

VARIETIES OF ALGEBRAS WITH PSEUDOINVOLUTION: CODIMENSIONS, COCHARACTERS AND COLENGTHS

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ABSTRACT. Let A be a finitely generated superalgebra with pseudoinvolution $*$ over an algebraically closed field F of characteristic zero. In this paper we develop a theory of polynomial identities for this kind of algebras. In particular, we shall consider three sequences that can be attached to $\text{Id}^*(A)$, the T_2^* -ideal of identities of A : the sequence of $*$ -codimensions $c_n^*(A)$, the sequence of $*$ -cocharacter $\chi_{\langle n \rangle}^*(A)$ and the $*$ -colength sequence $l_n^*(A)$.

Our purpose is threefold. First we shall prove that the $*$ -codimension sequence is eventually non-decreasing, i.e., $c_n^*(A) \leq c_{n+1}^*(A)$, for n large enough. Secondly, we study superalgebras with pseudoinvolution having the multiplicities of their $*$ -cocharacter bounded by a constant. Among them, we characterize the ones with multiplicities bounded by 1. Finally, we classify superalgebras with pseudoinvolution A such that $l_n^*(A)$ is bounded by 3. In the last section we relate the $*$ -colengths with the polynomial growth of the $*$ -codimensions: we show that $l_n^*(A)$ is bounded by a constant if and only if $c_n^*(A)$ grows at most polynomially.

1. Introduction

Let $A = A_0 \oplus A_1$ be an associative superalgebra over an algebraically closed field F of characteristic zero. A pseudoinvolution on A is a graded linear map $*$: $A \rightarrow A$ such that for all $a, b \in A_0 \cup A_1$, $a^{**} = (-1)^{|a|}a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, where $|c|$ denotes the homogeneous degree of c . Notice that, according to the previous definition, if $a \in A_1$, then $a^{**} = -a$ and so there are no symmetric or skew-symmetric elements of homogeneous degree 1. Moreover, since we are assuming F to be algebraically closed of characteristic zero, there exists an element i such that $i^2 = -1$ and we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i},$$

where A_0^+ and A_0^- denote the sets of symmetric and skew elements of A_0 and A_1^i and A_1^{-i} denote the sets of i -symmetric and i -skew elements of A_1 , respectively. Here it is worth mentioning that an element $a \in A_1$ is said i -symmetric if $a^* = ia$ whereas it is called i -skew when $a^* = -ia$.

The existence of pseudoinvolutions of the first kind, i.e., pseudoinvolutions fixing the ground field F , was proven in [16] by Jaber. Later on, Martinez and Zelmanov in [21] classified the irreducible bimodules over simple finite dimensional Jordan superalgebras via pseudoinvolutions.

The present paper contains several results of PI-theory (theory of algebras with polynomial identities) concerning finitely generated superalgebras with pseudoinvolution, or simply $*$ -algebras. If A is such an algebra and $\text{Id}^*(A)$ denotes its T_2^* -ideal of $*$ -identities, then one can attach to them three sequences. The first one is the sequence of $*$ -codimensions, denoted by $c_n^*(A)$, that gives a quantitative measure of how many identities are satisfied by A . This sequence was introduced by Regev in [27], in the setting of associative algebras without additional structure, and in the same paper it was proved that it is exponentially bounded, provided that the algebra satisfies a non-trivial polynomial identity. A similar result was achieved in [14] for $*$ -algebras and their $*$ -codimension sequence. In the same paper the authors also proved quite a few properties about $*$ -algebras whose $*$ -codimensions grow polynomially. Moreover, in [12] the author captured the exponential growth of $c_n^*(A)$ giving a positive answer to the Amitsur conjecture for $*$ -algebras. Remark that such a conjecture was firstly proved in the setting of ordinary polynomial identities in a celebrated paper by Giambruno and Zaicev (see [8]).

The second sequence meant to be attached to a $*$ -algebra is the sequence of $*$ -cocharacters that one can construct in the following way. Let $P_{n_1, \dots, n_4}(A)$ be the space of multilinear $*$ -polynomials in n_1 even symmetric variables, n_2 even skew variables, n_3 odd i -symmetric variables and n_4 odd i -skew variables, modulo $\text{Id}^*(A)$. Then the group $S_{n_1} \times \dots \times S_{n_4}$ acts on $P_{n_1, \dots, n_4}(A)$ by permutation of the variables of the same homogeneous degree which are all (i -)symmetric or all (i -)skew at the same time with respect to $*$. The $\langle n \rangle$ -th $*$ -cocharacter $\chi_{\langle n \rangle}^*(A)$, where $\langle n \rangle = (n_1, \dots, n_4)$ and $n = n_1 + \dots + n_4$, is the $S_{n_1} \times \dots \times S_{n_4}$ -character of $P_{n_1, \dots, n_4}(A)$ corresponding to this action. One of the main goal of PI-theory is to decompose $\chi_{\langle n \rangle}^*(A)$ into the sum of irreducible characters, i.e. to write

$$\chi_{\langle n \rangle}^*(A) = \sum_{(\lambda) \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)},$$

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where $\langle \lambda \rangle = (\lambda(1), \dots, \lambda(4))$ is a multipartition of $\langle n \rangle$ and $m_{\langle \lambda \rangle}$ is the multiplicity of the corresponding irreducible character. In the past years much effort was put into the study of the cocharacters of algebras in various settings. For instance, in [2, 15, 26, 28] conditions ensuring that the multiplicities are bounded by a constant were given for graded algebras, superalgebras with superinvolution, superalgebras and algebras with involution, respectively.

The last sequence that we will consider is strictly related to the $*$ -cocharacter. The so-called $*$ -colength sequence, denoted by $l_n^*(A)$, is the sum of the multiplicities in the decomposition of the $\langle n \rangle$ -th $*$ -cocharacter, for all $n = n_1 + \dots + n_4 \geq 1$. Such a sequence was extensively studied for instance in [19, 20, 25, 29], where algebras with colengths bounded by 2, 3 or 4 were classified.

Since there is a one-to-one correspondence between T_2^* -ideals and $*$ -varieties, it is convenient to translate all the objects we have defined above in the language of $*$ -varieties. Recall that, roughly speaking, a $*$ -variety \mathcal{V} is the class of all $*$ -algebras sharing the same T_2^* -ideal of identities. Thus if $\mathcal{V} = \text{var}^*(A)$ is the $*$ -variety generated by A , then we set $\text{Id}^*(\mathcal{V}) = \text{Id}^*(A)$, $c_n^*(\mathcal{V}) = c_n^*(A)$ and so on.

The goal of this paper is threefold. We first study the $*$ -codimension sequence by proving that, similarly to the ordinary case, it is eventually non-decreasing, i.e., $c_n^*(A) \leq c_{n+1}^*(A)$ for n large enough.

Secondly, we study $*$ -varieties of algebras with multiplicities of their $*$ -cocharacter bounded by a constant and, among them, we characterize the ones with multiplicities bounded by 1.

Finally, we classify $*$ -varieties having the $*$ -colength sequence bounded by 3 and more generally, we relate the $*$ -colengths with the polynomial growth of the $*$ -codimensions by proving that the $*$ -colengths are bounded by a constant if and only if $c_n^*(A)$ is polynomially bounded.

2. Preliminaries

Throughout this paper F will denote an algebraically closed field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra over F , i.e., an algebra graded by \mathbb{Z}_2 , the cyclic group of order two. The elements of A_0 and A_1 are called homogeneous elements of degree 0 (or even elements) and of degree one (or odd elements), respectively.

Let now consider a countable set of variables $X = \{x_1, x_2, \dots\}$. Write $X = Y \cup Z$ as the disjoint union of two subsets, requiring that the variables of Y are of even homogeneous degree and the variables of Z are of odd homogeneous degree. If we denote by \mathcal{F}_0 the subspace of $F\langle Y \cup Z \rangle$ spanned by all monomials in the variables of $Y \cup Z$ having an even number of variables of Z and by \mathcal{F}_1 the subspace spanned by all monomials having an odd number of variables of Z , then it is clear that $F\langle Y \cup Z \rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a \mathbb{Z}_2 -grading. Thus we can refer to $F\langle Y \cup Z \rangle$ as the free associative superalgebra.

Now suppose that the superalgebra A is endowed with a pseudoinvolution $*$. Recall that a pseudoinvolution on A is a graded linear map $*$: $A \rightarrow A$ such that $a^{**} = (-1)^{|a|}a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, for any homogeneous elements $a, b \in A_0 \cup A_1$. Here $|c|$ denotes the homogeneous degree of $c \in A_0 \cup A_1$.

Clearly, if $a \in A_1$, then $a^{**} = -a$ and so there are no symmetric or skew-symmetric elements of homogeneous degree 1. Moreover, there exists an element i such that $i^2 = -1$. Furthermore, since $\text{char}F = 0$, we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i},$$

where $A_0^+ = \{a \in A_0 \mid a^* = a\}$ and $A_0^- = \{a \in A_0 \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A_0 and $A_1^i = \{a \in A_1 \mid a^* = ia\}$ and $A_1^{-i} = \{a \in A_1 \mid a^* = -ia\}$ denote the sets of i -symmetric and i -skew elements of A_1 , respectively.

We shall write $F\langle Y \cup Z, * \rangle$ for the free superalgebra with pseudoinvolution on the countable set $Y \cup Z$ over F . It is useful to regard $F\langle Y \cup Z, * \rangle$ as generated by (even) symmetric and skew variables and by (odd) i -symmetric and i -skew variables: if for $j = 1, 2, \dots$, we let $y_j^+ = y_j + y_j^*$, $y_j^- = y_j - y_j^*$, $z_j^i = z_j - iz_j^*$ and $z_j^{-i} = z_j + iz_j^*$, thus $F\langle Y \cup Z, * \rangle = F\langle y_1^+, y_1^-, z_1^i, z_1^{-i}, y_2^+, y_2^-, z_2^i, z_2^{-i}, \dots \rangle$.

A polynomial $f(y_1^+, \dots, y_m^+, y_1^-, \dots, y_n^-, z_1^i, \dots, z_r^i, z_1^{-i}, \dots, z_s^{-i}) \in F\langle Y \cup Z, * \rangle$ is a $*$ -polynomial identity of A (or simply a $*$ -identity), and we write $f \equiv 0$, if

$$f(u_1^+, \dots, u_m^+, u_1^-, \dots, u_n^-, v_1^i, \dots, v_r^i, v_1^{-i}, \dots, v_s^{-i}) = 0,$$

for all $u_1^+, \dots, u_m^+ \in A_0^+$, $u_1^-, \dots, u_n^- \in A_0^-$, $v_1^i, \dots, v_r^i \in A_1^i$ and $v_1^{-i}, \dots, v_s^{-i} \in A_1^{-i}$.

We denote by $\text{Id}^*(A) = \{f \in F\langle Y \cup Z, * \rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^* -ideal of $*$ -identities of A , i.e., $\text{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, * \rangle$ invariant under all graded endomorphisms of $F\langle Y \cup Z \rangle$ commuting with the pseudoinvolution $*$.

Given polynomials $f_1, \dots, f_n \in F\langle Y \cup Z, * \rangle$ we shall denote by $\langle f_1, \dots, f_n \rangle_{T_2^*}$ the T_2^* -ideal generated by f_1, \dots, f_n . Moreover, in order to simplify the notation, we shall denote by y any even variable, by z any odd variable and by x an arbitrary variable.

As in the ordinary case, in characteristic zero, every $*$ -identity is equivalent to a system of multilinear $*$ -identities. Hence, in order to study $\text{Id}^*(A)$, one can define

$$P_n^* = \text{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, w_j = y_j^+ \text{ or } w_j = y_j^- \text{ or } w_j = z_j^i \text{ or } w_j = z_j^{-i}, j = 1, \dots, n\}$$

as the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^i, z_1^{-i}, \dots, y_n^+, y_n^-, z_n^i, z_n^{-i}$ (i.e., y_j^+ or y_j^- or z_j^i or z_j^{-i} appears in each monomial at degree 1) and then analyze $P_n^* \cap \text{Id}^*(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \text{Id}^*(A)}, \quad n \geq 1,$$

is called the n -th $*$ -codimension of A .

