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COMPARISON RESULTS FOR A LINEAR ELLIPTIC EQUATION WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this paper we study a linear elliptic equation having mixed boundary conditions, defined in a connected open set Ω of \mathbb{R}^n . We prove a comparison result with a suitable "symmetrized" Dirichlet problem which cannot be uniformly elliptic depending on the regularity of $\partial\Omega$. Regularity results for non-uniformly elliptic equations are also given.

1. INTRODUCTION

Let Ω be a connected bounded open set of \mathbb{R}^n whose boundary $\partial\Omega$ is made of two manifolds Γ_0 and Γ_1 , having Γ_0 positive $(n-1)$ -dimensional Hausdorff measure. We consider the mixed problem

$$\begin{cases} -(a_{ij}(x)u_{x_i})_{x_j} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_1, \end{cases} \quad (1.1)$$

where $a_{ij}(x)$, $i, j = 1, \dots, n$, are bounded, measurable functions on Ω satisfying the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 \quad \text{for a. e. } x \in \Omega, \forall \xi \in \mathbb{R}^n \quad (1.2)$$

and $f \in L^p(\Omega)$, $p > \frac{2n}{n+2}$ if $n > 2$, $p > 1$ if $n = 2$.

It is well known that a way to obtain sharp estimates for solutions of elliptic problems is the comparison with solutions of suitable symmetrized problems using Schwarz symmetrization (see [18], [19], [3], [1]). In this order of ideas we are able to compare the solution of problem (1.1) with the solution of a Dirichlet problem with spherically symmetric data, defined in

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a ball having the same measure as Ω . This problem cannot be uniformly elliptic depending on the shape of Ω .

To be more precise let us define (see also [14]) the function

$$\lambda(t) = \inf_E P_\Omega(E), \quad t \geq 0, \quad (1.3)$$

where $P_\Omega(E)$ is the perimeter of E relative to Ω and the set E varies in the class of measurable subsets of Ω such that $|E| = t$ and $\partial E \cap \Gamma_0$ does not contain any set of positive $(n-1)$ -dimensional Hausdorff measure. Let us suppose that

$$\sup_{t>0} \frac{t^\alpha}{\lambda(t)} < \infty, \quad 1 - \frac{1}{n} \leq \alpha \leq \frac{3}{2} - \frac{1}{n}. \quad (1.4)$$

We observe that in (1.4), $\alpha < 1 - \frac{1}{n}$ cannot hold for any set Ω , since this condition is denied by the classical isoperimetric inequality. On the other hand (1.4) implies a relative isoperimetric inequality of the kind

$$|E|^\alpha \leq Q P_\Omega(E), \quad \forall E \subset \Omega, \quad E \text{ measurable}. \quad (1.5)$$

If $\alpha = 1 - \frac{1}{n}$, (1.4) means that a relative isoperimetric inequality holds (see e. g. [11], [17], [8]). Depending on the shape of Γ_1 , condition (1.4) can fail for $\alpha = 1 - \frac{1}{n}$; this happens, for instance, when Γ_1 has some cuspidal points (see §2 for an example).

We consider the problem

$$\begin{cases} -(\nu(\omega_n |x|^n) w_{x_i})_{x_i} = n^2 \omega_n^{2/n} f^\# & \text{in } \Omega^\# \\ w = 0 & \text{on } \partial\Omega^\#, \end{cases} \quad (1.6)$$

where $f^\#$ is the spherically symmetric rearrangement of f (see §2 for the definition), $\Omega^\#$ is the ball centered at the origin, having the same area as Ω , and $\nu(t) = t^{-2+2/n} \lambda^2(t)$. If (1.4) holds true, we prove the following estimates

$$\begin{aligned} u^\#(x) &\leq w(x) \quad \text{a.e. } x \in \Omega^\#, \\ \int_\Omega |Du|^2 dx &\leq n^2 \omega_n^{2/n} \int_{\Omega^\#} \nu |Dw|^2 dx, \end{aligned} \quad (1.7)$$

where $w \in H_0^1(\nu, \Omega^\#)$ ¹ is the solution of problem (1.6). We explicitly observe that problem (1.6) is uniformly elliptic when in (1.4) $\alpha = 1 - \frac{1}{n}$. In this case estimates (1.7) give the same upper bound as in [17] where problem (1.1) is compared with a problem defined in a sector with mixed boundary conditions. When $\alpha > 1 - \frac{1}{n}$ the weak assumption on the regularity of $\partial\Omega$ implies that problem (1.6) is not uniformly elliptic and the estimates of the solution are given in terms of the function $\lambda(t)$. In §2, we give an example of a domain Ω whose boundary is characterized by the behavior of the function $\lambda(t)$ as a power of t with exponent greater than $1 - \frac{1}{n}$. Comparison (1.7) allows us to obtain estimates for $u(x)$ in terms of the solution of problem (1.6). Let us observe that under the assumption (1.4), $\frac{1}{\nu}$ belongs to a Marcinkiewicz space M^t . Then §3 is devoted to prove regularity results for a class of degenerate problems, when $\frac{1}{\nu} \in M^t$ (see [14] for Neumann problem). These estimates allow us to obtain existence and regularity results for problem (1.1) under the assumption (1.4). When $\frac{1}{\nu}$ is in Lorentz or Lebesgue spaces a priori estimates for problem (1.6) are given in [4], [20], [7].

