{\sigma(i_2)}^{\eta_{\sigma(i_2)}} w_3 \ldots w_{r+1},$$

where $\eta_i = -\varepsilon_i$ for all i, that is, $\eta_i = +$ if $\varepsilon_i = -$ and $\eta_i = -$ if $\varepsilon_i = +$. Notice that the use of such map (in the ordinary case) started with Kemer (see [20]).

In [1, Lemma 2] and [10, Lemma 1], the authors gave the following basic properties of the map $\tilde{}$.

Lemma 1. *The map* $\tilde{}: P^\sharp_{m,n,p,q} \to P^\sharp_{m,n,q,p}$ *has the following properties.*

(1) *If* $f \in P^\sharp_{m,n,p,q}$, *then* $\tilde{\tilde{f}} = f$.
(2) *If* $f \in P^\sharp_{m,n,p,q}$, *then for any set of variables* $Z = \{z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-\}$, f *is alternating on* Z *if and only if* $\tilde{f}$ *is symmetric on* Z.
(3) *If* A *is a superalgebra with superinvolution, then* $f \in Id^\sharp(A)$ *if and only if* $\tilde{f} \in Id_2^*(G(A))$.

Here $Id_2^*(G(A))$ *means the* T*-ideal of* $*$*-graded polynomial identities of* $G(A)$.

Another basic result we shall need is the Wedderburn–Malcev theorem for finite dimensional superalgebras with superinvolution. First we recall some definitions. An ideal (subalgebra) I of a superalgebra A with superinvolution $\sharp$ is a $\sharp$-superideal ($\sharp$-superalgebra) if it is a graded ideal (subalgebra) and $I^\sharp = I$.

We also say that the algebra A is a simple $\sharp$-superalgebra if $A^2 \neq 0$ and A has nontrivial $\sharp$-superideals.

Theorem 2 ([7], Theorem 4.1)**.** *Let A be a finite dimensional superalgebra with superinvolution over a field F of characteristic zero. Then there exist simple $\sharp$-superalgebras $B_1, \ldots, B_m \subset A$ such that*

$$A = B_1 \oplus \ldots \oplus B_m + J(A),$$

where the Jacobson radical $J(A)$ is a $\sharp$-superideal.

In the light of the previous theorem, it is clear that we need a classification of the finite dimensional simple $\sharp$-superalgebras over an algebraically closed field F (see [2, 17, 26]). In order to describe such a result we first recall same important facts.

It is well known (see [30]) that if F is algebraically closed, a simple superalgebra A is isomorphic to one of the following types:

(i) Given $k+l\geq 1$, $k\geq l\geq 0$,

$$M_{k,l}(F)=\left\{\begin{pmatrix} X & Y\\ Z & T\end{pmatrix} | X\in M_k(F), Y\in M_{k\times l}(F), Z\in M_{l\times k}(F), T\in M_l(F)\right\}$$
$$=(M_{k,l}(F))_0\oplus (M_{k,l}(F))_1$$

where $(M_{k,l}(F))_0=\left\{\begin{pmatrix} X & 0\\ 0 & T\end{pmatrix}\right\}$ and $(M_{k,l}(F))_1=\left\{\begin{pmatrix} 0 & Y\\ Z & 0\end{pmatrix}\right\}$;

(ii) $Q(n)=M_n(F\oplus cF)=Q(n)_0\oplus Q(n)_1$, where $Q(n)_0=M_n(F)$ and $Q(n)_1=cM_n(F)$ with $c^2=1$.

If A is a superalgebra, we denote by A^{sop} the superalgebra which has the same graded vector space structure as A but the product in A^{sop} is given on homogeneous elements a,b by

$$a\circ b=(-1)^{|a||b|}ba.$$

The direct sum $R=A\oplus A^{\mathrm{sop}}$ is a superalgebra by setting $R_0=A_0\oplus A_0^{\mathrm{sop}}$ and $R_1=A_1\oplus A_1^{\mathrm{sop}}$ and it is endowed with the exchange superinvolution

$$(a,b)^{\mathrm{ex}}=(b,a).$$

Recall that if A and B are two algebras (superalgebras) endowed with involutions (superinvolutions) $*$ and $\star$, respectively, then $(A,*)$ and $(B,\star)$ are isomorphic, as algebras (superalgebras) endowed with involution (superinvolution), if there exist an isomorphism of algebras (superalgebras) $\phi: A\to B$ such that $\phi(a^*)=\phi(a)^\star$, for all $a\in A$.

Theorem 3 ([2, 17, 26]). *Let A be a finite dimensional $\sharp$-simple superalgebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following:*

(1) *$M_{k,2s}(F)$ with the orthosymplectic superinvolution* osp *defined by*

$$\begin{pmatrix} X & Y\\ Z & T\end{pmatrix}^{\mathrm{osp}}=\begin{pmatrix} I_k & 0\\ 0 & Q\end{pmatrix}^{-1}\begin{pmatrix} X & -Y\\ Z & T\end{pmatrix}^{t}\begin{pmatrix} I_k & 0\\ 0 & Q\end{pmatrix}$$

where t denotes the usual matrix transpose, $Q=\begin{pmatrix} 0 & I_s\\ -I_s & 0\end{pmatrix}$ and I_k, I_s are the identity matrices of orders k and s, respectively;

(2) *$M_{k,k}(F)$ with the transpose superinvolution* trp *defined by*

$$\begin{pmatrix} X & Y\\ Z & T\end{pmatrix}^{\mathrm{trp}}=\begin{pmatrix} T^t & -Y^t\\ Z^t & X^t\end{pmatrix};$$

(3) *$M_{k,l}(F)\oplus M_{k,l}(F)^{\mathrm{sop}}$ with the exchange superinvolution;*

(4) *$Q(n)\oplus Q(n)^{\mathrm{sop}}$ with the exchange superinvolution.*

According to the previous result, $Z(A)\cong F$ if either $A\cong (M_{k,2s}(F),\mathrm{osp})$ or $A\cong(M_{k,k}(F),\mathrm{trp})$, $Z(A)\cong F\oplus F$ if $A\cong M_{k,l}(F)\oplus M_{k,l}(F)^{\mathrm{sop}}$ and $Z(A)\cong (F\oplus cF)\oplus(F\oplus cF)$ if $A\cong Q(n)\oplus Q(n)^{\mathrm{sop}}$.

3. On central $*$-polynomials

In this section we shall introduce the main object of the paper. Let A be an algebra with involution $*$. A polynomial $f(x_1^+,\dots,x_r^+,x_1^-,\dots,x_s^-)\in F\langle X,*\rangle$ is a central $*$-polynomial of A if $f(a_1^+,\dots,a_r^+,a_1^-,\dots,a_s^-)\in Z(A)$ for all $a_1^+,\dots,a_r^+\in A^+$, $a_1^-,\dots,a_s^-\in A^-$. Clearly in case f takes only the zero value, f is a $*$-identity of A whereas if f takes a non-zero value in $Z(A)$, we say that f is a proper central $*$-polynomial. Let $\mathrm{Id}^{*,z}(A)$ denote the set of central $*$-polynomials of A. Notice that $\mathrm{Id}^{*,z}(A)$ is a T_*-space of $F\langle X,*\rangle$, i.e., a vector space invariant under all endomorphisms of the free algebra commuting with the involution $*$.

If we set

$$P_n^{*,z}(A)=\frac{P_n^*}{P_n^*\cap \mathrm{Id}^{*,z}(A)},$$

then the quotient space

$$\Delta_n^*(A)=\frac{P_n^*\cap \mathrm{Id}^{*,z}(A)}{P_n^*\cap \mathrm{Id}^*(A)}$$

corresponds to the space of proper central $*$-polynomials in n fixed variables. The following remark can be easily checked.

Remark 1. *Let A and B two $*$-algebras. If $Id^*(A) = Id^*(B)$, then $Id^{*,z}(A) = Id^{*,z}(B)$ and $\Delta_n^*(A) = \Delta_n^*(B)$, for all $n \geq 1$.*

We call $c_n^{*,z}(A) = \dim_F P_n^{*,z}(A)$ and $\delta_n^*(A) = \dim_F \Delta_n^*(A)$, $n = 1, 2, \ldots,$ the sequences of central $*$-codimensions and proper central $*$-codimensions of A, respectively. Because of relation (1), it easily seen that

$$c_n^*(A) \geq c_n^{*,z}(A) \text{ and } c_n^*(A) \geq \delta_n^*(A). \tag{3}$$

Remark that for all $0 \leq r \leq n$ if we set

$$\Delta^*_{r,n-r}(A) = \frac{P^*_{r,n-r} \cap \mathrm{Id}^{*,z}(A)}{P^*_{r,n-r} \cap \mathrm{Id}^*(A)}$$

and $\delta^*_{r,n-r}(A) = \dim_F \Delta^*_{r,n-r}(A)$, then

$$\delta_n^*(A) = \sum_{r=0}^{n} \binom{n}{r} \delta^*_{r,n-r}(A).$$

Notice that the spaces $P_n^{*,z}(A)$ and $\Delta_n^{*,z}(A)$ have an induced structure of left $\mathbb{Z}_2 \wr S_n$-module. We denote by $\chi_n^{*,z}(A)$ and $\chi_n^*(\Delta(A))$ the corresponding characters, respectively. By compete reducibility we decompose such characters into irreducibles:

$$\chi_n^{*,z}(A) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}\chi_{\lambda,\mu}, \qquad \chi_n^*(\Delta(A)) = \sum_{|\lambda|+|\mu|=n} m''_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to (λ, μ) and $m'_{\lambda,\mu}$, $m''_{\lambda,\mu}$ are the corresponding multiplicities.

Clearly,

$$\chi_n^*(A) = \chi_n^{*,z}(A) + \chi_n^*(\Delta(A)),$$

and $m_{\lambda,\mu} = m'_{\lambda,\mu} + m''_{\lambda,\mu}$, for all $\lambda \vdash r$, $\mu \vdash n-r$, $r = 0, \ldots, n$. Hence $m'_{\lambda,\mu} \leq m_{\lambda,\mu}$ and $m''_{\lambda,\mu} \leq m_{\lambda,\mu}$ and we write

$$\chi_n^{*,z}(A) \leq \chi_n^*(A), \qquad \chi_n^*(\Delta(A)) \leq \chi_n^*(A). \tag{4}$$

Notice that since the three codimensions do not change by extension of the base field, from now on we shall assume that the base field F is algebraically closed.