Let $n \geq 1$ and write $n = n_1 + \dots + n_4$ as a sum of four non-negative integers. We denote by $P_{n_1, \dots, n_4} \subseteq P_n^*$ the vector space of multilinear polynomials in which the first n_1 variables are symmetric, the next n_2 variables are skew, the next n_3 variables are i -symmetric and the last n_4 variables are i -skew. Now if we set $P_{n_1, \dots, n_4}(A) = \frac{P_{n_1, \dots, n_4}}{P_{n_1, \dots, n_4} \cap \text{Id}^*(A)}$ and $c_{n_1, \dots, n_4}(A) = \dim_F P_{n_1, \dots, n_4}(A)$, it is immediate to see that

$$c_n^*(A) = \sum_{n_1 + \dots + n_4 = n} \binom{n}{n_1, \dots, n_4} c_{n_1, \dots, n_4}(A),$$

where $\binom{n}{n_1, \dots, n_4} = \frac{n!}{n_1! \dots n_4!}$ stands for the multinomial coefficient. Hence the growth of $c_n^*(A)$ is related to the growth of multinomial coefficients and of $c_{n_1, \dots, n_4}(A)$, for any $n = n_1 + \dots + n_4$. In [14, Corollary 2.1] it was proved that such a sequence is exponentially bounded, provided that A satisfies a non-trivial polynomial identity.

An important tool that one can use in order to get information about the $*$ -codimension sequence is the so-called $*$ -cocharacter of a $*$ -algebra A defined as follows. Let the group $S_{n_1} \times \dots \times S_{n_4}$ act on P_{n_1, \dots, n_4} by permuting the variables with the same homogeneous degree and the same symmetry with respect to $*$. Then $P_{n_1, \dots, n_4}(A)$ inherits the structure of left $S_{n_1} \times \dots \times S_{n_4}$ -module and its character arises. For all $n \geq 1$ and $n_1 + \dots + n_4 = n$ we call such character the $\langle n \rangle$ -th $*$ -cocharacter, where $\langle n \rangle = (n_1, \dots, n_4)$, and we denote it by $\chi_{\langle n \rangle}^*(A)$. Moreover, by complete reducibility and according to the representation theory of GL_n ,

$$(1) \quad \chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)},$$

where $\langle \lambda \rangle = (\lambda(1), \dots, \lambda(4)) \vdash \langle n \rangle$ means that $\lambda(i) \vdash n_i$ for all $1 \leq i \leq 4$, and $m_{\langle \lambda \rangle}$ is the multiplicity of the corresponding irreducible character. Furthermore, such multiplicities equal the maximal number of linearly independent highest weight vectors associated to the multipartition $\langle \lambda \rangle$. Recall that a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row of the corresponding Young tableaux (see for instance [3, Chapter 12]).

Another numerical sequence that can be attached to a $*$ -algebra A is the sequence of $*$ -colengths. If the $\langle n \rangle$ -th cocharacter of A has a decomposition as in (1), then the n -th colength of A is defined as

$$l_n^*(A) = \sum_{\substack{\langle \lambda \rangle \vdash \langle n \rangle \\ n_1 + \dots + n_4 = n}} m_{\langle \lambda \rangle}.$$

We conclude this section by recalling some basic results concerning the sequences of cocharacters and colengths.

Remark 1. Let A and B be two $*$ -algebras such that

$$\chi_{\langle n \rangle}^*(A) = \sum m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)} \quad \text{and} \quad \chi_{\langle n \rangle}^*(B) = \sum m'_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)}.$$

1. If $B \in \text{var}^*(A)$, then $m'_{\langle \lambda \rangle} \leq m_{\langle \lambda \rangle}$, for all $\langle \lambda \rangle \vdash \langle n \rangle$ and $l_n^*(B) \leq l_n^*(A)$, for all n .
2. The direct sum $A \oplus B$ is also a $*$ -algebra, with pseudoinvolution induced by those defined on A and B . Moreover, if

$$\chi_{\langle n \rangle}^*(A \oplus B) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} \bar{m}_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)}$$

is the decomposition of the $\langle n \rangle$ -th cocharacter of $A \oplus B$, then $\bar{m}_{\langle \lambda \rangle} \leq m_{\langle \lambda \rangle} + m'_{\langle \lambda \rangle}$, for all $\langle \lambda \rangle \vdash \langle n \rangle$.

3. On finite dimensional $*$ -algebras

In this section we investigate the structure and the properties of $*$ -varieties generated by finite dimensional $*$ -algebras. Such a restriction is motivated by the following theorem stating that any finitely generated $*$ -algebra has the same polynomial identities of a suitable finite dimensional one.

Theorem 2. [14, Lemma 4.3]. Let F be an algebraically closed field of characteristic zero. If \mathcal{V} is $*$ -variety generated by a finitely generated $*$ -algebra B over F , satisfying an ordinary identity, then $\mathcal{V} = \text{var}^*(C)$, for some finite dimensional $*$ -algebra C over F .

In light of the previous theorem, from now on it suffices to study only finite dimensional $*$ -algebras and we start by presenting a Wedderburn-Malcev decomposition for such kind of algebras. Recall that an ideal (subalgebra) I of a superalgebra A with pseudoinvolution $*$ is a $*$ -ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^* = I$. The algebra A is a $*$ -simple algebra if $A^2 \neq 0$ and A has no non-trivial $*$ -ideals.

Theorem 3. [14, Theorem 5.1] *Let A be a finite dimensional $*$ -algebra over an algebraically closed field F of characteristic 0. Then there exists a semisimple $*$ -subalgebra B such that $A = B + J(A)$ and $J(A)$ is a $*$ -ideal of A . Moreover $B = B_1 \oplus \cdots \oplus B_k$, where B_1, \dots, B_k are $*$ -simple algebras.*

Due to Theorem 3, in order to study the structure of any finite dimensional $*$ -algebra, it is useful to present the classification of the $*$ -simple ones.

Let $M_n(F)$ be the algebra of $n \times n$ matrices. It is well known (see, for instance [10, Theorem 3.6.8]) that on such an algebra it is possible to define, up to isomorphism, two involutions: the transpose t and the symplectic s . The last one is defined on $2l \times 2l$ matrices by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where the original matrix is partitioned into $l \times l$ blocks.

Moreover, if $n = k + h$, $k \geq h$, then the matrix algebra $M_n(F)$ becomes a superalgebra, denoted by $M_{k,h}(F)$, with grading

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F), T \in M_h(F) \right\},$$

$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k \times h}(F), Z \in M_{h \times k}(F) \right\}.$$

For all $k \geq h \geq 0$, the superalgebra $M_{k,h}(F)$ can be endowed with the pseudotranspose pseudoinvolution pt :

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{pt} = \begin{pmatrix} X^t & Z^t \\ -Y^t & T^t \end{pmatrix}.$$

Notice that, in case $h = 0$, then the matrix superalgebra $M_{k,0}(F)$ has trivial grading and the pseudotranspose pseudoinvolution pt coincides with the usual transpose involution t . Moreover, in case k is even, the superalgebra $M_{k,0}(F)$ can be endowed also with the symplectic (pseudo)involution s .

If $h = k$, the superalgebra $M_{k,k}(F)$ can be endowed with the pseudosymplectic pseudoinvolution ps defined by:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{ps} = \begin{pmatrix} T^t & iY^t \\ iZ^t & X^t \end{pmatrix}.$$

Furthermore, if B is a superalgebra, we denote by B^{sop} the superalgebra with the same graded vector space structure as B but with product given on homogeneous elements a, b by

$$a \circ b = (-1)^{|a||b|}ba.$$

The direct sum $R = B \oplus B^{sop}$ is a superalgebra with $R_0 = B_0 \oplus B_0^{sop}$ and $R_1 = B_1 \oplus B_1^{sop}$ and it can be endowed with the pseudoinvolution pe_x , called pseudoexchange and defined on homogeneous elements by

$$(a, b)^{pe_x} = \left((-1)^{|a||b|}b, a \right).$$

Finally, we have to consider also the superalgebra $Q(n) = M_n(F) + cM_n(F)$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$, with $c^2 = 1$. According to [14, Lemma 5.2], on such a superalgebra it is possible to define the following pseudoinvolutions:

- $\sharp_t^+ : Q(n) \rightarrow Q(n)$ such that $(a + cb)^{\sharp_t^+} = a^t + icb^t$, for all $a + cb \in Q(n)$.
- $\sharp_t^- : Q(n) \rightarrow Q(n)$ such that $(a + cb)^{\sharp_t^-} = a^t - icb^t$, for all $a + cb \in Q(n)$.
- $\sharp_s^+ : Q(n) \rightarrow Q(n)$ such that $(a + cb)^{\sharp_s^+} = a^s + icb^s$, for all $a + cb \in Q(n)$, n even.
- $\sharp_s^- : Q(n) \rightarrow Q(n)$ such that $(a + cb)^{\sharp_s^-} = a^s - icb^s$, for all $a + cb \in Q(n)$, n even.

The following theorem gives the classification of the finite dimensional $*$ -simple algebras over an algebraically closed field F . Recall that if A and B are two superalgebras endowed with pseudoinvolutions $*$ and \star , respectively, then $(A, *)$ and (B, \star) are isomorphic, as $*$ -algebras, if there exists an isomorphism of superalgebras $\psi : A \rightarrow B$ such that $\psi(a^*) = \psi(a)^\star$, for all $a \in A$.

Theorem 4. *Let A be a finite dimensional $*$ -simple algebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

1. $M_{k,l}(F)$ with the pseudotranspose pseudoinvolution pt , $k \geq h > 0$;
2. $M_{k,0}(F)$ with the transpose (pseudo)involution t , $k > 0$;

3. $M_{k,0}(F)$ with the symplectic (pseudo)involution s , $k = 2l > 0$;
4. $M_{k,k}(F)$ with the pseudosymplectic pseudoinvolution ps , $k > 0$;
5. $Q(n)$ with the pseudoinvolution \sharp_t^+ or \sharp_t^- , $n \geq 1$;
6. $Q(n)$ with the pseudoinvolution \sharp_s^+ or \sharp_s^- , $n \geq 2$ even;
7. $M_{k,l}(F) \oplus M_{k,l}(F)^{sop}$ with the pseudoexchange pseudoinvolution pex , $k \geq h \geq 0$;
8. $Q(n) \oplus Q(n)^{sop}$ with the pseudoexchange pseudoinvolution pex , $n \geq 1$.

Concerning the Jacobson radical, we now recall a very useful result about its structure. If $A = B + J$ is the Wedderburn-Malcev decomposition of a $*$ -algebra A , then by using the left and right multiplication by the unit element of B , we can decompose J into the direct sum of B -bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i, k \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{k,l}J_{im}$, where $\delta_{k,l}$ is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B . For a proof of this result see [9, Lemma 2]. Notice that if $*$ is a pseudoinvolution on A , then $J_{00}^* = J_{00}$, $J_{11}^* = J_{11}$, $J_{01}^* = J_{10}$ and $J_{10}^* = J_{01}$.

4. Non-decreasing sequences

Main goal of this section is to prove that the $*$ -codimension sequence of a finitely generated $*$ -algebra is eventually non-decreasing.

In general this is not to take too much for granted, since for instance for any $k \geq 1$, we can construct a suitable $*$ -algebra with k drops in its $*$ -codimension sequence, as highlighted in the following proposition.

Proposition 5. *For any $k \geq 1$, there exists a $*$ -algebra A such that $c_{n_1}^*(A) > c_{n_1+1}^*(A), \dots, c_{n_k}^*(A) > c_{n_k+1}^*(A)$, for some integers $n_1 < \dots < n_k$.*

Proof. Let $F_0[\vartheta]$ be the ring of polynomials in the variable ϑ with zero constant term, endowed with trivial grading and trivial pseudoinvolution. Moreover, if we let $R_q = F_0[\vartheta]/(\vartheta^{q+1})$, then $R_q^q \neq 0$ and $R_q^{q+1} = 0$.

Now fix an integer $r > 1$ and choose positive integers $n_1 < \dots < n_{k+1}$ in the following way. For $1 \leq j \leq k+1$, without loss of generality let us suppose that $r \geq k - j$ and consider the superalgebra $M_{r,k-j}(F)$ with the pseudotranspose pseudoinvolution. Then, the algebra $B_j = M_{r,k-j}(R_{n_j})$ of $(r+k-j) \times (r+k-j)$ matrices over the ring R_{n_j} naturally inherits both the \mathbb{Z}_2 -grading and the pseudoinvolution $*$, requiring that for all $a \in R_{n_j}$, $a^* = a$.

Thus B_1, \dots, B_{k+1} are $*$ -algebras and it is clear that

$$c_n^*(B_j) = \begin{cases} c_n^*(M_{r,k-j}(R_{n_j})) & \text{if } n \leq n_j, \\ 0 & \text{if } n > n_j. \end{cases}$$

Moreover, notice that B_j has the same multilinear $*$ -polynomial identities of degree less than or equal to n_j as the ones of $M_{r,k-j}(F)$ and by [12, Theorem 4.2] the $*$ -codimension sequence of $M_{r,k-j}(F)$ grows exponentially as $(r+k-j)^2$. Therefore, we can find integers $n_1 < \dots < n_{k+1}$ such that

$$c_{n_j}^*(B_j) > c_{n_j+1}^*(B_{j+1}),$$

for all $1 \leq j \leq k$.