Let us remark that, when $\Gamma_0 = \partial\Omega$, (1.4) holds with $\alpha = 1 - \frac{1}{n}$, since in (1.3) $P_\Omega(E) = P_{\mathbb{R}^n}(E)$ and then (1.4) gives the classical isoperimetric inequality. In this case the comparison result (1.7) is exactly the one obtained by Talenti in [18] for the Dirichlet problem.

When $\mathcal{H}^{n-1}(\Gamma_0) \rightarrow 0$, that is problem (1.1) approaches a Neumann problem, (1.4) does not hold since we can take sets $E \subset \Omega$ having measure near to $|\Omega|$ and relative perimeter arbitrarily small. In this case results in this order of ideas have been obtained in [15], where the Neumann problem is compared with two Dirichlet problems defined in two balls having measure $\frac{|\Omega|}{2}$.

2. COMPARISON RESULTS

We begin this section by recalling some definitions that will be useful in the following. Let Ω be an open, bounded set of \mathbb{R}^n and let us consider a measurable function $u : \Omega \rightarrow \mathbb{R}$. The distribution function of u is defined by

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geq 0,$$

¹Let $\nu \geq 0$, $\nu^{-1} \in L^t(\Omega)$, for some $t \geq 1$; $H_0^1(\nu, \Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1(\nu, \Omega)} = \left(\int_\Omega \nu |Du|^2 dx \right)^{1/2}.$$

while the decreasing rearrangement of u is defined as the distribution function of μ , i.e.,

$$u^*(s) = \sup \{t \geq 0 : \mu(t) \geq s\}, \quad s \in [0, |\Omega|].$$

By using the previous notations we also introduce the decreasing spherically symmetric rearrangement of u as follows

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#,$$

where $\Omega^\#$ denotes the ball centered at the origin, having the same measure as Ω , and ω_n is the measure of the unitary ball in \mathbb{R}^n .

The theory of rearrangements is well-known and exhaustive treatments of it can be found, for example, in [18], [5], [13].

We are interested in proving a comparison result between the solution of problem (1.1) and the solution of the symmetrized problem (1.6) defined in $\Omega^\#$. Our main result is the following:

Theorem 2.1. *Let Ω be a connected bounded open set of \mathbb{R}^n such that its boundary $\partial\Omega$ consists of two manifolds, Γ_0 and Γ_1 , with $\mathcal{H}^{n-1}(\Gamma_0) > 0$ and let us suppose that (1.4) holds true. Let u and w be solutions of problems (1.1) and (1.6) with $\nu(\omega_n |x|^n) = \omega_n^{-2+2/n} |x|^{2-2/n} \lambda^2(\omega_n |x|^n)$, respectively, and let f be so regular in order to guarantee the existence of w . Then*

$$(i) \quad u^\#(x) \leq w(x) \quad \text{a.e. } x \in \Omega^\#,$$

$$(ii) \quad \int_{\Omega} |Du|^2 dx \leq n^2 \omega_n^{2/n} \int_{\Omega^\#} \nu |Dw|^2 dx.$$

Proof. For $h > 0$ and $t \geq 0$ we define

$$\varphi_h(x) = \begin{cases} h \operatorname{sign} u & \text{if } |u| > t+h \\ (|u|-t) \operatorname{sign} u & \text{if } t < |u| \leq t+h \\ 0 & \text{otherwise.} \end{cases}$$

We use φ_h as test function in (1.1), then we get

$$\frac{1}{h} \int_{t < |u| \leq t+h} a_{ij} u_{x_i} u_{x_j} dx = \int_{|u| > t+h} f \operatorname{sign} u dx + \frac{1}{h} \int_{t < |u| \leq t+h} f(|u|-t) \operatorname{sign} u dx.$$

Using (1.2), letting h goes to 0, we have

$$-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \leq \int_{|u|>t} |f| dx. \quad (2.1)$$

