THREE SOLUTIONS TO MIXED BOUNDARY VALUE PROBLEM DRIVEN BY $p(z)$-LAPLACE OPERATOR

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Abstract. We prove the existence of at least three weak solutions to a mixed Dirichlet-Neumann boundary value problem for equations driven by the $p(z)$-Laplace operator in the principal part. Our approach is variational and use three critical points theorems.

1. Introduction

Let $M \subset \mathbb{R}^N$ ($N \geq 3$) be an open bounded domain with smooth boundary. In this article we consider the following mixed Dirichlet-Neumann boundary value problem driven by the $p(z)$-Laplace operator:

$$(P_{\xi,\mu}) \begin{cases} -\text{div} (|\nabla u(z)|^{p(z)-2}\nabla u(z)) + a(z)|u(z)|^{p(z)-2}u(z) = \xi g(z, u(z)) & \text{in } M, \\ u = 0 & \text{on } M_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu h(\gamma(u)) & \text{on } M_2, \end{cases}$$

where $p \in C(M)$ is a function with some regularities satisfying

$$N < p^- := \inf_{z \in M} p(z) \leq p(z) \leq p^+ := \sup_{z \in M} p(z) < +\infty,$$

$M_1, M_2$ are smooth $(N-1)$-dimensional submanifolds of $\partial M$ and $\Gamma$ is a smooth $(N-2)$-dimensional submanifolds of $\partial M$ with $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = \partial M$, $\overline{M_1} \cap \overline{M_2} = \Gamma$, $a \in L^\infty(M)$ with $a_0 := \text{ess inf}_{z \in M} a(z) > 0$ is the potential function, $g : M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $z \in \mathbb{R}$, $z \to g(z, y)$ is measurable and for a.a. $z \in M$, $y \to g(z, y)$ is continuous), $h : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $\gamma : W^{1,p}(M) \to L^p(\partial M)$ is the trace operator, $\xi > 0$ and $\mu \geq 0$ are real parameters, and $\nu$ is the outer unit normal to $\partial M$.

We recall that the $p(z)$-Laplace operator drives processes of physical interest, as stated in Diening-Harjulehto-Hästö-Růžička [7]. Existence and multiplicity results for problems involving the $p(z)$-Laplace operator were obtained by Papageorgiou-Vetro [12], Rodrigues [13], Vetro [14] (Dirichlet condition), Deng-Wang [6], Heidarkhani-Afrouzi-Hadjian [10], Pan-Afrouzi-Li [11] (Neumann condition). In [12] the authors consider a $(p(z), q(z))$-equation with reaction $g$ which depends on the solution but does not satisfy the Ambrosetti-Rabinowitz condition. Their approach is variational based on critical point theory together with Morse theory (critical groups). These

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tools are used also in [13]. This time, the author studies a nonlinear eigenvalue problem for \( p(z) \)-Laplacian-like operator, originated from a capillary phenomena. The reaction is superlinear at infinity and the author proves the existence of infinite many pairs of solutions. [14] also considers a problem driven by the \( p(z) \)-Laplacian-like operator. There, the reaction satisfies a sub-critical growth condition and the results deal with the existence of one and three solutions (via critical point theory). In [10, 11] the authors prove the existence of three solutions for \( p(z) \)-Laplace problems with potential function. In particular, [10] considers small perturbations of nonhomogeneous Neumann condition. A similar problem is discussed in [6], where the authors use sub-supersolution method and strong comparison principle. For mixed boundary value problems there are the recent works of Barletta-Livrea-Papageorgiou [1], Bonanno-D’Agui-Sciammetta [2] (for the constant \( p \)-Laplace operator).

Here, we give two existence theorems of three weak solutions to problem \( (P_{\xi,\mu}) \) (that is, mixed boundary value problem with variable exponent version of the \( p \)-Laplace operator), by using a variational approach and critical point theorems. Here the reaction \( g : M \times \mathbb{R} \to \mathbb{R} \) is \( L^1 \)-Carathéodory (that is, \( g \) is Carathéodory and for any \( s > 0 \) there exists \( l_s \in L^1(M) \) with \( |g(z, y)| \leq l_s(z) \), for a.e. \( z \in M \) and for all \( |y| \leq s \)). So, the three critical points results of Bonanno-Marano [4] and Bonanno-Candito [3] apply to energy functionals associated to problem \( (P_{\xi,\mu}) \).