Now let $B = B_1 \oplus \dots \oplus B_{k+1}$, then

$$c_{n_j}^*(B) = c_{n_j}^*(B_j) > c_{n_j+1}^*(B_{j+1}) = c_{n_j+1}^*(B),$$

for all $1 \leq j \leq k$, as claimed. \square

Theorem 6. *Let A be a finitely generated $*$ -algebra over an algebraically closed field F of characteristic zero. Then the $*$ -codimension sequence is eventually non-decreasing, i.e., for n large enough,*

$$c_n^*(A) \leq c_{n+1}^*(A).$$

Proof. Since A is finitely generated, by Theorem 2 there exists a finite dimensional $*$ -algebra B such that $\text{Id}^*(A) = \text{Id}^*(B)$, so let prove the statement for the $*$ -algebra B . Write $B = C + J$, where C is a maximal semisimple $*$ -subalgebra of B , $J = J(B)$ is its Jacobson radical, $J^{t-1} \neq 0$ and $J^t = 0$ for some $t \geq 0$. We shall show that $c_n^*(B) \leq c_{n+1}^*(B)$ for all $n \geq t$ and this will complete the proof.

If B is a nilpotent $*$ -algebra, that is $C = 0$, then $c_n^*(A) = 0$ for all $n \geq t$ and we have nothing to prove. So assume $C \neq 0$. Given $n \geq t$, let $c_{n_1, \dots, n_4}(B) = r$, where $n_1 + \dots + n_4 = n$, and let f_1, \dots, f_r be the $*$ -polynomials of P_{n_1, \dots, n_4}^* that are linearly independent modulo $P_{n_1, \dots, n_4}^* \cap \text{Id}^*(B)$. In order to simplify the notation, let us assume that $f_i = f_i(x_1, \dots, x_n)$ where x_1, \dots, x_n are either symmetric or skew (even) variables or i -symmetric or i -skew (odd) variables.

For all $1 \leq i \leq r$, consider the $*$ -polynomials of $P_{n_1+1, n_2, n_3, n_4}^*$

$$h_i = h_i(x_1, \dots, x_n, y_{n+1}^+) = \sum_{j=1}^n f_i(x_1, \dots, x_{j-1}, y_{n+1}^+ x_j + x_j y_{n+1}^+, x_{j+1}, \dots, x_n).$$

We shall prove that h_1, \dots, h_r are linearly independent modulo $P_{n_1+1, n_2, n_3, n_4}^* \cap \text{Id}^*(B)$.

To this end, suppose by contradiction that there exist not all zero scalars $\alpha_1, \dots, \alpha_r$ such that

$$h = \sum_{i=1}^r \alpha_i h_i \in \text{Id}^*(B).$$

Since f_1, \dots, f_r are linearly independent modulo $P_{n_1, \dots, n_4}^* \cap \text{Id}^*(B)$, then the $*$ -polynomial $f = \sum_{i=1}^r \alpha_i f_i$ is not a $*$ -identity of B and there exist elements $a_1, \dots, a_n \in B_0^+ \cup B_0^- \cup B_1^i \cup B_1^{-i}$ such that $f(a_1, \dots, a_n) \neq 0$. In particular, if $a_j \in J$ for some j , then we also assume that $a_j \in J_{lm}$ for some $l, m \in \{0, 1\}$.

If e denotes the unit element of C and $b \in J_{00} \cup J_{01} \cup J_{10} \cup J_{11}$, then we get that

$$eb + be = \begin{cases} 2b & \text{if } b \in C \cup J_{11}, \\ b & \text{if } b \in J_{01} \cup J_{10}, \\ 0 & \text{if } b \in J_{00}. \end{cases}$$

Hence, since by assumption $n \geq t$ and $J^t = 0$, $a_j \in C$ for some j and

$$h_i(a_1, \dots, a_n, e) = \alpha f_i(a_1, \dots, a_n),$$

where α is a positive integer. Therefore,

$$h(a_1, \dots, a_n, e) = \sum_{i=1}^r \alpha_i h_i(a_1, \dots, a_n, e) = \alpha \left(\sum_{i=1}^r \alpha_i f_i(a_1, \dots, a_n) \right) \neq 0,$$

a contradiction. Thus h_1, \dots, h_r are linearly independent modulo $P_{n_1+1, n_2, n_3, n_4}^* \cap \text{Id}^*(B)$ and in particular

$$c_{n_1, \dots, n_4}(B) \leq c_{n_1+1, n_2, n_3, n_4}(B).$$

By using the latter one plus the combinatorial identity

$$\binom{n+1}{n_1+1, n_2, n_3, n_4} = \binom{n}{n_1, \dots, n_4} + \sum_{i=2}^4 \binom{n}{n_1, \dots, n_i-1, \dots, n_4},$$

it readily follows that

$$c_n^*(B) = \sum_{n_1+\dots+n_4=n} \binom{n}{n_1, \dots, n_4} c_{n_1, \dots, n_4}(B) \leq \sum_{n_1+\dots+n_4=n} \binom{n+1}{n_1+1, n_2, n_3, n_4} c_{n_1+1, n_2, n_3, n_4}(B) \leq c_{n+1}^*(B)$$

and we are done. \square

5. Varieties of almost polynomial growth

This section is devoted to the study of $*$ -varieties of almost polynomial growth. Recall that if \mathcal{V} is such a $*$ -variety, then $c_n^*(\mathcal{V})$ is exponentially bounded but for any proper subvariety \mathcal{U} , $c_n^*(\mathcal{U})$ grows polynomially. In what follows, we refer to the growth of a $*$ -variety as the asymptotic behaviour of its $*$ -codimension sequence.

In [14] the authors introduced five suitable $*$ -algebras and they proved that they generate the only $*$ -varieties of almost polynomial growth. Here we present such results and, along the way, we shall add some information concerning the $\langle n \rangle$ -th cocharacters and the $*$ -colengths of these algebras.

Let $F \oplus F$ be the two dimensional group algebra of \mathbb{Z}_2 . We denote by D the superalgebra $F \oplus F$ with trivial grading and exchange (pseudo)involution ex given by $(a, b)^{ex} = (b, a)$, for all $(a, b) \in D$.

Now consider a non-trivial grading on $F \oplus F$, by setting $D_0 = F(1, 1)$ and $D_1 = F(1, -1)$. On this superalgebra it is possible to define the following pseudoinvolutions:

- 1) $ex_i : (a, b) \mapsto i^{|(a,b)|} (b, a)$, for all $(a, b) \in D_0 \cup D_1$;
- 2) $ex_{-i} : (a, b) \mapsto (-i)^{|(a,b)|} (b, a)$, for all $(a, b) \in D_0 \cup D_1$.

In order to simplify the notation, we shall denote by D^i and D^{-i} the superalgebra $F \oplus F = F(1, 1) \oplus F(1, -1)$ with the pseudoinvolution ex_i and ex_{-i} , respectively.

In the following theorem we collect some results concerning the algebras above (see [14, Theorems 3.1, 3.3]). Recall that $[x_1, x_2] = x_1 x_2 - x_2 x_1$ stands for the commutator between the variables x_1 and x_2 .

Theorem 7. *Let D, D^i, D^{-i} be the $*$ -algebras presented above.*

1. $\text{Id}^*(D) = \langle [x_1, x_2], z^i, z^{-i} \rangle_{T_2^*}$.
2. $\text{Id}^*(D^i) = \langle [x_1, x_2], y^-, z^i \rangle_{T_2^*}$.

3. $Id^*(D^{-i}) = \langle [x_1, x_2], y^-, z^{-i} \rangle_{T_2^*}$.
4. The algebras D , D^i and D^{-i} generate $*$ -varieties of almost polynomial growth.

Now we shall prove the following.

Proposition 8. For any $n \geq 1$, $l_n^*(D) = l_n^*(D^i) = l_n^*(D^{-i}) = n + 1$.

Proof. For any $A \in \{D, D^i, D^{-i}\}$, let us assume that the $\langle n \rangle$ -th cocharacter of A is of the form

$$\chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

Now, by taking into account the T_2^* -ideals of the identities of the above algebras, it is not difficult to see that

$$m_{\langle \lambda \rangle} = \begin{cases} 1 & \text{if } \begin{cases} \langle \lambda \rangle = ((n_1), (n_2), \emptyset, \emptyset), n_1 + n_2 = n & \text{in case } A = D, \\ \langle \lambda \rangle = ((n_1), \emptyset, \emptyset, (n_4)), n_1 + n_4 = n & \text{in case } A = D^i, \\ \langle \lambda \rangle = ((n_1), \emptyset, (n_3), \emptyset), n_1 + n_3 = n & \text{in case } A = D^{-i}, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $l_n^*(D) = l_n^*(D^i) = l_n^*(D^{-i}) = n + 1$ and the proof is complete. \square

Now let us consider the following subalgebra of $UT_4(F)$, the algebra of 4×4 upper-triangular matrices:

$$M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34}.$$

Such an algebra can be endowed with the reflection involution \circ , i.e., the involution obtained by reflecting a matrix along its secondary diagonal. Hence, if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$ then

$$a^\circ = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}.$$

If we regard M as a superalgebra with trivial grading, then the above involution is, clearly, a pseudoinvolution and we can see M as a $*$ -algebra.

Next we consider a non-trivial grading on M : we denote by M^{sup} the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$, $M_1 = Fe_{12} \oplus Fe_{34}$ and endowed with the following pseudoinvolution $\bar{\circ}$:

$$a^{\bar{\circ}} = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} - \gamma e_{34}.$$

In the following theorem we collect some results concerning the above $*$ -algebras (see [14, Theorem 3.1, 3.2]).

Theorem 9. Let M and M^{sup} be the $*$ -algebras presented above.

1. $Id^*(M) = \langle y_1^-, y_2^-, z^i, z^{-i} \rangle_{T_2^*}$.
2. $Id^*(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$.
3. The algebras M and M^{sup} generate $*$ -varieties of almost polynomial growth.

Since the $*$ -algebra M can be seen as an algebra with involution, by writing in the language of $*$ -algebras the results of [23] (see in particular Corollary 1), we get the following proposition.

Proposition 10. For any $n \geq 1$, we have that $l_n^*(M) = \begin{cases} \frac{3n^2+4}{4} & \text{if } n \text{ is even,} \\ \frac{3n^2+5}{4} & \text{if } n \text{ is odd.} \end{cases}$

Finally, we have the following result concerning the $*$ -algebra M^{sup} .

Proposition 11. For any $n \geq 1$, we have that $l_n^*(M^{sup}) = \begin{cases} \frac{n^2+2n+2}{2} & \text{if } n \text{ is even,} \\ \frac{n^2+2n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$

Proof. Assume that the $\langle n \rangle$ -th cocharacter of M^{sup} is of the form

$$\chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

By using the same approach of [7, Theorem 6.4] it easily follows that $m_{\langle \lambda \rangle} = 1$ if $\langle \lambda \rangle = ((n), \emptyset, \emptyset, \emptyset)$; $m_{\langle \lambda \rangle} = q+1$ if either $\langle \lambda \rangle = ((p+q, p), \emptyset, (1), \emptyset)$ or $\langle \lambda \rangle = ((p+q, p), \emptyset, \emptyset, (1))$; $m_{\langle \lambda \rangle} = 0$ in all other cases. As a direct consequence we get the desired result about the $*$ -colength sequence. \square

The above $*$ -algebras characterize the $*$ -varieties of polynomial growth.

Theorem 12. [14, Theorem 6.1] Let F be an algebraically closed field of characteristic zero. A $*$ -variety \mathcal{V} has polynomial growth if and only if $D, D^i, D^{-i}, M, M^{sup} \notin \mathcal{V}$.

As a consequence we have the following corollary.

Corollary 13. [14, Corollaries 6.1, 6.2] The $*$ -algebras $D, D^i, D^{-i}, M, M^{sup}$ are the only ones generating $*$ -varieties of almost polynomial growth. Moreover, there is no $*$ -varieties of intermediate growth between polynomial and exponential.

6. Multiplicities of the cocharacters bounded by a constant

Main goal of this section is to find conditions ensuring that the multiplicities of the $*$ -cocharacters of a given $*$ -algebra are bounded by a constant. The first subsection contains information about $*$ -varieties not containing M and M^{sup} , while in the second one we will present the main theorem of this section.

6.1. Varieties not containing the $*$ -algebras M and M^{sup} .

In the following lemma we shall deal with $*$ -varieties not containing the algebra M .