We now proceed to evaluate the left hand side of (2.1) by the following inequalities:

$$-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \leq (-\mu'(t))^{\frac{1}{2}} \left(-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \right)^{\frac{1}{2}}$$

and

$$-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx = P_{\Omega}(\{x \in \Omega : |u| > t\}) \geq \lambda(\mu(t)), \quad (2.2)$$

where $\mu(t)$ denotes the distribution function of $u(x)$. We gather

$$\frac{\lambda^2(\mu(t))}{(-\mu'(t))} \leq -\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \leq \int_{|u|>t} |f| dx \leq \int_0^{\mu(t)} f^*(r) dr; \quad (2.3)$$

the last inequality in (2.3) is the Hardy-Littlewood theorem. We rewrite (2.3) in the form

$$1 \leq \lambda^{-2}(\mu(t)) (-\mu'(t)) \int_0^{\mu(t)} f^*(r) dr,$$

and, by integrating between 0 and t , we obtain

$$t \leq \int_0^t \lambda^{-2}(\mu(\tau)) (-\mu'(\tau)) \int_0^{\mu(\tau)} f^*(r) dr = \int_{\mu(t)}^{\Omega} \lambda^{-2}(\tau) d\tau \int_0^{\tau} f^*(r) dr.$$

By definition of decreasing rearrangement, we then get

$$u^*(s) \leq \int_s^{|\Omega|} \lambda^{-2}(\tau) d\tau \int_0^{\tau} f^*(r) dr = w^*(s), \quad s \in [0, |\Omega|], \quad (2.4)$$

where w is the solution of problem (1.6), i.e., (i).

In order to prove (ii), using Hölder inequality, (2.2) and (2.3), we obtain

$$\begin{aligned} \int_{\Omega} |Du|^2 dx &\leq \int_0^{|\Omega|} \left\{ \frac{1}{(-\mu'(t))} \left(-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \right) \right\} (-d\mu(t)) \\ &\leq \int_0^{|\Omega|} \left\{ \frac{1}{\lambda(\mu(t))} \left(-\frac{d}{dt} \int_{|u|>t} |Du|^2 dx \right) \right\}^2 (-d\mu(t)) \\ &\leq \int_0^{|\Omega|} \left\{ \frac{1}{\lambda(s)} \int_0^s f^*(r) dr \right\}^2 ds = n^2 \omega_n^{2/n} \int_{\Omega^\#} \nu |Dw|^2 dx, \end{aligned}$$

where $\nu = \nu(\omega_n |x|^n)$. \square

Remark 1. From (i) we can deduce the following estimates

$$\operatorname{ess\,sup} |u| \leq \operatorname{ess\,sup} |w|; \quad (2.5)$$

$$\int_{\Omega} |u|^q dx \leq \int_{\Omega^{\#}} |w|^q dx, \quad q > 0.$$

Remark 2. If $\alpha = 1 - \frac{1}{n}$, a relative isoperimetric inequality holds true

$$|E|^{1-\frac{1}{n}} \leq Q P_{\Omega}(E), \quad \forall E \subset \Omega, E \text{ measurable},$$

where $Q = Q(\Gamma_1, E)$. Then we can write explicitly $\lambda(t)$ in (2.4) if we are able to evaluate the constant Q . In [17] Pacella and Tricarico proved that, if $\mathcal{H}^{n-1}(\Gamma_1) > 0$, then

$$Q = (n\beta_n^{1/n})^{-1},$$

where β_n is the Lebesgue measure of the unitary sector² in \mathbb{R}^n , with amplitude $\beta \in (0, \frac{\pi}{2}]$. In this case we can write (2.4) as follows

$$u^*(s) \leq \frac{1}{n^2 \beta_n^{2/n}} \int_s^{|\Omega|} dr r^{-2+\frac{2}{n}} \int_0^r f^*(\tau) d\tau, \quad s \in (0, |\Omega|)$$

and our comparison results reduce to those ones contained in [17], between a solution u of (1.1) and the solution of a symmetrized mixed problem in a sector with amplitude β .

In what follows we give an example of a domain Ω where hypotheses (1.4) is satisfied for $\alpha > 1 - \frac{1}{n}$ computing $\lambda(t)$ and evaluating $\sup_{t>0} \frac{t^\alpha}{\lambda(t)}$. We consider the set $\Omega \subset \mathbb{R}^2$ delimited by the x -axis, the parabola $y = x^2$ and the circle centered on the x -axis and orthogonal to the parabola in the point $(\frac{1}{2}, \frac{1}{4})$ (see figure), that is the "cone"

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : 0 < x < \frac{\sqrt{2}}{4} + \sqrt{2}, 0 < y < x^2, 0 < \left(x - \frac{1}{4}\right)^2 + y^2 < \frac{1}{8} \right\}.$$

Let $\Gamma_0 = \{(x, y) \in \partial\Omega : (x - \frac{1}{4}t)^2 + y^2 = \frac{1}{8}\}$ and $\Gamma_1 = \partial\Omega - \Gamma_0$. We will prove that

$$\sup_{t>0} \frac{t^\alpha}{\lambda(t)} < \infty \quad \text{if } \alpha = \frac{2}{3}.$$