Let B be a finite dimensional superalgebra with superinvolution over F such that $Id^*(A) = Id^*(G(B))$. Since F is algebraically closed, B has a Wedderburn–Malcev decomposition $B_1 \oplus \cdots \oplus B_m + J$ where B_i are $\sharp$-simple superalgebras listed in Theorem 3.

We say that B is reduced if for some permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$ we have that $B_{i_1}JB_{i_2}J \ldots JB_{i_m} \neq 0$. Moreover, we give also the following definition.

Definition 1. *A semisimple subalgebra $\overline{B} = B_{i_1} \oplus \cdots \oplus B_{i_k}$ of B, where $i_1, \ldots i_k \in \{1, \ldots, m\}$ are distinct, is centrally admissible in $G(B)$ if there exists a multilinear proper central $*$-polynomial $f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_s^-)$ of $G(B)$ with $r + s \geq k$ such that*

$$f(a_1^+, \ldots, a_{k_1}^+, b_1^+, \ldots, b_{r-k_1}^+, a_1^-, \ldots, a_{k_2}^-, b_1^-, \ldots, b_{s-k_2}^-) \neq 0$$

for some $a_1^+ \in G(B_{i_1})^+, \ldots, a_{k_1}^+ \in G(B_{i_{k_1}})^+, a_1^- \in G(B_{i_{k_1+1}})^-, \ldots, a_{k_2}^- \in G(B_{i_k})^-$, $b_1^+, \ldots, b_{r-k_1}^+ \in G(B)^+$, $b_1^-, \ldots b_{s-k_2}^- \in G(B)^-$, $k_1 + k_2 = k$.

Remark that if the semisimple subalgebra $\overline{B}$ of B is centrally admissible in $G(B)$ of maximal dimension, then $\widehat{B} = \overline{B} + J$ is reduced. In fact, without loss of generality let us assume that $\overline{B} = B_1 \oplus \cdots \oplus B_k$ and let $f = f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_s^-)$ be a proper multilinear central polynomial of Definition 1. Since $G(B_i)G(B_j) = 0$ for any $i \neq j$, then $G(B_{i_1})G(J)G(B_{i_2})G(J) \ldots G(J)G(B_{i_m}) \neq 0$ for some permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$. Thus it follows that $B_{i_1}JB_{i_2}J \ldots JB_{i_k} \neq 0$, i.e., the superalgebra with superinvolution $\widehat{B} = \overline{B} + J$ is reduced.

In what follows we shall prove that $\delta_n^*(G(B))$ is bounded from above and from below, up to a polynomial factor, by d^n where d is the maximal dimension of a centrally admissible subalgebra in $G(B)$. This result will lead to the main theorem of the paper.

4. An upper bound for $\delta_n^*(A)$

In this section an upper bound for the proper central $*$-codimension sequence will be found.

If $\lambda \vdash n$, we define an infinite hook $H(d,l)$ as follows:

$$H(d,l) = \bigcup_{n\geq 1} \{\lambda = (\lambda_1, \lambda_2, \dots,) \vdash n \mid \lambda_{d+1} \leq l\}.$$

We say that λ lies in the hook $H(d,l)$, and we write $\lambda \in H(d,l)$, if the corresponding Young diagram is contained in $H(d,l)$. Moreover, we shall denote by d_λ the degree of the corresponding character χ_λ.

The next technical lemma describes the $*$-cocharacter of the Grassmann envelope of a superalgebra with superinvolution in terms of the shape of the Young diagrams associated to the irreducibles appearing in its decomposition.

Lemma 2. *Let A be a finite dimensional superalgebra with superinvolution $\sharp$ and Jacobson radical $J = J(A)$ and let B be the maximal semisimple $\sharp$-subalgebra of A appearing in the Wedderburn–Malcev decomposition, i.e. $A = B + J$, with $\dim_F B_0^+ = p_1$, $\dim_F B_0^- = p_2$, $\dim_F B_1^+ = q_1$ and $\dim_F B_1^- = q_2$. Moreover, suppose that $J^m = 0$ for some $m \geq 1$. If*

$$\chi_n^*(G(A)) = \sum_{\lambda,\mu} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

is the $*$-cocharacter of $G(A)$, then $\lambda \in H(p_1,q_2)\cup S_1$ and $\mu \in H(p_2,q_1)\cup S_2$, where $S_1 = ((p_1+q_2+m)^{p_1+q_2+m})$ and $S_2 = ((p_2+q_1+m)^{p_2+q_1+m})$.

Moreover, any character $\chi_{\lambda,\mu}$ appearing with a nonzero multilicity in $\chi_n^*(G(A))$ has at most $m-1$ boxes outside $H(p_1,q_2)$ and $H(p_2,q_1)$.

Proof. By taking into account (2), in order to reach the desired conclusion, we can study $\chi_{r,n-r}^*(A)$ for all $0 \leq r \leq n$. Moreover, as we saw in the previous section, $G(A)$ can be regarded as a superalgebra with graded involution, where $G(A)_0^+ = A_0^+ \otimes G_0$, $G(A)_0^- = A_0^- \otimes G_0$, $G(A)_1^+ = A_1^- \otimes G_1$ and $G(A)_1^- = A_1^+ \otimes G_1$. Thus, if we consider $P_{n_1,\dots,n_4}^*(G(A))$, the $S_{n_1} \times \ldots \times S_{n_4}$-character arises and we call it graded $*$-cocharacter of $G(A)$. One can induce the $*$-cocharacter of $G(A)$ from the graded one in the following way.

Let $(\lambda,\mu) \vdash (r, n-r)$ where $0 \leq r \leq n$, be a multipartition whose corresponding $S_r \times S_{n-r}$-character $\chi_\lambda \otimes \chi_\mu$ appears with a nonzero multiplicity in the decomposition of the $*$-cocharacter $\chi_{r,n-r}^*(G(A))$. Then there exists a multilinear $*$-polynomial $h(x_1^+,\dots,x_r^+,x_{r+1}^-,\dots,x_n^-)$ such that

$$h(a_1^+,\dots,a_{n_1}^+,b_1^+,\dots,b_{n_3}^+,c_1^-,\dots,c_{n_2}^-,d_1^-,\dots,d_{n_4}^-) \neq 0,$$

for some non-negative integers $n_1,\dots,n_4$, $r = n_1+n_3$, $n-r = n_2+n_4$ and for some $a_1^+,\dots,a_{n_1}^+ \in G(A)_0^+$, $b_1^+,\dots,b_{n_3}^+ \in G(A)_1^+$, $c_1^-,\dots,c_{n_2}^- \in G(A)_0^-$ and $d_1^-,\dots,d_{n_4}^- \in G(A)_1^-$. Thus $F(S_r \times S_{n-r})h$ is an irreducible $S_r \times S_{n-r}$-module with character $\chi_\lambda \otimes \chi_\mu$.

Now let us consider the S_{n_1}-permutation action on the variables $x_1^+,\dots,x_{n_1}^+$, the S_{n_3}-permutation action on the variables $x_{n_1+1}^+,\dots,x_{n_1+n_3}^+$, the S_{n_2}-permutation action on the variables $x_{n_1+n_3+1}^-,\dots,x_{n_1+n_2+n_3}^-$ and the S_{n_4}-permutation action on the variables $x_{n_1+n_2+n_3+1}^-,\dots,x_{n_1+n_2+n_3+n_4}^-$. It is clear that if $I = \{i_1,\dots,i_t\}$ is a set of indices, here we are identifying S_I, the symmetric group acting on I, with S_t.

It turns out that $F(S_r \times S_{n-r})h$ becomes a left FH-module, where $H = S_{n_1} \times \ldots \times S_{n_4}$, which is not irreducible in general. Hence, let M be one of its irreducible components associated to the H-character $\chi_{\lambda_1} \otimes \ldots \otimes \chi_{\lambda_4}$, where $\lambda_i \vdash n_i$, $1 \leq i \leq 4$.

By using Froboenius reciprocity theorem, $\chi_\lambda \otimes \chi_\mu$ is one of the irreducible $S_r \times S_{n-r}$-character induced by $\chi_{\lambda_1} \otimes \ldots \otimes \chi_{\lambda_4}$. Let $g = g(x_1^+,\dots,x_r^+,x_1^-,\dots,x_{n-r}^-)$ be a generator of the H-module M. By the representation theory of the symmetric group (see [14, Chapter 2] for further details), there exist four essential idempotents $e_{T_{\lambda_i}}$, $1 \leq i \leq 4$, acting non trivially on g, thus let $f = e_{T_{\lambda_1}} \cdots e_{T_{\lambda_4}} g$.

From [14, Lemma 2.5.1], it follows that

$$f' = \sum_{\substack{\sigma \in C_{T_{\lambda_1}} \\ \tau \in C_{T_{\lambda_2}}}} (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\sigma\tau f \tag{5}$$

is nonzero and it generates the same module.

Now let us apply to f' the map $\tilde{}$ in order to get

$$\tilde{f}' = \tilde{f}'(y_1^+, \ldots, y_{n_1}^+, y_1^-, \ldots, y_{n_2}^-, z_1^+, \ldots, z_{n_3}^+, z_1^-, \ldots, z_{n_4}^-).$$

By Lemma 1, $\tilde{f}'$ is not a $\sharp$-identity of A.

In order to simplify the notation, in what follows we write $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jt_j})$, where t_j is the height of λ_j, i.e. the number of the rows of λ_j, for all $1 \le j \le 4$.

Let us consider the conjugate partitions λ_1' and λ_2' and let us suppose that $\lambda_{11}', \ldots, \lambda_{1i}' > p_1$, $\lambda_{1,i+1}' \le p_1$, $\lambda_{21}', \ldots, \lambda_{2j}' > p_2$ and $\lambda_{2,j+1}' \le p_2$ for some i and j. Similarly, let us suppose $\lambda_{31}, \ldots, \lambda_{3k} > q_2$, $\lambda_{3,k+1} \le q_2$, $\lambda_{41}, \ldots, \lambda_{4l} > q_1$ and $\lambda_{4,l+1} \le q_1$ for some k and l. Then by (5) one gets that $\tilde{f}'$ is alternating on i sets $Y_1^+, \ldots, Y_i^+$ of even symmetric variables with $|Y_1^+| = \lambda_{11}' > p_1, \ldots, |Y_i^+| = \lambda_{1i}' > p_1$ and on j sets $Y_1^-, \ldots, Y_j^-$ of even skew variables with $|Y_1^-| = \lambda_{21}' > p_2, \ldots, |Y_j^-| = \lambda_{2i}' > p_2$.