Lemma 14. *Let A be a $*$ -algebra such that $M \notin \text{var}^*(A)$. Then*

- 1) $(M_{k,k}(F), ps) \notin \text{var}^*(A)$, for any $k > 1$.
- 2) $(Q(n) \oplus Q(n)^{sop}, pex) \notin \text{var}^*(A)$, for any $n > 1$.
- 3) $(Q(n), \#_s^\pm) \notin \text{var}^*(A)$, for any $n = 2m \geq 4$.
- 4) $(Q(n), \#_t^\pm) \notin \text{var}^*(A)$, for any $n \geq 2$.
- 5) $(M_{k,0}(F), t) \notin \text{var}^*(A)$, for any $k > 1$.
- 6) $(M_{k,0}(F) \oplus M_{k,0}(F)^{sop}, pex) \notin \text{var}^*(A)$, for any $k > 1$.

Proof. Let us start by proving item 1). Suppose by contradiction that $(M_{k,k}(F), ps) \in \text{var}^*(A)$ and let

$$C = \text{span}_F \left\{ \underbrace{e_{11} + e_{k+1,k+1}}_a, \underbrace{e_{22} + e_{k+2,k+2}}_b, \underbrace{e_{12}}_c, \underbrace{e_{k+2,k+1}}_{c^*} \right\} \subseteq (M_{k,k}(F), ps)$$

be the subalgebra spanned by the above four even elements and with induced pseudoinvolution.

Now let $\{e_{11} + e_{44}, e_{22} + e_{33}, e_{12}, e_{34}\}$ be a basis of M . Then the map $\varphi : M \rightarrow C$ given by

$$\varphi(e_{11} + e_{44}) = a, \quad \varphi(e_{22} + e_{33}) = b, \quad \varphi(e_{12}) = c, \quad \varphi(e_{34}) = c^*,$$

is clearly an isomorphism of superalgebras. Moreover, for all $X \in M$, $\varphi(X^\circ) = \varphi(X)^{ps}$ and so M and C are isomorphic as $*$ -algebras. This implies $M \cong C \in \text{var}^*((M_{k,k}(F), ps)) \subseteq \text{var}^*(A)$, a contradiction.

Now, items 2) and 3) can be proved with the same proof of item 1). It is sufficient to consider, respectively:

$$C' = \text{span}_F \left\{ \underbrace{(e_{11}, e_{11})}_a, \underbrace{(e_{22}, e_{22})}_b, \underbrace{(e_{12}, 0)}_c, \underbrace{(0, e_{12})}_{c^*} \right\} \subseteq (Q(n) \oplus Q(n)^{sop}, pex)$$

and

$$C'' = \text{span}_F \left\{ \underbrace{e_{11} + e_{m+1,m+1}}_a, \underbrace{e_{22} + e_{m+2,m+2}}_b, \underbrace{e_{1,m+2}}_c, \underbrace{-e_{2,m+1}}_{c^*} \right\} \subseteq (Q(n), \#_s^\pm).$$

In order to prove item 4) we shall show that $M \in \text{var}^*(Q(n), \#_t^\pm)$. If not, we would have that $\text{Id}^*((Q(n), \#_s^\pm)) \not\subseteq \text{Id}^*(M)$ and so there would exist a non-zero multilinear polynomial f such that $f \in \text{Id}^*((Q(n), \#_s^\pm))$ and $f \notin \text{Id}^*(M)$. In order to reach a contradiction we need only to show that f is actually the zero polynomial. Since $\text{Id}^*(M) = \langle y_1^- y_2^-, z^i, z^{-i} \rangle_{T_2^*}$, then according to [23] we may assume that f is a polynomial of the type

$$f = \sum_I \alpha_{I,J} y_{i_1}^+ \cdots y_{i_k}^+ y_{j_1}^- \cdots y_{j_{n-k-1}}^+ \pmod{\text{Id}^*(M)},$$

with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$, $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{n-k-1}\}$.

Suppose that there exists $\alpha_{I,J} \neq 0$, for some I, J . Then by making the evaluation $y_{i_1}^+ = \cdots = y_{i_k}^+ = e_{11}$, $y_{j_1}^+ = \cdots = y_{j_{n-k-1}}^+ = e_{22}$ and $y^- = e_{12} - e_{21}$, one gets $\alpha_{I,J} e_{12} - \alpha_{J,I} e_{21} = 0$. Thus $\alpha_{I,J} = \alpha_{J,I} = 0$, a contradiction. Therefore we have proved that f is the zero polynomial and we are done.

The same approach used for item 4) it is sufficient to prove also the last two items. In particular, we can use the same evaluation for item 5) whereas, for item 6), we consider the evaluation $y_{i_1}^+ = \cdots = y_{i_k}^+ = (e_{11}, e_{11})$, $y_{j_1}^+ = \cdots = y_{j_{n-k-1}}^+ = (e_{22}, e_{22})$ and $y^- = (e_{12}, -e_{12})$. \square

Here we want to highlight that the $*$ -algebras $(Q(1), \#_t^+)$ and $(Q(1), \#_t^-)$ are T_2^* -equivalent to D^{-i} and D^i , respectively, whereas $(F \oplus F^{sop}, pex)$ is T_2^* -equivalent to D . Recall that given two $*$ -algebras A and B , we say that A is T_2^* -equivalent to B , and we write $A \sim_{T_2^*} B$, in case $\text{Id}^*(A) = \text{Id}^*(B)$.

Now we shall focus our attention on $*$ -varieties not containing M^{sup} .

Lemma 15. *Let A be a $*$ -algebra such that $M^{sup} \notin \text{var}^*(A)$. Then*

- 1) $(M_{k,h}(F), pt) \notin \text{var}^*(A)$, for any $k \geq h > 0$.
- 2) $(M_{k,h}(F) \oplus M_{k,h}(F)^{\text{sup}}, pex) \notin \text{var}^*(A)$, for any $k \geq h \geq 1$.

Proof. In order to prove item 1) we shall show that $M^{\text{sup}} \in \text{var}^*(M_{k,h}(F), pt)$. If not, we would have that $\text{Id}^*((M_{k,h}(F), pt)) \not\subseteq \text{Id}^*(M^{\text{sup}})$ and so there would exist a non-zero multilinear polynomial f such that $f \in \text{Id}^*((M_{k,h}(F), pt))$ and $f \notin \text{Id}^*(M^{\text{sup}})$. In order to reach a contradiction we need only to show that f is actually the zero polynomial. Since $\text{Id}^*(M^{\text{sup}}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$, then f is a polynomial of the type

$$f = \sum_I \alpha_{I,J} y_{i_1}^+ \cdots y_{i_k}^+ z y_{j_1}^+ \cdots y_{j_{n-k-1}}^+ \pmod{\text{Id}^*(M^{\text{sup}})},$$

with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$, $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{n-k-1}\}$.

Suppose that there exists $\alpha_{I,J} \neq 0$, for some I, J . Then by making the evaluation $y_{i_1}^+ = \cdots = y_{i_k}^+ = e_{11}$, $y_{j_1}^+ = \cdots = y_{j_{n-k-1}}^+ = e_{k+h, k+h}$ and $z^i = e_{1, k+h} + i e_{k+h, 1}$ (resp., $z^{-i} = e_{1, k+h} - i e_{k+h, 1}$), one gets $\alpha_{I,J} e_{1, k+h} \pm i \alpha_{J,I} e_{k+h, 1} = 0$. Thus $\alpha_{I,J} = \alpha_{J,I} = 0$, a contradiction. Therefore we have proved that f is the zero polynomial and we are done.

Now let us prove item 2). Suppose by contradiction that $B = (M_{k,h}(F) \oplus M_{k,h}(F)^{\text{sup}}, pex) \in \text{var}^*(A)$. Let $l = k + h$ and let $C = C_0 \oplus C_1$ be a subalgebra of B with

$$C_0 = \text{span}_F \left\{ \underbrace{(e_{11}, e_{11})}_a, \underbrace{(e_{ll}, e_{ll})}_b \right\} \quad \text{and} \quad C_1 = \text{span}_F \left\{ \underbrace{(e_{1l}, 0)}_c, \underbrace{(0, e_{1l})}_{c^*} \right\}.$$

The map $\varphi : M^{\text{sup}} \rightarrow C$ given by

$$\varphi(e_{11} + e_{44}) = a, \quad \varphi(e_{22} + e_{33}) = b, \quad \varphi(e_{12}) = -c, \quad \varphi(e_{34}) = c^*,$$

is clearly an isomorphism of superalgebras such that, for all $X \in M^{\text{sup}}$, $\varphi(X^{\bar{0}}) = \varphi(X)^{\text{pex}}$. Hence M^{sup} and C are isomorphic as $*$ -algebras and this implies that $M^{\text{sup}} \cong C \in \text{var}^*(B) \subseteq \text{var}^*(A)$, a contradiction. \square

Furthermore, we have the following remark.

Remark 16. Let us consider the $*$ -algebra $(M_{2l,0}(F), s)$ endowed with the symplectic (pseudo)involution s .

- $(M_{2,0}(F), s) \in \text{var}^*((M_{2l,0}(F), s))$, for any $l \geq 1$.
- $(M_{2,0}(F), s) \in \text{var}^*((Q(2), \#_s^{\pm}))$.

Proof. The first item is proved in [11, Lemma 4.8] whereas the second one is proved in [11, Lemma 4.9]. \square

Corollary 17. Let A be a finite dimensional $*$ -algebra with Wedderburn-Malcev decomposition $A = A_1 \oplus \cdots \oplus A_m + J$. If $M, M^{\text{sup}}, (M_{2,0}(F), s), (M_{1,1}(F), ps) \notin \text{var}^*(A)$, then for each $1 \leq i \leq m$, A_i is isomorphic to one of the following algebras:

1. F with trivial grading and trivial (pseudo)involution.
2. D with trivial grading and exchange (pseudo)involution ex .
3. $D^i = F(1, 1) \oplus F(1, -1)$ with pseudoinvolution ex_i .
4. $D^{-i} = F(1, 1) \oplus F(1, -1)$ with pseudoinvolution ex_{-i} .
5. $Q(1) \oplus Q(1)^{\text{sup}}$ with pseudoexchange pseudoinvolution pex .

Proof. The result follows by putting together Lemmas 14, 15 and Remark 16. \square

Finally, by Corollary 17 and by using the same approach employed, for instance, in [15, Theorem 4.3], we get the main result of this section.

Theorem 18. Let A be a finite dimensional $*$ -algebra such that $M, M^{\text{sup}}, (M_{2,0}(F), s), (M_{1,1}(F), ps) \notin \text{var}^*(A)$. Then $\text{var}^*(A) = \text{var}^*(B_1 \oplus \cdots \oplus B_m)$, where, for each $i = 1, \dots, m$, B_i is isomorphic to one of the following algebras:

1. $F + J_i$ and trivial (pseudo)involution on F ,
2. $D + J_i$ and exchange (pseudo)involution ex on D ,
3. $D^i + J_i$ and pseudoinvolution ex_i on D^i ,
4. $D^{-i} + J_i$ and pseudoinvolution ex_{-i} on D^{-i} ,
5. $Q(1) \oplus Q(1)^{\text{sup}} + J_i$ and pseudoexchange pseudoinvolution pex on $Q(1) \oplus Q(1)^{\text{sup}}$,

where J_i is the Jacobson radical of B_i .

6.2. Classifying $*$ -algebras with bounded multiplicities of the cocharacter.

We are now in a position to prove one of the main theorems of this paper. We start with the following lemma.

Lemma 19. *Let $A = C + J$ be a finite dimensional $*$ -algebra, where $J = J(A)$ is its Jacobson radical and C is a $*$ -simple subalgebra of A isomorphic to either*

1. F with trivial grading and trivial (pseudo)involution or
2. D with trivial grading and exchange (pseudo)involution or
3. $D^i = F(1, 1) \oplus F(1, -1)$ with pseudoinvolution ex_i or
4. $D^{-i} = F(1, 1) \oplus F(1, -1)$ with pseudoinvolution ex_{-i} or
5. $Q(1) \oplus Q(1)^{sup}$ with pseudoinvolution pex .

If the $\langle n \rangle$ -cocharacter of A has decomposition as in (1), then there exists a constant N such that $m_{\langle \lambda \rangle} \leq N$, for all $\langle \lambda \rangle \vdash \langle n \rangle$.

Proof. Assume first that C is isomorphic to F or D . Since they have trivial grading, they are essentially algebras with involution and the result is given in [30, Lemma 5.2]. Moreover, we can deal with the algebras D^i and D^{-i} using the same technique employed by Otera in [26, Lemma 7]. Finally, the last case, can be treated as [15, Lemma 5.1]. \square

Theorem 20. *Let A be a finitely generated superalgebra over an algebraically closed field F of characteristic zero with pseudoinvolution $*$ satisfying an ordinary polynomial identity and let its $\langle n \rangle$ -cocharacter be as in (1). If $M, M^{sup}, (M_{2,0}(F), s), (M_{1,1}(F), ps) \notin \text{var}^*(A)$ then there exists a constant N such that, for all $\langle \lambda \rangle \vdash \langle n \rangle$, we have the inequality*

$$m_{\langle \lambda \rangle} \leq N.$$

Moreover, there exists a constant q such that $m_{\langle \lambda \rangle} = 0$ whenever

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

Proof. By Theorem 2 we may assume that A is finite dimensional. Now, by using Remark 1, Theorem 18 and Lemma 19, we get the first part of the statement. The second part of the theorem follows by using the same argument of [11, Theorem 5.2]. \square

7. Multiplicities of cocharacters bounded by one

In this section we study $*$ -algebras whose multiplicities of the $*$ -cocharacter are bounded by 1. We will prove that such $*$ -algebras must satisfy a suitable list of $*$ -identities. In what follows, we denote by $[x_1, x_2] = x_1x_2 - x_2x_1$, the usual commutator among two variables, and by $x_1 \circ x_2 = x_1x_2 + x_2x_1$, the Jordan product among x_1 and x_2 .