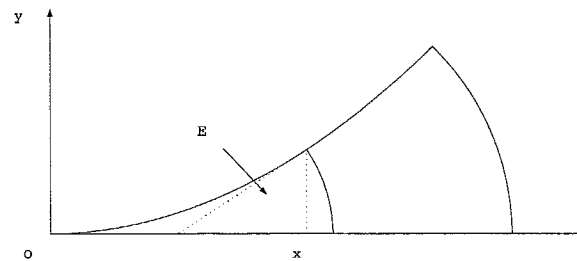
First of all we prove the existence of a set $E \subseteq \Omega$ having measure t and minimum perimeter between all the measurable subsets of Ω such that $\partial E \cap \Gamma_0$ does not contain any subset of positive 1-dimensional Hausdorff measure. The existence of such a set is proved in [11] if the boundary of Ω is locally

²The sector $A(\beta, R)$ in \mathbb{R}^n , $\beta \in [0, \pi[$ and $R > 0$ is defined as the set

$$A(\beta, R) = \{x \in \mathbb{R}^n : 0 < |x| < R, (x, \xi) > \cos \beta \cdot |x|\}$$

where ξ is the vector $(1, 0, \dots, 0)$ and (\cdot, \cdot) denotes the scalar product in \mathbb{R}^n .

We call unitary sector with amplitude β the set $A(\beta, 1)$.



Lipschitz. Since $\partial\Omega$ has only one point where it is not Lipschitz, i.e., $(0, 0)$, and $(0, 0)$ does not belong to $\partial E \cap \Omega$, we can adapt the proof contained in [11] to our case and by means of a simple extension argument (see [10]) we can prove that such a set exists. It is easy to realize that $\partial E \cap \Omega$ must be concave towards E and $\partial E \cap \Omega$ must be a curve having one terminal point on $y = 0$ and the other one on $y = x^2$. The regular case is studied in [8].

Then we can suppose that $\partial E \cap \Omega$ is the curve of polar equation $\rho = \rho(\vartheta)$, $\vartheta \in [0, \gamma_t]$. If we impose that $|E| = t$ and $\partial E \cap \Omega$ has minimum length, we find that $\rho(\vartheta)$ satisfies the following Euler equation

$$\frac{\rho}{\sqrt{\rho^2 + \rho'^2}} - \frac{d}{d\vartheta} \frac{\rho'}{\sqrt{\rho^2 + \rho'^2}} + 2\mu\rho = 0, \quad (\mu = \text{Lagrange multiplier}) \quad (2.6)$$

the boundary condition

$$\rho'(0) = 0 \quad (2.7)$$

and the transversality condition

$$\rho'(\gamma_t) = -\frac{\sin^2 \gamma_t}{\cos \gamma_t (1 + \sin^2 \gamma_t)}. \quad (2.8)$$

Since solutions of equation (2.6) are a family of circles, the unique solution of (2.6) that satisfies (2.7) and (2.8) is the circular arc orthogonal to $\partial\Omega$ for $\vartheta = 0$ and $\vartheta = \gamma_t$ (see figure).

Let $x_0 = x_0(t)$ be the first cartesian coordinate of the point $(\rho(\gamma_t), \gamma_t)$; it is easy to find that $\partial E \cap \Omega$ is the circular arc centered in $(\frac{x_0}{2}, 0)$ and x_0 is

the unique solution of

$$|E| = \frac{x_0^2}{8} \arctan(2x_0) (1 + 4x_0^2) + \frac{x_0^3}{12} = t.$$

The length of $\partial E \cap \Omega$ is $\frac{x_0}{2} \arctan(2x_0) \sqrt{1 + 4x_0^2}$ and the quotient

$$\frac{t^{2/3}}{\lambda(t)} = \frac{\left[\frac{x_0^2}{8} \arctan(2x_0) (1 + 4x_0^2) + \frac{x_0^3}{12} \right]^{2/3}}{\frac{x_0}{2} \arctan(2x_0) \sqrt{1 + 4x_0^2}}$$

is a strictly increasing function of x_0 and then of t . Then $\sup \frac{t^{2/3}}{\lambda(t)}$ is achieved when $t = |\Omega|$ and it is $\frac{1}{\sqrt[3]{9}} \frac{(3\pi+2)^{2/3}}{\sqrt{2\pi}}$.

3. REGULARITY RESULTS FOR SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

In this section we consider the problem

$$\begin{cases} -(a_{ij}(x)z_{x_i})_{x_j} = f & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where Ω is an open, bounded subset of \mathbb{R}^n , $a_{ij}(x)$ are such that

$$a_{ij}(x)\xi_i\xi_j \geq \nu(x)|\xi|^2, \quad \text{a. e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n \quad (3.2)$$

and $\nu \in L^1(\Omega)$ is a non-negative function.

We investigate the regularity of solutions of (3.1) when $\frac{1}{p}$ belongs to the Marcinkiewicz space M^t and f is in a Lorentz space. Let us first recall the definition of these spaces.