2. Mathematical background

Let \((E, E^*)\) be a Banach topological pair. Here, we use the variable exponent Lebesgue spaces \( L^{p(z)}(M) \), \( L^{p(z)}(\partial M) \), and the generalized Lebesgue-Sobolev space \( W^{1, p(z)}(M) \). These spaces (referring to the norms below) are separable, reflexive and uniformly convex Banach spaces (see Fan-Zhang [8]). Precisely, we have

\[
L^{p(z)}(M) = \left\{ u : M \to \mathbb{R} : u \text{ is measurable and } \int_M |u(z)|^{p(z)} \, dz < +\infty \right\},
\]

with the norm

\[
\|u\|_{L^{p(z)}(M)} := \inf \left\{ \xi > 0 : \int_M \left| \frac{u(z)}{\xi} \right|^{p(z)} \, dz \leq 1 \right\} \quad \text{(i.e., Luxemburg norm)}.
\]

On the other hand, we have

\[
L^{p(z)}(\partial M) = \left\{ u : \partial M \to \mathbb{R} : u \text{ is measurable and } \int_{\partial M} |u(z)|^{p(z)} \, d\sigma < +\infty \right\},
\]

where \( \sigma \) is the surface measure on \( \partial M \). This time, we consider the norm

\[
\|u\|_{L^{p(z)}(\partial M)} := \inf \left\{ \xi > 0 : \int_M \left| \frac{u(z)}{\xi} \right|^{p(z)} \, d\sigma \leq 1 \right\}.
\]

Also, the generalized Lebesgue-Sobolev space \( W^{1, p(z)}(M) \) is defined as

\[
W^{1, p(z)}(M) := \{ u \in L^{p(z)}(M) : |\nabla u| \in L^{p(z)}(M) \},
\]

and we take the norm
\[ \|u\|_{W^{1,p}(M)} = \|u\|_{L^p(M)} + \|\nabla u\|_{L^p(M)}, \]

which is equivalent to the norm
\[ \|u\| := \inf \left\{ \xi > 0 : \int_M \left( a(z) \left| \frac{u(z)}{\xi} \right|^{p(z)} + \left| \nabla u(z) \right|^{p(z)} \right)dz \leq 1 \right\}, \]

(for details we refer to D’Aguì-Sciammetta [5]). So, we work with the norm \( \|u\| \) instead of \( \|u\|_{W^{1,p}(M)} \) on \( W^{1,p}(M) \). In proving our theorems, we make use of the following result, which links \( \|u\| \) to \( \rho(u) = \int_M \left( a(z) \left| u(z) \right|^{p(z)} + \left| \nabla u(z) \right|^{p(z)} \right)dz \) (see Fan-Zhao [9]).

**Theorem 1.** If \( u \in W^{1,p}(M) \), one has

(i) \( \|u\| < 1 (= 1, > 1) \iff \rho(u) < 1 (= 1, > 1); \)

(ii) if \( \|u\| > 1 \), then \( \|u\|^{p^{-}} \leq \rho(u) \leq \|u\|^{p^{+}}; \)

(iii) if \( \|u\| < 1 \), then \( \|u\|^{p^{-}} \leq \rho(u) \leq \|u\|^{p^{+}}. \)

For notational simplicity, by \( E \) we denote the set
\[ E = W^{1,p}_{0,M_1}(M) = \{ u \in W^{1,p}(M) : u_{|M_1} = 0 \}, \]

where we consider the norm \( \|u\| \).