Lemma 21. *Let A be a $*$ -algebra and let us consider its $\langle n \rangle$ -th $*$ -cocharacter*

$$\chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

If $m_{\langle \lambda \rangle} \leq 1$ for all $\langle \lambda \rangle \vdash \langle n \rangle$ and $n \geq 1$, then A satisfies at least one identity of each group:

- (i) $y_1^+ z_2^i \equiv 0$ or $y_1^+ \circ z_2^i \equiv 0$ or $[y_1^+, z_2^i] \equiv 0$,
- (ii) $y_1^+ z_2^{-i} \equiv 0$ or $y_1^+ \circ z_2^{-i} \equiv 0$ or $[y_1^+, z_2^{-i}] \equiv 0$,
- (iii) $y_1^- z_2^i \equiv 0$ or $y_1^- \circ z_2^i \equiv 0$ or $[y_1^-, z_2^i] \equiv 0$,
- (iv) $y_1^- z_2^{-i} \equiv 0$ or $y_1^- \circ z_2^{-i} \equiv 0$ or $[y_1^-, z_2^{-i}] \equiv 0$,
- (v) $z_1^i z_2^{-i} \equiv 0$ or $z_1^i \circ z_2^{-i} \equiv 0$ or $[z_1^i, z_2^{-i}] \equiv 0$,
- (vi) $y_1^+ y_2^- \equiv 0$ or $y_1^+ \circ y_2^- \equiv 0$ or $[y_1^+, y_2^-] \equiv 0$.

Proof. We shall prove that A satisfies at least one identity of the group (i), the other ones will follow with similar arguments.

Let us consider the regular representation P_2^* and its decomposition into irreducible $S_{n_1} \times \cdots \times S_{n_4}$ -submodules, where $n_1 + \cdots + n_4 = 2$. In particular let us study the submodule $M_{\langle \lambda \rangle}$ where $\langle \lambda \rangle = ((1), \emptyset, (1), \emptyset)$. Since $m_{\langle \lambda \rangle} \leq 1$, then there exist scalars α and β such that $(\alpha, \beta) \neq (0, 0)$ and

$$(2) \quad \alpha y_1^+ z_2^i + \beta z_2^i y_1^+ \equiv 0$$

on A . By applying the pseudoinvolution $*$ to the previous $*$ -identity and dividing by i , we also get

$$(3) \quad \alpha z_2^i y_1^+ + \beta y_1^+ z_2^i \equiv 0$$

on A .

If either $\alpha = 0$ or $\beta = 0$, then $y_1^+ z_2^i \equiv 0$ that is equivalent to $z_2^i y_1^+ \equiv 0$. Otherwise, summing (2) and (3) we get the *-identity

$$(\alpha + \beta)y_1^+ z_2^i + (\alpha + \beta)z_2^i y_1^+ \equiv 0.$$

Hence, if $\alpha + \beta \neq 0$, it readily follows that $y_1^+ \circ z_2^i \equiv 0$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and by substituting in (2), we get $[y_1^+, z_2^i] \equiv 0$.

Thus we have proved that A satisfies at least one identity of group (i) as claimed. \square

We need also the following technical lemma.

Lemma 22. *Let A be a *-algebra and let*

$$\chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

be its $\langle n \rangle$ -th *-cocharacter. If A satisfies at least one identity of each group:

- (i) $y_1^+ z_2^i \equiv 0$ or $y_1^+ \circ z_2^i \equiv 0$ or $[y_1^+, z_2^i] \equiv 0$,
- (ii) $y_1^+ z_2^{-i} \equiv 0$ or $y_1^+ \circ z_2^{-i} \equiv 0$ or $[y_1^+, z_2^{-i}] \equiv 0$,
- (iii) $y_1^- z_2^i \equiv 0$ or $y_1^- \circ z_2^i \equiv 0$ or $[y_1^-, z_2^i] \equiv 0$,
- (iv) $y_1^- z_2^{-i} \equiv 0$ or $y_1^- \circ z_2^{-i} \equiv 0$ or $[y_1^-, z_2^{-i}] \equiv 0$,
- (v) $z_1^i z_2^{-i} \equiv 0$ or $z_1^i \circ z_2^{-i} \equiv 0$ or $[z_1^i, z_2^{-i}] \equiv 0$,

then $m_{\langle \lambda \rangle} \leq 1$ if either $\langle \lambda \rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$ or $\langle \lambda \rangle = (\emptyset, \emptyset, \emptyset, \lambda(4))$, for all $n \geq 1$.

Proof. We will prove the statement in case $\langle \lambda \rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$. The other one will follow with similar arguments.

In order to simplify the notation, write the first group of polynomials as

$$(4) \quad \alpha y_1^+ z_2^i + \beta z_2^i y_1^+ \equiv 0,$$

where $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (1, 1)$ or $(\alpha, \beta) = (1, -1)$. Analogously, write the third group of polynomials as

$$(5) \quad \alpha' y_1^- z_2^i + \beta' z_2^i y_1^- \equiv 0,$$

where $(\alpha', \beta') = (1, 0)$ or $(\alpha', \beta') = (1, 1)$ or $(\alpha', \beta') = (1, -1)$.

Let us assume first that $(\alpha, \beta) = (1, 0)$, then $y_1^+ z_2^i \equiv 0$. Remark that $z_1^i \circ z_3^i$ is an even symmetric variable, thus we can substitute it inside the previous *-identity and get the following consequence

$$z_1^i z_3^i z_2^i + z_3^i z_1^i z_2^i \equiv 0.$$

By taking the * of the previous one and dividing by i , we get also

$$z_2^i z_3^i z_1^i + z_2^i z_1^i z_3^i \equiv 0.$$

Hence it follows that

$$P_{0,0,n,0}^*(A) = \text{span}_F \{z_1^i \cdots z_n^i\} \pmod{\text{Id}^*(A)}, \text{ if } n \geq 3,$$

$$P_{0,0,2,0}^*(A) = \text{span}_F \{z_1^i z_2^i, z_2^i z_1^i\} \pmod{\text{Id}^*(A)}.$$

If $n = 2$ we have nothing to prove. Now, if $n \geq 3$, then $\dim_F P_{0,0,n,0}^*(A) \leq 1$ and only $\chi_{(\emptyset, \emptyset, (n), \emptyset)}$ can participate in the cocharacter. Since $m_{(\emptyset, \emptyset, (n), \emptyset)} \leq \deg \chi_{(\emptyset, \emptyset, (n), \emptyset)} = 1$, we are done.

If $(\alpha', \beta') = (1, 0)$ then $y_1^- z_2^i \equiv 0$ and we can replace $[z_1^i, z_3^i]$ instead of y_1^- obtaining

$$z_1^i z_3^i z_2^i - z_3^i z_1^i z_2^i \equiv 0.$$

As in the previous case, we get the claim.

Now assume $(\alpha, \beta) = (\alpha', \beta') = (1, 1)$. Then $y_1^+ z_2^i + z_2^i y_1^+ \equiv 0$ and $y_1^- z_2^i + z_2^i y_1^- \equiv 0$. Let us replace in the first *-identity the variable y_1^+ by $z_1^i \circ z_3^i$ and in the second *-identity the variable y_1^- by $[z_1^i, z_3^i]$. If we sum such two consequences, we get that

$$z_1^i z_3^i z_2^i + z_2^i z_1^i z_3^i \equiv 0$$

on A . An easy computation shows that

$$P_{0,0,n,0}^*(A) = \text{span}_F \{z_1^i \cdots z_{n-1}^i z_n^i, z_1^i \cdots z_{n-2}^i z_n^i z_{n-1}^i\} \pmod{\text{Id}^*(A)}$$

and so $\dim_F P_{0,0,n,0}^*(A) \leq 2$. Thus only $\chi_{(\emptyset, \emptyset, (n), \emptyset)}$ and $\chi_{(\emptyset, \emptyset, (1^n), \emptyset)}$ can participate in the character of $P_{0,0,n,0}^*(A)$. Since $\deg \chi_{(\emptyset, \emptyset, (n), \emptyset)} = \deg \chi_{(\emptyset, \emptyset, (1^n), \emptyset)} = 1$, also in this case we are done.

Let now $(\alpha, \beta) = (1, 1)$ and $(\alpha', \beta') = (1, -1)$. Then, by making the same substitution of variables as before and by summing the corresponding consequences, we get the *-identity

$$(6) \quad z_1^i z_3^i z_2^i + z_2^i z_3^i z_1^i \equiv 0.$$

Notice that, using the previous *-identity, we get that $(z_1^i z_2^i z_3^i)^* = i z_3^i z_2^i z_1^i \equiv -i z_1^i z_2^i z_3^i$, thus $z_1^i z_2^i z_3^i$ is an odd i -skew variable. By taking into account the group (v) of identities, we have to consider three more possibilities.

If $z_1^i z_2^{-i} \equiv 0$, then $z_1^i \cdots z_4^i \equiv 0$ and $P_{0,0,n,0}(A) = 0$ for all $n \geq 4$. Furthermore, if $n = 2$ there is nothing to prove and if $n = 3$ then

$$P_{0,0,3,0}^*(A) = \text{span}_F\{z_1^i z_2^i z_3^i, z_1^i z_3^i z_2^i, z_2^i z_1^i z_3^i\} \pmod{\text{Id}^*(A)}.$$

Therefore, in the cocharacter decomposition only $\chi_{(\emptyset, \emptyset, (3), \emptyset)}$, $\chi_{(\emptyset, \emptyset, (1^3), \emptyset)}$ and $\chi_{(\emptyset, \emptyset, (2, 1), \emptyset)}$ can appear. Notice that the multiplicity of the latter one cannot be greater than 2, since $\deg \chi_{(\emptyset, \emptyset, (2, 1), \emptyset)} = 2$ and $\dim_F P_{0,0,3,0}^*(A) \leq 3$. Moreover, $\deg \chi_{(\emptyset, \emptyset, (3), \emptyset)} = \deg \chi_{(\emptyset, \emptyset, (1^3), \emptyset)} = 1$ and also in this case the claim is established.

Now let $[z_1^i, z_2^{-i}] \equiv 0$, then

$$(7) \quad z_1^i z_2^i z_3^i z_4^i \equiv z_4^i z_1^i z_2^i z_3^i.$$

If $n \leq 3$, the previous argument applies, so let us suppose $n \geq 4$. If in particular $n > 4$ then by the identities (6) and (7) we easily get that

$$P_{0,0,n,0}^*(A) = \text{span}_F\{z_1^i \cdots z_n^i\} \pmod{\text{Id}^*(A)},$$

and we have nothing to prove.

If $n = 4$, then by using the same *-identities,

$$P_{0,0,4,0}^*(A) = \text{span}_F\{z_1^i z_2^i z_3^i z_4^i, z_1^i z_3^i z_2^i z_4^i, z_1^i z_2^i z_4^i z_3^i\} \pmod{\text{Id}^*(A)}.$$

In the corresponding character, the irreducibles $\chi_{(\emptyset, \emptyset, (4), \emptyset)}$, $\chi_{(\emptyset, \emptyset, (1^4), \emptyset)}$, $\chi_{(\emptyset, \emptyset, (2, 2), \emptyset)}$, $\chi_{(\emptyset, \emptyset, (3, 1), \emptyset)}$ and $\chi_{(\emptyset, \emptyset, (2, 1^2), \emptyset)}$ can occur. Notice that $\deg \chi_{(\emptyset, \emptyset, (2, 2), \emptyset)} = 2$ and $\deg \chi_{(\emptyset, \emptyset, (3, 1), \emptyset)} = \deg \chi_{(\emptyset, \emptyset, (2, 1^2), \emptyset)} = 3$, thus they cannot appear with multiplicity greater than 2, since $\dim_F P_{0,0,4,0}^*(A) \leq 3$. Moreover, as before, $\deg \chi_{(\emptyset, \emptyset, (4), \emptyset)} = \deg \chi_{(\emptyset, \emptyset, (1^4), \emptyset)} = 1$ and there is nothing to prove.

