

# Strongly super-Poisson statistics replaced by a wide-pulse Poisson process: the billiard random generator

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## Abstract

In this paper we present a study on random processes consisting of delta pulses characterized by strongly super-Poisson statistics and calculate its spectral density. We suggest a method for replacing a strongly super-Poisson process with a wide-pulse Poisson process, while demonstrating that these two processes can be set in such a way to have similar spectral densities, the same mean values, and the same correlation times. We also present a billiard system that can be used to generate random pulse noise of arbitrary statistical properties. The particle dynamics is considered in terms of delta and wide pulses simultaneously. The results of numerical experiments with the billiard system are in a good agreement with the analytical findings.

*Key words:* renewal process, fluctuation phenomena, stochastic processes, billiard-like systems, super-Poisson statistics, hardware random number generator

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## 1. Introduction

Random fluctuations represent a main topic and a non-negligible ingredient when modeling the dynamics of both physical and biological systems, ranging from Josephson junctions [1, 2, 3] bioinformatics [4, 5], to natural systems [6, 7, 8, 9, 10], specifically in population dynamics [11, 12, 13, 14], infective disease and epidemics [15, 16, 17], molecular motors [18, 19, 20, 21], bacterial growth in food products [22], inception and development of diseases due to genetic mutations [23, 24, 25].

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<sup>\*</sup>Fully documented templates are available in the elsarticle package on CTAN.

The renewal pulse process consists of points (or delta-pulses) randomly located on the time axes with independent identically distributed waiting times (WTs) between neighboring pulses. The WT probability density function (PDF) is the only basic property needed to define the process. The renewal process describes a sequence of recurrent events, whose effect is to reset to zero the system's memory [26, 27, 28, 29, 30, 31, 32]. A positive correlation between the neighboring pulses indicates that the presence of a pulse at a certain time instant increases the probability that another pulse appears during the immediately successive time interval. In this case, the pulse train appears as a sequence of clusters, each one consisting of several pulses close to each other, with different clusters being widely distanced from each other. Therefore, the variance of WTs is larger than for a Poisson process with the same average WT. This properties characterize the super-Poisson statistics [33, 34]. They are also related to Hawkes (or self-exciting) processes [35].

In this paper we study the strongly super-Poisson process (SSPP), which can be used to simulate the registration by a counter (or detector) of interacting particles [36, 37] or photons of squeezed states of light [38], signals in a neuron network [39], networks of queues [40], and to investigate dynamics of a termite population [41], persistency and/or intermittent structures in different time scales of seismogenesis [42], presence of crucial events in music [43]. In epidemiology, the super-Poisson process well describes the statistics of appearance of new infection cases (people becoming ill) in the presence of highly infectious diseases such as COVID-19 [41, 44]. Since they are characterized by a larger variance, super-Poisson processes are more complex than sub-Poisson processes. Reliable analysis requires more statistics, which implies a much larger number of experiments and longer numerical simulations. Therefore, super-Poisson processes are less studied and a simplified model of such processes can be very useful.

In the case of a SSPPs the pulses are grouped in such a way to form clusters, that is as packages consisting of many pulses very close to each other. These clusters can be easily distinguished since the distances between them are quite large in comparison with their size. We demonstrate that a SSPP can be replaced with a Poisson cluster process [45]. This means that the times at which clusters appear are distributed according to a Poisson statistics. We analyse the possibility to replace a SSPP of single delta-pulses with the corresponding Poisson sequence of wide pulses, where each wide pulse corresponds to a cluster. This idea was suggested in Ref. [41], where each pulse in a Poisson process was modeled as a delta-pulse with a large amplitude.

In this paper, we consider a more general model of a wide-pulse Poisson process (WPPP) in such a way that the spectral density of the WPPP resembles that of the corresponding SSPP.

We also present a billiard system as an example of generator of random pulses with arbitrary statistical properties [46]. This system is characterized by the presence of one-particle or more non-interacting free point particles undergoing elastic collisions with boundaries. For dispersing Sinai billiards the following strong chaotic properties hold: a) central limit theorem; b) exponential decay

of correlations; c) ergodicity, in the presence of mixing, of the particle flow [47, 48].