If we put

$$\bar{f}(s) = \frac{1}{s} \int_0^s f^*(r) dr$$

we say that a function $f \in L(p, q)$, $1 < p < \infty$, $1 \leq q \leq \infty$, if the quantity

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left(\bar{f}(s) s^{1/p} \right)^q \frac{ds}{s} \right)^{1/q} & 1 \leq q < \infty \\ \sup_{s>0} \bar{f}(s) s^{1/p} & q = \infty \end{cases}$$

is finite. We remark that $L(p, p) = L^p(\Omega)$ and for $1 < q < p < r < \infty$ the following inclusions hold

$$L^r(\Omega) \subset L(p, 1) \subset L(p, q) \subset L^p(\Omega) \subset L(p, r) \subset L(p, \infty) \subset L^q(\Omega).$$

Now let z be a measurable function on Ω , $1 < t < \infty$ and $\frac{1}{t} + \frac{1}{t'} = 1$. Let

$$\|z\|_{M^t} = \min \left\{ C \geq 0 : \int_K |z(x)| dx \leq C |K|^{1/t'} \text{ for all measurable } K \subset \Omega \right\};$$

M^t is the set of measurable functions z on Ω satisfying $\|z\|_{M^t} < \infty$. It is easy to verify that M^t is a Banach space under the norm $\|\cdot\|_{M^t}$ and $M^t = L(t, \infty)$.

Before entering into details, let us recall some other preliminary results. We first introduce a relation between nonnegative functions in $L^1(\Omega)$. We say that f is dominated by g , and we write $f \prec g$ if

$$\int_0^s f^*(r) dr \leq \int_0^s g^*(r) dr \quad \forall s \in [0, |\Omega|] \text{ and } \int_0^{|\Omega|} f^*(r) dr = \int_0^{|\Omega|} g^*(r) dr. \quad (3.3)$$

We explicitly observe that the definition given in (3.3) makes sense also if f and g are defined in different sets with the only restriction that these sets should have the same measure. Various properties and characterizations of such a relation are given, for example, in [9] and [2]. We only recall the following:

Theorem 3.1. *The following statements are equivalent:*

i) $f \prec g$;

ii) $\int_\Omega f(x) \eta(x) dx \leq \int_0^{|\Omega|} g^*(s) \eta^*(s) ds, \quad \int_\Omega f(x) dx = \int_\Omega g(x) dx$
for all non negative $\eta \in L^\infty(\Omega)$;

iii) $\int_0^{|\Omega|} f^*(s) \eta^*(s) ds \leq \int_0^{|\Omega|} g^*(s) \eta^*(s) ds, \quad \int_\Omega f(x) dx = \int_\Omega g(x) dx$
for all non negative $\eta \in L^\infty(\Omega)$.

Obviously the above theorem implies that

$$\|f\| \leq \|g\| \quad (3.4)$$

in any Lorentz space.

Now, let z be a measurable function in Ω . We consider for any $s \in [0, |\Omega|]$ a subset $E(s) \subset \Omega$ such that

1) $|E(s)| = s$;

2) $s_1 < s_2 \implies E(s_1) \subset E(s_2)$;

3) $E(s) = \{x \in \Omega : |z(x)| > t\}$ if $s = \mu(t)$.

For any $f \in L^1(\Omega)$, $f \geq 0$, there exists a function F such that

$$\int_{E(s)} f dx = \int_0^s F(t) dt.$$

In general, F is not a rearrangement of f , but $F \prec f$ (see [4], [3]). Roughly speaking, we say that F is built from f on the level sets of z .

The main tool to obtain our regularity result for solutions of problem (3.1) is the following theorem due to Alvino and Trombetti [4].

Theorem 3.2. Let $\nu \geq 0$, $\nu \in L^1(\Omega)$, $\frac{1}{\nu} \in L^t(\Omega)$ for some $t \geq 1$ and $f \in L^p(\Omega)$, $\frac{1}{p} = \frac{1}{2} - \frac{1}{2t} + \frac{1}{n}$. If $z \in H_0^1(\nu, \Omega)$ is a solution of (3.1), then

$$z^\#(x) \leq \frac{1}{n^2 \omega_n^{2/n}} \int_{\omega_n |x|^n}^{| \Omega |} \frac{r^{-2+2/n}}{\underline{\nu}(r)} dr \int_0^r f^*(s) ds \quad x \in \Omega^\#; \quad (3.5)$$

moreover, the following estimate holds

$$\int_\Omega \nu |Dz|^2 dx \leq \int_0^{| \Omega |} \frac{1}{\underline{\nu}(r)} \left(\frac{r^{-1+1/n}}{n \omega_n^{1/n}} \int_0^r f^*(s) ds \right)^2 dr, \quad (3.6)$$

where $\frac{1}{\underline{\nu}}$ is built from $\frac{1}{\nu}$ on the level sets of z .