Finally, if $z_1^i \circ z_2^{-i} \equiv 0$, then we get

$$z_1^i z_2^i z_3^i z_4^i \equiv -z_4^i z_1^i z_2^i z_3^i$$

and the previous arguments apply.

We are left to prove the statement when $(\alpha, \beta) = (1, -1)$ and either $(\alpha', \beta') = (1, 1)$ or $(\alpha', \beta') = (1, -1)$. However such cases are easily proven following step-by-step the proof of the cases $(\alpha, \beta) = (1, 1)$, $(\alpha', \beta') = (1, -1)$ and $(\alpha, \beta) = (\alpha', \beta') = (1, 1)$, respectively.

The proof is now complete. \square

We are now in a position to prove the main theorem of this section.

Theorem 23. *Let A be a *-algebra and let us consider its $\langle n \rangle$ -th *-cocharacter*

$$\chi_{\langle n \rangle}^*(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

Then $m_{\langle \lambda \rangle} \leq 1$ for all $\langle \lambda \rangle \vdash \langle n \rangle$ and $n \geq 1$, if and only if A satisfies at least one identity of each group:

- (i) $y_1^+ z_2^i \equiv 0$ or $y_1^+ \circ z_2^i \equiv 0$ or $[y_1^+, z_2^i] \equiv 0$,
- (ii) $y_1^+ z_2^{-i} \equiv 0$ or $y_1^+ \circ z_2^{-i} \equiv 0$ or $[y_1^+, z_2^{-i}] \equiv 0$,
- (iii) $y_1^- z_2^i \equiv 0$ or $y_1^- \circ z_2^i \equiv 0$ or $[y_1^-, z_2^i] \equiv 0$,
- (iv) $y_1^- z_2^{-i} \equiv 0$ or $y_1^- \circ z_2^{-i} \equiv 0$ or $[y_1^-, z_2^{-i}] \equiv 0$,
- (v) $z_1^i z_2^{-i} \equiv 0$ or $z_1^i \circ z_2^{-i} \equiv 0$ or $[z_1^i, z_2^{-i}] \equiv 0$,
- (vi) $y_1^+ y_2^- \equiv 0$ or $y_1^+ \circ y_2^- \equiv 0$ or $[y_1^+, y_2^-] \equiv 0$.

Proof. If $m_{\langle \lambda \rangle} \leq 1$ for all $\langle \lambda \rangle \vdash \langle n \rangle$ and $n \geq 1$, then the statement follows from Lemma 21.

Let us suppose that A satisfies at least one *-identity of each group (i)–(vi). Then by Lemma 22, $m_{\langle \lambda \rangle} \leq 1$ if either $\langle \lambda \rangle = (\emptyset, \emptyset, \lambda(3), \emptyset)$ or $\langle \lambda \rangle = (\emptyset, \emptyset, \emptyset, \lambda(4))$. Now notice that if we restrict the pseudoinvolution $*$ to A_0 , we get a subalgebra of A with involution, thus if

$$\chi_{r, n-r}(A_0) = \sum_{\substack{\lambda(1) \vdash r \\ \lambda(2) \vdash n-r}} m_{\lambda(1), \lambda(2)} \chi_{\lambda(1)} \otimes \chi_{\lambda(2)}$$

is the *-cocharacter of A_0 , then $m_{\lambda(1), \lambda(2)} = m_{\lambda(1), \lambda(2), \emptyset, \emptyset}$, for all $\lambda(1) \vdash r$ and $\lambda(2) \vdash n - r$. Since A_0 , as algebra with involution, satisfies at least one identity of the group (vi), then by [5, Theorem 3], $m_{\langle \lambda \rangle} \leq 1$ for all $\langle \lambda \rangle = (\lambda(1), \lambda(2), \emptyset, \emptyset)$.

Let now analyze irreducibles corresponding to multipartitions of the type $\langle \lambda \rangle = (\lambda(1), \emptyset, \lambda(3), \emptyset)$ where $\lambda(1) \vdash n_1 \neq 0$, $\lambda(3) \vdash n_3 \neq 0$ and $n_1 + n_3 = n$.

If $y_1^+ z_2^i \equiv 0$ then also $z_2^i y_1^+ \equiv 0$ and in this case $m_{\langle \lambda \rangle} = 0$. Hence, suppose that either $y_1^+ \circ z_2^i \equiv 0$ or $[y_1^+, z_2^i] \equiv 0$. In both cases, in a monomial of $P_{n_1, 0, n_3, 0}^*(A)$, modulo $\text{Id}^*(A)$, the even symmetric variables can be separated from the odd i -symmetric ones, so that

$$P_{n_1, 0, n_3, 0}^* \equiv \text{span}_F\{y_{i_1}^+ \cdots y_{i_{n_1}}^+ z_{j_1}^i \cdots z_{j_{n_3}}^i\}$$

or

$$P_{n_1,0,n_3,0}^* \equiv \text{span}_F \{z_{j_1}^i \cdots z_{j_{n_3}}^i y_{i_1}^+ \cdots y_{i_{n_1}}^+\}$$

modulo $\text{Id}^*(A)$. Therefore

$$m_{\langle \lambda \rangle} \leq \max\{m_{\lambda(1),\emptyset,\emptyset,\emptyset}, m_{\emptyset,\emptyset,\lambda(3),\emptyset}\} \leq 1.$$

Similar arguments hold also if $\langle \lambda \rangle = (\lambda(1), \emptyset, \emptyset, \lambda(4))$ or $\langle \lambda \rangle = (\emptyset, \lambda(2), \lambda(3), \emptyset)$ or $\langle \lambda \rangle = (\emptyset, \lambda(2), \emptyset, \lambda(4))$ or $\langle \lambda \rangle = (\emptyset, \emptyset, \lambda(3), \lambda(4))$.

Finally, let $\langle \lambda \rangle = (\lambda(1), \dots, \lambda(4))$. By using the $*$ -identities of the groups (i)–(iv), in a monomial of P_n^* , modulo $\text{Id}^*(A)$, we can separate the even variables from the odd ones, as we did before. Thus

$$m_{\langle \lambda \rangle} \leq \max\{m_{\lambda(1),\lambda(2),\emptyset,\emptyset}, m_{\emptyset,\emptyset,\lambda(3),\lambda(4)}\} \leq 1$$

and we are done. \square

The classification of T -ideals of varieties whose multiplicities of the cocharacters are bounded by one is originally motivated by the study of the distributivity of the so-called lattice of subvarieties. More precisely, if \mathcal{A} , \mathcal{B} and \mathcal{C} are subvarieties of the variety \mathcal{V} , then we say that the lattice of subvarieties of \mathcal{V} is distributive if

$$(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{C}).$$

In case of associative algebras without any additional structure, it was proved in [1] that such a lattice is distributive if and only if in the corresponding cocharacter sequence of \mathcal{V} , all the multiplicities are bounded by one. A similar result was achieved for superalgebras in [5] and, more generally, for algebras graded by a finite group in [6]. The same does not hold for algebras with involution and for superalgebras with graded involution or superinvolution: in fact, in [5] and [22], respectively, it was shown that in these cases there exist varieties with a distributive lattice of the subvarieties but whose multiplicities are equal to 2, for some suitable multipartitions.

In the setting of $*$ -varieties, it is clear that an analogous result to that of algebras with involution holds, since as we already saw in the proof of Theorem 23, if we restrict the pseudoinvolution to the even part of the superalgebra, we get an algebra with involution. Here we present the example given in [5, Section 4] in the language of $*$ -algebras for convenience of the reader.

Example. Let \mathcal{V} be the $*$ -variety such that $\text{Id}^*(\mathcal{V}) = \langle y_1^+ y_2^+, y_1^- y_2^-, x_1 x_2 x_3, z^i, z^{-i} \rangle_{T_2^*}$, where x_1, x_2, x_3 are any kind of variables. It is easy to check that the $*$ -cocharacter of \mathcal{V} is the following

$$\begin{aligned} \chi_{\langle 1 \rangle}^*(\mathcal{V}) &= \chi_{((1),\emptyset,\emptyset,\emptyset)} + \chi_{(\emptyset,(1),\emptyset,\emptyset)}, \\ \chi_{\langle 2 \rangle}^*(\mathcal{V}) &= 2\chi_{((1),(1),\emptyset,\emptyset)}, \\ \chi_{\langle n \rangle}^*(\mathcal{V}) &= 0, \text{ for all } n \geq 3. \end{aligned}$$

If one sets $I_1 = \langle y_1^+ \circ y_1^-, z^i, z^{-i} \rangle_{T_2^*}$, $I_2 = \langle [y_1^+, y_1^-], z^i, z^{-i} \rangle_{T_2^*}$, $I_3 = \langle x_1 x_2, z^i, z^{-i} \rangle_{T_2^*}$, $I_4 = \langle y_1^+, z^i, z^{-i} \rangle_{T_2^*}$, $I_5 = \langle y_1^-, z^i, z^{-i} \rangle_{T_2^*}$ and $I_6 = \langle x_1, z^i, z^{-i} \rangle_{T_2^*}$, then the $*$ -subvarieties of \mathcal{V} correspond to such T_2^* -ideals and they form a distributive lattice.

8. Colengths bounded by three

The purpose of this section is to give a classification of $*$ -varieties having colength sequence bounded by 3. To this end, we first recall the definition of elementary grading on a matrix algebra $M_n(F)$. Let (g_1, \dots, g_n) be a n -tuple of elements of \mathbb{Z}_2 , then the elementary grading induced by (g_1, \dots, g_n) is the \mathbb{Z}_2 -grading $M_n(F) = A_0 \oplus A_1$ where

$$\begin{aligned} A_0 &= \text{span}_F \{e_{ij} \mid g_i + g_j = 0\} \\ A_1 &= \text{span}_F \{e_{ij} \mid g_i + g_j = 1\}, \end{aligned}$$

where the e_{ij} 's denote the usual matrix units.

8.1. Some suitable $*$ -algebras.

First we shall introduce some suitable $*$ -algebras that will help us prove the main result of this part of the paper. In Corollary 13 we have seen that D , D^i , D^{-i} , M and M^{sup} are the only $*$ -algebras generating $*$ -varieties of almost polynomial growth. We start by recalling the definitions of some $*$ -algebras belonging to these varieties (see [14] for more details).

For any $k \geq 2$, let I_k be the $k \times k$ identity matrix and set $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$. We consider the following commutative subalgebra of the upper-triangular matrix algebra $UT_k(F)$:

$$C_k = \left\{ \alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F \right\}.$$

We shall denote by $C_{k,*}$ the algebra C_k with trivial grading and (pseudo)involution given by

$$\left(\alpha I_k + \sum_{1 \leq j < k} \alpha_j E_1^j \right)^* = \alpha I_k + \sum_{1 \leq j < k} (-1)^j \alpha_j E_1^j.$$

Instead, we shall use the symbols C_k^i and C_k^{-i} to denote the algebra C_k with elementary grading induced by $(0, 1, 0, 1, \dots) \in \mathbb{Z}_2^k$ and pseudoinvolution given by

$$\left(\alpha I_k + \sum_{1 \leq j < k} \alpha_j E_1^j \right)^* = \alpha I_k + \sum_{1 \leq j < k} (\pm i)^{f(j)} (-1)^j \alpha_j E_1^j,$$

where $f(j) = 0$ if j is even and $f(j) = 1$ if j is odd.

Next we exhibit finite dimensional $*$ -algebras belonging to the $*$ -variety generated by M or M^{sup} . Let

$$A_2 = \text{span}_F \{e_{11} + e_{44}, e_{12}, e_{34}\} \subseteq UT_4(F),$$

$$N_3 = \text{span}_F \{I, e_{23} + e_{45}, e_{12} - e_{56}, e_{13}, e_{46}\} \subseteq UT_6(F),$$

$$U_3 = \text{span}_F \{I, e_{23} + e_{45}, e_{12} + e_{56}, e_{13}, e_{46}\} \subseteq UT_6(F)$$

and consider the following $*$ -algebras:

- $A_{2,*}$, the algebra A_2 with trivial grading and reflection (pseudo)involution \circ ,
- $N_{3,*}$, the algebra N_3 with trivial grading and reflection (pseudo)involution \circ ,
- $U_{3,*}$, the algebra U_3 with trivial grading and reflection (pseudo)involution \circ ,
- A_2^{sup} , the algebra A_2 with elementary grading induced by $(0, 1, 0, 1)$ and pseudoinvolution $\bar{\circ}$,
- N_3^{sup} (U_3^{sup} , resp.), the algebra N_3 (U_3 , resp.) with elementary grading induced by $(0, 1, 1, 0, 0, 1)$ and pseudoinvolution $\bar{\circ}$ defined by

$$\bar{\circ}(I_6) = I_6, \quad \bar{\circ}(e_{23} + e_{45}) = e_{23} + e_{45}, \quad \bar{\circ}(e_{12} \pm e_{56}) = i(e_{12} \pm e_{56}), \quad \bar{\circ}(e_{13}) = -ie_{46}, \quad \bar{\circ}(e_{46}) = -ie_{13}.$$

The above $*$ -algebras have been extensively studied in [13, 14, 24, 25]. In the following proposition we collect the results concerning their $*$ -colength sequences.