We recall that a function \( u \in E \) satisfying
\[ \int_M \left| \nabla u(z) \right|^{p(z)-2} \nabla u(z) \nabla v(z)dz + \int_M a(z) |u(z)|^{p(z)-2}u(z)v(z)dz \]
\[ = \xi \int_M g(z, u(z))v(z)dz + \mu \int_{M_2} h(\gamma(u(z)))\gamma(v(z))dz, \]

for all \( v \in E \), means a weak solution of problem \( (P_{\xi, \mu}). \)

We mention the fact that \( W^{1,p}(M) \hookrightarrow W^{1,p^{-}}(M) \) continuously. Also, as \( N < p^{-} \), \( W^{1,p^{-}}(M) \hookrightarrow C_0(M) \) compactly, and hence \( W^{1,p}(M) \hookrightarrow C_0(M) \) compactly (so \( E \hookrightarrow C_0(M) \) compactly). If we put
\[ k = \sup_{u \in W^{1,p}(M) \setminus \{0\}} \frac{\sup_{z \in M} |u(z)|}{\|u\|}, \]

then
\[ \|u\|_{\infty} \leq k\|u\|, \]

with \( \|u\|_{\infty} \) to denote the usual norm in \( L^\infty(M). \)

The quantity
\[ k_b = 2^{\frac{p^{-}-1}{p}} \max \left[ \left( \frac{1}{\|a\|_1} \right)^{\frac{1}{p}}, \frac{\text{diam} (M)}{N^{\frac{1}{p^{-}}} \left( \frac{p^{-} - 1}{p^{-} - N} \right) \text{meas} (M)} \right], \]

where \( M \) is convex, \( \text{diam} (M) \) is the diameter of \( M \), \( \text{meas} (M) \) is the Lebesgue measure of \( M \), satisfies the inequality \( k_b(1 + \text{meas} (M)) \geq k \). This means that \( k_b(1 + \text{meas} (M)) \) is an upper bound of \( k \) (see [5]).
Next, let $G : M \times \mathbb{R} \to \mathbb{R}$ be the function given as
\[ G(z,t) = \int_0^t g(z,y) \, dy, \quad \text{for all } t \in \mathbb{R}, \ z \in M, \]
and $H : \mathbb{R} \to \mathbb{R}$ be the function given as
\[ H(t) = \int_0^t h(z) \, dz, \quad \text{for all } t \in \mathbb{R}. \]

Our approach is variational, which means that we study the critical points of the energy functional (say $I_\xi$) associated to problem $(P_\xi, \mu)$. So, we introduce the functional $B : E \to \mathbb{R}$ defined by
\[ B(u) = \int_M G(z,u(z)) \, dz + \frac{\mu}{\xi} \int_{\Gamma_2} H(\gamma(u(z))) \, d\sigma, \quad \text{for all } u \in E. \]
Clearly, $B \in C^1(E, \mathbb{R})$ and has a compact derivative given as
\[ B'(u)(v) = \int_M g(z,u(z))v(z) \, dz + \frac{\mu}{\xi} \int_{\Gamma_2} h(\gamma(u(z)))\gamma(v(z)) \, d\sigma, \quad \text{for all } u,v \in E. \]

Moreover, let $A : E \to \mathbb{R}$ be the functional given as
\[ A(u) = \int_M \frac{1}{p(z)} [ |\nabla u(z)|^{p(z)} + a(z)|u(z)|^{p(z)} ] \, dz, \quad \text{for all } u \in E, \]
with $A$ in $C^1(E, \mathbb{R})$. Note that $A$ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative $A' : E \to E^*$ is
\[ A'(u)(v) = \int_M [ |\nabla u(z)|^{p(z)-2}\nabla u(z)\nabla v(z) + a(z)|u(z)|^{p(z)-2}u(z)v(z) ] \, dz, \quad \text{for all } u,v \in E. \]

We recall the following property of $A'$ (see, for example, [13, Proposition 2.6]).

**Proposition 1.** The functional $A' : E \to E^*$ is a strictly monotone and bounded homeomorphism.

Finally, consider the functional $I_\xi : E \to \mathbb{R}$ defined by $I_\xi(u) = A(u) - \xi B(u)$ for all $u \in E$. We have
\[ \inf_{u \in E} A(u) = A(0) = B(0) = 0. \]
We mention that the critical points of $I_\xi$ are the weak solutions of problem $(P_\xi, \mu)$.

### 3. Three weak solutions of Bonanno-Marano type

We establish a theorem producing three weak solutions to problem $(P_\xi, \mu)$. So, we use the following three critical point result of Bonanno-Marano [4, Theorem 3.6].