We devise a Sinai-like billiard that consists of a set of strait lines as boundaries and a circle within these boundaries as a scatterer. We consider collisions of the particle with some suitably defined and marked sections of the boundary. Each collision corresponds to a delta-pulse. In the following we will show that it is possible to generate sub- and super-Poisson processes with different correlation properties by changing the shape of the billiard and the marked boundaries. The time intervals between successive collisions of the particle correspond to WTs of the pulse process. Different PDFs of WTs can be obtained experimentally and approximated by a gamma-distribution with the corresponding shape parameter. Moreover, for SSPP, once the geometrical properties of the billiard are known, it is also possible to obtain analytically the corresponding shape parameter. Numerical and analytical results are in a good agreement.

## 2. Materials and methods

### 2.1. SSPP with gamma-distribution for WT

Let us consider the point renewal process

$$\xi(t) = \sum_j \delta(t - t_j), \quad (1)$$

consisting of  $\delta$ -shape pulses with constant amplitude. Here  $t_j$ , which is a random variable, represents the time of the  $j$ -th pulse appearance. The WTs between two neighboring pulses,  $\vartheta_j = t_j - t_{j-1}$ , are independent identically distributed random variables. The mean of WTs,  $\langle \vartheta \rangle = T$ , is the conditional period of the process.

In the following, we use Gamma distribution for inter-pulse intervals  $\vartheta$  with  $n$  as a shape parameter,  $\alpha$  as a scale parameter,  $T = \alpha n$ , and the variance  $\sigma_\vartheta^2 = \alpha^2 n$  [49, 41, 50]. The shape parameter can be found as

$$n = \frac{T^2}{\sigma_\vartheta^2}. \quad (2)$$

If  $n < 1$ , Eq. (1) provides a process with super-Poisson statistics. The spectral density of the process  $\xi(t) - \langle \xi \rangle$  [41] can be written in the form

$$S_{\xi(t) - \langle \xi \rangle}(\omega) = \frac{1}{T} \frac{\sinh(n/2 \ln(1 + (\omega T/n)^2))}{\cosh(n/2 \ln(1 + (\omega T/n)^2)) - \cos(n \arccos((1 + (\omega T/n)^2)^{-1/2}))}. \quad (3)$$

In the case of Poisson statistics ( $n = 1$ ) we get a constant solution, corresponding to  $\delta$ -correlation. The integral of the correlation function is called noise intensity and is defined as

$$S_{\xi(t) - \langle \xi \rangle}(0) = \frac{\sigma_\vartheta^2}{T^3} = \frac{1}{Tn}. \quad (4)$$

In the case  $n \ll 1$ , we get strongly super-Poisson processes and Eq. (3) becomes

$$S_{\xi(t)-\langle\xi\rangle}^{SSPP}(\omega) = \frac{1}{Tn} \frac{\ln(1 + (\omega T/n)^2)}{(\ln(1 + (\omega T/n)^2))^2/4 - \arccos^2((1 + (\omega T/n)^2)^{-1/2})}, \quad (5)$$

91 which is an approximated expression of the spectral density.

92 The approximated expression, Eq. (5), shows a good agreement with the  
 93 exact expression, Eq. (3), of the spectral density for  $n \leq 0.05$ , as demonstrated  
 94 in Fig. 1 for different values of  $n < 1$ , namely for  $n = 0.05, 0.1, 0.4$ .

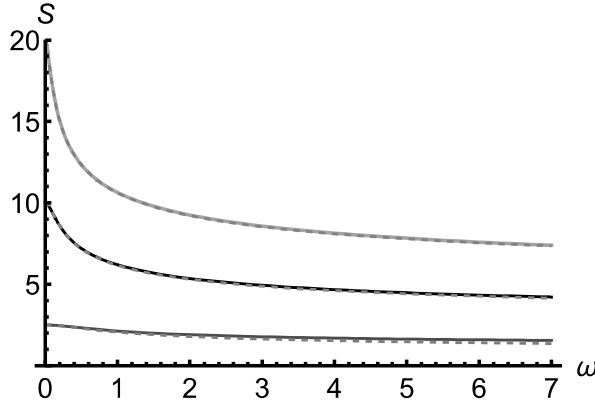


Figure 1: (color on line only) Solid and dashed lines represent the spectrum density, obtained for  $n = 0.05, 0.1, 0.4$  (from top to bottom) and  $T = 1$ , by using, respectively: i) the exact expression, Eq. (3); ii) the approximated expression, Eq. (5).

