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## ABSTRACT

Our aim is to investigate spaces with  $\sigma$ -discrete and meager dense sets, as well as selective versions of these properties. We construct numerous examples to point out the differences between these classes while answering questions of Tkachuk [22], Hutchison [13] and the authors of [7].

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## 1. Introduction

Topologists and analysts have often considered properties stating that a space has a *small* dense set. The most popular of them is *separability*, that is, the property of having a countable dense set. The famous *Suslin Problem* asked whether there is a non-separable linearly ordered space where families of pairwise disjoint open sets are at most countable and the still open *Separable Quotient Problem* asks whether every infinite dimensional Banach space has an infinite dimensional separable quotient.

Smallness conditions for dense sets other than separability have also been considered. A space is called *d-separable* if it has a dense set which is the countable union of discrete subsets. This property was introduced by Kurepa in his PhD dissertation under the name of *property  $K_0$*  as part of his study of the Suslin Problem. The latter can in fact be restated to ask whether there is a non-*d*-separable linearly ordered space where families of pairwise disjoint open sets are at most countable. *d*-separability has a much better behavior

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than separability: arbitrary products of  $d$ -separable spaces are  $d$ -separable, and for every space  $X$  there is a cardinal  $\kappa$  such that  $X^\kappa$  is  $d$ -separable. We will introduce a natural property called *nwd-separability* which is obtained by replacing *discrete* with *nowhere dense* in the definition. Every  $T_1$   $d$ -separable space without isolated points is *nwd-separable*, and *nwd-separability* shows a behavior which is somewhat close to that of  $d$ -separability.

A new class of smallness conditions for dense sets has been introduced as part of the program known as *selection principles in mathematics* and attracted a lot of attention recently, see [5], [7], [12] or [19] among others. The general idea is that a small dense set can be obtained by diagonalizing over a countable sequence of dense sets. In this way, one can define a selective strengthening of any of the properties we mentioned above: a space is *D-separable* [7] iff for every sequence  $\{D_n: n < \omega\}$  of dense sets there is a discrete set  $E_n \subset D_n$  for every  $n < \omega$  such that  $\bigcup_{n < \omega} E_n$  is dense. We define *NWD-separability* as a selective version of *nwd-separability* in a similar way and compare it with *D-separability*.

We devoted most of our efforts to point out various differences between these properties and to construct a great wealth of examples.

In Section 2, we introduce *nwd-separability* and start with pointing out some facts concerning products. We continue by proving that  $\omega^* \times 2^\omega$  is a compact space which is *nwd-separable* but not  $d$ -separable. The section ends with answering a question of Tkachuk [22] by showing that there is a Corson-compact space with non- $d$ -separable square.

Next, in Section 3 we begin to deal with selective versions of separability. We present a new construction of a countable  $M$ -separable, non- $R$ -separable space which also serves as an answer to a question of Hutchinson [13]. We present a general framework to deal with selective separability properties and conclude that the class of  $D$ - and *NWD-separable* spaces is close under finite unions.

In Section 4, our aim is to construct ZFC examples separating the newly introduced properties. We present an *NWD-separable* space which is not  $d$ -separable and countable, dense subsets of  $2^c$  which are not *NWD-separable*. We finish by investigating some related cardinal invariants and answering several questions from [7].

Section 5 is devoted to show, by forcing, that even in the class of first-countable spaces,  $d$ - and  $D$ -separability (*nwd*- and *NWD-separability*) can be different; compare this with the result that every separable Fréchet space is  $M$ -separable [4].

Finally, in Section 6 we finish with some positive results: we prove that every monotone normal, *nwd-separable* space is  $D$ -separable and show that the additional assumption of compactness even yields a  $\sigma$ -disjoint  $\pi$ -base. The last part of the section deals with the question whether  $\sigma(2^{\omega_1})$  is  $D$ -separable.

Throughout the paper, by *space* we mean a  $T_1$  topological space.  $Fn(\kappa, \mu; \lambda)$  denotes the set of all partial functions  $f$  from  $\kappa$  to  $\mu$  such that  $\text{dom } f$  has size less than  $\lambda$ . We refer the reader to [16] and [10] for set theoretical and topological notations and notions which are used in the paper but remained undefined.

## 2. Non-selective properties

We start by defining two natural weakenings of separability. The first one has been studied extensively in the past.

**Definition 2.1.** A space is  $d$ -separable (respectively, *nwd-separable*) if there are discrete (respectively, nowhere dense) sets  $\{D_n: n < \omega\}$  such that  $\bigcup_{n < \omega} D_n$  is dense.

Arhangel'skii [1] proved that arbitrary products of  $d$ -separable spaces are  $d$ -separable. Juhász and Szentmiklóssy [15] proved that for every space  $X$ , the space  $X^{d(X)}$  is  $d$ -separable. Moreover, they proved that for every compact space  $X$ , the countable power  $X^\omega$  is  $d$ -separable. However, the behavior of product spaces considering *nwd-separability* is much simpler:

**Proposition 2.2.** *If  $X$  is  $nwd$ -separable and  $Y$  is arbitrary then  $X \times Y$  is  $nwd$ -separable; thus finite products of  $nwd$ -separable spaces remain  $nwd$ -separable.*

$\prod\{X_\alpha: \alpha < \lambda\}$  is  $nwd$ -separable for arbitrary spaces  $X_\alpha$  with  $|X_\alpha| \geq 2$  and infinite  $\lambda$ .

**Proof.** Note that if  $E \subseteq X$  is nowhere dense then  $E \times Y$  is nowhere dense in  $X \times Y$ . Thus the first part clearly follows.

Now, observe that  $X = \prod\{X_n: n \in \omega\}$  is  $nwd$ -separable for arbitrary spaces  $X_n$  with  $|X_n| \geq 2$ ; indeed, fix some  $x_i \in X_i$  for  $i \in \omega$  and define  $D_n = \{y \in X: (\forall i \geq n) (y(i) = x_i)\}$ . Note that  $D_n$  is nowhere dense for each  $n \in \omega$  and  $\bigcup_{n < \omega} D_n$  is dense in  $X$ . Now consider an arbitrary infinite product  $X = \prod\{X_\alpha: \alpha < \lambda\}$  and note that  $X$  is homeomorphic to a countably infinite product.  $\square$

Note that finite powers can be non- $nwd$ -separable. A space is called an *almost  $P$ -space* if and only if every non-empty countable intersection of open sets has non-empty interior.

**Observation 2.3.** Let  $X$  be a regular countably compact almost  $P$ -space. Then every meager set is nowhere dense in  $X$ ; thus  $X$  is not  $nwd$ -separable.