Moreover, recalling the definition of the map $\tilde{}$ and using [14, Lemma 4.8.6], $\tilde{f}'$ is alternating on l sets $Z_1^+, \ldots, Z_l^+$ of odd symmetric variables with $|Z_1^+| = \lambda_{41} > q_1, \ldots, |Z_l^+| = \lambda_{4l} > q_1$ and on k sets $Z_1^-, \ldots, Z_k^-$ of odd skew variables with $|Z_1^-| = \lambda_{31} > q_2, \ldots, |Z_k^-| = \lambda_{3l} > q_2$.

Since $\tilde{f}'$ is not an identity and $\dim_F B_0^+ = p_1$, $\dim_F B_0^- = p_2$, $\dim_F B_1^+ = q_1$ and $\dim_F B_1^- = q_2$, it follows that there exists a nonzero evaluation in which at least

$$|Y_1^+| + \ldots + |Y_i^+| - ip_1 = \lambda_{11}' + \ldots + \lambda_{1i}' - ip_1$$

even symmetric elements are taken from the radical J; at least

$$|Y_1^-| + \ldots + |Y_j^-| - jp_2 = \lambda_{21}' + \ldots + \lambda_{2j}' - jp_2$$

even skew elements are taken form J; at least

$$|Z_1^+| + \ldots + |Z_l^+| - lq_1 = \lambda_{41} + \ldots + \lambda_{4l} - lq_1$$

odd symmetric elements are taken from J and at least

$$|Z_1^-| + \ldots + |Z_k^-| - kq_2 = \lambda_{31} + \ldots + \lambda_{3k} - kq_2$$

odd skew elements are taken from J.

Since $J^m = 0$, it readily follows that

$$\sum_{s=1}^{i} \lambda_{1s}' - ip_1 + \sum_{s=1}^{j} \lambda_{2s}' - jp_2 + \sum_{s=1}^{l} \lambda_{4s} - lq_1 + \sum_{s=1}^{k} \lambda_{3s} - kq_2 \le m - 1.$$

We now split λ_1 and λ_2 in the following way. Take $\bar{\lambda}_i \vdash n_i'$ and $\tilde{\lambda}_i \vdash n_i - n_i'$ such that $\bar{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{ip_i})$ and $\tilde{\lambda}_i = (\lambda_{i,p_i+1}, \lambda_{i,p_i+2}, \ldots)$, $1 \le i \le 2$. Then the S_{n_i}-character χ_{λ_i} is one of the irreducible characters induced from the irreducible $S_{n_i'}$-character $\chi_{\bar{\lambda}_i}$, $1 \le i \le 2$.

Similarly, let $\bar{\lambda}_3 \vdash n_3'$ where $\bar{\lambda}_3 = (\bar{\lambda}_{31}, \bar{\lambda}_{32}, \ldots)$ with $\bar{\lambda}_{31} = q_2, \ldots, \bar{\lambda}_{3k} = q_2$, $\bar{\lambda}_{3,k+1} = \lambda_{3,k+1}, \bar{\lambda}_{3,k+2} = \lambda_{3,k+2}, \ldots$ and let $\bar{\lambda}_4 \vdash n_4'$ where $\bar{\lambda}_4 = (\bar{\lambda}_{41}, \bar{\lambda}_{42}, \ldots)$ with $\bar{\lambda}_{41} = q_1, \ldots, \bar{\lambda}_{4l} = q_1$, $\bar{\lambda}_{4,l+1} = \lambda_{4,l+1}, \bar{\lambda}_{4,l+2} = \lambda_{4,l+2}, \ldots$ Thus the S_{n_s}-character χ_{λ_s} may be induced from the $S_{n_s'}$-character $\chi_{\bar{\lambda}_s}$, $3 \le s \le 4$.

Notice that the $S_r \times S_{n-r}$-character $\chi_\lambda \otimes \chi_\mu$ is one of the irreducible characters induced from the H'-character $\chi_{\bar{\lambda}_1} \otimes \ldots \otimes \chi_{\bar{\lambda}_4}$, where $H' = S_{n_1'} \times \ldots \times S_{n_4'}$, with $\chi_{\bar{\lambda}_1} \subseteq H(p_1, 0)$, $\chi_{\bar{\lambda}_2} \subseteq H(p_2, 0)$, $\chi_{\bar{\lambda}_3} \subseteq H(0, q_2)$, $\chi_{\bar{\lambda}_4} \subseteq H(0, q_1)$, and $n - n_1' - n_2' - n_3' - n_4' \le m - 1$.

Applying the Littlewood-Richardson rule, the $S_{n_1'+n_3'}$-character induced from $\chi_{\bar{\lambda}_1} \otimes \chi_{\bar{\lambda}_3}$ lies in the hook $H(p_1, q_2)$ whereas the $S_{n_2'+n_4'}$-character induced from $\chi_{\bar{\lambda}_2} \otimes \chi_{\bar{\lambda}_4}$ lies in the hook $H(p_2, q_1)$. Therefore, the number of boxes not belonging to $H(p_1, q_2)$ and $H(p_2, q_1)$ is at most $m - 1$, as claimed. □

We are now in a position to prove the main result of the section.

Lemma 3. *There exist constants C and t such that*

$$\delta_n^*(G(A)) \leq Cn^t d^n,$$

for all $n \geq 1$, where d is the maximal dimension of a centrally admissible subalgebra in $G(A)$.

Proof. Let B be a centrally admissible subalgebra in $G(A)$ of maximal dimension with superinvolution $\sharp$. We may assume $B = A_1 \oplus \ldots \oplus A_k$ and let $\widehat{B} = B + J$ be a reduced algebra such that $p_1 = \dim_F B_0^+$, $p_2 = \dim_F B_0^-$, $q_1 = \dim_F B_1^+$ and $q_2 = \dim_F B_1^-$.

By taking into account Lemma 2 and relation (2), the irreducibles appearing in the $*$-cocharacter $\chi^*_{r,n-r}(G(\widehat{B}))$ are of the form $\chi_\lambda \otimes \chi_\mu$ with $\lambda \in H(p_1, q_2) \cup S_1$ and $\mu \in H(p_2, q_1) \cup S_2$, where S_1 and S_2 are squares of fixed sizes depending only on the index of nilpotency of J.

Thus by [14, Lemma 6.2.4, Lemma 6.2.5], for all $0 \leq r \leq n$

$$\deg \chi_\lambda \otimes \chi\mu = d_\lambda d_\mu \leq C' n^{a'} (p_1 + q_2)^r (p_2 + q_1)^{n-r},$$

for some constants C', a'. Here $\lambda \vdash r$ and $\mu \vdash n - r$, $0 \leq r \leq n$.

Since by [3] the multiplicities are polynomially bounded, we get

$$c_n^*(G(\widehat{B})) = \sum_{r=0}^{n} \binom{n}{r} c^*_{r,n-r}(G(\widehat{B})) \leq \sum_{r=0}^{n} \binom{n}{r} Cn^a (p_1 + q_2)^r (p_2 + q_1)^{n-r} = Cn^a d^n,$$

for some constants C, a.

By taking into account (3), it readily follows that

$$\delta_n^*(G(\widehat{B})) \leq Cn^a d^n.$$

Now let $g = g(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_{n-r}^-)$ be a multilinear proper central $*$-polynomial of $G(A)$ generating the irreducible left submodule M of $\Delta_n^*(G(A))$. Then there exists a nonzero evaluation of g in some suitable centrally admissible subalgebra D. By the previous arguments we get

$$\dim_F M \leq C' n^{a'} (\dim_F D)^n \leq C' n^{a'} d^n, \qquad (6)$$

since B has maximal dimension.

Since the multiplicities in $\chi_n^*(G(A))$ are polynomially bounded, by (4) so they are the ones of $\chi_n^*(\Delta(G(A)))$. This fact together with (6) implies the desired conclusion. □

5. A lower bound for $\delta_n^*(A)$

In this section, we let A be a superalgebra with superinvolution $\sharp$ and $G(A)$ be its Grassmann envelope with induced involution $*$.

Furthermore, in what follows Y^+ stands for a set of even symmetric variables with respect to $\sharp$, Y^- stands for a set of even skew variables, Z^+ stands for a set of odd symmetric variables and Z^- stands for a set of odd skew variables. Given integers $d, l, t \geq 0$, we also define the partition

$$h(d, l, t) = (\underbrace{t + l, \ldots, t + l}_{d}, \underbrace{l, \ldots, l}_{t}).$$

Lemma 4. *Let A be a simple superalgebra with superinvolution and let $\dim_F A_0^+ = p_1$, $\dim_F A_0^- = p_2$, $\dim_F A_1^+ = q_1$ and $\dim_F A_1^- = q_2$. Then for t large enough, there exist partitions $\bar{\lambda}$ and $\bar{\mu}$ with*

$$h(p_1, q_2, 2t - p_1 - q_2) \leq \bar{\lambda} \leq h(p_1, q_2, 2t),$$

$$h(p_2, q_1, 2t - p_2 - q_1) \leq \bar{\mu} \leq h(p_2, q_1, 2t)$$

and tableaux $T_{\bar{\lambda}}$ and $T_{\bar{\mu}}$ such that $e_{T_{\bar{\lambda}}} e_{T_{\bar{\mu}}} h$ is a proper central $*$-polynomial of $G(A)$ for a suitable multilinear $*$-polynomial h.

Proof. According to [10, Lemma 4] for every $t \geq 1$, there exists a multilinear $\sharp$-polynomial

$$f(Y_1^+, \ldots, Y_{2t}^+, Y_1^-, \ldots, Y_{2t}^-, Z_1^+, \ldots, Z_{2t}^+, Z_1^-, \ldots, Z_{2t}^-)$$

which is alternanting on each set of variables $Y_i^+, Y_i^-, Z_i^+, Z_i^-$, where $|Y_i^+| = p_1$, $|Y_i^-| = p_2$, $|Z_i^+| = q_1$, $|Z_i^-| = q_2$, $1 \leq i \leq 2t$, and f takes invertible central values in A. By Lemma 1, the polynomial $\tilde{f}$ is alternanting on each Y_i^+, Y_i^-, and symmetric on each Z_i^+, Z_i^-, $1 \le i \le 2t$. Moreover, $\tilde{f}$ is not a $\mathbb{Z}_2$-graded $*$-identity of $G(A)$ and, since the number of odd variables is even, $\tilde{f}$ takes central values in $G(A)$.