**Theorem 2.** Let $(E, E^*)$ be a Banach topological pair with $E$ reflexive. Let $A : E \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative has a continuous inverse on $E^*$, $B : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $A(0) = B(0) = 0$. Assume that there exist $r > 0$ and $w \in E$, with $0 < r < A(w)$, such that
(i) \( \sigma = \frac{1}{r} \sup_{A(u) \leq r} B(u) < \frac{B(w)}{A(w)} = \rho; \)

(ii) for each \( \xi \in \left[ \frac{1}{\rho}, \frac{1}{\sigma} \right] \), \( I_\xi := A - \xi B \) is coercive.

Then, for each \( \xi \in \left[ \frac{1}{\rho}, \frac{1}{\sigma} \right] \), \( I_\xi \) has at least three distinct critical points in \( E \).

Here, we define \( \eta : \overline{M} \to \mathbb{R} \) by \( \eta(z) = \rho(z, \partial M) \), where \( \rho \) means the Euclidean distance. Let \( D = \eta(z_0) \) with \( z_0 \in M \) point of maximum for \( \eta \) so that \( B(z_0, D) = \{ z \in \mathbb{R}^N : \rho(z_0, z) < D \} \subset M \). Fixed \( s \in [1, +\infty[ \), we set \( s_D = s^{-1} \) and \( \kappa_D = \frac{s}{(s-1)D} \) (note that \( (1-s_D)D\kappa_D = 1 \)). For each \( \alpha \geq 1 \), we consider a function \( w_\alpha : M \to \mathbb{R} \) defined by

\[
(2) \quad w_\alpha(z) = \begin{cases} 
0 & z \in M \setminus B(z_0, D), \\
\alpha & z \in B(z_0, s_D D), \\
\alpha \kappa_D(D - |z - z_0|) & z \in B(z_0, D) \setminus B(z_0, s_D D).
\end{cases}
\]

Now, we set \( \alpha := d \geq 1 \) (so we fix \( w_d : M \to \mathbb{R} \) and \( c \geq k \) with

\[
A(w_d)p^+ \int_M \max_{|y| \leq c} G(z, y)dz < \left( \frac{c}{k} \right)^{p^-} \int_{B(z_0, s_D D)} G(z, d)dz.
\]

Also, we choose

\[
\xi \in \Omega := \left[ \frac{A(w_d)}{\int_{B(z_0, s_D D)} G(z, d)dz}, \frac{\left( \frac{c}{k} \right)^{p^-}}{p^+ \int_M \max_{|y| \leq c} G(z, y)dz} \right]
\]

so that

\[
(3) \quad \delta := \min \left\{ \frac{c^{p^-} - \xi^{p^+}k^{p^+} \int_M \max_{|y| \leq c} G(z, y)dz}{p^+k^{p^+}\sigma(M_2)\max_{|y| \leq c} H(y)}, \frac{1}{2k^{p^-}p^+\sigma(M_2)\limsup_{|y| \to +\infty} \frac{H(y)}{|y|^{p^-}}} \right\},
\]

with \( \sigma(M_2) := \int_{M_2} d\sigma \) and, as usual, we take \( r/0 = +\infty \).

Our first result is the following proposition, where we use the hypothesis:

(h) \( h : \mathbb{R} \to \mathbb{R} \) satisfies

\[
\limsup_{|z| \to +\infty} \frac{H(z)}{|z|^{p^-}} < +\infty.
\]

Recall that \( h \) is continuous, too.

Proposition 2. If (h) holds, then we can find \( \delta > 0 \) as in (3) such that, for each \( \mu \in [0, \delta] \), the functional \( I_\xi(u) = A(u) - \xi B(u) \), for all \( u \in E \) (\( \xi \in \Omega \)) is coercive whenever

\[
\limsup_{|z| \to +\infty} \frac{\sup_{y \in M} G(z, y)}{|y|^{p^-}} < \frac{1}{2c^{p^-}\text{meas}(M)} \int_M \max_{|y| \leq c} G(z, y)dz.
\]
Proof. Suppose

\[
\limsup_{|y| \to +\infty} \sup_{z \in M} \frac{G(z, y)}{|y|^{p^-}} > 0,
\]

so that we can find \( l > 0 \) with

\[
\limsup_{|y| \to +\infty} \sup_{z \in M} \frac{G(z, y)}{|y|^{p^-}} < l < \int_M \max_{|y| \leq c} G(z, y) \frac{dz}{2c^{p^-} \text{meas} (M)},
\]

\[
\Rightarrow \quad G(z, y) \leq l |y|^{p^-} + C_l, \quad \text{for each } y \in \mathbb{R} \text{ and } z \in M \text{ (for some } C_l > 0).}
\]