**Proof.** Fix nowhere dense sets  $E_n \subseteq X$  for  $n \in \omega$  and a non-empty open  $V \subseteq X$ . Construct a decreasing sequence of open sets  $U_n \subseteq V \setminus E_n$  such that  $\overline{U_{n+1}} \subseteq U_n$ . Then  $\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} \overline{U_n} \neq \emptyset$  by  $X$  being countably compact and thus there is a non-empty open  $U \subseteq \bigcap_{n \in \omega} U_n$  by  $X$  being an almost  $P$ -space. Thus  $U \cap \bigcup_{n \in \omega} E_n = \emptyset$  which shows that  $\bigcup_{n \in \omega} E_n$  is not dense in  $V$ .  $\square$

Thus  $(\omega^*)^n$  is not  $nwd$ -separable for  $n \in \omega$  since every finite power of  $\omega^*$  is a compact, almost  $P$ -space. J.T. Moore [18] showed that there is an  $L$ -space, i.e. hereditarily Lindelöf, non-separable space, with a  $d$ -separable square. Thus, there are non- $d$ -separable spaces with  $d$ -separable square. Todorćević [24] got the idea that the Ellentuck topology can show the same situation for  $nwd$ -separability, and his conjecture was correct:

**Example 2.4.** The Ellentuck topology  $\mathcal{X} = [\omega]^\omega$  is a first-countable, non- $nwd$ -separable space with  $nwd$ -separable square.

**Proof.** Recall that the standard basis for the Ellentuck topology is

$$\{[s, X]: s \in [\omega]^{<\omega}, X \in [\omega]^\omega\}$$

where  $[s, X] = \{Y \in [\omega]^\omega: Y \text{ is an end-extension of } s, Y \setminus s \subseteq X\}$ . It is well known, though not trivial, that every meager set in the Ellentuck topology is nowhere dense; thus  $\mathcal{X}$  is not  $nwd$ -separable.

Let us construct a  $D_{s,t} \subseteq \mathcal{X}^2$  for  $s, t \in [\omega]^{<\omega}$  as follows: for  $A_0, A_1 \in [\omega]^\omega$  we say that  $A_0$  and  $A_1$  are *merged* iff for all  $i < 2$  and  $n, m \in A_i$  with  $n < m$  there is  $k \in A_{1-i}$  such that  $n < k < m$ . Let

$$D_{s,t} = \{(A, B) \in [s, \omega] \times [t, \omega]: A \setminus s \text{ and } B \setminus t \text{ are merged}\}.$$

It can be easily seen that  $D_{s,t}$  is nowhere dense and that the meager set  $\bigcup\{D_{s,t}: s, t \in [\omega]^{<\omega}\}$  is dense in  $\mathcal{X}^2$ .  $\square$

Our next aim is to show that  $d$ -separability and  $nwd$ -separability are different properties even in the realm of compact spaces.

**Example 2.5.** The space  $X = \omega^* \times 2^\omega$  is a compact  $nwd$ -separable space which is not  $d$ -separable.

**Proof.** Indeed, let  $D = \{x_n: n \in \omega\}$  be a countable dense subset of  $2^\omega$ . Then  $\bigcup_{n < \omega} \omega^* \times \{x_n\}$  is a  $\sigma$ -nowhere dense and dense subset of  $X$ .

To see that  $X$  is not  $d$ -separable, suppose by contradiction that  $\{D_n: n < \omega\}$  is a countable family of discrete sets whose union is dense in  $X$  and let  $\mathcal{B} = \{B_m: m < \omega\}$  be a countable base for  $2^\omega$ ,  $\tau$  be the topology of  $\omega^*$  and  $\pi: \omega^* \times 2^\omega \rightarrow \omega^*$  be the projection onto the first coordinate. Let

$$D_{nm} = \{\pi(z): z \in D_n \wedge (\exists U \in \tau) ((U \times B_m) \cap D_n = \{z\})\}.$$

Then  $D_{nm}$  is a discrete subset of  $\omega^*$  and  $\bigcup\{D_{nm}: (n, m) \in \omega \times \omega\}$  is dense in  $\omega^*$ . But this contradicts [Observation 2.3](#).  $\square$

Recall that a space  $X$  is called a *Corson compactum* if it is compact and there is a cardinal  $\kappa$  such that  $X$  can be embedded in  $\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa: |\{\alpha < \kappa: x(\alpha) \neq 0\}| \leq \omega\}$ . In [\[22\]](#) Tkachuk asked if the square of every Corson compactum is  $d$ -separable. We are going to show that Todorćević's classical example of a Corson compactum is a counterexample to Tkachuk's question.

Given a tree  $(T, \leq)$ , we let  $T \otimes T = \{(s, t): s, t \in T \wedge ht(s) = ht(t)\}$  and order  $T \otimes T$  as follows  $(s, t) \leq (s', t')$  if and only if  $s \leq s'$  and  $t \leq t'$ .

Now fix a stationary co-stationary subset  $A$  of  $\omega_1$  and let  $T$  be the tree of all countable compact subsets of  $A$  ordered by  $s \leq t$  if and only if  $s$  is an initial part of  $t$ .

**Theorem 2.6.** (Todorćević, [\[23\]](#))  $T \otimes T$  is Baire in the final parts topology.

Let  $P(T)$  be the set of all paths in  $T$  with the topology inherited from  $2^T$ . It is easy to see that  $P(T)$  is closed and thus  $P(T)$  is compact. From the fact that  $A$  is co-stationary it follows that every path is countable in  $T$  and hence  $P(T) \subseteq \Sigma(2^T)$ . Thus  $P(T)$  is a Corson compactum.

**Theorem 2.7.** The square of Todorćević's Corson compactum  $P(T)$  is not  $nwd$ -separable (hence not  $d$ -separable either).

**Proof.** Given  $t \in T$ , we let  $U_t$  be the set of all paths passing through  $t$ . We note that  $\{U_s \times U_t: (s, t) \in T \otimes T\}$  is a  $\pi$ -base for  $P(T) \times P(T)$ .

Now suppose that  $P(T)^2$  has a dense set of the form  $\bigcup_{n < \omega} D_n$ , where each  $D_n$  is nowhere dense.

**Claim.**  $W_n = \{(s, t) \in T \otimes T: (U_s \times U_t) \cap D_n = \emptyset\}$  is open dense in the final parts topology on  $T \otimes T$ .

**Proof.** To prove that  $W_n$  is open, note that if  $(U_s \times U_t) \cap D_n = \emptyset$  and  $(s', t') \geq (s, t)$  then we also have  $(U_{s'} \times U_{t'}) \cap D_n = \emptyset$ .

To prove that  $W_n$  is dense, let  $(s, t) \in T \otimes T$  be arbitrary and note that  $(U_s \times U_t) \setminus \overline{D_n}$  is a non-empty open set, so we can find  $s'$  and  $t'$  such that  $U_{s'} \times U_{t'} \subset (U_s \times U_t) \setminus \overline{D_n}$ . But then  $W_n \ni (s', t') \geq (s, t)$ .  $\square$

By the Baire property of  $T \otimes T$  we can choose  $(s, t) \in \bigcap_{n < \omega} W_n$ , but then we see that  $(U_s \times U_t) \cap (\bigcup_{n < \omega} D_n) = \emptyset$ , which is a contradiction.  $\square$

### 3. Selective properties

The following properties are the first selection principles for dense sets to have been considered and were introduced in [\[20\]](#) under a different name.

**Definition 3.1.** A space is called *M-separable* (*R-separable*), if given a sequence  $\{D_n: n < \omega\}$  of dense sets there are finite (one-point) sets  $F_n \subset D_n$  such that  $\bigcup_{n < \omega} F_n$  is dense in  $X$ .

The standard way of constructing an  $M$ -separable non- $R$ -separable space uses function spaces via the following theorem.