Let $n_1 = 2tp_1$, $n_2 = 2tp_2$, $n_3 = 2tq_2$, $n_4 = 2tq_1$ and let $\lambda_1 = ((2t)^{p_1}) \vdash n_1$, $\lambda_2 = ((2t)^{p_2}) \vdash n_2$, $\lambda_3 = (q_2^{2t}) \vdash n_3$, $\lambda_4 = (q_1^{2t}) \vdash n_4$. Then there exist $e_{T_{\lambda_1}}, \dots, e_{T_{\lambda_4}}$ such that $g = e_{T_{\lambda_1}} \dots e_{T_{\lambda_4}} \tilde{f}$ is not a $\mathbb{Z}_2$-graded $*$-identity of $G(A)$. Moreover, g takes central values in $G(A)$.

Set $M = FHg$ as the left $H = S_{n_1} \times \dots \times S_{n_4}$-module generated by g and let M' the $S_r \times S_{n-r}$-module induced by M, where $r = n_1 + n_3$ and $n - r = n_2 + n_4$. If $\overline{M}$ is an irreducible submodule of M', then by the Littlewood-Richardson rule, $\overline{M}$ is associated to a pair of partition $\bar{\lambda}$ and $\bar{\mu}$ such that

$$h(p_1, q_2, 2t - p_1 - q_2) \le \bar{\lambda} \le h(p_1, q_2, 2t),$$

$$h(p_2, q_1, 2t - p_2 - q_1) \le \bar{\mu} \le h(p_2, q_1, 2t),$$

and there exist Young tableaux $T_{\bar{\lambda}}$ and $T_{\bar{\mu}}$ such that $e_{T_{\bar{\lambda}}} e_{T_{\bar{\mu}}} h$ is a proper central $*$-polynomial of $G(A)$ for some multilinear $*$-polynomial $h \in M'$. □

From now until the end of the section, let $B = A_1 \oplus \dots \oplus A_k$ be centrally admissible in $G(A)$ of maximal dimension, $\dim_F B = d$, where $A_1, \dots, A_k$ are simple superalgebras with superinvolution. Moreover, let $l_i = \dim_F((A_i)_0^+ \oplus (A_i)_1^-)$, $m_i = \dim_F((A_i)_0^- \oplus (A_i)_1^+)$ for all $1 \le i \le k$, so that $d = \sum_{i=1}^k (l_i + m_i)$.

Let $g = g(x_1^+, \dots, x_r^+, x_1^-, \dots, x_{s-r}^-)$ be a multilinear proper $*$-polynomial of $G(A)$ such that

$$g(a_1^+, \dots, a_{k_1}^+, a_{k_1+1}^+, \dots, a_r^+, a_1^-, \dots, a_{k_2}^-, a_{k_2+1}^-, \dots, a_{s-r}^-) \ne 0,$$

for some $a_1^+ \in G(A_1)^+, \dots, a_{k_1}^+ \in G(A_{k_1})^+$, $a_1^- \in G(A_{k_1+1})^-, \dots, a_{k_2}^- \in G(A_k)^-$, $k = k_1 + k_2$. Hence $g \notin \mathrm{Id}^*(G(\widehat{B}))$ where $\widehat{B} = B + J$.

Remark 2. *Let A_i be a simple superalgebra with superinvolution. Then for all $b \in Z(A_i)_0^\varepsilon$, $\varepsilon \in \{+, -\}$ and $b \ne 0$, there exists an element $c \in Z(A_i)_0^\varepsilon$ such that $cb = 1_{A_i}$.*

Proof. The proof easily follows by noticing that either $Z(A_i)_0 \cong F$ or $Z(A_i)_0 \cong F \oplus F$ according to Theorem 3. □

In what follows, we denote by f^ε a multilinear $*$-polynomial symmetric or skew with respect to $*$ according to $\varepsilon \in \{+, -\}$ and by $\circ$ the Jordan product among two variables, i.e., $x_1 \circ x_2 = \frac{1}{2}(x_1 x_2 + x_2 x_1)$.

Lemma 5. *Let $f_1^{\varepsilon_1}, \dots, f_k^{\varepsilon_k}$, $\varepsilon_i \in \{+, -\}$, $1 \le i \le k$, be multilinear $*$-polynomials on distinct sets of variables such that $f_i^{\varepsilon_i}$ is a proper central $*$-polynomial of $G(A_i)$ for all i and let $y_1^+, \dots, y_r^+$, $y_1^-, \dots, y_{s-r}^-$, $z_1^{\varepsilon_1}, \dots, z_k^{\varepsilon_k}$ be new variables. Then the $*$-polynomial*

$$g' = g(y_1^+ \circ z_1^{\varepsilon_1} f_1^{\varepsilon_1}, \dots, y_{k_1}^+ \circ z_{k_1}^{\varepsilon_{k_1}} f_{k_1}^{\varepsilon_{k_1}}, y_{k_1+1}^+, \dots, y_r^+, y_1^- \circ z_{k_1+1}^{\varepsilon_{k_1+1}} f_{k_1+1}^{\varepsilon_{k_1+1}}, \dots, y_{k_2}^- \circ z_k^{\varepsilon_k} f_k^{\varepsilon_k}, y_{k_2+1}^-, \dots, y_{s-r}^-)$$

is a proper central $*$-polynomial of $G(\widehat{B})$.

Proof. It is clear that g' is a central $*$-polynomial of $G(\widehat{B})$ since g is central, thus we have only to prove that $g' \notin \mathrm{Id}^*(G(\widehat{B}))$.

Since g is multilinear and $g(a_1^+, \dots, a_r^+, a_1^-, \dots, a_{s-r}^-) \ne 0$, then without loss of generality we may assume $a_i^+ = b_i \otimes v_i$, $1 \le i \le r$, $a_j^- = c_j \otimes w_j$, $1 \le j \le s - r$, where $b_i, c_j \in \widehat{B}$ and $v_i, w_j \in G$ are distinct homogeneous elements.

Thus

$$g(a_1^+, \dots, a_r^+, a_1^-, \dots, a_{s-r}^-) = \tilde{g}(b_1, \dots, b_r, c_1, \dots, c_{s-r}) \otimes v_1 \cdots v_r w_1 \cdots w_{s-r} \ne 0.$$

For all $1 \le i \le k$, let φ be the evaluation such that $\varphi(f_i^{\varepsilon_i})$ is a nonzero central element of $G(A_i)$. Notice that $Z(G(A_i)) = Z(A_i)_0 \otimes G_0$, hence by Remark 2 φ can be extended to an evaluation so that

$$\varphi(z_i^{\varepsilon_i} f_i^{\varepsilon_i}) = 1_{A_i} \otimes h_i \ne 0,$$

where $h_i \in G_0$, $\varphi(y_p^+) = a_p^+$, $1 \le p \le r$ and $\varphi(y_q^-) = a_q^-$, $1 \le q \le s - r$. It readily follows that

$$g(a_1^+ \circ (1_{A_1} \otimes h_1), \dots, a_{k_1}^+ \circ (1_{A_{k_1}} \otimes h_{k_1}), a_{k_1+1}^+, \dots, a_r^+, a_1^- \circ (1_{A_{k_1+1}} \otimes h_{k_1+1}), \dots, a_{k_2}^- \circ (1_{A_k} \otimes h_k), a_{k_2+1}^-, \dots, a_{s-r}^-)$$
$$= \tilde{g}(b_1, \dots, b_r, c_1, \dots, c_{s-r}) \otimes h_1 \cdots h_k v_1 \cdots v_r w_1 \cdots w_{s-r} \ne 0.$$

This completes the proof. □

We now introduce a technique of gluing Young tableaux that one can find in [14, Chapter 6].

Let $\lambda_1 \vdash n_1, \ldots, \lambda_k \vdash n_k$ be partitions such that for all $1 \leq i \leq k$,

$$h(p_i, q_i, t_i - s_i) \leq \lambda_i \leq h(p_i, q_i, t_i), \tag{7}$$

and for all $1 \leq i \leq k-1$,

$$t_i - s_i \geq \max\{t_{i+1} + p_{i+1}, t_{i+1} + q_{i+1}\}, \tag{8}$$

where p_i, q_i, t_i, s_i, $1 \leq i \leq k$, are fixed integers.

As we know, we can associate a partition to the corresponding Young diagram. Then, by the above relations, we can glue the first row of λ_{i+1} to the $(p_i+1)-$th row of λ_i, the second row of λ_{i+1} to the $(p_i+2)-$th row of λ_i and so on. In this way we get a new partition, that we denote by $\lambda_i \star \lambda_{i+1}$, of the integer $n_i + n_{i+1}$. Remark that if $\lambda = \lambda_1 \star \ldots \star \lambda_k$, then

$$h(p, q, t_k - s_k) \leq \lambda \leq h(p, q, t),$$

where $p = p_1 + \ldots + p_k$, $q = q_1 \ldots + q_k$ and $t \geq \max\{t_1 + p_1 - p, t_1 + q_1 - q\}$.

As we did with Young diagrams, we can glue also Young tableaux in a similar way. Let $T_{\lambda_1}, \ldots, T_{\lambda_k}$ be tableaux corresponding to the partitions $\lambda_1, \ldots, \lambda_k$, respectively. Now we define new tableaux as follows: set $T'_{\lambda_1} = T_{\lambda_1}$ and for all $2 \leq i \leq k$, let T'_{λ_i} be the tableau obtained from T_{λ_i} by adding the integer $n_1 + \ldots + n_{i-1}$ to each entry of T_{λ_i}. Then we denote by

$$T_\lambda = T'_{\lambda_1} \star \ldots \star T'_{\lambda_k}$$

the Young tableau obtained by gluing together the tableaux $T'_{\lambda_1}, \ldots, T'_{\lambda_k}$ according to the previous procedure. It is clear that such a tableau has distinct entries $1, 2, \ldots, n$, where $n = n_1 + \ldots + n_k$.