Since \((\xi k)^{p^-} > \xi p^+ \int_M \max_{|y| \leq c} G(z, y) dz\), we have

\[
(4) \quad \xi \int_M G(z, u(z)) dz \leq \xi l \int_M |u(z)|^{p^-} dz + \xi C_l \text{meas} (M)
\]

\[
\leq \frac{(\xi k)^{p^-} \text{meas} (M)}{p^+ \int_M \max_{|y| \leq c} G(z, y) dz} \left( l \int_M |u(z)|^{p^-} dz + C_l \text{meas} (M) \right)
\]

\[
\leq \frac{(\xi k)^{p^-} \text{meas} (M)}{p^+ \int_M \max_{|y| \leq c} G(z, y) dz} (lk^{p^-} ||u||^{p^-} + C_l) \quad \text{for all } u \in E \text{ (by (1))}
\]

So, as \( \delta > \mu \), we get

\[
1 > 2\mu k^{p^-} p^+ \sigma(M_2) \limsup_{|y| \to +\infty} \frac{H(y)}{|y|^{p^-}},
\]

\[
\Rightarrow \quad H(y) \leq \frac{|y|^{p^-}}{2\mu k^{p^-} p^+ \sigma(M_2)} + C_\mu, \quad \text{for all } y \in \mathbb{R} \text{ (for some } C_\mu > 0).}
\]

By (1), we obtain

\[
(5) \quad \int_{M_2} H(\gamma(u(z))) d\sigma \leq \frac{1}{2\mu k^{p^-} p^+ \sigma(M_2)} \int_{M_2} |u(z)|^{p^-} dz + C_\mu \sigma(M_2)
\]

\[
\leq \frac{1}{2\mu p^+} ||u||^{p^-} + C_\mu \sigma(M_2), \quad \text{for all } u \in E.
\]

If \( ||u|| \geq 1 \), (4) and (5) lead to

\[
I_\xi(u) = A(u) - \xi B(u)
\]

\[
\geq \frac{1}{p^+} ||u||^{p^-} - \frac{(\xi k)^{p^-} \text{meas} (M)}{p^+ \int_M \max_{|y| \leq c} G(z, y) dz} (lk^{p^-} ||u||^{p^-} + C_l) - \frac{1}{2p^+} ||u||^{p^-} - \mu C_\mu \sigma(M_2)
\]

\[
= \frac{1}{p^+} \left( \frac{1}{2} - \frac{e^{p^-} \text{meas} (M)}{\int_M \max_{|y| \leq c} G(z, y) dz} l \right) ||u||^{p^-} - \frac{(\xi k)^{p^-} \text{meas} (M)}{p^+ \int_M \max_{|y| \leq c} G(z, y) dz} - \mu C_\mu \sigma(M_2).
\]
By the choice of \( l \), we get
\[
\frac{1}{2} - \frac{c^p^- \text{meas}(M)}{\int_M \max_{|y| \leq c} G(z, y) dz} l > 0
\]
\( \Rightarrow \) \( I_\xi \) is coercive.

On the other hand, if
\[
\limsup_{|y| \to +\infty} \frac{\sup_{z \in M} G(z, y)}{|y|^{p^-}} \leq 0,
\]
we can find a positive constant \( C \) with \( G(z, y) \leq C \) for all \( y \in \mathbb{R} \) and \( z \in M \). So, following the same lines as above, we deduce that
\[
I_\xi(u) \geq \frac{1}{2p^+} \|u\|^{p^-} - \frac{(c_k)^{p^-} \text{meas}(M) C}{p^+ \int_M \max_{|y| \leq c} G(z, y) dz} - \mu C \sigma(M_2)
\]
\( \Rightarrow I_\xi \) is (again) coercive.