**Theorem 3.2.** ([5, Theorems 21 and 57]) *Let  $X$  be a Tychonoff space. Then  $C_p(X)$  is  $M$ -separable ( $R$ -separable) if and only if  $C_p(X)$  is separable and  $X^n$  is Menger (Rothberger) for every  $n < \omega$ .*

Then it would suffice to take  $X = 2^\omega$  in the above theorem. The Cantor set is, in fact, known to be Menger, but not Rothberger. Indeed, any Rothberger subset of the reals has strong measure zero.

We would like to show an alternative, more combinatorial, construction of an  $M$ -separable non- $R$ -separable space. We will use this example later to answer a question of Hutchison.

**Example 3.3.** A countable  $M$ -separable non- $R$ -separable space  $X$ .

**Proof.** Let  $X = Fn(\omega, \omega; \omega)$ , that is the set of all finite partial functions from  $\omega$  to  $\omega$ . Provide  $X$  with the following topology. A basic neighborhood of the point  $F \in X$  is a set of the form

$$V(F, \mathcal{F}) = \{G \in X: G \supset F \wedge (\forall f \in \mathcal{F}) (\forall n \in \text{dom } G \setminus \text{dom } F) (G(n) \neq f(n))\}$$

where  $\mathcal{F} \in [\omega^\omega]^{<\omega}$ .

**Claim 3.3.1.**  $X$  is not  $R$ -separable.

**Proof.** Let  $D_n = \{F \in X: n \in \text{dom } F\}$ . Then  $D_n$  is dense in  $X$ . Suppose by contradiction that we can find points  $F_n \in D_n$  such that  $D = \{F_n: n < \omega\}$  is dense in  $X$ . Let  $f \in \omega^\omega$  be the function defined by  $f(n) := F_n(n)$ . Then  $V(\emptyset, \{f\})$  is easily seen to miss  $D$ .  $\square$

**Claim 3.3.2.** Let  $D \subset X$  and  $k < \omega$  be such that:

$$D \cap V(\emptyset, \mathcal{F}) \neq \emptyset \quad \text{for each } \mathcal{F} \in [\omega^\omega]^k. \quad (1)$$

Then there is a finite subset  $D'$  of  $D$  such that:

$$D' \cap V(\emptyset, \mathcal{F}) \neq \emptyset \quad \text{for each } \mathcal{F} \in [\omega^\omega]^k.$$

**Proof.** For  $F \in X$  let

$$W(F) = \{(f_1, \dots, f_k) \in (\omega^\omega)^k: \forall 1 \leq j \leq k (\forall i < |F|) f_j(i) \neq F(i)\}.$$

Let  $T$  be the cofinite topology on  $\omega$ . Then  $T$  is compact, and  $W(F)$  is an open subset of the compact space  $(T^k)^\omega$ .

By (1), the set  $\{W(F): F \in D\}$  is an open cover of  $(T^\omega)^k$ . So there is a finite set  $D' \subset D$  such that  $\{W(F): F \in D'\}$  covers  $(T^\omega)^k$ . Then  $D'$  satisfies the requirements of the claim.  $\square$

**Claim 3.3.3.**  $X$  is  $M$ -separable.

**Proof.** Enumerate  $X \times \omega$  as  $\{(F_n, k_n): n < \omega\}$ . Using Claim 3.3.2 we choose, for each  $n < \omega$ , a finite subset  $D'_n$  of the set  $\{F \in D_n: F_n \subset F\}$  such that

$$D'_n \cap V(F_n, \mathcal{F}) \neq \emptyset \quad \text{for each } \mathcal{F} \in [\omega^\omega]^{k_n}.$$

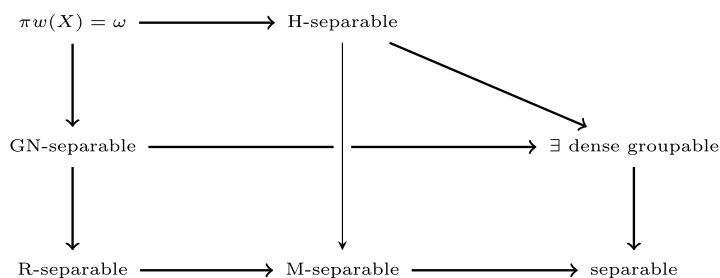


Fig. 1. Implications between the properties.

Then  $D = \bigcup \{D'_n : n \in \omega\}$  is dense. Indeed, if  $F \in X$  and  $\mathcal{F} \in [\omega^\omega]^{<\omega}$ , then pick  $n \in \omega$  with  $F_n = F$  and  $k_n = |\mathcal{F}|$ . Then there is  $d \in D'_n \subset D$  with  $d \in V(F, \mathcal{F})$ .  $\square$

In [11], Gruenhage, Natkaniec and Piotrowski say that a space  $X$  satisfies property  $(GC)$  if there is a disjoint, countable collection  $\mathcal{N}$  of nowhere dense sets such that, for every non-empty open set  $U \subset X$  we have  $|\{N \in \mathcal{N} : U \cap N = \emptyset\}| < \omega$ . In her PhD thesis [13] Hutchison proves that every space having a dense metrizable subset satisfies  $(GC)$  and that property  $(GC)$  is equivalent to having a  $\sigma$ -disjoint  $\pi$ -base in the realm of linearly ordered topological spaces. This notion is strictly intertwined with the notion of *groupable* dense set, which is the basis for another selective version of separability.

**Definition 3.4.** A dense set  $D \subset X$  is called *groupable* if it admits a partition  $\mathcal{A} = \{A_n : n < \omega\}$  into finite sets such that every non-empty open subset of  $X$  meets all but finitely elements of  $\mathcal{A}$ .

**Definition 3.5.** A topological space  $X$  is called  $(GN)$ -separable (from Gerlits and Nagy) if for every sequence  $\{D_n : n < \omega\}$  of dense sets there are points  $d_n \in D_n$  such that  $\{d_n : n < \omega\}$  is a groupable dense set.

We say that  $X$  is *H-separable* if for each sequence  $\{D_n : n < \omega\}$  of dense sets, one can pick finite sets  $F_n \subset D_n$  so that for every non-empty open set  $O \subset X$ , the intersection  $O \cap F_n$  is non-empty for all but finitely many  $n$ .

$GN$ -separability was introduced by Di Maio, Kočinac and Meccariello [8] under the name of *selection principle*  $S_1(\mathcal{D}, \mathcal{D}^{gp})$  while  $H$ -separability was introduced by Bella, Bonanzinga and Matveev in [5].

Clearly, every space having a groupable dense set satisfies property  $(GC)$ , and actually, a space has a groupable dense set if and only if it satisfies property  $(GC)$  witnessed by a collection of finite sets. Hutchison asked if one could add  $(GN)$ -separability to this equivalence, for the class of countable spaces. As a partial result, she noted that in a space satisfying  $(GC)$  witnessed by a collection of finite sets every dense set is groupable. We are going to give a negative answer to her question. Actually, we can prove a bit more (see Fig. 1 on the relationships among the properties we defined above).

**Theorem 3.6.** *There is a countable, H-separable, non-R-separable space  $X$ .*

**Proof.** Let  $X$  be the space from Example 3.3.

Assume that  $\{D_n : n < \omega\}$  is a sequence of dense sets.