The spectral width  $\Delta\omega$  is the frequency at which the spectral density is equal to half of its maximum value (see Eq. (4)). To get an explicit expression of  $\Delta\omega$  we consider Eq. (5), which in the limit  $\omega T \gg n$  becomes

$$S_{\xi(t)-\langle\xi\rangle}^{SSPP}(\omega \gg n) = \frac{1}{Tn} \frac{\ln(\alpha\omega)}{\ln^2(\alpha\omega)/2 - \pi^2/8} = \frac{1}{2Tn}. \quad (6)$$

Finally, from Eq. (6) we get

$$\Delta\omega = \beta n/T, \quad \beta = \exp(2 + \sqrt{4 - \pi^2/4}) \approx 25.5. \quad (7)$$

95 The spectral width is small for SSPP, therefore the correlation time  $\tau$  tends  
 96 to become larger [26, 27]. This explains why it is hard using an SSPP as a  
 97 noise source in stochastic differential equations. The methods used to analyt-  
 98 ically studying these equations hold only if the correlation time of the noise  
 99 is significantly less than the characteristic time scale of the system dynam-  
 100 ics [26, 29, 30, 31, 41].

101 Fig. 2 shows the width of the spectral density for different values of the  
 102 shape parameter. Note that the curves are characterized by a fast exponential  
 103 decrease for small frequencies, while they tend to maintain a constant value at

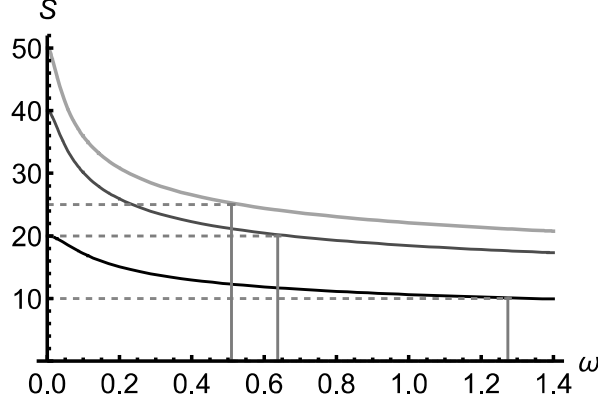


Figure 2: (color on line only) Spectrum densities  $S_{\xi(t)-\langle\xi\rangle}$  (solid lines) and corresponding spectral widths  $\Delta\omega$  (dashed and vertical lines) obtained for different  $n = 0.02, 0.025, 0.05$  (from top to bottom) and  $T = 1$ .

larger frequencies. As a result, we can approximate the function by a sum of an exponential function and a constant function, while considering this expression as the spectrum density of the WPPP corresponding to this SSPP.

## 2.2. The strongly super-Poisson process replaced by the corresponding wide-pulse Poisson process

Let us consider a WPPP, indicated by  $\zeta(t)$ , with pulses of identical shape  $G(t)$ , corresponding to clusters of  $\delta$ -pulses of the original SSPP,  $\xi(t)$  (see Eq. (1)). The mathematical expression of this process can be written as follows

$$\zeta(t) = \sum_j G(t - t_m) = \int G(t - t_1) \eta(t_1) dt_1, \quad (8)$$

with  $\eta(t)$  being the pulse process with  $\delta$ -shape pulses distributed according to the Poisson statistics. Correspondingly, we obtain the spectrum density of  $\zeta(t)$

$$S_\zeta(\omega) = S_\eta(\omega) \langle |F(\omega)|^2 \rangle, \quad (9)$$

where  $F(\omega)$  is the Fourier transform of  $G(t)$ .

Let  $N$  be the average number of pulses in a cluster. The average size of the pulse-cluster is  $\langle \int G(t') dt' \rangle = F(0) = N$ , with the average distance between clusters being

$$T_P = NT. \quad (10)$$

One of the main characteristics of a pulse process is the variance of the WTs.

Since the WTs are distributed exponentially for WPPP,  $\sigma_P^2 = T_P^2 = N^2 T^2$  and  $S_\eta(\omega) = T_P^{-1}$  (according to Eq. (4)). The spectral densities of both processes  $S_\zeta(0) = N^2/T_P = N/T$  (Eq. (9)) and  $S_{\xi(t)-\langle\xi\rangle}(0)$  (Eq. (4)) coincide for  $\omega = 0$ .