Proposition 24. *For n large enough, we have that*

- $l_n^*(C_{k,*}) = l_n^*(C_k^i) = l_n^*(C_k^{-i}) = k$.
- $l_n^*(N_{3,*}) = l_n^*(U_{3,*}) = 4$.
- $l_n^*(A_{2,*}) = l_n^*(A_2^{sup}) = l_n^*(N_3^{sup}) = l_n^*(U_3^{sup}) = 5$.

Now, we shall consider the direct sum of some of the above $*$ -algebras.

Proposition 25. *For n large enough, we have that*

- $l_n^*(C_{2,*} \oplus C_2^i) = l_n^*(C_{2,*} \oplus C_2^{-i}) = l_n^*(C_2^i \oplus C_2^{-i}) = 3$,
- the $*$ -algebras $C_{2,*} \oplus C_3^i$, $C_{2,*} \oplus C_3^{-i}$, $C_{3,*} \oplus C_2^i$, $C_{3,*} \oplus C_2^{-i}$, $C_2^i \oplus C_3^{-i}$, $C_3^i \oplus C_2^{-i}$, $C_{2,*} \oplus C_2^i$, $\oplus C_2^{-i}$ have $*$ -colength sequence greater than 3.

In the final part of this subsection we shall focus our attention on finite dimensional Grassmann algebras endowed with some suitable pseudoinvolution. Recall that the Grassmann algebra with 1 on a k -dimensional vector space over F , $k \geq 1$, is the algebra

$$G_k = \langle 1, e_1, \dots, e_k \mid e_i e_j = -e_j e_i \rangle.$$

Consider the following map defined on G_k :

- $\tau: e_j \mapsto -(i)^{|e_j|} e_j$,
- $\psi: e_j \mapsto (i)^{|e_j|} e_j$,
- $\rho: e_j \mapsto (-1)^j (i)^{|e_j|} e_j$.

Now we are ready to introduce the following $*$ -algebras:

- $G_{2,\tau}$, the Grassmann algebra G_2 with trivial grading and pseudoinvolution τ ,
- $G_{3,\tau}$, the Grassmann algebra G_3 with trivial grading and pseudoinvolution τ ,
- $G_{2,\tau}^{gr}$, the Grassmann algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and pseudoinvolution τ ,
- $G_{2,\psi}^{gr}$, the Grassmann algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and pseudoinvolution ψ ,
- $G_{2,\rho}^{gr}$, the Grassmann algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and pseudoinvolution ρ ,
- $G_{2,\tau}^{gr,i}$, the Grassmann algebra G_2 with grading $(F1 + Fe_1, Fe_2 + Fe_1e_2)$ and pseudoinvolution τ ,

- $G_{2,\rho}^{gr_i}$, the Grassmann algebra G_2 with grading $(F1 + Fe_1, Fe_2 + Fe_1e_2)$ and pseudoinvolution ρ .

The following result is easily proved.

Proposition 26. *For n large enough, we have that*

- $l_n^*(G_{2,\tau}) = 3$,
- $l_n^*(A) > 3$, for any $A \in \{G_{3,\tau}, G_{2,\tau}^{gr}, G_{2,\psi}^{gr}, G_{2,\rho}^{gr}, G_{2,\tau}^{gr_i}, G_{2,\rho}^{gr_i}, G_{2,\tau} \oplus C_{3,*}, G_{2,\tau} \oplus C_2^i, G_{2,\tau} \oplus C_2^{-i}\}$.

8.2. Preliminary lemmas.

In this part, we shall present several results that will help us prove the main theorem of this section. In Section 5 we have introduced the $*$ -algebras D , D^i and D^{-i} . We start by considering their direct sum

$$S = D \oplus D^i \oplus D^{-i}.$$

Lemma 27. $Id^*(S) = \langle [x_1, x_2], y^- z^i, y^- z^{-i}, z^i z^{-i} \rangle_{T_2^*}$.

Proof. Let $I = \langle [x_1, x_2], y^- z^i, y^- z^{-i}, z^i z^{-i} \rangle_{T_2^*}$. Since $Id^*(S) = Id^*(D) \cap Id^*(D^i) \cap Id^*(D^{-i})$ it is easy to check that $I \subseteq Id^*(S)$. In order to prove the opposite inclusion, let f be a multilinear $*$ -identity of S of degree n . After reducing f modulo I , we obtain that either $f = \alpha y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^-$ or $f = \alpha y_1^+ \cdots y_{n_1}^+ z_1^i \cdots z_{n_2}^i$ or $f = \alpha y_1^+ \cdots y_{n_1}^+ z_1^{-i} \cdots z_{n_2}^{-i}$, where $n_1 + n_2 = n$ and $n_1, n_2 \geq 0$. In the first case, let us consider the evaluation $y_k^+ = ((1, 1), 0, 0)$, $1 \leq k \leq n_1$, and $y_j^- = ((1, -1), 0, 0)$, $1 \leq j \leq n_2$. We obtain that $f = \alpha ((1, -1)^n, 0, 0) = 0$ and so $\alpha = 0$. In all the other cases, by considering the evaluations $z_j^i = (0, 0, (1, -1))$ or $z_j^{-i} = (0, (1, -1), 0)$, respectively, $j = 1, \dots, n$, we get $\alpha = 0$. Thus f modulo I is the zero polynomial, i.e. $f \in I$ and the proof is complete. \square

Lemma 28. *Let $A = F + J$ be a finite dimensional $*$ -algebra and let $J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$.*

1. If $A_{2,*} \notin var^*(A)$, then $(J_{10})_0 = (J_{01})_0 = 0$.
2. If $A_2^{sup} \notin var^*(A)$, then $(J_{10})_1 = (J_{01})_1 = 0$.

Proof. The first item is proved in [25, Lemma 8.1]. With a similar approach we get also the second item. \square

Lemma 29. *Let $B = F + J_{11}$ be a finite dimensional $*$ -algebra.*

1. If $C_{k,*} \notin var^*(A)$, $k \geq 2$, then $(y^-)^{k-1} \equiv 0$ on B .
2. If $C_k^i \notin var^*(A)$, $k \geq 2$, then $(z^i)^{k-1} \equiv 0$ on B .
3. If $C_k^{-i} \notin var^*(A)$, $k \geq 2$, then $(z^{-i})^{k-1} \equiv 0$ on B .

Proof. The first item is proved in [25, Lemma 8.2]. With a similar approach we get also the last two items. \square

Lemma 30. *Let $B = F + J_{11}$ be a finite dimensional $*$ -algebra.*

1. If $[y_1^+, y_2^+] \neq 0$ on B then $U_{3,*} \in var^*(B)$.
2. If $[y^+, y^-] \neq 0$ on B then $N_{3,*} \in var^*(B)$.
3. If $[y^+, z^i] \neq 0$ on B then $U_3^{sup} \in var^*(B)$.
4. If $[y^+, z^{-i}] \neq 0$ on B then $N_3^{sup} \in var^*(B)$.

Proof. The first two items are proved in [25, Lemma 8.3]. Similarly we can deal with the last two items. \square

In the following lemmas we shall focus our attention on the Grassmann algebras we have introduced in the previous subsection.

Lemma 31. *Let $B = F + J_{11}$ be a finite dimensional $*$ -algebra.*

1. If $[y_1^-, y_2^-] \neq 0$ on B then $G_{2,\tau} \in var^*(B)$.
2. If $y_1^- \circ y_2^- \equiv 0$ on B and $y_1^- y_2^- y_3^- \neq 0$ on B then $G_{3,\tau} \in var^*(B)$.

Proof. The first item is proved in [25, Lemma 8.4] whereas the proof of the second one can be found in [20, Lemma 4.8]. \square

Lemma 32. *Let $B = F + J_{11}$ be a finite dimensional $*$ -algebra.*

1. If $z^i z^{-i} \neq 0$ on B then $G_{2,\rho}^{gr} \in var^*(B)$.
2. If $[z_1^i, z_2^i] \neq 0$ on B then $G_{2,\psi}^{gr} \in var^*(B)$.
3. If $[z_1^{-i}, z_2^{-i}] \neq 0$ on B then $G_{2,\tau}^{gr} \in var^*(B)$.
4. If $y^- z^{-i} \neq 0$ on B then $G_{2,\tau}^{gr_i} \in var^*(B)$.

5. If $y^- z^i \neq 0$ on B then $G_{2,\rho}^{gr_i} \in \text{var}^*(B)$.

Proof. Let us prove the first item (the other ones follow analogously).

Since $z^i z^{-i} \neq 0$ on B , then there exist two non-zero elements $a \in (J_{11})_1^i$ and $b \in (J_{11})_1^{-i}$ such that $ab \neq 0$. We consider R to be the subalgebra of B generated by $1, a, b$ over F and let I be the T_2^* -ideal generated by $a^2, b^2, ab + ba$. Hence, if we write $\bar{1} = 1 + I, \bar{a} = a + I, \bar{b} = b + I$ and $\bar{a}\bar{b} = ab + I$, then the $*$ -algebra $\bar{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$. It is easy to check that the linear map $\varphi : \bar{R} \rightarrow G_{2,\rho}^{gr}$ such that $\varphi(\bar{1}) = 1, \varphi(\bar{a}) = e_1$ and $\varphi(\bar{b}) = e_2$, is an isomorphism of $*$ -algebras. It follows that $G_{2,\rho}^{gr} \in \text{var}^*(B)$ and the proof is complete. \square

In order to prove the main result of this section we need just one more preliminary lemma.

Lemma 33. *Let $B = F + J_{11}$ be a $*$ -algebra such that $B \in \text{var}^*(S)$, where $S = D \oplus D^i \oplus D^{-i}$. Then*

$$B \sim_{T_2^*} V_1 \oplus V_2 \oplus V_3,$$

where $V_1 \in \text{var}^*(D), V_2 \in \text{var}^*(D^{-i})$ and $V_3 \in \text{var}^*(D^i)$.

Proof. Let us consider $V_1 = F + (J_{11})_0, V_2 = F + (J_{11})_0^+ + (J_{11})_1^i$ and $V_3 = F + (J_{11})_0^+ + (J_{11})_1^{-i}$. By using the identities of the algebra S (see Lemma 27), it is easy to see that V_1, V_2 and V_3 are $*$ -subalgebras of B . Hence

$$\text{Id}^*(B) \subseteq \text{Id}^*(V_1 \oplus V_2 \oplus V_3).$$

In order to prove the opposite inclusion, let $f \in \text{Id}^*(V_1 \oplus V_2 \oplus V_3)$ be a multilinear $*$ -polynomial of degree n . Since $\text{Id}^*(S) \subseteq \text{Id}^*(B)$, we can write f , modulo the $*$ -identities of B , as either

$$\begin{aligned} f &= \alpha y_1^+ \cdots y_n^+ \text{ or} \\ f &= \beta y_{i_1}^+ \cdots y_{i_p}^+ y_{j_1}^- \cdots y_{j_q}^- \text{ or} \\ f &= \gamma y_{i_1}^+ \cdots y_{i_p}^+ z_{j_1}^i \cdots z_{j_q}^i \text{ or} \\ f &= \delta y_{i_1}^+ \cdots y_{i_p}^+ z_{j_1}^{-i} \cdots z_{j_q}^{-i}, \end{aligned}$$

where $p + q = n$ and $i_1 < \cdots < i_p, j_1 < \cdots < j_q$.

If f is of the first type, by making the evaluation $y_i^+ = (1_F, 1_F, 1_F)$, for all $i = 1, \dots, n$, we get $\alpha = 0$. If f is of one of the other types, we also get that $f \in \text{Id}^*(B)$, since $f \in \text{Id}^*(V_1 \oplus V_2 \oplus V_3) = \text{Id}^*(V_1) \cap \text{Id}^*(V_2) \cap \text{Id}^*(V_3)$. Hence we have proved that $\text{Id}^*(B) = \text{Id}^*(V_1 \oplus V_2 \oplus V_3)$.

The proof is now complete since clearly $V_1 \in \text{var}^*(D_*), V_2 \in \text{var}^*(D^{-i})$ and $V_3 \in \text{var}^*(D^i)$. \square

8.3. Classification of the $*$ -varieties with colengths bounded by three.

Finally we are ready to prove the main result of this section. We shall denote by N a nilpotent $*$ -algebra and by C a commutative non-nilpotent algebra with trivial grading and trivial pseudoinvolution.