Enumerate  $X \times \omega$  as  $\{(F_m, k_m) : m < \omega\}$ . Using Claim 3.3.2 from Example 3.3 we choose, for each  $n < \omega$  and  $m \leq n$ , a finite subset  $D_m^n$  of the set  $\{F \in D_n : F_m \subset F\}$  such that

$$D_m^n \cap V(F_m, \mathcal{F}) \neq \emptyset \quad \text{for each } \mathcal{F} \in [\omega^\omega]^{k_m}.$$

Let  $D_n = \bigcup \{D_m^n : m \leq n\}$ .

If  $V(F, \mathcal{F})$  is a basic open set,  $F_m = F$  and  $k_m = |\mathcal{F}|$ , then  $V(F, \mathcal{F}) \cap D_m^n \neq \emptyset$  for  $m \leq n$ , and so  $V(F, \mathcal{F}) \cap D_n \neq \emptyset$ .

Thus  $X$  is  $H$ -separable.  $\square$

**Definition 3.7.** We say that a set  $X \subset 2^{\omega_1}$  is a *very strong HFD* iff for each sequence  $\{A_n: n \in \omega\}$  of pairwise disjoint, non-empty finite subsets of  $X$  there is  $\beta < \omega_1$  such that for all  $s \in Fn(\omega_1 \setminus \beta, 2; \omega)$  there are infinitely many  $n$  with  $A_n \subset [s]$ , where  $[s] = \{x \in 2^{\omega_1}: s \subset x\}$ .

The following proposition is straightforward.

**Proposition 3.8.** *A very strong HFD cannot contain a groupable dense set.*

**Proposition 3.9.** *A dense HFD  $X \subset 2^{\omega_1}$  is  $R$ -separable.*

**Proof.** Let  $\mathcal{D}$  be a countable family of dense subsets of  $X$  and fix an enumeration  $\langle D_{n,m}: n, m \in \omega \rangle$  of  $\mathcal{D}$  without repetition. Let  $\vartheta$  be a large enough regular cardinal and let  $M$  be a countable elementary submodel of  $\mathcal{H}_\vartheta$  which contains  $X$  and  $\langle D_{n,m}: n, m \in \omega \rangle$ . Let  $\beta = M \cap \omega_1$  and enumerate  $Fn(\beta, 2; \omega)$  as  $\{s_n: n \in \omega\}$ . For all  $n \in \omega$  choose  $x_{n,m} \in [s_n] \cap D_{n,m}$  such that the function  $m \mapsto x_{n,m}$  is in  $M$ .

We claim that  $D = \{x_{n,m}: n, m < \omega\}$  is dense. Indeed, let  $s \in Fn(\omega_1, 2; \omega)$  and note that  $s \upharpoonright \beta = s_n$  for some  $n$ . The set  $D_n = \{x_{n,m}: m \in \omega\} \in M$ , so there is  $\beta_n \in M$  such that  $[t] \cap D_n \neq \emptyset$  for all  $t \in Fn(\omega_1 \setminus \beta_n, 2; \omega)$  as  $X$  is an *HFD*. But  $s \setminus s_n \in Fn(\omega_1 \setminus \beta, 2; \omega) \subset Fn(\omega_1 \setminus \beta_n, 2; \omega)$  hence  $[s \setminus s_n] \cap D_n \neq \emptyset$ . Since  $D_n \subset [s_n]$  we have that  $[s] \cap D_n \neq \emptyset$  as well.  $\square$

**Theorem 3.10.** *After adding  $\omega_1$  Cohen reals to any model there is a countable  $R$ -separable space without a groupable dense subset.*

**Proof.** Let  $\mathcal{G}$  be a generic filter for the poset  $Fn(\omega \times \omega_1, 2; \omega)$  and note that  $g = \bigcup \mathcal{G}$  is a function from  $\omega \times \omega_1$  to 2. Define  $x_n \in {}^{\omega_1}2$  by the formula  $x_n(\alpha) = g(n, \alpha)$  for  $n \in \omega$  and let  $X = \{x_n: n \in \omega\}$ .

We claim that  $X$  is a very strong *HFD*. Indeed, assume that  $\{A_n: n \in \omega\}$  is a sequence of pairwise disjoint, non-empty finite subsets of  $X$  in  $V[\mathcal{G}]$ . Let  $\{B_n: n \in \omega\}$  be the sequence of pairwise disjoint, non-empty finite subsets of  $\omega$  such that  $A_n = \{x_i: i \in B_n\}$ . Fix a countable ordinal  $\beta < \omega_1$  such that  $\{B_n: n \in \omega\} \in V[\mathcal{G} \upharpoonright (\omega \times \beta)]$ . Since  $Fn(\omega \times \omega_1, 2; \omega)$  and  $Fn(\omega \times (\omega_1 \setminus \beta), 2; \omega)$  are isomorphic, we can assume that  $\beta = 0$ , i.e.  $\{B_n: n \in \omega\} \in V$ . Let  $s \in Fn(\omega_1, 2; \omega)$  be arbitrary. Then the set

$$D_s = \{p \in Fn(\omega \times \omega_1, 2; \omega): p \Vdash \exists n \in \omega \ A_n \subset [s]\} \quad (2)$$

is dense. Indeed, for each  $q \in Fn(\omega \times \omega_1, 2; \omega)$  there is  $n \in \omega$  such that  $\text{dom}(q) \cap B_n \times \omega_1 = \emptyset$ . Define  $p \supset q$  as follows:  $\text{dom}(p) = \text{dom}(q) \cup B_n \times \text{dom}(s)$ , and  $q(i, \alpha) = s(\alpha)$  for  $i \in B_n$  and  $\alpha \in \text{dom}(s)$ . Then  $q \Vdash A_n \subset [s]$ , i.e.  $q \in D_s$ ; this proves that  $D_s$  is dense.

Thus  $D_s \cap \mathcal{G} \neq \emptyset$  and hence in  $V[\mathcal{G}]$  there is  $n \in \omega$  with  $A_n \subset [s]$ . So  $X$  is a very strong *HFD*.

$X$  is  $R$ -separable by Proposition 3.9 and  $X$  cannot contain a groupable dense set by Proposition 3.8.  $\square$

In [12, Example 3.2] Gruenhage and Sakai constructed a maximal  $R$ -separable space from CH, and it is straightforward that such a space also satisfies the requirements of Theorem 3.10.

**Question 3.11.** (1) Is there a ZFC example of a crowded  $R$ -separable space without a groupable dense subset?  
(2) Is there a ZFC example of a (countable)  $GN$ -separable space with uncountable  $\pi$ -weight?



To get a consistent example for [Question 3.11\(2\)](#), any counterexample to Malykhin's problem would do. Indeed, any countable Fréchet–Urysohn non-metrizable group has uncountable  $\pi$ -weight and every countable Fréchet–Urysohn space without isolated points is  $GN$ -separable, by [\[12\]](#).

Our next goal is to analyze selective versions of  $d$ -separability and  $nwd$ -separability.

**Definition 3.12.** A space  $X$  is called  $D$ -separable (respectively,  $NWD$ -separable) if for every sequence  $\{D_n: n < \omega\}$  of dense sets there are discrete (respectively, nowhere dense) sets  $E_n \subset D_n$  such that  $\bigcup_{n < \omega} E_n$  is dense in  $X$ .