Thus we obtain  $N = n^{-1}$ . This result is confirmed by the fact that the gamma distribution is infinitely divisible. As a consequence, a process consisting of each  $N$ -th pulse in SSPP is a Poisson process, if  $N \in \mathbb{Z}$ . Knowing the parameter values of SSPP we can therefore calculate those of WPPP and find  $G(t)$ .

The spectral density of the SSPP process (Eq. (3)) decreases rapidly at low frequencies and slowly at high frequencies. The part of  $S_{\xi(t)-\langle\xi\rangle}(\omega)$ , which can be considered informative from a physical point of view, corresponds to smaller frequencies, i.e.  $\omega \leq \Delta\omega$ , with  $\Delta\omega$  being given by Eq. (7). Higher frequencies correspond to delta-function pulse (zero width), even if in real processes the pulses obviously possess a finite width. As a result, the spectral density can be approximated as a sum of an exponential function and a constant function

$$\tilde{S}_{\xi(t)-\langle\xi\rangle}(\omega) = \frac{1}{nT} \left( (1 - \sqrt{0.5}) \exp \left[ -\frac{\omega T}{7n} \right] + \sqrt{0.5} \right)^2, \quad (11)$$

where the parameter values are obtained by exploiting the condition of equality of the spectral widths  $\Delta\omega$  obtained for the original spectral density (Eq. (3)) and for the approximated expression (Eq. (11)), and numerical approximation of  $S_{\xi(t)-\langle\xi\rangle}$  inside this frequency interval. Fig. 3 shows the spectral densities for

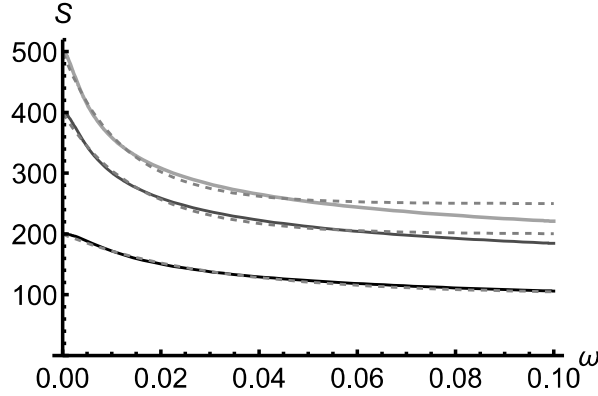


Figure 3: (color on line only) Spectrum densities  $S_{\xi(t)-\langle\xi\rangle}$  (solid lines) and their approximated version  $\tilde{S}_{\xi(t)-\langle\xi\rangle}$  (dashed lines) obtained for  $n = 0.02, 0.025, 0.05$  (from top to bottom) and  $T = 1$ .

different  $n$  and their approximation  $\tilde{S}_{\xi(t)-\langle\xi\rangle}$ . This simplified expression for the spectral density can be considered as the spectral density of a wide pulse. By using Eq. (9) and Fourier transforming, we get

$$G(t) = \frac{1}{n} \left( \frac{(1 - \sqrt{0.5})T}{\pi 7n} \frac{1}{(t)^2 + (T/7n)^2} + \sqrt{0.5} \delta(t) \right), \quad (12)$$

which provides the shape of the wide pulses.

121 In Fig. 4 the pulses of SSPP (green solid vertical lines of height 1) are  
 122 replaced by a WPPP consisting of a delta-pulse (blue higher vertical line) and  
 123 a continuous part given by a Cauchy-Lorentz distribution (black solid line).

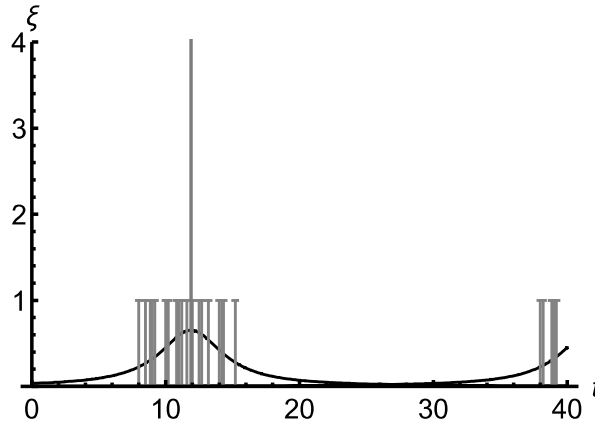


Figure 4: (color on line only) The pulses of SSPP (solid vertical lines of height 1) are replaced by WPPP consisting of a delta-pulse (blue higher vertical line) and a continuous part given by a Cauchy-Lorentz distribution (black solid line). Here  $n = 0.05$  and  $T = 1$ .