Furthermore, in [12, Lemma 14] it was proved that

$$e_{T_\lambda} = e_{T'_{\lambda_1}} \cdots e_{T'_{\lambda_k}} + b, \tag{9}$$

where $b \in \mathrm{span}_F\{\sigma \in S_n \mid \sigma(N_i) \nsubseteq N_i \text{ for some } 1 \leq i \leq k\}$, where N_i denotes the set of integers filled in the tableau $e_{T'_{\lambda_i}}$.

Next we apply the gluing technique to pairs of Young tableaux. For all $1 \leq i \leq k$ consider $(\lambda_i, \mu_i) \vdash (n_1^{(i)}, n_2^{(i)})$ and suppose that $\lambda_1, \ldots \lambda_k$ and $\mu_1, \ldots, \mu_k$ are glueable, i.e. they satisfy conditions (7) and (8). Then, by the previous arguments, we can construct (λ', μ') where $\lambda' = \lambda_1 \star \ldots \star \lambda_k$ and $\mu' = \mu_1 \star \ldots \star \mu_k$.

We set $n_0 = 0$, $n_1^{(0)} = 0$, $n_2^{(0)} = 0$, $n_1 = n_1^{(1)} + \ldots + n_1^{(k)}$ and $n_2 = n_2^{(1)} + \ldots + n_2^{(k)}$. Let $(T_{\lambda_i}, T_{\mu_i})$ be the pair of tableaux corresponding to (λ_i, μ_i). If we add the integer $n_0 + n_1^{(0)} + n_1^{(1)} \ldots + n_1^{(i-1)}$ to all entries of T_{λ_i} and the integer $n_0 + n_2^{(0)} + n_2^{(1)} \ldots + n_2^{(i-1)}$ to all entries of T_{μ_i}, $1 \leq i \leq k$, we get new pairs of tableaux which we denote by $(T'_{\lambda_i}, T'_{\mu_i})$. Then we define

$$T_{\lambda'} = T'_{\lambda_1} \star \ldots \star T'_{\lambda_k} \ \text{ and } T_{\mu'} = T'_{\mu_1} \star \ldots \star T'_{\mu_k}.$$

We denote by $N_1^{(i)}$ and $N_2^{(i)}$ the sets of integers filled in T'_{λ_i} and T'_{μ_i}, respectively, $1 \leq i \leq k$. Moreover, let

$$N_1 = N_1^{(1)} \cup \ldots \cup N_1^{(k)};$$
$$N_2 = N_2^{(1)} \cup \ldots \cup N_2^{(k)}.$$

By generalizing (9), we have the following.

Lemma 6. *Let $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_k$ partitions satisfying conditions (7) and (8), furthermore let $(T_{\lambda_i}, T_{\mu_i})$ be pair of tableaux corresponding to (λ_i, μ_i), $1 \leq i \leq k$. If $T_{\lambda'} = T'_{\lambda_1} \star \ldots \star T'_{\lambda_k}$ and $T_{\mu'} = T'_{\mu_1} \star \ldots \star T'_{\mu_k}$, then*

$$e_{T_{\lambda'}} e_{T'_{\mu'}} = e_{T'_{\lambda_1}} \cdots e_{T'_{\lambda_k}} e_{T'_{\mu_1}} \cdots e_{T'_{\mu_k}} + \tau,$$

where $\tau \in span_F\{\sigma \in S_{n_1} \times S_{n_2} \mid \sigma(N_j^{(i)}) \nsubseteq N_j^{(i)}$ for some $j \in \{1, 2\}$ and $1 \leq i \leq k\}$.

In the next Lemma we apply the gluing technique in order to construct a proper central $*$-polynomial that will be very useful. To this end, for all $1 \leq i \leq k$, let $\dim(A_j)_0^+ = p_{1j}$, $\dim(A_j)_0^- = p_{2j}$, $\dim(A_j)_1^+ = q_{1j}$ and $\dim(A_j)_1^- = q_{2j}$. Moreover, let $P_1 = \dim B_0^+ = \sum_{j=1}^k p_{1j}$, $P_2 = \dim B_0^- = \sum_{j=1}^k p_{2j}$, $Q_1 = \dim B_1^+ = \sum_{j=1}^k q_{1j}$ and $Q_2 = \dim B_1^- = \sum_{j=1}^k q_{2j}$.

Lemma 7. *For some positive integers $t_1, \dots, t_k$, there exist partitions $\lambda' \vdash n_1$ and $\mu' \vdash n_2$, where $n_1 + n_2 = \sum_{i=1}^k t_i(l_i + m_i)$, Young tableaux $T_{\lambda'}$, $T_{\mu'}$ and a multilinear proper central $*$-polynomial g' of $G(\widehat{B})$ such that*

$$h(P_1, Q_2, 2t_1 - 4\dim \widehat{B}) \leq \lambda' \leq h(P_1, Q_2, 2t_1),$$
$$h(P_2, Q_1, 2t_1 - 4\dim \widehat{B}) \leq \mu' \leq h(P_2, Q_1, 2t_1) \tag{10}$$

and $e_{T_{\lambda'}} e_{T_{\mu'}} g'$ is also a proper central $*$-polynomial of $G(\widehat{B})$.

Proof. Let $u_j = \dim A_j$, for all $1 \leq j \leq k$. By Lemma 4, for any integer t_j large enough there exist partitions λ_j and μ_j, pair of tableaux $(T_{\lambda_j}, T_{\mu_j})$ and multilinear proper central $*$-polynomials h_j of $G(A_j)$ such that

$$h(p_{1j}, q_{2j}, 2t_j - u_j) \leq \lambda_j \leq h(p_{1j}, q_{2j}, 2t_j),$$
$$h(p_{2j}, q_{1j}, 2t_j - u_j) \leq \mu_j \leq h(p_{2j}, q_{1j}, 2t_j)$$

and $e_{T_{\lambda_j}} e_{T_{\mu_j}} h_j$ is a proper central $*$-polynomial of $G(A_j)$.

Now let $t_1 = t \geq 2k \dim \widehat{B}$ be an arbitrary integer and for $2 \leq l \leq k$ define

$$r_l = u_{l-1} + \max\{p_{1l}, p_{2l}, q_{1l}, q_{2l}\}.$$

Also set $r'_l = r_l$ if r_l is even and $r'_l = r_l + 1$ if r_l is odd. Finally, for all $1 \leq l \leq k-1$ define

$$2t_{l+1} = 2t_l - r'_{l+1}.$$

Hence for all $1 \leq l \leq k$ we get

$$2t_l - u_l = 2t_{l+1} + r'_{l+1} - u_l \geq 2t_{l+1} + r_{l+1} - u_l = 2t_{l+1} + \max\{p_{1l}, p_{2l}, q_{1l}, q_{2l}\}.$$

Thus conditions (7) and (8) hold and we can glue the partition $\lambda_1, \dots, \lambda_k$ and the partitions $\mu_1, \dots, \mu_k$. If $\lambda' = \lambda_1 \star \cdots \star \lambda_k$ and $\mu' = \mu_1 \star \cdots \star \mu_k$, then

$$h(P_1, Q_2, 2t_k - u_k) \leq \lambda' \leq h(P_1, Q_2, T_1)$$

and

$$h(P_2, Q_1, 2t_k - u_k) \leq \mu' \leq h(P_2, Q_1, T_2),$$

where $T_1 \geq \max\{2t_1 + p_{11} - P_1, 2t_1 + q_{21} - Q_2\}$ and $T_2 \geq \max\{2t_1 + p_{21} - P_2, 2t_1 + q_{11} - Q_1\}$.

Now we compute

$$\begin{aligned} 2t_1 - 2t_k &= \sum_{j=1}^{k-1}(2t_j - 2t_{j+1}) = \sum_{j=1}^{k-1} r'_{j+1} \leq k + \sum_{j=1}^{k-1} r_{j+1} \\ &= k + \sum_{j=1}^{k-1} (u_j + \max\{p_{1,j+1}, p_{2,j+1}, q_{1,j+1}, q_{2,j+1}\}) \leq k + 2\dim \widehat{B} \leq 3\dim \widehat{B}. \end{aligned}$$

Thus,

$$2t_k - u_k \geq 2t_1 - 3\dim \widehat{B} - u_k \geq 2t_1 - 4\dim \widehat{B}.$$

Recalling that $t_1 = t$, we get

$$h(P_1, Q_2, 2t - 4\dim \widehat{B}) \leq \lambda' \leq h(P_1, Q_2, 2t),$$
$$h(P_2, Q_1, 2t - 4\dim \widehat{B}) \leq \mu' \leq h(P_2, Q_1, 2t).$$

Now remark that, by Lemma 4 $\deg h_j = 2t_j(l_j + m_j)$, thus we can assume $\lambda' \vdash n_1$ and $\mu' \vdash n_2$, where $n_1 + n_2 = \sum_{i=1}^k t_i(l_i + m_i)$.

Following the previous construction, let consider $T_{\lambda'} = T'_{\lambda_1} \star \cdots \star T'_{\lambda_k}$ and $T_{\mu'} = T'_{\mu_1} \star \cdots \star T'_{\mu_k}$. Moreover, for all $1 \leq j \leq k$ let $N_1^{(j)}$ and $N_2^{(j)}$ be the sets of integers filled in T'_{λ_j} and T'_{μ_j}, respectively.

If $\bar{h}_j = e_{T_{\lambda_j}} e_{T_{\mu_j}} h_j$, then it is clear that at least one among $\bar{h}_j + \bar{h}_j^*$ and $\bar{h}_j - \bar{h}_j^*$ is a proper central $*$-polynomial of $G(\widehat{B})$. Hence, for all $1 \leq j \leq k$, we choose $f_j^+ = \bar{h}_j + \bar{h}_j^*$, if $\bar{h}_j + \bar{h}_j^* \notin \mathrm{Id}^*(G(\widehat{B}))$ otherwise we consider $f_j^- = \bar{h}_j - \bar{h}_j^*$. Furthermore, we rename the variables of $f_j^{\varepsilon_j}$, $\varepsilon_j \in \{+, -\}$, so that the symmetric variables are indexed by the integers of $N_1^{(j)}$ and the skew variables are indexed by the integers of $N_2^{(j)}$.