Observe that if  $X$  is  $D$ -separable (respectively,  $NWD$ -separable) then every dense subset of  $X$  is  $D$ -separable (respectively,  $NWD$ -separable) as well. Also, if  $X$  is  $D$ -separable (respectively,  $NWD$ -separable) then it is also  $d$ -separable (respectively,  $nwd$ -separable).

$D$ -separability was already investigated in [\[12\]](#) and [\[2\]](#). Let us now introduce a general framework for dealing with selection principles for dense sets.

**Definition 3.13.** For each topological space  $X$ , let  $\mathbb{A}_X \subset \mathcal{P}(X)$ . We say that  $X$  is  $\mathbb{A}$ -separable iff for each sequence  $\{D_n: n \in \omega\}$  of dense subsets of  $X$  there are  $A_n \in \mathcal{P}(D_n) \cap \mathbb{A}_X$  for  $n \in \omega$  such that  $\bigcup \{A_n: n \in \omega\}$  is dense in  $X$ .

The formulation and the proof of the following result is based on [\[12, Theorem 2.2\]](#).

**Theorem 3.14.** Assume that for each topological space  $X$  we have  $\mathbb{A}_X \subset \mathcal{P}(X)$  such that

- (a)  $\mathbb{A}_X$  is an ideal,
- (b) if  $Z \subset X$ , then  $\mathbb{A}_Z \subset \mathbb{A}_X$ ,
- (c) if  $U \subset X$  is open, then  $\mathbb{A}_U = \{A \cap U: A \in \mathbb{A}_X\}$ .

Then the union of two  $\mathbb{A}$ -separable spaces is  $\mathbb{A}$ -separable.

**Proof.** First we need some easy observations.

**Observation 3.15.** If  $X$  is  $\mathbb{A}$ -separable, and  $U \subset X$  is open, then  $U$  is also  $\mathbb{A}$ -separable.

Indeed, if  $\{D_n: n \in \omega\}$  are dense subsets of  $U$ , then  $E_n = D_n \cup (X \setminus \bar{U})$  are dense subsets of  $X$  for  $n \in \omega$ , so there are sets  $A_n \in \mathbb{A}$  with  $A_n \subset D_n \cup (X \setminus \bar{U})$  such that  $A = \bigcup \{A_n: n \in \omega\}$  is dense in  $X$ . Let  $B_n = A_n \cap D_n$ . Observe that  $B_n \in \mathbb{A}_U$  by (c). Since  $\bigcup \{B_n: n \in \omega\} = A \cap U$ , the set  $\bigcup \{B_n: n \in \omega\}$  is dense in  $U$ , which proves the observation.

We need the following lemma which corresponds to [\[12, Lemma 2.1\]](#).

**Lemma 3.16.** A topological space  $X$  is  $\mathbb{A}$ -separable iff for every decreasing sequence  $\{D_n: n \in \omega\}$  of dense subsets of  $X$ , there are sets  $E_n \subset D_n$  from  $\mathbb{A}_X$  for  $n \in \omega$  such that  $\bigcup \{E_n: n \in \omega\}$  is dense in  $X$ .

**Proof.** Let  $\{C_m: m \in \omega\}$  be a sequence of dense subsets of  $X$ . For each  $n \in \omega$ , let  $D_n = \bigcup \{C_m: m \geq n\}$ . The sequence  $\{D_n: n \in \omega\}$  is decreasing, so there are sets  $E_n \subset D_n$  from  $\mathbb{A}_X$  for  $n \in \omega$  such that  $\bigcup \{E_n: n \in \omega\}$  is dense in  $X$ . Let  $F_m = C_m \cap \bigcup \{E_n: n \leq m\}$ . Then  $F_m \subset C_m$  and  $F_m \in \mathbb{A}_X$  by (a), and  $\bigcup \{F_m: m < \omega\} = \bigcup \{E_n: n < \omega\}$ , so  $\bigcup \{F_m: m < \omega\}$  is dense.  $\square$

Assume  $X = Y \cup Z$ , where  $Y$  and  $Z$  are  $\mathbb{A}$ -separable. Assume that  $\{D_n: n \in \omega\}$  is a sequence of dense subsets of  $X$ . By [Lemma 3.16](#) we can assume that the sequence is decreasing.



Put  $U_n = X \setminus \overline{Y \cap D_n}$ . Then  $\{U_n: n \in \omega\}$  is an increasing family of open sets in  $X$ .

Fix an  $n \in \omega$ . Clearly  $Z \cap U_n$  is dense in  $U_n$ . Since  $U_n$  is open, the subspace  $U_n \cap Z$  of  $Z$  is  $\mathbb{A}$ -separable.

For  $k \geq n$ ,  $U_n \cap D_k \subset U_n \cap D_n \subset U_n \cap Z \subset U_n$ , so the set  $U_n \cap D_k$  is dense in  $Z \cap U_n$ . Since  $Z \cap U_n$  is  $\mathbb{A}$ -separable, there are sets  $F_{n,k} \subset U_n \cap D_k$  from  $\mathbb{A}_{Z \cap U_n}$  for  $k \geq n$  such that  $\bigcup \{F_{n,k}: k \geq n\}$  is dense in  $U_n$ .

Since  $\mathbb{A}_{Z \cap U_n} \subset \mathbb{A}_Z$  by (b), we have  $\{F_{n,k}: n \leq k < \omega\} \subset \mathbb{A}_Z$ .

Now put  $F_k = \bigcup \{F_{n,k}: n \leq k\}$  for  $k \in \omega$ . Then  $F_k \subset D_k$  and  $F_k \in \mathbb{A}_Z$  by (a) for  $k < \omega$ , and  $\bigcup \{F_k: k \in \omega\}$  is dense in  $\bigcup \{U_n: n \in \omega\}$ .

Let  $V = X \setminus \overline{\bigcup \{U_n: n \in \omega\}}$ . For each  $n \in \omega$ ,  $D_n \cap V \subset D_n \setminus U_n \subset \overline{D_n \cap Y}$ , so  $D_n \cap V \cap Y$  is dense in  $V$ .

Since  $Y \cap V$  is  $\mathbb{A}$ -separable by [Observation 3.15](#), there are sets  $G_n \subset Y \cap V \cap D_n$  with  $G_n \in \mathbb{A}_{Y \cap V}$  for  $n \in \omega$  such that  $\bigcup \{G_n: n \in \omega\}$  is dense in  $Y \cap V$ , and so it is also dense in  $V$ . Since  $\mathbb{A}_Z \cup \mathbb{A}_Y \subset \mathbb{A}_X$  by (b), we have  $F_n \cup G_n \in \mathbb{A}_X$  by (a). Thus  $F_n \cup G_n$  is a subset of  $D_n$  from  $\mathbb{A}_X$  and  $\bigcup \{F_n \cup G_n: n \in \omega\}$  is dense in  $X$ .  $\square$

### Corollary 3.17.

- (1) [\[7\]](#) The union of two  $D$ -separable spaces is  $D$ -separable.
- (2) The union of two  $NWD$ -separable spaces is  $NWD$ -separable.

## 4. Examples in ZFC and related results

The first part of this section deals with the construction of an  $NWD$ -separable space which is not  $d$ -separable.