124 We note that the correlation time divided by the period is an indicator of  
 125 how computationally complex the renewal processes are. Obtaining statistically  
 126 meaningful numerical samples requires longer simulations, when the renewal  
 127 process approaches strong super-Poisson statistics. We recall that the corre-  
 128 lation times of an SSPP and that of the corresponding WPPP are the same,  
 129 while the period of the SSPP is much smaller than that of the WPPP. As a  
 130 consequence, the SSPP from a computational point of view is  $N$  times more  
 131 demanding than WPPP.

### 133 2.3. The billiard-like generator of random pulse processes with regulated period- 134 icity

135 In this section we show a generator of random point processes based on a  
 136 billiard system (see Fig. 5). Simulations are carried out after drawing the bound-  
 137 aries (walls) of the billiard and marking some of them. The particle moves and  
 138 collides with these walls. Each collision with the marked boundary is recorded  
 139 and correspondingly a pulse appears on the graph. After a certain time one  
 140 observes a pulse sequence  $\xi(t)$ . The statistical properties of  $\xi(t)$  are determined  
 141 by the shape of the billiard and the marked sections of the boundaries. Fig. 5  
 142 shows a billiard in the shape of a crocodile. The subregion marked by the dashed  
 143 red-white line corresponds to its eye. The light green circle in its front left paw  
 144 is a scatterer that makes chaotic the movement of the point-like particle (black  
 145 dot) as proved in [47, 48, 51].

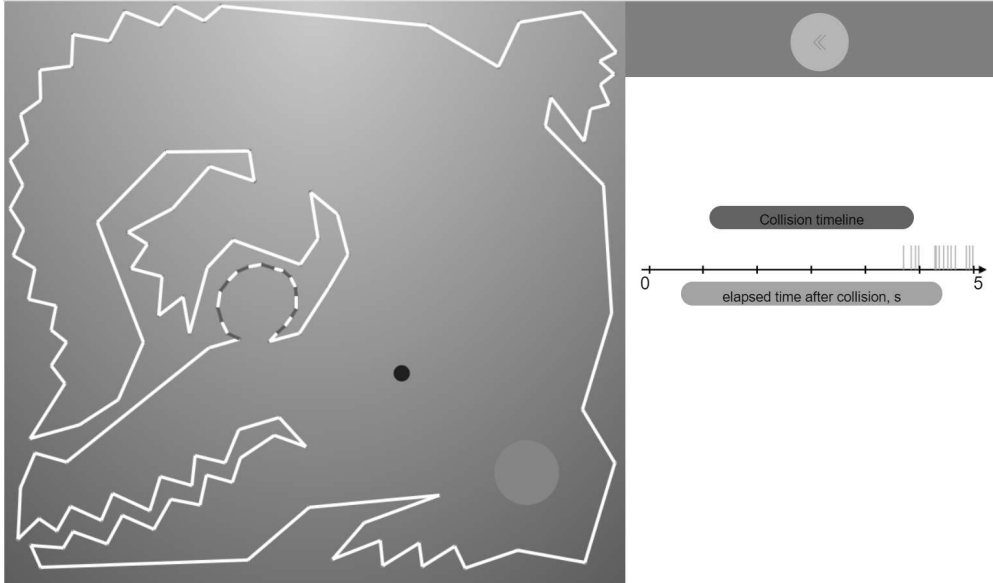


Figure 5: (color on line only) The crocodile-shaped billiard with the subregion marked by dashed red-white line (corresponding to the crocodile's eye) and a short part of the pulse sequence containing one cluster.

146 The SSPP can be obtained by considering the small subregion (crocodile's  
 147 eye) with marked boundaries (dashed red-white line) of length  $L$ . The particle  
 148 moves randomly in the billiard and rarely enters the subregion through a narrow  
 149 passage with a width of  $l_0 \ll L$ . Then it collides with the marked boundaries  
 150 several times, generating pulses of SSPP, and finally goes back and exits. This  
 151 set of collisions within the subregion corresponds to a cluster of pulses of a  
 152 SSPP and to a single broad pulse of a WPPP. In the case of WPPP, we are  
 153 not interested in individual collisions, but only in the fact that the particle is  
 154 located inside the subregion.