By (3.6), if we assume that (3.2) holds with

$$\begin{aligned} \nu &\in L^1(\Omega), \nu(x) \geq 0 \text{ for a. e. } x \in \Omega \\ \frac{1}{\nu} &\in M^t \text{ for some } t > 1 \end{aligned} \quad (3.7)$$

and we suppose that

$$f \in L(p, 2) \quad \text{with} \quad \frac{1}{p} = \frac{1}{2} + \frac{1}{n} - \frac{1}{2t} \quad (3.8)$$

we obtain an estimate of the norm $\|z\|_{H_0^1(\nu, \Omega)}$ in terms of the norm of f and $\frac{1}{\nu}$.

More precisely we have the following:

Theorem 3.3. If $z \in H_0^1(\nu, \Omega)$ is solution of (3.1) under the assumptions (3.2), (3.7), (3.8), then

$$\int_\Omega \nu |Dz|^2 dx \leq \frac{1}{n^2 \omega_n^{2/n}} \left[\frac{p}{2(p-1)} + 2 - \frac{2}{n} \right] \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2. \quad (3.9)$$

Remark. We explicitly observe that under the assumption (1.4) the function $\frac{1}{\nu(x)} = \omega_n^{2-2/n} |x|^{-2+2/n} \lambda^{-2}(\omega_n |x|^n) \in M^t(\Omega^\#)$ and hence by iii) of Theorem 2.1 and (3.9) we obtain the following estimate for the solution u of problem (1.1) with $f \in L(p, 2)$

$$\int_\Omega |Du|^2 dx \leq \left[\frac{p}{2(p-1)} + 2 - \frac{2}{n} \right] \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2.$$

The a priori estimate implies the solvability of problem (1.1) under assumption (1.4).

Proof. By (3.6) we have

$$\int_\Omega \nu |Dz|^2 dx \leq \int_0^{| \Omega |} \left(\frac{d}{dr} \int_0^r \frac{1}{\underline{\nu}(\sigma)} d\sigma \right) \frac{r^{-2+2/n}}{n^2 \omega_n^{2/n}} \left(\int_0^r f^*(s) ds \right)^2 dr.$$

Integrating by parts we get

$$\begin{aligned} \|z\|_{H_0^1(\nu, \Omega)}^2 &\leq \frac{1}{n^2 \omega_n^{2/n}} \left\{ | \Omega |^{-2+2/n} \left(\int_0^{| \Omega |} \frac{1}{\underline{\nu}(s)} ds \right) \left(\int_0^{| \Omega |} f^*(s) ds \right)^2 \right. \\ &\quad - \lim_{r \rightarrow 0} \left(\int_0^r \frac{1}{\underline{\nu}(s)} ds \right) r^{-2+2/n} \left(\int_0^r f^*(s) ds \right)^2 \\ &\quad + \left(2 - \frac{2}{n} \right) \int_0^{| \Omega |} dr \left(\int_0^r \frac{1}{\underline{\nu}(s)} ds \right) r^{-3+2/n} \left(\int_0^r f^*(\sigma) d\sigma \right)^2 \\ &\quad \left. - 2 \int_0^{| \Omega |} dr \left(\int_0^r \frac{1}{\underline{\nu}(s)} ds \right) r^{-2+2/n} \left(\int_0^r f^*(s) ds \right) f^*(r) \right\} \\ &= I_1 - I_2 + I_3 - I_4. \end{aligned} \quad (3.10)$$

By recalling the definition of norm in M^t and (3.4) we get

$$\begin{aligned} &\left(\int_0^r \frac{1}{\underline{\nu}(s)} ds \right) r^{-2+2/n} \left(\int_0^r f^*(s) ds \right)^2 \\ &\leq \left\| \frac{1}{\nu} \right\|_{M^t} r^{-1-1/t+2/n} \left(\int_0^r f^*(s) ds \right)^2 \leq \frac{p}{2(p-1)} \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2; \end{aligned} \quad (3.11)$$

thus, $0 \leq I_2 < \infty$. By the same arguments we get

$$I_1 \leq \frac{1}{n^2 \omega_n^{2/n}} \frac{p}{2(p-1)} \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2. \quad (3.12)$$

On the other hand

$$\begin{aligned} I_3 &\leq \frac{1}{n^2 \omega_n^{2/n}} \left(2 - \frac{2}{n} \right) \left\| \frac{1}{\nu} \right\|_{M^t} \int_0^{| \Omega |} r^{2/n-1/t} \bar{f}(r)^2 dr \\ &= \frac{1}{n^2 \omega_n^{2/n}} \left(2 - \frac{2}{n} \right) \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2, \end{aligned} \quad (3.13)$$

and, reasoning in the same way, $0 \leq I_4 < \infty$. Thus, by (3.12) and (3.13), disregarding non-positive terms in the right-hand side of (3.10), we get