\( \square \)

We are ready to establish the existence of three weak solutions. To this aim we suppose that there are \( d \geq 1 \) and \( c \geq k \) with

\[
A(w_d) > \left( \frac{c}{k} \right)^{p^-},
\]
where \( w_d : M \to \mathbb{R} \) is given as in (2), satisfying

\( (S_1) \) \( p^+ A(w_d) \int_M \max_{|y| \leq c} G(z, y) dz < \left( \frac{c}{R} \right)^{p^-} \int_{B(z_0, \delta D)} G(z, d) dz; \)

\( (S_2) \) \limsup_{|y| \to +\infty} \frac{\sup_{z \in M} G(z, y)}{|y|^{p^-}} < \frac{\int_M \max_{|y| \leq c} G(z, y) dz}{2e^{p^-} \text{meas}(M)}; \)

\( (S_3) \) \( G(z, y) > 0 \) for all \( z \in M, y \in [0, d]. \)

**Theorem 3.** If \((h), (S_1)-(S_3)\) hold, then we can find \( \delta > 0 \) as in (3) such that, for each \( \mu \in [0, \delta] \), problem \((P_{\xi, \mu})\) has at least three weak solutions in \( E (\xi \in \Omega). \)

**Proof.** We set
\[
r := \frac{1}{p^+} \left( \frac{c}{k} \right)^{p^-},
\]
so that, by (6), we have
\[
A(w_d) > \left( \frac{c}{k} \right)^{p^-} > r.
\]

By Theorem 1, for all \( u \in E \) such that \( u \in A^{-1}([\infty, r]) \), we obtain
\[
\min \{ \|u\|^{p^+}, \|u\|^{p^-} \} \leq rp^+,
\]
\( \Rightarrow \|u\| \leq \max \left\{ (p^+ r)^{\frac{1}{p^+}}, (p^+ r)^{\frac{1}{p^-}} \right\} = \frac{c}{k}, \)
\( \Rightarrow \max_{z \in M} |u(z)| \leq k \|u\| \leq c \) (by (1)).
Also, we have

\[ B(w_d) = \int_M G(z, w_d(z)) \, dz + \frac{\mu}{\xi} \int_{M_2} H(\gamma(w_d(z))) \, d\sigma. \]

So, we deduce that

\[
\frac{1}{r} \sup_{A(u) \leq r} B(u) \leq \frac{\int_M \max_{|y| \leq c} G(z, y) \, dz + \frac{\mu}{\xi} \int_M \max_{|y| \leq c} H(y) \, d\sigma}{\frac{1}{p^r} \left( \frac{c}{k} \right)^{p^r}}
\]

\[ = p^+ \left( \frac{k}{c} \right)^{p^r} \left[ \int_M \max_{|y| \leq c} G(z, y) \, dz + \frac{\mu}{\xi} \sigma(M_2) \max_{|y| \leq c} H(y) \right]. \]

Now, if \( \max_{|y| \leq c} H(y) = 0 \), we have

\[
\frac{1}{r} \sup_{A(u) \leq r} B(u) < \frac{1}{\xi},
\]

and if \( \max_{|y| \leq c} H(y) > 0 \), it turns out to be true as

\[
\mu < \frac{c^{p^r} - \xi p^+ k^{p^r} \int_M \max_{|y| \leq c} G(z, y) \, dz}{p^+ k^{p^r} \sigma(M_2) \max_{|y| \leq c} H(y)}.
\]

By \((S_3)\) we get

\[
B(w_d) \geq \int_{B(z_0, sD)} G(z, d) \, dz
\]

\[
\Rightarrow \frac{B(w_d)}{A(w_d)} \geq \frac{\int_{B(z_0, sD)} G(z, d) \, dz}{A(w_d)} > \frac{1}{\xi}
\]

\[
\Rightarrow \frac{B(w_d)}{A(w_d)} > \frac{1}{r} \sup_{A(u) \leq r} B(u),
\]

\[
\Rightarrow \text{Theorem 2}(i) \text{ is true.}
\]

By Proposition 2, we know that Theorem 2(ii) holds true. Since all the regularity hypotheses of Theorem 2 on \( A \) and \( B \) are true, then Theorem 2 gives us the existence of at least three critical points of \( I_\xi \), which are three weak solutions of \( (P_{\xi,\mu}) \). \( \square \)

4. THREE WEAK SOLUTIONS OF BONANNO-CANDITO TYPE

In this section, we do not use hypothesis \((h)\) in establishing the existence of three weak solutions. Here, we assume that \( g \) and \( h \) are nonnegative. We apply the following three critical points result of Bonanno-Candito [3, Theorem 3.3].