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We consider billiards with a finite space. Thus the distribution of particle's positions results to be homogeneous for all regions of the billiard [47, 48]. The velocity modulus,  $v$ , of the particle is constant (perfectly elastic collisions). The dynamics of the particle can be described by the two coordinates  $x$  and  $y$ , and the angle  $\psi$  between the velocity vector and a fixed direction. The joint probability distribution of these random values at the steady state is homogeneous and reads

$$w(x, y, \psi) = \frac{1}{2\pi\Omega}, \quad (13)$$

where  $\Omega$  is the area of an accessible billiard region. The probability per unit time of a collision with a part of the boundary of small length  $dL$  with angle  $\phi$  is determined by the projection  $v \cos \phi$  of the velocity on the direction perpen-



pendicular to the boundary of the scatterer [52]. Consequently, the probability per unit time of a collision with a small part of the boundary  $dL$  with any angle  $\phi$  is

$$P(dL) = dL \int_{-\pi/2}^{\pi/2} w(x, y, \psi) v \cos \phi d\phi = \frac{dL v}{\pi \Omega}. \quad (14)$$

The probability per unit time of a collision with a part of the boundary of length  $L$  is therefore given by

$$P(L) = \frac{Lv}{\pi \Omega}. \quad (15)$$

Therefore, we can find the average time between the entries of a particle into the subregion, which corresponds to  $T_P$ , and the number  $N$  of collisions inside the subregion. The average period of collisions with the marked boundary inside the subregion is

$$T = \frac{1}{P(L)} = \frac{\pi \Omega}{Lv} \quad (16)$$

The period of WPPP is

$$T_P = \frac{1}{P(l_0)} = \frac{\pi \Omega_-}{l_0 v}, \quad (17)$$

where  $\Omega_-$  is the area of the billiard region without the subregion. As a result, according to Eq. (10), the shape parameter  $n$  and the corresponding average number,  $N$ , of pulses in the cluster are given by

$$n = \frac{1}{N} = \frac{T_P}{T} = \frac{\Omega l_0}{\Omega_- L} \approx \frac{l_0}{L}, \quad (18)$$

where we assumed the area of the subregion to be small enough to consider  $\Omega_- \approx \Omega$ .

These formulas can be generalised for the case of non-point particle. Let  $a$  be the diameter of the particle. The dynamics of such a particle corresponds to the dynamics of a point-particle in a smaller billiard. The pass width is redefined as  $l_0 - a$  instead of  $l_0$  for the marked boundary. Moreover, the length of the subregion's boundary, depending on its shape, is redefined as follows: i)  $L - \pi a$  for a circle-shaped subregion; ii)  $L - 4a$  for a rectangle-shaped subregion.

### 3. Results and discussion

In this section we present the results obtained by computer simulations for the time evolution of the billiard-like system and compare them with the analytical findings obtained in Sec.2 for the SSPP. To generate the pulse process we consider collisions of the particle with the marked boundary of the subregion. We find the average waiting time  $T$  and the variance  $\sigma_\vartheta^2$ , and obtain the shape parameter  $n$  through Eq. (2).

In Fig. 6 dots and vertical bars indicate the values of  $n$  and the corresponding indeterminacy at different particle diameters  $a$  for the crocodile-shaped billiard. The analytical results coming from

$$n = \frac{1}{N} = \frac{l_0 - a}{L - \pi a}, \quad (19)$$

in the case of circular-shaped subregion are shown by the dashed line.

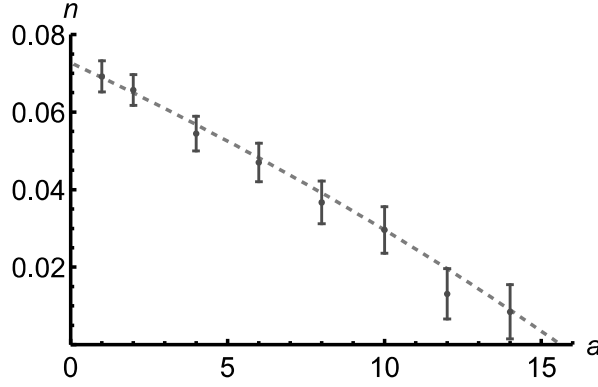


Figure 6: (color on line only) Dependence of  $n$  on the particle diameter  $a$  for crocodile-shaped billiard. Results obtained from Eq. (19) (dashed line) and numerically (vertical bars). Here  $l_0=0.64$  and  $L=8.8$ .