$$\|z\|_{H_0^1(\nu, \Omega)}^2 \leq \frac{1}{n^2 \omega_n^{2/n}} \left[\frac{p}{2(p-1)} + 2 - \frac{2}{n} \right] \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{p,2}^2,$$

that is, (3.9). \square

Using the pointwise estimate (3.9) it is possible to prove regularity results for solutions of problem (3.1) in Lorentz spaces. We know that, if $f \in L(p, 2)$, with $\frac{1}{p} = \frac{1}{2} + \frac{1}{n} - \frac{1}{2t}$, then the solution $z \in H_0^1(\Omega)$ belongs, by Sobolev embeddings, to the Lorentz space $L(\beta, 2)$, with $\frac{1}{\beta} = \frac{1}{2} - \frac{1}{n} + \frac{1}{2t}$. The theorem that follows shows in Lorentz spaces how the summability of z improves by improving the summability of f until we have $z \in L^\infty(\Omega)$ when $f \in L(q, 1)$ with $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$. Such results extend, in the linear case, the results contained in [7] where the case $\frac{1}{p} \in L(t, h)$, $1 \leq h \leq t$, is considered. The following theorem holds.

Theorem 3.4. *Let $z \in H_0^1(\nu, \Omega)$ be a solution of (3.1) under the assumptions (3.2), (3.7) and let $f \in L(q, k)$, with $\frac{1}{q} < \frac{1}{2} + \frac{1}{n} - \frac{1}{2t}$. Then the following results hold:*

i) *if $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$, $k = 1$, then $z \in L^\infty(\Omega)$. Besides*

$$\|z\|_\infty \leq \frac{1}{n^2 \omega_n^{2/n}} \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{q,1};$$

ii) *if $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$, $k > 1$, then $z \in L_{\Phi_k}$ where L_{Φ_k} denotes the Orlicz space generated by the function $\Phi_k(s) = \exp(|s|^{k'}) - 1$. Besides*

$$\int_\Omega e^{\alpha z^{k'}} dx \leq C, \quad (3.14)$$

where α is a constant depending on $n, k, q, \left\| \frac{1}{\nu} \right\|, \|f\|$ and C is a constant depending on $n, k, q, \left\| \frac{1}{\nu} \right\|, \|f\|$ and $|\Omega|$;

iii) *if $\frac{1}{q} > \frac{2}{n} - \frac{1}{t}$, $k \geq 1$, then $z \in L(\beta, k)$ with $\frac{1}{\beta} = \frac{1}{q} + \frac{1}{t} - \frac{2}{n}$. Besides*

$$\|z\|_{\beta,k} \leq \left(\frac{\beta^2}{\beta - 1} \right) \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{q,k}^*.$$

Proof. i) From (3.5) we get

$$\|z\|_\infty \leq \frac{1}{n^2 \omega_n^{2/n}} \int_0^{|\Omega|} \frac{r^{-2+2/n}}{\underline{\nu}(r)} dr \int_0^r f^*(s) ds;$$

since the function $\frac{1}{\underline{\nu}}$ is dominated by $\frac{1}{\nu}$ and the function

$$g(r) = r^{-2+2/n} \int_0^r f^*(s) ds$$

is decreasing, from Theorem 3.1 we deduce

$$\begin{aligned} \|z\|_\infty &\leq \frac{1}{n^2 \omega_n^{2/n}} \int_0^{|\Omega|} \left(\frac{1}{\underline{\nu}(r)} \right)^* r^{-2+2/n} \left(\int_0^r f^*(s) ds \right) dr \\ &\leq \frac{1}{n^2 \omega_n^{2/n}} \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \int_0^{|\Omega|} r^{2/n-1/t} \bar{f}(r) \frac{dr}{r} = \frac{1}{n^2 \omega_n^{2/n}} \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \|f\|_{q,1}. \end{aligned}$$

ii) From (3.5), integrating by parts we get

$$\begin{aligned} z^*(s) &\leq \frac{1}{n^2 \omega_n^{2/n}} \left(\int_0^{|\Omega|} \frac{1}{\underline{\nu}(r)} dr \right) |\Omega|^{-2+2/n} \left(\int_0^{|\Omega|} f^*(r) dr \right) \quad (3.15) \\ &\quad - \left(\int_0^s \frac{1}{\underline{\nu}(r)} dr \right) s^{-2+2/n} \left(\int_0^s f^*(r) dr \right) \\ &\quad + \left(2 - \frac{2}{n} \right) \int_s^{|\Omega|} \left(\int_0^r \frac{1}{\underline{\nu}(\sigma)} d\sigma \right) r^{-3+2/n} \left(\int_0^r f^*(\sigma) d\sigma \right) dr \\ &\quad - \int_s^{|\Omega|} \left(\int_0^r \frac{1}{\underline{\nu}(\sigma)} d\sigma \right) r^{-2+2/n} f^*(r) dr \\ &= I_1 - I_2 + I_3 - I_4. \end{aligned}$$