Theorem 34. *Let A be a $*$ -algebra over an algebraically closed field F of characteristic zero. The following conditions are equivalent.*

1. $l_n^*(A) \leq 3$.

2. The following $*$ -algebras do not belong to the $*$ -variety generated by A :

$$C_{4,*}, C_4^i, C_4^{-i}, A_{2,*}, N_{3,*}, U_{3,*}, A_2^{sup}, N_3^{sup}, U_3^{sup}, G_{3,\tau}, G_{2,\psi}^{gr}, G_{2,\tau}^{gr}, G_{2,\rho}^{gr}, G_{2,\tau}^{gr_i}, G_{2,\rho}^{gr_i}, \\ C_{2,*} \oplus C_3^i, C_{2,*} \oplus C_3^{-i}, C_{3,*} \oplus C_2^i, C_{3,*} \oplus C_2^{-i}, C_2^i \oplus C_3^{-i}, C_3^i \oplus C_2^{-i}, C_{2,*} \oplus C_2^i \oplus C_2^{-i}, G_{2,\tau} \oplus C_{3,*}, G_{2,\tau} \oplus C_2^i, G_{2,\tau} \oplus C_2^{-i}.$$

3. The $*$ -algebra A is T_2^* -equivalent to N or to $B \oplus N$, where B is one of the following algebras:

$$C, C_{2,*}, C_{3,*}, C_2^i, C_3^i, C_2^{-i}, C_3^{-i}, G_{2,\tau}, C_{2,*} \oplus C_2^i, C_{2,*} \oplus C_2^{-i}, C_2^i \oplus C_2^{-i}.$$

Proof. By using item 1 of Remark 1 and Propositions 24, 25, 26, one can easily prove that 3. implies 1. and that 1. implies 2.

Now, let us assume 3. Since $C_{4,*}, C_4^i, C_4^{-i}, A_{2,*}, A_2^{sup} \notin \text{var}^*(A)$ and they belong to $\text{var}^*(D), \text{var}^*(D^i), \text{var}^*(D^{-i}), \text{var}^*(M)$ and $\text{var}^*(M^{sup})$, respectively, it follows that $D, D^i, D^{-i}, M, M^{sup} \notin \text{var}^*(A)$. Hence, by Theorem 12, the $*$ -codimensions of A are polynomially bounded. Moreover, by [14, Theorem 4.1], since $D \notin \text{var}^*(A)$, we may assume that A is a finite dimensional algebra. As a consequence, it is not difficult to see that A can be decomposed as

$$A = B_1 \oplus \cdots \oplus B_m,$$

where either B_i is nilpotent or $B_i = F + J(B_i), 1 \leq i \leq m$.

Notice that if B_i is nilpotent for all i , then A is nilpotent and we are done in this case. Therefore, we may assume that there exists i such that $B_i = F + J(B_i)$, with $J(B_i) = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$. Since

$A_{2,*}, A_2^{sup} \notin \text{var}^*(A)$, by Lemma 28, it follows that $J_{10} = J_{01} = 0$ and so $B_i = (F + J_{11}) \oplus J_{00}$ is a direct sum of $*$ -algebras, where J_{00} is nilpotent. In order to complete the proof we may focus our attention on $*$ -algebras of the type $B = F + J_{11}$.

Since $U_{3,*}, N_{3,*}, U_3^{sup}, N_3^{sup} \notin \text{var}^*(B)$, by Lemma 30, it follows that the $*$ -polynomials $[y_1^+, y_2^+]$, $[y^+, y^-]$, $[y^+, z^i]$, $[y^+, z^{-i}]$ are $*$ -identities of B . Similarly, since $G_{2,\rho}^{gr}, G_{2,\psi}^{gr}, G_{2,\tau}^{gr}, G_{2,\tau}^{gr}, G_{2,\rho}^{gr} \notin \text{var}^*(B)$, by Lemma 32, we get that $z^i z^{-i}$, $[z_1^i, z_2^i]$, $[z_1^{-i}, z_2^{-i}]$, $y^- z^{-i}$, $y^- z^i \equiv 0$ on B .

First we shall suppose that $[y_1^-, y_2^-]$ is a $*$ -identity of B . In this case, we have that $[x_1, x_2], y^- z^i, y^- z^{-i}, z^i z^{-i}$ are $*$ -identities of B . It follows that $B \in \text{var}^*(S)$ (see Lemma 27). By Lemma 33, we get that $B \sim_{T_2^*} V_1 \oplus V_2 \oplus V_3$, where $V_1 \in \text{var}^*(D)$, $V_2 \in \text{var}^*(D^{-i})$ and $V_3 \in \text{var}^*(D^i)$. Moreover, since B has polynomial growth, we must have that also V_1, V_2 and V_3 have polynomial growth. In conclusion, since

$$C_{4,*}, C_4^i, C_4^{-i}, C_{2,*} \oplus C_3^i, C_{2,*} \oplus C_3^{-i}, C_{3,*} \oplus C_2^i, C_{3,*} \oplus C_2^{-i}, C_2 \oplus C_3^i, C_3 \oplus C_2^{-i}, C_{2,*} \oplus C_2^i \oplus C_2^{-i} \notin \text{var}^*(B),$$

as a consequence of the results in [14, Sections 7,9], we get that B is T_2^* -equivalent to one of the following algebras:

$$C, \quad C_{2,*}, \quad C_{3,*}, \quad C_2^i, \quad C_3^i, \quad C_2^{-i}, \quad C_3^{-i}, \quad C_{2,*} \oplus C_2^i, \quad C_{2,*} \oplus C_2^{-i}, \quad C_2^i \oplus C_2^{-i}.$$

Now let $[y_1^-, y_2^-] \neq 0$ on B . Then, by Lemma 31, $G_{2,\tau} \in \text{var}^*(B)$. We claim that actually $B \sim_{T_2^*} G_{2,\tau}$. Notice that, since $G_{2,\tau} \oplus C_{3,*}, G_{2,\tau} \oplus C_2^i, G_{2,\tau} \oplus C_2^{-i} \notin \text{var}^*(B)$, then we must have that $C_{3,*}, C_2^i, C_2^{-i} \notin \text{var}^*(B)$. As a consequence, by Lemma 29, it follows that $(y^-)^2 \equiv z^i \equiv z^{-i} \equiv 0$ on B . After linearizing $(y^-)^2 \equiv 0$, we get $y_1^- y_2^- + y_2^- y_1^- \equiv 0$ on B . Finally, since $G_{3,\tau} \notin \text{var}^*(B)$, by Lemma 31, we obtain that $y_1^- y_2^- y_3^- \equiv 0$ on B . In conclusion, recalling that (see [25, Lemma 7.1])

$$\text{Id}^*(G_{2,\tau}) = \langle [y^+, x], y_1^- y_2^- + y_2^- y_1^-, y_1^- y_2^- y_3^-, z^i, z^{-i} \rangle_{T_2^*}$$

we get that $\text{Id}^*(G_{2,\tau}) \subseteq \text{Id}^*(B)$ and the claim is proved.

For each case above, since $A = B_1 \oplus \dots \oplus B_m$, we get the desired conclusion. \square

9. Relating polynomial codimension growth and colengths

As a last result of the paper, we want to establish a relation among the $*$ -codimension and the $*$ -colength sequences. The following lemma, concerning the $*$ -algebra $D = F \oplus F$, will be useful in what follows.

Lemma 35. *The $*$ -algebra D belongs to the $*$ -variety generated by $(M_{2,0}(F), s)$ and to the $*$ -variety generated by $(M_{1,1}(F), ps)$.*

Proof. Let us prove the result for $(M_{1,1}(F), ps)$. The case of $(M_{2,0}(F), s)$ is given in [11, Lemma 4.10].

Let $D' = \text{span}_F \{e_{11} + e_{22}, e_{11} - e_{22}\}$ be a subalgebra of $(M_{1,1}(F), ps)$ spanned by two elements of homogeneous degree zero and with induced pseudoinvolution. Now let $\{(1, 1), (1, -1)\}$ be a basis of the $*$ -algebra D . The linear map $\varphi : D \rightarrow D'$ given by

$$\varphi((1, 1)) = e_{11} + e_{22}, \quad \varphi((1, -1)) = e_{11} - e_{22},$$

is clearly an isomorphism of superalgebras. Moreover,

$$\varphi((1, 1)^{ex}) = \varphi(1, 1) = e_{11} + e_{22} = (\varphi(1, 1))^{ps},$$

$$\varphi((1, -1)^{ex}) = \varphi(-1, 1) = -e_{11} + e_{22} = (\varphi(1, -1))^{ps},$$

and so φ is an isomorphism of $*$ -algebras and the proof is complete. \square

We conclude this paper with the following result, relating the growth of the $*$ -codimension sequence of a finite dimensional $*$ -algebra A with its $*$ -colengths. Recall that, for all $n \geq 1$ and for all n_1, \dots, n_4 such that $n_1 + \dots + n_4 = n$, the n -th $*$ -colength is defined as

$$l_n^*(A) = \sum_{\substack{\langle \lambda \rangle \vdash \langle n \rangle \\ n_1 + \dots + n_4 = n}} m_{\langle \lambda \rangle},$$

where $m_{\langle \lambda \rangle}$ are the multiplicities of the irreducibles appearing in the (n_1, \dots, n_4) -cocharacter of A .

We want to highlight that a similar result concerning algebras with ordinary polynomial identities was proved by Kemer in [17], by Vieira in [29] in the setting of superalgebras, in [20] for algebras with involution, in [11] for superalgebras with graded involution and in [15] for superalgebras with superinvolution.

Theorem 36. *Let A be a finite dimensional algebra with pseudoinvolution $*$ over an algebraically closed field of characteristic zero. Then $c_n^*(A)$ is polynomially bounded if and only if $l_n^*(A) \leq h$, for some constant h and for all $n \geq 1$.*

Proof. First let us assume that $l_n^*(A) \leq h$, for some h . By using Remark 1 and Propositions 8, 10 and 11, we get that $D, D^i, D^{-i}, M, M^{sup} \notin \text{var}^*(A)$. Thus by Theorem 12, $c_n^*(A)$ must be polynomially bounded and this direction is proved.

Conversely, let us suppose that $c_n^*(A)$ is polynomially bounded, say $c_n^*(A) \approx an^k$, for some integers k and $a > 0$. Consider the $\langle n \rangle$ -cocharacter of A :

$$\chi_{\langle n \rangle}(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

By Theorem 12, it follows that $D, D^i, D^{-i}, M, M^{sup} \notin \text{var}^*(A)$. Moreover, by Lemma 35, we have that also $(M_{2,0}(F), s), (M_{1,1}(F), s) \notin \text{var}^*(A)$. Hence Theorem 20 applies and so there exists a constant M such that $m_{\langle \lambda \rangle} \leq M$, for all multipartition $\langle \lambda \rangle$. Furthermore, by Theorem 20, there exists q such that $m_{\langle \lambda \rangle} = 0$ whenever

$$(8) \quad (|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

On the other hand, since $D \notin \text{var}^*(A)$, by Lemma 5.1 and Theorem 2.1 of [4], there exists $s \geq 1$ such that

$$(9) \quad y_1^- w_1 y_2^- w_2 \cdots y_s^- w_s \equiv 0 \text{ on } A,$$

where the w_i 's are (eventually empty) words in even variables (symmetric or skew). Moreover, since $c_n^*(A) \approx an^k$, we get that $z_1 z_2 \cdots z_{k+1} \equiv 0$ on A , where the z_i 's are any odd variables (the result follows using the same idea of [18, Theorem 5.1]). Clearly we obtain that

$$(10) \quad z_1 w_1 z_2 w_2 \cdots z_{k+1} w_{k+1} \equiv 0 \text{ on } A,$$

where the w_i 's are (eventually empty) words in any kind of variables. From (8), (9) and (10), we get that $m_{\langle \lambda \rangle} = 0$, for any multipartition $\langle \lambda \rangle$ such that $|\lambda(1)| - \lambda(1)_1 \geq q$ or $|\lambda(2)| \geq sk$ or $|\lambda(3)| + |\lambda(4)| \geq k + 1$. Thus $m_{\langle \lambda \rangle} = 0$, $\langle \lambda \rangle \vdash \langle n \rangle$, for $n = n_1 + \cdots + n_4$ large enough. Hence only a finite number of multipartitions $\langle \lambda \rangle$ satisfies the conditions $|\lambda(1)| - \lambda(1)_1 < q$, $|\lambda(2)| < sk$ and $|\lambda(3)| + |\lambda(4)| < k + 1$ and since $m_{\langle \lambda \rangle} \leq M$, for all $\langle \lambda \rangle$, it follows that, for any $\langle n \rangle$,

$$l_n^*(A) \leq h, \text{ for some constant } h.$$

□

Remark that, due to Theorem 2, the previous result holds also in the case of finitely generated $*$ -algebras.

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