By Lemma 5, the polynomial

$$g' = g(y_1^+ \circ z_1^{\varepsilon_1} f_1^{\varepsilon_1}, \dots, y_{k_1}^+ \circ z_{k_1}^{\varepsilon_{k_1}} f_{k_1}^{\varepsilon_{k_1}}, y_{k_1+1}^+, \dots, y_r^+, y_1^- \circ z_{k_1+1}^{\varepsilon_{k_1+1}} f_{k_1+1}^{\varepsilon_{k_1+1}}, \dots, y_{k_2}^- \circ z_k^{\varepsilon_k} f_k^{\varepsilon_k}, y_{k_2+1}^-, \dots, y_{s-r}^-)$$

is a proper central $*$-polynomial of $G(\widehat{B})$ and $\deg g' = n + k + s$, where $n = n_1 + n_2$. We claim that $\bar{g} = e_{T_{\lambda'}} e_{T_{\mu'}} g'$ is also a proper central $*$-polynomial of $G(\widehat{B})$.

Notice that $\bar{g}$ is by construction a central $*$-polynomial, thus we are left to prove that $\bar{g} \notin \mathrm{Id}^*(G(\widehat{B}))$. To this end, let ψ be a non-zero evaluation of g'. By Lemma 6

$$\psi(\bar{g}) = \psi(e_{T'_{\lambda_1}} \cdots e_{T'_{\lambda_k}} e_{T'_{\mu_1}} \cdots e_{T'_{\lambda_k}} g') + \psi(\tau g'),$$

where $\tau \in \mathrm{span}_F\{\sigma \in S_{n_1} \times S_{n_2} \mid \sigma(N_1^{(j)}) \nsubseteq N_1^{(j)}, \sigma(N_2^{(j)}) \nsubseteq N_2^{(j)}$ for some $1 \le j \le k\}$. It readily follows that since $G(A_i)G(A_j) = 0$ if $i \ne j$, then $\psi(\tau g') = 0$. Moreover, since $e_{T'_{\lambda_j}}^2 = \alpha_j e_{T'_{\lambda_j}}$, $e_{T'_{\mu_j}}^2 = \beta_j e_{T'_{\mu_j}}$, for some non-zero $\alpha_j, \beta_j \in \mathbb{Z}$, and $f_j^{\varepsilon_j} = e_{T'_{\lambda_j}} e_{T'_{\mu_j}} (h_j \pm h_j^*)$, we get

$$e_{T'_{\lambda_1}} \cdots e_{T'_{\lambda_k}} e_{T'_{\mu_1}} \cdots e_{T'_{\lambda_k}} g' = \alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_k g',$$

hence

$$\psi(\bar{g}) = \psi(e_{T'_{\lambda_1}} \cdots e_{T'_{\lambda_k}} e_{T'_{\mu_1}} \cdots e_{T'_{\lambda_k}} g') = \alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_k \psi(g') \ne 0,$$

as claimed. □

The following Lemma gives us the required lower bound for the proper central $*$-codimension sequence.

Lemma 8. *Let $G(A)$ be the Grassmann envelope of a finite dimensional superalgebra with superinvolution A. If d is the maximal dimension of a centrally admissible $*$-subalgebra in $G(A)$, then for n large enough there exist constants $C > 0$ and a such that*

$$\delta_n^*(G(A)) \ge C n^a d^n.$$

Proof. By Lemma 7, for some positive integers $t = t_1, t_2, \dots, t_k$ there exist partitions $\lambda' \vdash n_1$ and $\mu' \vdash n_2$, Young tableaux $T_{\lambda'}$ and $T_{\mu'}$ and a multilinear $*$-polynomial g' such that (10) holds and $e_{T_{\lambda'}} e_{T_{\mu'}} g'$ is a proper central $*$-polynomial of $G(\widehat{B})$ with $\deg g' = n + k + s$ and $n = n_1 + n_2 = \sum_{i=1}^k t_i(l_i + m_i)$.

By the branching rule (see [19, Theorem 2.4.3]), there exist partitions $\widehat{\lambda}$ and $\widehat{\mu}$ and Young tableaux $T_{\widehat{\lambda}}$ and $T_{\widehat{\mu}}$ such that

$$\widehat{\lambda} \vdash \widehat{n}_1 \ge |\lambda'| \ge |h(P_1, Q_2, 2t - 4\dim \widehat{B})|,$$
$$\widehat{\mu} \vdash \widehat{n}_2 \ge |\mu'| \ge |h(P_2, Q_1, 2t - 4\dim \widehat{B})|$$

and $e_{T_{\widehat{\lambda}}} e_{T_{\widehat{\mu}}} g'$ is a proper central $*$-polynomial of $G(\widehat{B})$. Notice that $\widehat{n}_1$ and $\widehat{n}_2$ are the number of symmetric and skew variables of g', respectively, thus $n + k + s = \widehat{n}_1 + \widehat{n}_2$.

Moreover, by taking into account (10), we get that

$$\begin{aligned}
&n_1 + n_2 + k + s - |h(P_1, Q_2, 2t - 4\dim \widehat{B})| - |h(P_2, Q_1, 2t - 4\dim \widehat{B})| \\
&= n_1 + n_2 + k + s - P_1 Q_2 - (2t - 4\dim \widehat{B})(P_1 + Q_2) - P_2 Q_1 - (2t - 4\dim \widehat{B})(P_2 + Q_1) \\
&\le 4\dim \widehat{B}(P_1 + Q_2) + 4\dim \widehat{B}(P_2 + Q_1) + k + s = 4d\dim \widehat{B} + k + s
\end{aligned}$$

is a constant that does not depent on t. Hence Lemma 6.2.4 of [14] applies and there exists a constant c such that

$$d_{\widehat{\lambda}} \ge \widehat{n}_1^{-c} d_{h(P_1, Q_2, 2t - 4\dim \widehat{B})}$$

and

$$d_{\widehat{\mu}} \ge \widehat{n}_2^{-c} d_{h(P_2, Q_1, 2t - 4\dim \widehat{B})}.$$

It follows that

$$\begin{aligned}
\delta^*_{\widehat{n}_1, \widehat{n}_2}(G(\widehat{B})) &\ge d_{\widehat{\lambda}} d_{\widehat{\mu}} \ge (\widehat{n}_1 \widehat{n}_2)^{-c} d_{h(P_1, Q_2, 2t - 4\dim \widehat{B})} d_{h(P_2, Q_1, 2t - 4\dim \widehat{B})} \\
&\ge K_1 (\widehat{n}_1 \widehat{n}_2)^{C_1} (P_1 + Q_2)^{2t(P_1 + Q_2)} (P_2 + Q_1)^{2t(P_2 + Q_1)},
\end{aligned}$$

where $K_1 > 0$ and C_1 are constants. The latter inequality holds by Lemma 6.2.5 of [14].

Recalling Stirling formula $n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)$, we get

$$\begin{aligned}\frac{(\widehat{n}_1+\widehat{n}_2)!}{\widehat{n}_1!\widehat{n}_2!} &\geq \frac{(2t(P_1+Q_2)+2t(P_2+Q_1))!}{(2t(P_1+Q_2))!(2t(P_2+Q_1))!}\\ &\simeq \frac{\sqrt{2\pi 2td}}{\sqrt{12\pi^2t^2(P_1+Q_2)(P_2+Q_1)}}\cdot\frac{(2td)^{2td}}{(2t(P_1+Q_2))^{2t(P_1+Q_2)}(2t(P_2+Q_1))^{2t(P_2+Q_1)}}\\ &= K_2\frac{d^{2td}}{(P_1+Q_2)^{2t(P_1+Q_2)}(P_2+Q_1)^{2t(P_2+Q_1)}},\end{aligned}$$

for some constant $K_2 > 0$.

Hence

$$\begin{aligned}\delta_n^*(G(\widehat{B})) &\geq \binom{\widehat{n}_1+\widehat{n}_2}{\widehat{n}_1,\widehat{n}_2}\delta^*_{\widehat{n}_1,\widehat{n}_2}(G(\widehat{B}))\\ &\geq K_1(\widehat{n}_1\widehat{n}_2)^{C_1}(P_1+Q_2)^{2t(P_1+Q_2)}(P_2+Q_1)^{2t(P_2+Q_1)}K_2\frac{d^{2td}}{(P_1+Q_2)^{2t(P_1+Q_2)}(P_2+Q_1)^{2t(P_2+Q_1)}}\\ &= K_3(\widehat{n}_1\widehat{n}_2)^{C_1}d^{2td} \geq K_3(\widehat{n}_1+\widehat{n}_2)^{C_1}d^{2td} = K_3(n+k+s)^{C_1}d^{2td}.\end{aligned}$$

Notice that $n+k+s-2td$ is a constant, in fact by the previous arguments we got that

$$\widehat{n}_1 - P_1Q_2 - (2t-4\dim\widehat{B})(P_1+Q_2) = \gamma_1,$$
$$\widehat{n}_2 - P_2Q_1 - (2t-4\dim\widehat{B})(P_2+Q_1) = \gamma_2,$$

where γ_1 and γ_2 are constants. Adding the previous equations we get

$$\widehat{n}_1+\widehat{n}_2-2td = \gamma_1+\gamma_2+P_1Q_2+P_2Q_1-4d\dim\widehat{B},$$

a constant as claimed. Hence

$$\delta^*_{n+k+s}(G(\widehat{B})) \geq K_3(n+k+s)^{C_1}d^{2td} \geq K'(n+k+s)^{C_1}d^{n+k+s}, \tag{11}$$

for some constant $K' > 0$.

We now generalize the result for all positive integer N. To this end, we remark that if in g' we replace the variable $x_1^{\varepsilon_1}$, $\varepsilon_1 \in \{+,-\}$, with the Jordan product $x_1^{\varepsilon_1}\circ x_{s+1}^+$, then we get another proper central $*$-polynomial of $G(\widehat{B})$ since $G(A_1)$ is a unitary $*$-algebra and we can specialize x^+_{s+1} by 1_{A_1}. Thus, for all $M \geq \deg g = s$,

$$\delta^*_{M+1}(G(\widehat{B})) \geq \delta^*_M(G(\widehat{B})).$$

Now let $t_1\dots,t_k$ be positive integers for which we can apply Lemma 7. Notice that, according to the proof of such a lemma, if we choose $t_1 \geq 2k\dim\widehat{B}$ then the integers $t_2,\dots,t_k$ are automatically fixed. Moreover, for the first part of this proof, the statement is true for $\tilde{n}(t_1) = \sum_{i=1}^k t_i(l_i+m_i)+k+s$. Now let $N>0$ be an integer such that $\tilde{n}(t_1) \leq N \leq \tilde{n}(t_1+1)$ that is

$$\sum_{i=1}^k t_i(l_i+m_i)+k+s < N < \sum_{i=1}^k (t_i+1)(l_i+m_i)+k+s.$$

Notice that

$$p = N - \sum_{i=1}^k t_i(l_i+m_i)+k+s < \sum_{i=1}^k (l_i+m_i) = d,$$

thus p is bounded by a constant. Finally we get

$$\delta^*_N(G(\widehat{B})) \geq \alpha_0(N-p)^{\beta_0}d^{N-p} > \alpha_0(N-d)^{\beta_0}d^{N-d} = \alpha_0 d^{-d}(N-d)^{\beta_0}d^N \geq \alpha_1 N^{\beta_1}d^N,$$

for some constants $\alpha_0,\alpha_1,\beta_0,\beta_1$.