Let us consider the first and second integrals in the right-hand side of (3.15), i.e., I_1, I_2 . By assumptions we have

$$I_1, I_2 \leq \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} |\Omega|^{2/n-1/t-1} \left(\int_0^{|\Omega|} f^*(r) dr \right) \leq C_1 \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \|f\|_{q,k}.$$

On the other hand $-\infty < I_2 \leq 0$. Moreover, by Hölder inequality we get

$$\begin{aligned} I_3 &\leq \left(2 - \frac{2}{n} \right) \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \int_s^{|\Omega|} r^{-2-1/t+2/n} \left(\int_0^r f^*(\sigma) d\sigma \right) dr \\ &\leq \left(2 - \frac{2}{n} \right) \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \left(\log \frac{|\Omega|}{s} \right)^{1-1/k} \left(\int_s^{|\Omega|} r^{2k/n-k/t} \bar{f}(r)^k \frac{dr}{r} \right)^{1/k} \\ &= \left(\log \frac{|\Omega|}{s} \right)^{1-1/k} \left(2 - \frac{2}{n} \right) \left\| \frac{1}{\underline{\nu}} \right\|_{M^t} \|f\|_{q,k} \end{aligned}$$

and, reasoning in the same way, we prove that $-\infty < I_4 \leq 0$. Thus, disregarding non-positive terms in (3.15) we gather

$$z^*(s) \leq C_1 + C_2 \left(\log \frac{|\Omega|}{s} \right)^{1-1/k}$$

and then there exists $\alpha > 0$ such that

$$\int_0^{|\Omega|} e^{[\alpha z^*(s)]^{k'}} ds < +\infty.$$

iii) Reasoning as in the previous case, disregarding non-positive terms in (3.15), we can write

$$\begin{aligned} z^*(s) &\leq \frac{1}{n^2 \omega_n^{2/n}} \left(\int_0^{|\Omega|} \frac{1}{\nu(r)} dr \right) |\Omega|^{-2+2/n} \left(\int_0^{|\Omega|} f^*(r) dr \right) \\ &\quad + \frac{1}{n^2 \omega_n^{2/n}} \left(2 - \frac{2}{n} \right) \int_s^{|\Omega|} \left(\int_0^r \frac{1}{\nu(\sigma)} d\sigma \right) r^{-3+2/n} \left(\int_0^r f^*(\sigma) d\sigma \right) dr \\ &\leq C_1 \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{q,k} + C_2 \left\| \frac{1}{\nu} \right\|_{M^t} \int_s^{|\Omega|} r^{-2-1/t+2/n} \left(\int_0^r f^*(\sigma) d\sigma \right) dr. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{z}(s) &= \frac{1}{s} \int_0^s z^*(r) dr \\ &\leq C_1 \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{q,k} + C_2 \frac{1}{s} \int_0^s ds \int_r^{|\Omega|} \sigma^{-2-1/t+2/n} \int_0^\sigma f^*(\rho) d\rho \\ &\leq C_1 \left\| \frac{1}{\nu} \right\|_{M^t} \|f\|_{q,k} + C_2 \left\| \frac{1}{\nu} \right\|_{M^t} \int_0^{|\Omega|} \frac{r^{2/n-1/t}}{\max\{s, r\}} \bar{f}(r) dr. \end{aligned}$$

By definition of norm in Lorentz space $L(\beta, k)$, using Theorem 319 in [12] (the estimate is obvious if $k = 1$), we have

$$\begin{aligned} \left(\|z\|_{\beta,k}^* \right)^k &\leq C_1 \left\| \frac{1}{\nu} \right\|_{M^t}^k \|f\|_{q,k}^k \\ &\quad + C_2 \left\| \frac{1}{\nu} \right\|_{M^t}^k \int_0^\infty \left(\int_0^{|\Omega|} \frac{s^{1/\beta-1/k}}{\max\{s, r\} r^{1/\beta-1/k}} r^{2/n+1/\beta-1/t-1/k} \bar{f}(r) dr \right)^k ds \\ &\leq C_1 \left\| \frac{1}{\nu} \right\|_{M^t}^k \|f\|_{q,k}^k + C_2 \left\| \frac{1}{\nu} \right\|_{M^t}^k \left(\frac{\beta^2}{\beta-1} \right)^k \int_0^\infty \bar{f}(r)^k r^{k(2/n+1/\beta-1/t)-1} dr \\ &= C_1 \left\| \frac{1}{\nu} \right\|_{M^t}^k \|f\|_{q,k}^k + C_2 \left(\frac{\beta^2}{\beta-1} \right)^k \left\| \frac{1}{\nu} \right\|_{M^t}^k \left(\|f\|_{q,k} \right)^k, \end{aligned}$$

where $\frac{1}{q} = \frac{1}{\beta} + \frac{2}{n} - \frac{1}{t}$. \square

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