Given a cardinal  $\kappa$ , we say that a  $\pi$ -base  $\mathcal{U}$  is  $\kappa$ -deep iff for each decreasing sequence  $\{U_n\}_{n \in \kappa} \subset \mathcal{U}$  we have  $\text{int}(\bigcap_{n \in \kappa} U_n) \neq \emptyset$ .

**Lemma 4.1.** *For each infinite cardinal  $\kappa$ , there is a crowded regular space of size  $2^\kappa$  which has a  $\kappa$ -deep  $\pi$ -base.*

**Proof.** We claim that a space with the claimed properties is

$$X = \Sigma_\kappa(2^{\kappa^+}) = \{f \in 2^{\kappa^+}: |f^{-1}\{1\}| \leq \kappa\},$$

endowed with the  $\kappa$  supported box product topology.

If  $s \in 2^{<\kappa^+}$ , then let  $B(s) = \{f \in X: s \subset f\}$ . Put

$$\mathcal{U} = \{B(s): s \in 2^{<\kappa^+}\}.$$

Then  $\mathcal{U}$  is actually a base of  $X$ . To show that  $\mathcal{U}$  is deep, assume that  $\langle B(s_n): n \in \kappa \rangle$  is decreasing. Then we have  $s_0 \subset s_1 \subset \dots$ , and so  $s = \bigcup_{n \in \kappa} s_n$  is a function, and  $B(s) \subset \bigcap_{n \in \kappa} B(s_n)$ .  $\square$

**Lemma 4.2.** *Assume that  $X$  has an  $\omega$ -deep  $\pi$ -base. Then  $Y = X \times \mathbb{Q}$  is  $NWD$ -separable.*

**Proof.** We need two claims.

**Claim 4.2.1.** *If  $S \subset Y$  is dense, then for each non-empty open  $U \subset X$  and  $p < q \in \mathbb{Q}$  there is non-empty open  $V \subset U$  and  $p < r < q \in \mathbb{Q}$  such that*

$$\pi_r(S) \stackrel{\text{def}}{=} \{x \in X: \langle x, r \rangle \in S\}$$

*is dense in  $V$ .*

**Proof.** Assume on the contrary that the sets  $\pi_r(S)$  are nowhere dense. Enumerate  $(p, q) \cap \mathbb{Q}$  as  $\{r_n: n \in \omega\}$ . Construct a decreasing sequence  $\{U_n\}_{n \in \omega} \subset \mathcal{U}$  such that  $U_n \cap \pi_{r_n}(S) = \emptyset$ . Then the set  $W = \text{int}(\bigcap_{n \in \omega} W_{r_n})$  is non-empty. Thus  $S \cap (W \times (p, q)) = \emptyset$ , a contradiction.  $\square$

**Claim 4.2.2.** *If  $\{S_n: n < \omega\} \subset Y$  are dense, then for each a non-empty open  $U \subset X$  there is non-empty open  $V \subset U$  and there is a sequence  $\{r_n^V: n < \omega\} \subset \mathbb{Q}$  such that*

$$\bigcup_{n \in \omega} S_n \cap (X \times \{r_n^V\}) \text{ is dense in } V \times \mathbb{Q}. \quad (3)$$

**Proof.** Enumerate the pairs  $\{\langle p, q \rangle: p, q \in \mathbb{Q}, p < q\}$  as  $\{\langle p_n, q_n \rangle: n < \omega\}$ .

Construct a decreasing sequence of open sets  $U_0 \supset U_1 \supset U_1 \supset \dots$  from  $\mathcal{U}$  and distinct rational numbers  $r_n$  such that

- (1)  $U_0 \subset U$ ,
- (2)  $r_n \in (p_n, q_n)$ ,
- (3)  $\pi_{r_n}(S_n)$  is dense in  $U_{n+1}$ .

The construction can be carried out by the previous claim. Then  $V = \text{int}(\bigcap_{n \in \omega} U_n) \neq \emptyset$  works if we take  $r_n^V = r_n$ .  $\square$

Let  $\mathcal{V}$  be a maximal disjoint family of open sets in  $X$  such that every  $V \in \mathcal{V}$  satisfies the requirements of the previous claim. Then  $\bigcup \mathcal{V}$  is dense in  $X$  by the previous claim.

Let

$$A_n = \bigcup_{V \in \mathcal{V}} S_n \cap (V \times \{r_n^V\}). \quad (4)$$

Then  $A_n$  is nowhere dense because for  $V \in \mathcal{V}$  we have  $A_n \cap (V \times \mathbb{Q}) \subset X \times \{r_n^V\}$ . Moreover  $A = \bigcup_{n \in \omega} A_n$  is dense, because  $A \cap (V \times \mathbb{Q})$  is dense in  $V \times \mathbb{Q}$  by (3) and (4).  $\square$

**Example 4.3.** There is an *NWD*-separable, but not *d*-separable space of size  $\mathfrak{c}$ .

**Proof.** We show that if  $Y = X \times \mathbb{Q}$  where  $X$  is the space from Lemma 4.1, then  $Y$  is *NWD*-separable, but not *d*-separable.

By Lemma 4.2 the space  $Y$  is *NWD*-separable.

If  $\{D_n: n \in \omega\}$  are discrete in  $Y$ , then

$$\pi_q(D_n) \stackrel{\text{def}}{=} \{x \in X: \langle x, q \rangle \in D_n\} \quad (5)$$

is discrete, and so nowhere dense in  $X$  for  $q \in \mathbb{Q}$ ,  $n \in \omega$ . Since  $X$  has an  $\omega$ -deep  $\pi$ -base, there is an open  $U \subset X$  with  $(U \times \mathbb{Q}) \cap \bigcup_{n \in \omega} D_n = \emptyset$ , and so  $\bigcup_{n \in \omega} D_n$  is not dense.  $\square$

In Theorem 5.2 we show that it is consistent that  $2^\omega$  is large, but there is an *NWD*-separable non-*D*-separable space of size  $\aleph_1$ .

**Question 4.4.** Is there an *NWD*-separable non-*D*-separable space of size  $\aleph_1$  in ZFC? Is there at least one whose size is bounded in ZFC?

At least the first question seems to require techniques different from those of this paper. Indeed, a space having an  $\omega$ -deep subbase is Baire, and Shelah and Todorćević [21] showed that the existence of a Baire space of size  $\aleph_1$  is independent from ZFC.

Let us continue with another example: a countable, not *NWD*-separable space. The following result was proved in [7] using a direct construction; the  $\mathcal{D}$ -forced technology of [14] can be used to give an alternative proof.

**Example 4.5.**  $2^{\mathfrak{c}}$  has a countable, dense, not *NWD*-separable subspace.

**Proof.** By [14, Theorem 4.9] there is a countable, dense nodec subspace  $X$  of  $2^{\mathfrak{c}}$  such that  $X$  can be partitioned into submaximal dense subspaces  $\mathcal{D} = \{D_n: n \in \omega\}$ , and  $X$  is  $\mathcal{D}$ -forced, i.e. if  $D \subset X$  is somewhere dense, then  $D \supset D_n \cap U$  for some  $n \in \omega$  and non-empty open set  $U$ .