Hence for all n large enough

$$\delta^*_n(G(A)) \geq \delta^*_n(G(\widehat{B})) \geq Cn^a d^n,$$

for some constants $C>0$ and a, and we are done. □

We are now in a position to prove the main result.

Theorem 4. *Let $G(A)$ be the Grassmann envelope of a finite dimensional superalgebra with superinvolution A over an algebraically closed field of characteristic zero. If $G(A)$ has centrally admissible $*$-subalgebras, then for n large enough, there exist constants $C_1 > 0$, C_2, a_1, a_2 such that*

$$C_1 n^{a_1} d^n \leq \delta_n^*(G(A)) \leq C_2 n^{a_2} d^n,$$

where d is the maximal dimension of a centrally admissible $*$-subalgebra in $G(A)$.

Proof. The proof follows immediately from Lemmas 3 and 8. □

In case $G(A)$ has proper central $*$-polynomials but no centrally admissible $*$-subalgebras, the following proposition holds.

Proposition 1. *If $G(A)$ has proper central $*$-polynomials but has no centrally admissible $*$-subalgebras, then for n large enough $\delta_n^*(G(A)) = 0$.*

Proof. If A is nilpotent we have nothing to prove, so let us suppose $A = \bar{A} + J$, where $\bar{A}$ is maximal semisimple $*$-subalgebra of A, be not nilpotent and let $f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_{n-r}^-)$ be a proper central $*$-polynomial of $G(A)$. Then for all $1 \leq i \leq r$ and for all $1 \leq j \leq n-r$ there exist $a_i^+ \in G(A)^+$ and $a_j^- \in G(A)^-$ such that $f(a_1^+, \ldots, a_r^+, a_1^-, \ldots, a_{n-r}^-) \neq 0$. Thus it must be $a_i^+, a_j^- \in G(J)$ for all i, j and $J^n \neq 0$. It follows that $\delta_N^*(G(A)) = 0$ as soon as $J^N = 0$. □

Corollary 1. *If R is a $*$-algebra over a field of characteristic zero, then the proper central $*$-exponent $\exp_*^\delta(R)$ exists and is a non-negative integer. Moreover, $\exp_*^\delta(R) \leq \exp^*(R)$.*

6. Minimal superalgebras with superinvolution and Grassmann envelope

In this section we shall introduce a family of varieties of superalgebras with superinvolution, called minimal varieties, that plays an important role in the next section, where we will study the central exponent of a $*$-algebra.

Minimal varieties were completely described in [13] and in [4] in the settings of ordinary polynomial identities and identities with involution, respectively. Here we present the definition and some basics results concerning superalgebras with superinvolution and their relation with the Grassmann envelope.

First we notice that for all simple superalgebra with superinvolution A_i, there exist graded idempotents $e_1^{(i)}, \ldots, e_{n_i}^{(i)}$ such that the unit element of A_i equals $e_1^{(i)} + \cdots + e_{n_i}^{(i)}$ and each of them generates a left minimal ideal of A_i. Thus we call them graded minimal idempotents.

Definition 2. *Let A be a finite dimensional superalgebra with superinvolution $\sharp$. Then A is said to be minimal if either A is $\sharp$-simple or $A = A_1 \oplus \cdots \oplus A_m + J$, where A_i are $\sharp$-simple, $1 \leq i \leq m$ and there are elements $e_i \in (A_i)_0$, $1 \leq i \leq m$, and $w_{i,i+1} \in J$, $1 \leq i \leq m-1$, satisfying the following conditions:*

1. *e_i is a graded minimal idempotent of A_i;*
2. *$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}$;*
3. *$w_{12} w_{23} \cdots w_{m-1,m} \neq 0$;*
4. *$\{w_{i,i+1}, w_{i,i+1}^\sharp \mid 1 \leq i \leq m-1\}$ generates J as bilateral $\sharp$-ideal of A.*

The following lemma relates the $*$-exponent of a variety of algebras with involution with the dimension of the maximal semisimple subalgebra of a minimal superalgebra with superinvolution.

Lemma 9. *Let $\mathcal{V}$ be a variety of $*$-algebras over an algebraically closed field F. If $\exp^*(\mathcal{V}) \geq 2$, then there exists a minimal $\sharp$-algebra $A = A_1 \oplus \cdots \oplus A_m + J$ such that $G(A) \in \mathcal{V}$ and $\exp^*(\mathcal{V}) = \dim(A_1 \oplus \cdots \oplus A_m)$.*

Proof. By [1], there exists a finite dimensional superalgebra with superinvolution B such that $\mathcal{V} = \mathrm{var}^*(G(B))$. Let $\sharp$ be such a superinvolution and $B = B_{ss} + J$ where $B_{ss} = A_1 \oplus \cdots \oplus A_n$ is a semisimple $\sharp$-subalgebra of B of maximal dimension and A_i is simple for all $1 \leq i \leq n$.

Let $d = \exp^*(\mathcal{V})$. By [10], there exists $m \in \{1, \ldots, n\}$ such that $A_{i_1} J A_{i_2} J \cdots J A_{i_m} \neq 0$ and $d = \dim(A_{i_1} \oplus \cdots \oplus A_{i_m})$, for some indices $i_1, \ldots, i_m$. Without loss of generality, we can assume that $A_1 J A_2 J \cdots J A_m \neq 0$, thus there exist $x_1, \ldots, x_{m-1} \in J$ and $a_i \in A_i$, $1 \leq i \leq m$ such that $a_1 x_1 a_2 x_2 \cdots x_{m-1} a_m \neq 0$. We assume, as we may, $x_1, \ldots, x_{m-1}$ and $a_1, \ldots, a_m$ to be homogeneous and let $1_{A_1}, \ldots, 1_{A_m}$ be the unit elements of $A_1, \ldots, A_m$, respectively. We get

$$1_{A_1}(a_1x_1a_2)1_{A_2}(x_2a_3)1_{A_3}\cdots 1_{A_{m-1}}(x_{m-1}a_m)1_{A_m} \neq 0. \tag{12}$$

Notice that for all $1 \leq i \leq m$, $1_{A_i} = e_1^{(i)} + \ldots + e_{n_i}^{(i)}$, where the $e_j^{(i)}$'s are minimal idempotents of A_i. Thus by (12) we get that there exist $e_1 \in A_1, \ldots, e_m \in A_m$ such that

$$e_1y_1e_2y_2\cdots y_{m-1}e_m \neq 0,$$

where $y_1 = a_1x_1a_2$, $y_2 = x_2a_3, \ldots y_{m-1} = x_{m-1}a_m$.

Set $w_{12} = e_1y_1e_2$, $w_{23} = e_2y_2e_3, \ldots, w_{m-1,m} = e_{m-1}y_{m-1}e_m$. It is clear that for all $1 \leq i \leq m-1$ $w_{i,i+1} \in J_0 \cup J_1$, since $y_1, \ldots, y_{m-1} \in J$ are homogeneous. Moreover, since $e_ie_j = 0$ for all $i \neq j$, then condition 2. of Definition 2 holds.

Let consider A as the superalgebra with superinvolution generated by $A_1, \ldots, A_m$ and by $w_{12}, \ldots w_{m-1,m}$, $w_{12}^\sharp, \ldots w_{m-1,m}^\sharp$. By construction, $w_{12}, \ldots w_{m-1,m} \neq 0$ and $A = A_{ss} + J'$, where $A_{ss} = A_1 \oplus \cdots \oplus A_m$ and J' is generated by the elements $w_{12}, \ldots w_{m-1,m}$, $w_{12}^\sharp, \ldots w_{m-1,m}^\sharp$ as $\sharp$-ideal of A.

Hence $\exp^*(\mathcal{V}) = \dim A_{ss} = d$ and $G(A) \subseteq G(B) \in \mathcal{V}$. It follows that $G(A) \in \mathcal{V}$ and we are done. ☐

We give next the definition of minimal variety and right after we will prove a theorem that connect such a definition together with the one of minimal algebra.

Definition 3. *Let $\mathcal{V}$ be a variety algebras with involution $*$. Then $\mathcal{V}$ is minimal of $*$-exponent d if $\exp^*(\mathcal{V}) = d$ and for any proper subvariety $\mathcal{W}$, $\exp^*(\mathcal{W}) < d$.*

Theorem 5. *If $\mathcal{V}$ is a minimal $*$-variety of $*$-exponent $d \geq 2$, then there exists a minimal superalgebra with superinvolution A such that $\mathcal{V} = var^*(G(A))$.*

Proof. Let B be a $*$-algebra such that $\mathcal{V} = \mathrm{var}^*(B)$. By Lemma 9, there exists a minimal $\sharp$-superalgebra with superinvolution A such that $G(A) \in \mathcal{V}$ and $\exp^*(G(A)) = d$. Since $\mathcal{V}$ is minimal with respect to its $*$-exponent, it follows that $\mathcal{V} = \mathrm{var}^*(G(A))$. ☐

We are now in a position to prove that in case of minimal $\sharp$-superalgebras, the corresponding Grassmann envelope has no proper central $*$-polynomials.

Lemma 10. *Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a minimal superalgebra with superinvolution $\sharp$. If $m \geq 2$ then $G(A)$ has no proper central $*$-polynomials.*

Proof. Since A is minimal, we notice that $Z(A) \cap J = 0$. Moreover $Z(G(A)) = Z(A)_0 \otimes G_0$ and so

$$Z(G(A)) \cap G(J) = 0. \tag{13}$$

Let $f = f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_{n-r}^-)$ be a multilinear proper central $*$-polynomial of $G(A)$. If we evaluate each variable of f in some $G(A_i)$, then its value lies in $Z(G(A_i)) = Z(A_i)_0 \otimes G_0$, where either $Z(A_i)_0 \cong F$ or $Z(A_i)_0 \cong F \oplus F$.