Figure 7 presents the same dependence for a Sinai billiard-like system. The latter system is a square with a circle in the middle and a small opening which leads to a small circle.

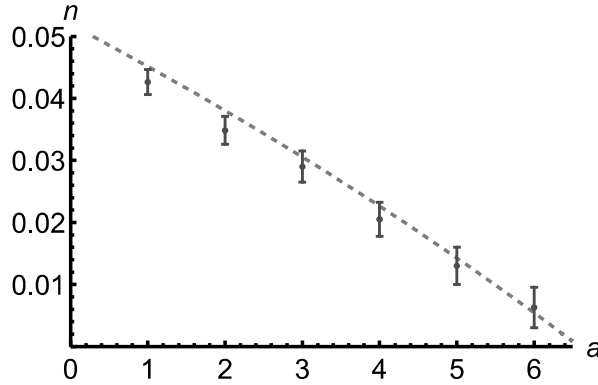


Figure 7: (color on line only) Dependence of  $n$  on the particle diameter  $a$  for a Sinai billiard-like system, i.e., a square with a circle inside and a small opening leading to a small circle-shaped subregion. Results obtained from Eq. (19) (dashed red-white line) and numerically (vertical bars). Here  $l_0=0.27$  and  $L=5.2$ .

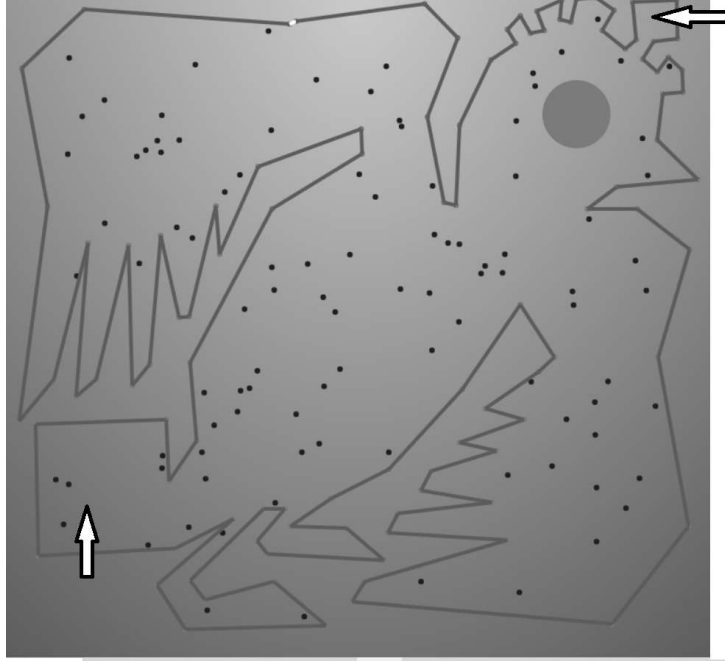


Figure 8: (color on line only) Bird-shaped billiard with square subregions marked by white arrows.

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Fig. 8 shows a bird-shaped billiard with square subregions, marked by white arrows, in the head's crest (top right) and the tail (bottom left). We note that the scatterer resembles an eye. As mentioned above, the statistics of collisions depends on the boundaries chosen. For example, when the subregion coincides with the whole perimeter of the billiard, the PDF of the WTs is close to an exponential function (results not shown), and the process can be characterized by Poisson statistics.

On the contrary, when only the boundaries of a small square subregion are marked, one gets a SSPP whose shape parameter is given by

$$n = \frac{l_0 - a}{L - 4La}. \quad (20)$$

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Fig. 9 shows the dependence of  $n$  on the particle diameter  $a$  for the bird-shaped billiard and a square-shaped subregion as the head's crest.

Fig. 10 shows the dependence of  $n$  on particle diameter  $a$  for the bird-shape billiard and square shaped subregion as its tail. Discrepancies between numerical and analytical results are due to the size of the marked subregion, which is not small enough to make events rare, when the particle enters the subregion.