Then  $X$  is not *NWD*-separable. Indeed, if  $E_n \subset D_n$  is nowhere dense, then  $E = \bigcup_{n \in \omega} E_n$  is not dense, because it cannot contain any  $D_n \cap U$ .  $\square$

**Example 4.6.** There is an *nwd*-separable, but not *d*-separable and not *NWD*-separable space of size  $2^{2^{\mathfrak{c}}}$ .

**Proof.** By Lemma 4.1, there is a crowded regular space  $X$  of size  $2^{2^{\mathfrak{c}}}$  which has a  $2^{\mathfrak{c}}$ -deep  $\pi$ -base. Let  $Y = 2^{\mathfrak{c}}$  with the product topology.

We claim that  $X \times Y$  has the required properties.

**Claim 4.6.1.**  $X \times Y$  is *nwd*-separable.

Indeed, let  $D = \{d_n: n \in \omega\}$  be dense in  $Y$ . Then  $S_n = X \times \{d_n\}$  is nowhere dense in  $X \times Y$ , but  $\bigcup_{n \in \omega} S_n = X \times D$  is dense in  $X \times Y$ .

**Claim 4.6.2.**  $X \times Y$  is not *d*-separable.

If  $\{D_n: n \in \omega\}$  are discrete in  $Y$ , then

$$\pi_y(D_n) \stackrel{\text{def}}{=} \{x \in X: \langle x, y \rangle \in D_n\} \quad (6)$$

is discrete, and so nowhere dense in  $X$  for  $y \in Y$ ,  $n \in \omega$ . Since  $X$  has a  $|Y|$ -deep  $\pi$ -base, there is an open  $U \subset X$  with  $(U \times Y) \cap \bigcup_{n \in \omega} D_n = \emptyset$ , and so  $\bigcup_{n \in \omega} D_n$  is not dense.

**Claim 4.6.3.**  $X \times Y$  is not *NWD*-separable.

By [14, Theorem 4.9] there is a countable, dense nodec subspace  $T$  of  $2^{\mathfrak{c}}$  such that  $T$  can be partitioned into submaximal dense subspaces  $\mathcal{D} = \{D_n: n \in \omega\}$ , and  $T$  is  $\mathcal{D}$ -forced, i.e. if  $D \subset T$  is somewhere dense, then  $D \supset D_n \cap V$  for some  $n \in \omega$  and non-empty open set  $V$ .

Let  $E_n = X \times D_n$  for  $n \in \omega$ . We show that if  $F_n \subset E_n$  is nowhere dense, then  $F = \bigcup_{n \in \omega} F_n$  cannot be dense.

Let  $\{B_i: i < \mathfrak{c}\}$  be a base of  $Y$ , and fix a  $\mathfrak{c}$ -deep  $\pi$ -base  $\mathcal{U}$  of  $X$ .

By induction on  $n$  construct a decreasing sequence  $\{U_n: -1 \leq n < \omega\} \subset \mathcal{U}$  as follows. Let  $U_{-1} \in \mathcal{U}$  be arbitrary non-empty. Assume that  $U_{n-1}$  is constructed.

By transfinite induction construct a decreasing sequence  $\{U_i^n: i < \mathfrak{c}\} \subset \mathcal{U} \cap \mathcal{P}(U_{n-1})$  such that for all  $i < \mathfrak{c}$  there is a non-empty  $V_i^n \subset B_i$  such that

$$F_n \cap (U_i^n \times V_i^n) = \emptyset.$$

Since  $\mathcal{U}$  is  $\mathfrak{c}$ -deep, there is a non-empty open  $U_n \subset X$  such that  $U \subset U_i^n$  for all  $i < \mathfrak{c}$ .

Finally there is  $U \in \mathcal{U}$  such that  $U \subset U_n$  for all  $n \in \omega$ .

Since  $U \subset U_i^n$ , we have  $F_n \cap (U \times V_i^n) = \emptyset$ . Write  $V_n = \bigcup_{i < \mathfrak{c}} V_i^n$  and  $G_n = D_n \setminus V_n$  for  $n < \omega$ . Then  $V_n$  is open dense in  $Y$  and  $F_n \cap (U \times V_n) = \emptyset$ , so

$$F_n \cap (U \times Y) \subset U \times G_n. \quad (7)$$

Since  $G = \bigcup_{n \in \omega} G_n$  cannot contain  $V \cap D_n$  for a non-empty open  $V$ , so  $G$  is nowhere dense in  $T$  because  $T$  is  $\mathcal{D}$ -forced. In particular,  $G$  is nowhere dense in  $Y$  so there is a non-empty open  $V$  with  $V \cap G = \emptyset$ .

But then, by (7),

$$\begin{aligned} F \cap (U \times V) &= \bigcup_{n \in \omega} F_n \cap (U \times V) = \bigcup_{n \in \omega} (F_n \cap (U \times Y)) \cap (U \times V) \\ &\subset \bigcup_{n \in \omega} (U \times G_n) \cap (U \times V) = (U \times G) \cap (U \times V) = U \times (G \cap V) = \emptyset, \end{aligned} \quad (8)$$

so  $F$  cannot be dense, which was to be proved.  $\square$

Motivated by Example 4.5, one can define the following cardinal invariants:

**Definition 4.7.** ([7]) Let

$$\begin{aligned} \mathfrak{d}\mathfrak{s} &= \min\{\kappa: 2^\kappa \text{ is not } D\text{-separable}\}, \\ \mathfrak{cd}\mathfrak{s} &= \min\{\kappa: 2^\kappa \text{ contains a countable non-}D\text{-separable subspace}\}. \end{aligned} \quad (9)$$

We have  $\mathfrak{cd}\mathfrak{s} \leq 2^\omega$  by Example 4.5. Moreover, as

$$\mathfrak{d} = \min\{\kappa: 2^\kappa \text{ contains a countable non-}M\text{-separable dense subspace}\}$$

was shown in [5], we also have  $\mathfrak{d} \leq \mathfrak{cd}\mathfrak{s}$ .

In [7] the authors proved that the space  $X^{2^{d(X)}}$  is never  $D$ -separable. In particular, if  $X$  is separable, then  $X^{2^\omega}$  is not  $D$ -separable. This exponent appears far from optimal and we can in fact improve it for separable spaces; the next theorem also solves Question 44 from [7], while we note that an alternative solution to this question was provided in [2].

**Theorem 4.8.** *If  $X$  is a separable space with  $|X| \geq 2$  then some dense subspace  $Y$  of  $X^{\omega_1}$  is not  $d$ -separable; hence  $X^{\omega_1}$  is not  $D$ -separable for any separable  $X$  with  $|X| \geq 2$ . Hence  $\mathfrak{d}\mathfrak{s} = \omega_1$ .*

**Proof.** J.T. Moore in [17] constructed an  $L$ -space  $L = \{f_\alpha: \alpha < \omega_1\} \subset \omega^{\omega_1}$  such that

$$|L \cap [\varepsilon]| = \omega_1 \quad \text{for each finite function } \varepsilon \in Fn(\omega_1, \omega; \omega). \quad (10)$$

Let  $D = \{d_n: n < \omega\}$  be dense in  $X$ .

For  $\alpha < \omega_1$  define  $y_\alpha \in D^{\omega_1}$  as follows:

$$y_\alpha(\beta) = \begin{cases} d_{f_\alpha(\beta)} & \text{if } \beta < \alpha, \\ d_0 & \text{if } \beta \geq \alpha. \end{cases} \quad (11)$$

Let  $Y = \{y_\alpha: \alpha < \omega_1\}$ .