Furthermore, denoted by 1_{A_i} the unit element of A_i, we have

$$(\alpha 1_{A_i} \otimes h_1)(w_{i,i+1} \otimes h_2) = \alpha w_{i,i+1} \otimes h_1h_2$$

whereas

$$(w_{i,i+1} \otimes h_2)(\alpha 1_{A_i} \otimes h_1) = 0,$$

for all $h_1 \in G_0$, $h_2 \in G$ and for all $\alpha \in F$ or $\alpha \in F \oplus F$ according if $Z(A_i)_0 \cong F$ or $Z(A_i)_0 \cong F \oplus F$.

Similarly,

$$(w_{i,i+1}^\sharp \otimes h_2)(\alpha 1_{A_i} \otimes h_1) = \alpha w_{i,i+1}^\sharp \otimes h_2h_1$$

and

$$(\alpha 1_{A_i} \otimes h_1)(w_{i,i+1}^\sharp \otimes h_2) = 0.$$

Moreover,

$$(w_{m-1,m} \otimes h_2)(\alpha 1_{A_m} \otimes h_1) = \alpha w_{m-1,m} \otimes h_2h,$$
$$(\alpha 1_{A_m} \otimes h_1)(w_{m-1,m} \otimes h_2) = 0$$

and

$$(\alpha 1_{A_m} \otimes h_1)(w^\sharp_{m-1,m} \otimes h_2) = \alpha w^\sharp_{m-1,m} \otimes h_1 h_2,$$
$$(w^\sharp_{m-1,m} \otimes h_2)(\alpha 1_{A_m} \otimes h_1) = 0.$$

Since $m \geq 2$, from the above relations we get that f is a $*$-identity of $G(A_i)$ for all $1 \leq i \leq m$.

Finally, if we evaluate at least one variable of f in $G(J)$ we get zero since such a value will lie in $G(J)$ and (13) holds. Thus $f \in \mathrm{Id}^*(G(A))$ and we are done. □

7. The central exponent

In this section we prove that for any $*$-algebra R $\exp^z_*(R)$ exists, moreover we compare it with $\exp^*(R)$.

Since $c^{*,z}_n(R) \leq c^*_n(R)$, it is clear that

$$\overline{\exp}^z_*(R) = \limsup_{n \to +\infty} \sqrt[n]{c^{*,z}_n(R)} \leq \exp^*(R).$$

In order to get our goal we shall also prove that

$$\underline{\exp}^z_*(R) = \liminf_{n \to +\infty} \sqrt[n]{c^{*,z}_n(R)} \geq \exp^*(R).$$

Theorem 6. *Let R be a $*$-algebra over a field of characteristic zero such that $\exp^*(R) \geq 2$. Then either $c^{*,z}_n(R) = 0$ for all $n \geq 0$ or*

$$C_1 n^{t_1} \exp^*(R)^n \leq c^{*,z}_n(R) \leq C_2 n^{t_2} \exp^*(R)^n,$$

for some constants $C_1 > 0$, C_2, t_1, t_2.

Proof. If $\exp^*(R) = 2$ and R is commutative, then it is clear that $c^{*,z}_n(R) = 0$ for all $n \geq 0$. So let us suppose $\exp^*(R) \geq 2$ and R non-commutative.

As in the previous sections, we may assume that F is algebraically closed and $R = G(A)$ for some finite dimensional superalgebra with superinvolution A. By [10, Theorem 3], there exist constants $C_1 > 0$, C_2, t_1, t_2 such that

$$C_1 n^{t_1} d^n \leq c^*_n(G(A)) \leq C_2 n^{t_2} d^n, \tag{14}$$

where $d = \exp^*(G(A))$. Hence by (3), we also get $c^{*,z}_n(R) \leq C_2 n^{t_2} d^n$.

Now let $\mathcal{V} = \mathrm{var}^*(G(A))$ be the $*$-variety generated by $G(A)$. Recall that by [1, Theorem 5], any T_*-ideal is finitely generated, hence $\mathcal{V}$ contains a subvariety $\mathcal{W}$ which is minimal of exponent $d = \exp^*(\mathcal{W}) = \exp^*(\mathcal{V})$. Hence by Theorem 5 there exists a minimal superalgebra with superinvolution B such that $\mathcal{W} = \mathrm{var}^*(G(B))$.

Suppose that B is not simple, then by Lemma 10

$$\mathrm{Id}^{*,z}(G(A)) \subseteq \mathrm{Id}^{*,z}(G(B)) = \mathrm{Id}^*(G(B)).$$

Thus $c^{*,z}_n(G(A)) \geq c^*_n(G(B))$ and since $c^*_n(G(B))$ has a lower bound as in (14), we get that also $c^{*,z}_n(G(A))$ has too.

Now suppose that B is simple. It is clear that $\mathrm{Id}^{*,z}(G(A)) \subseteq \mathrm{Id}^{*,z}(G(B))$, hence $c^{*,z}_n(G(A)) \geq c^{*,z}_n(G(B))$ and we have only to prove that $c^{*,z}_n(G(B)) \geq C_1 n^{t_1} d^n$, for some constants $C_1 > 0$ and t_1.

Let $N = c^*_n(G(B))$ and let $f_1, \ldots, f_N \in P^*_n$ be multilinear $*$-polynomials linearly independent modulo $\mathrm{Id}^*(G(B))$. We claim that $f_1 x^+_{n+1}, \ldots, f_N x^+_{n+1}$ are linearly independent modulo $\mathrm{Id}^{*,z}(G(B))$. To this end, suppose that there exist not all zero scalars $\alpha_1, \ldots, \alpha_N \in F$ such that $\alpha_1 f_1 x^+_{n+1} + \cdots + \alpha_N f_N x^+_{n+1} \in \mathrm{Id}^{*,z}(G(B))$.

Denote by $f = \alpha_1 f_1 + \cdots + \alpha_N f_N$, then it is clear that $f \notin \mathrm{Id}^*(G(B))$. If f is a central $*$-polynomial, then it is a proper central $*$-polynomial and its values are taken in $Z(G(B)) = Z(B)_0 \otimes G_0$. Now let $a \in B^+_0 \cup B^-_1$ such that $a \notin Z(B)_0$. If we evaluate x^+_{n+1} in $a \otimes h$, for some suitable $h \in G$, then $f x^+_{n+1}$ evaluates in $a' \otimes h'$, for some $a' \in B$ and $h' \in G$, which is not a central element of $G(B)$.

Moreover, if f is not a central $*$-polynomial, then $f x^+_{n+1}$ has a non central evaluation by specializing x^+_{n+1} with $1_B \otimes h$, for some $h \in G_0$. Hence we reached a contradiction and we have $\alpha_1 = \ldots = \alpha_N = 0$.

So we have proved that $c^*_n(G(B)) \leq c^{*,z}_{n+1}(G(B))$ and the conclusion follows from [10, Theorem 3] and from $d = \exp^*(G(B))$.

□

Notice that in case $\exp^*(R) \leq 1$, by [25, Theorems 1 and 2] we may assume that R is a finite dimensional $*$-algebra. In the next lemma we analyze the structure of some finite dimensional $*$-algebra.

Lemma 11. *Let R be a finite dimensional $*$-algebra over an algebraically closed field such that* $\exp^*(R) = 1$. *If $c_n^{*,z}(R) = 0$ for some $n \geq 2$, then $R = C \oplus N$, where C is a commutative $*$-algebra and N is a nilpotent $*$-algebra.*

Proof. Consider the Wedderburn-Malcev decomposition $R = A + J$ where $A = A_1 \oplus \cdots \oplus A_m$ is a semisimple $*$-algebra with A_i's simple $*$-algebras and J is the Jacbson radical of R. Since $\exp^*(R) = 1$, then all $A_1, \ldots, A_m$ are one dimensional.

Let $J_0 = \{j \in J \mid je = ej = 0\}$ and $J_1 = \{j \in J \mid je = ej = j\}$, where e is the unit element of A_1. Since $c_n^{*,z}(R) = 0$ for some $n \geq 2$, any $*$-polynomial of degree n is central. In particular, $x_1 \cdots x_n$ is a central $*$-polynomial of R, where x_i's are symmetric or skew variables. Thus it follows that J_0 and J_1 are $*$-ideals of R and $J = J_0 \oplus J_1$. Also $A_i J_1 = 0$ for all $2 \leq i \leq m$ since $eA_i = 0$. Hence $R = (A_1 + J_1) \oplus (A_2 + \ldots A_m + J_0)$. Moreover, since $j = e^{n-1} j$ lies in the center of R for any $j \in J_1$, $A_1 + J_1$ is a commutative $*$-algebra. Applying the same arguments to the $*$-algebra $A_2 + \ldots A_m + J_0$, we finally get that $R = C_1 \oplus \cdots \oplus C_m \oplus N$ where $C_1, \ldots C_m$ are commutative $*$-algebras and $N \subset J$ is a nilpotent $*$-algebra. □

Putting together the previous results we get the following.

Theorem 7. *If R is any $*$-algebra over a field of characteristic zero, then its central exponent* $\exp_*^z(R)$ *exists. Moreover, if* $\exp^*(R) \geq 3$, *then* $\exp_*^z(R) = \exp^*(R)$. *When* $\exp^*(R) \leq 2$, *then* $\exp_*^z(R) = \exp^*(R)$ *or* 0.

Proof. First suppose $\exp^*(R) \geq 2$. Then by Theorem 6, $\exp_*^z(R)$ exists and $\exp_*^z(R) = \exp^*(R)$ or $\exp_*^z(R) = 0$ in case R is commutative.

Let now us assume that $\exp^*(R) = 1$. As we said before, we may assume that R is a finite dimensional $*$-algebra. Moreover, recalling that the codimensions do not change by extending the base field, we may assume that the field is algebraically closed. Notice that the sequence of central $*$-codimesions of R is also polynomially bounded, hence $\exp_*^z(R) \leq 1$. Thus if $c_n^{*,z}(R) \neq 0$ for all n, it readily follows that $\exp_*^z(R) = 1$. Otherwise, If $c_n^{*,z}(R) = 0$ for some $n \geq 2$, then by Lemma 11 it follows that $\exp_*^z(R) = 0$.

Finally, when $\exp^*(R) = 0$, then R is nilpotent and $\exp_*^z(R) = 0$. This complete the proof. □

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