**Theorem 4.** Let \((E, E^*)\) be a Banach pair with \( E \) reflexive. Let \( A : E \to \mathbb{R} \) be a convex, coercive and continuously Gâteaux differentiable functional whose derivative has a continuous inverse on \( E^* \), \( B : E \to \mathbb{R} \) be a continuously Gâteaux differentiable functional whose derivative is compact with

\[
\inf_{u \in E} A(u) = A(0) = B(0) = 0.
\]
If there exist \( r_1, r_2 > 0 \) and \( w \in E \), with \( 4r_1 < 2A(w) < r_2 \), satisfying

(i) \( \frac{1}{r_1} \sup_{A(u)<r_1} B(u) < \frac{2B(w)}{3A(w)} \);

(ii) \( \frac{1}{r_2} \sup_{A(u)<r_2} B(u) < \frac{B(w)}{3A(w)} \);

(iii) \( \inf_{s \in [0,1]} B(su_1 + (1-s)u_2) \geq 0 \), for all \( u_1, u_2 \in E \), with \( B(u_1) \geq 0 \) and \( B(u_2) \geq 0 \), which are local minima of \( I_\xi = A - \xi B \), for each \( \xi \in \hat{\Omega} \), where

\[ \hat{\Omega} := \left\{ \frac{3A(w)}{2}, \min \left\{ \frac{r_1}{\sup_{A(u)<r_1} B(u)}, \frac{r_2}{\sup_{A(u)<r_2} B(u)} \right\} \right\}, \]

then \( I_\xi \) has at least three distinct critical points in \( A^{-1}([-\infty, r_2]) \).

Next, we suppose that there are \( d \geq 1 \) and \( c_1, c_2 > 0 \), with \( \min\{c_1, c_2\} \geq k \), such that

\[ \frac{3}{2} \frac{A(w_d)}{\int_{B(z_0, sDD)} G(z, d)dz} < \min \left\{ \left( \frac{c_1}{k} \right)^{p^-}, \frac{1}{p\int M G(z, c_1)dz}, \frac{2p}{p\int M G(z, c_2)dz} \right\}, \]

where \( w_d : M \to \mathbb{R} \) is given as in (2), satisfying

\[ (S'_1) \frac{2}{p^+} \left( \frac{c_1}{k} \right)^{p^-} < A(w_d) < \frac{1}{2p^+} \left( \frac{c_1}{k} \right)^{p^-}; \]

\[ (S'_2) \max \left\{ \frac{\int M G(z, c_1)dz}{\left( \frac{c_1}{k} \right)^{p^-}}, \frac{\int M G(z, c_2)dz}{\left( \frac{c_1}{k} \right)^{p^-}} \right\} < \frac{2}{3} \frac{\int_{B(z_0, sDD)} G(z, d)dz}{\Phi(w_d)}; \]

\[ (S'_3) g(z, y) \geq 0 \) for each \( (z, y) \in M \times \mathbb{R}. \)

Here, we consider

\[ \xi \in \hat{\Omega} := \left[ \frac{3}{2} \frac{A(w_d)}{\int_{B(z_0, sDD)} G(z, d)dz}, \frac{1}{p\int M G(z, c_1)dz}, \frac{\left( \frac{c_1}{k} \right)^{p^-}}{2p\int M G(z, c_2)dz} \right], \]

so that

\[ \delta^* := \min \left\{ \left( \frac{c_1}{k} \right)^{p^-} - \xi p\int M G(z, c_1)dz, \left( \frac{c_1}{k} \right)^{p^-} - \frac{2p\int M G(z, c_2)dz}{\sigma(M_2)H(c_1)}, \frac{(c_1)^{p^-} - 2\xi p\int M G(z, c_2)dz}{2p\sigma(M_2)H(c_2)} \right\}. \]

**Theorem 5.** If \((S'_1)-(S'_3)\) hold, then we can find \( \delta^* > 0 \) as in (7) such that, for each \( \mu \in [0, \delta^*] \), problem \((P_\xi, \mu) \ (\xi \in \hat{\Omega}) \) has at least three distinct weak solutions \( u_*, u^*, \tilde{u} \), whose values range is the interval \([0, c_2]\).