As one can see, the main results are independent of the details of the shape

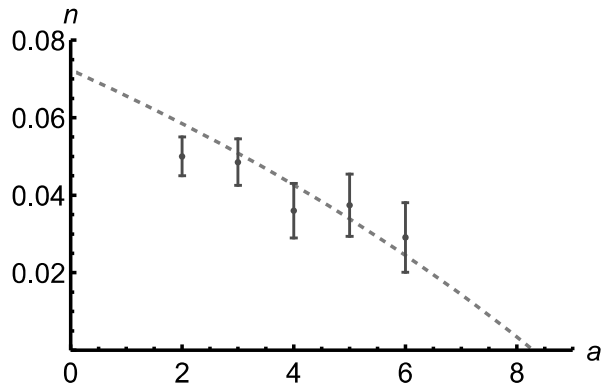


Figure 9: (color on line only) The dependence of  $n$  on particle diameter  $a$  for the billiard in the shape of a bird with the subregion as a square in the crest on its head (top right) from Eq. (20) (dashed line) and numerical results (vertical bars). Here  $l_0=0.33$  and  $L=4.7$ .

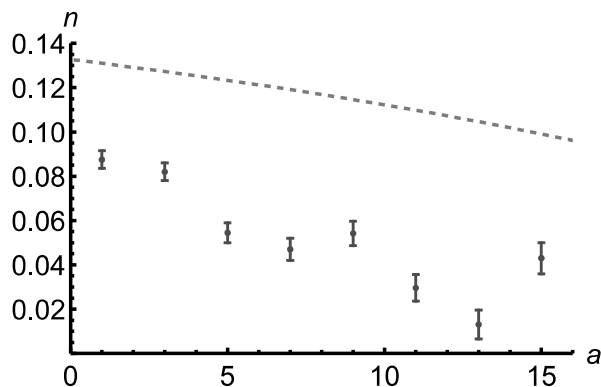


Figure 10: (color on line only) Dependence of  $n$  on the particle diameter  $a$  for the billiard in the shape of a bird with the subregion as a square in its tail from Eq. (20) (dashed line) and numerical results (vertical bars). Here  $l_0=1.5$  and  $L=11$ .

189 of both the billiard and the subregion, but depends only on the ratio of the  
 190 width of the opening to the whole length of the marked boundaries.

191 Moreover, we have shown that strongly super-Poisson statistics under the  
 192 condition  $n \leq 0.05$ , corresponding to  $N \geq 20$ , can be effectively replaced by  
 193 a wide-pulse Poisson process or by a numerical model based on a billiard-like  
 194 random generator.

195 This point represents the main topic of our work, since it is now possible  
 196 to give a quantitative definition of *strongly* super-Poisson process, a concept  
 197 until now introduced only from a qualitative point of view. *Strongly* indicates  
 198 therefore a super-Poisson process characterized by a mean number of pulses in  
 199 a cluster,  $N$ , bigger than 20.

200 Super-Poisson processes are more difficult to study and therefore are less

investigated than the sub-Poisson ones, which are quasi-periodical with a periodicity parameter defined as  $1 - \sigma_{\vartheta}^2/T^2$ . We recall that a Poisson process can be obtained from a sub-Poisson one for periodicity parameter equal to zero [53]. This parameter takes on negative values for super-Poisson processes, which can be therefore assigned to a negative periodicity. The results obtained here by using gamma-distributed WT's can be generalised to other distributions, since in the limit of strongly super-Poisson process the structure within a cluster does not influence the Poisson statistics of the times at which clusters appear, and the parameter  $n$ , defined by Eq. (2), is no longer a shape parameter. The strongly super-Poisson process could also be used to simulate coughing, hiccups, finding chanterelles. In this work, we considered processes with a large but finite variance. Anyway, this approach can be also used to analyse heavy-tailed (power law) distributed WT's and fractional Poisson processes [54, 55, 56, 57].

The method proposed for generating pulse processes with arbitrary correlation properties can be also applied to design hardware random number generators.

Here we obtained results for random processes, which are a particular case of time-dependent functions. Meanwhile, these findings can be generalised to systems whose super-Poisson structure is in the space, not in the time [34]. This super-Poisson structures can be observed in matter close to a critical point, such as critical opalescence, which is the result of large density fluctuations. Finally, we note that super-Poisson processes can be used to describe structures of polymers and viruses, and spatial distribution of stars in Space, plants on Earth, social group structure and decorative elements in an ornament.

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