**Proof.** For reader convenience we set \( r_1 := \frac{1}{p^+} \left( \frac{c_1}{k} \right)^{p^-} \) and \( r_2 := \frac{1}{p^+} \left( \frac{c_1}{k} \right)^{p^-} \). So, by \((S'_1)\) we have \( 4r_1 < 2A(w_d) < r_2 \). As \( \delta^* > \mu \) and \( H(z) \geq 0 \) for \( z > 0 \), we obtain

\[ \frac{1}{r_1} \sup_{A(u) \leq r_1} B(u) \leq \sup_{A(u) \leq r_1} \frac{\int M G(z, u(z))dz + \frac{\mu}{\xi} \int M \gamma(u(z)))d\sigma}{\frac{1}{p^+} \left( \frac{c_1}{k} \right)^{p^-}} \]

\[ = p^+ \left( \frac{k}{c_1} \right)^{p^-} \left[ \int_M G(z, c_1)dz + \frac{\mu}{\xi} \sigma(M_2)H(c_1) \right] \]
Also, we have

\[
\frac{2}{r_2} \sup_{A(u) \leq r_2} B(u) \leq 2 \sup_{A(u) \leq r_2} \frac{\int_M G(z, u(z))dz + \frac{\mu}{\xi} \int_{M_2} H(\gamma(u(z)))d\sigma}{\frac{1}{p^+} \left( \frac{c_2}{\xi} \right)^p} \\
\leq 2 \sup_{A(u) \leq r_2} \frac{\int_M G(z, u(z))dz}{\frac{1}{p^+} \left( \frac{c_2}{\xi} \right)^p} \\
< \frac{1}{\xi} \frac{2 \int_{B(z_0, sD)} G(z, d)dz}{A(w_d)} \leq \frac{2 B(w_d)}{3 A(w_d)}.
\]

This means that Theorem 4(i)-(ii) hold true.

Next, consider two local minima of \( I_\xi \), say \( u_* \), \( u^* \) \in \( E \). Clearly, \( u_* \), \( u^* \) are critical points of \( I_\xi \), and hence weak solutions of \( (P_{\xi, u^*}) \). We have to show that \( u_* \), \( u^* \geq 0 \). Let \( w \) be a weak solution of \( (P_{\xi, u}) \) so that

\[
\int_M \left| \nabla w \right|^{p(z)-2} \nabla w \cdot \nabla v|dz| + \int_M a(z)w^{p(z)-2}w|v|dz = \xi \int_M g(z, w)v dz + \mu \int_{M_2} h(\gamma(w))\gamma(v)d\sigma
\]

for all \( v \in E \). So, if we choose \( v = \min\{w, 0\} = w^- \in E \), we get

\[
\int_M \left| \nabla w^- \right|^{p(z)}dz + \int_M a(z)w^-^{p(z)}dz = \xi \int_M g(z, w^-)w^-dz + \mu \int_{M_2} h(\gamma(w^-))\gamma(w^-)d\sigma \leq 0
\]

(recall the sign assumptions on the data).

This leads to \( \|w^-\| = 0 \), which is absurd, and hence \( u_* \), \( u^* \) are nonnegative. So, we have

\[
su_* + (1 - s)u^* \geq 0 \quad \text{for all } s \in [0, 1],
\]

\[
\Rightarrow B(su_* + (1 - s)u^*) \geq 0 \quad \text{for all } s \in [0, 1],
\]

\[
\Rightarrow \quad \text{Theorem 4(iii) is true.}
\]

Since all the regularity hypotheses of Theorem 4 on \( A \) and \( B \) remain true, we conclude that \( (P_{\xi, u}) \) has at least three distinct weak solutions for each \( \xi \in \widehat{\Omega} \). \( \square \)

**References**


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