Then  $Y$  is dense in  $X^{\omega_1}$  by (10). Moreover, if  $D \subset Y$  is discrete, then  $D$  is countable because  $L$  is hereditarily Lindelöf and the map  $f_\alpha \rightarrow y_\alpha$  is continuous. So  $Y$  is not  $d$ -separable, because  $d(Y) = \omega_1$  by (11).  $\square$

However, the following remains open:

**Conjecture 4.9.** *The space  $X^{d(X)^+}$  is never  $D$ -separable.*

The next corollary solves Question 45 from [7]:

**Corollary 4.10.**  *$\text{MA} + \neg\text{CH}$  implies  $\mathfrak{d}\mathfrak{s} < \mathfrak{c}\mathfrak{d}\mathfrak{s}$ .*

**Proof.**  $\mathfrak{d}\mathfrak{s} = \aleph_1$  is true in every model of ZFC. Since  $\mathfrak{d} \leq \mathfrak{c}\mathfrak{d}\mathfrak{s}$  and  $\mathfrak{d} = \mathfrak{c}$  in every model of MA the statement of the corollary follows from the failure of CH.  $\square$

The authors of [7] ask *what is  $\mathfrak{c}\mathfrak{d}\mathfrak{s}$ ?* Up to this point it was even unknown whether  $\mathfrak{c}\mathfrak{d}\mathfrak{s}$  could consistently be less than the continuum, so the following theorem may be considered a partial answer to Question 43 of [7].

**Theorem 4.11.** *If  $\text{cof}(\mathcal{M}) = \omega_1$  then  $2^{\omega_1}$  has a countable dense subspace which is not NWD-separable, in particular, we have  $\mathfrak{c}\mathfrak{d}\mathfrak{s} = \omega_1$ .*

We recall some definitions from [14].

Let  $S$  be a set, and

$$\mathbb{B} = \{\langle B_\zeta^i : i < \lambda \rangle : \zeta < \mu\}$$

be a family of partitions of  $S$ . We say that  $\mathbb{B}$  is *independent* iff

$$\mathbb{B}[\varepsilon] \stackrel{\text{def}}{=} \bigcap \{B_\zeta^{\varepsilon(\zeta)} : \zeta \in \text{dom } \varepsilon\} \neq \emptyset$$

for each  $\varepsilon \in \text{Fn}(\mu, \lambda; \omega)$ .  $\mathbb{B}$  is *separating* iff for each  $\{\alpha, \beta\} \in [S]^2$  there are  $\zeta < \mu$  and  $\rho \neq \nu < \lambda$  such that  $\alpha \in B_\zeta^\rho$  and  $\beta \in B_\zeta^\nu$ .

We shall denote by  $\tau_{\mathbb{B}}$  the (obviously 0-dimensional) topology on  $S$  generated by the subbase  $\{B_\zeta^0, B_\zeta^1 : \zeta < \mu\}$ , moreover we set  $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}} \rangle$ . Clearly, the family  $\{\mathbb{B}[\varepsilon] : \varepsilon \in \text{Fn}(\mu, 2; \omega)\}$  is a base for the space  $X_{\mathbb{B}}$ . Note that  $X_{\mathbb{B}}$  is Hausdorff iff  $\mathbb{B}$  is separating.

**Observation 4.12.** Let  $\lambda$  be an infinite cardinal. Then, up to homeomorphism, there is a natural one-to-one correspondence between countable dense subspaces  $X$  of  $2^\lambda$  and spaces of the form  $X_{\mathbb{B}} = \langle \omega, \tau_{\mathbb{B}} \rangle$ , where  $\mathbb{B} = \{\langle B_\xi^0, B_\xi^1 \rangle : \xi < \lambda\}$  is a separating and independent family of 2-partitions of  $\omega$ .

**Proof of Theorem 4.11.** By the Hewitt–Marczewski–Pondiczery Theorem, [10, Theorem 2.3.15], there is a separating, independent family

$$\{\langle A_\zeta^i : i < \omega \rangle : \zeta < 2^\omega\}$$

of partitions of  $\omega$ . In particular, there are partitions

$$\{\langle F_j : j < \omega \rangle\} \cup \{\langle B_\zeta^i : i < 2 \rangle : \zeta < \omega_1\}$$

of  $\omega$  such that

$$\mathbb{B} = \{\langle B_\zeta^i: i < 2 \rangle: \zeta < \omega_1\}$$

is separating and

$$F_j \cap \mathbb{B}[\varepsilon] \neq \emptyset \quad (12)$$

for all  $j \in \omega$  and  $\varepsilon \in Fn(\omega_1, 2; \omega)$ . We can assume that

$$\forall x \neq y \in \omega \exists n < \omega, i < 2 \quad (x \in B_n^i \wedge y \in B_n^{1-i}). \quad (13)$$

Next, let us fix any partition  $\{I_\nu: \omega \leq \nu < \omega_1\}$  of  $\omega_1 \setminus \omega$  into uncountable pieces with  $(\nu + 1) \cap I_\nu = \emptyset$  and then by transfinite recursion on  $\omega \leq \nu < \omega_1$  define

- sequences  $\langle A_\alpha^k: k < \omega \rangle$  for  $\alpha \in I_\nu$ ,
- partitions  $\langle C_\nu^0, C_\nu^1 \rangle$  of  $\omega$ ,

such that the inductive hypothesis

$$\forall \varepsilon \in Fn(\omega_1, 2; \omega) \forall j < \omega \quad |F_j \cap \mathbb{B}_\nu[\varepsilon]| = \omega \quad (\phi_\nu)$$

holds, where

$$\mathbb{B}_\nu = \{\langle C_\sigma^0, C_\sigma^1 \rangle: \omega \leq \sigma < \nu\} \cup \{\langle B_\sigma^0, B_\sigma^1 \rangle: \sigma \in \omega \cup (\omega_1 \setminus \nu)\}.$$

Note that  $(\phi_\nu)$  simply says that every set  $F_j$  is dense in the space  $X_{\mathbb{B}_\nu}$ . We shall then conclude that  $\mathbb{C} = \mathbb{B}_{\omega_1}$  is as required.

Let us observe first that  $(\phi_\omega)$  holds because (12) holds and  $\mathbb{B}[\varepsilon] = \mathbb{B}_\omega[\varepsilon]$ .

Clearly, if  $\nu$  is a limit ordinal and  $(\phi_\zeta)$  holds for each  $\zeta < \nu$  then  $(\phi_\nu)$  also holds. So the induction hypothesis is preserved in limit steps.

Now consider a  $\nu < \omega_1$  and assume that  $(\phi_\nu)$  holds.

Note that  $\{A_\nu^j: j \in \omega\}$  is already defined and let

$$C_\nu^0 = B_\nu^0 \cup \bigcup_{j < \omega} A_\nu^j; \quad C_\nu^1 = B_\nu^1 \setminus \bigcup_{j < \omega} A_\nu^j; \quad (14)$$

let

$$\mathbb{B}'_\nu = \{\langle C_\sigma^0, C_\sigma^1 \rangle: \omega \leq \sigma < \nu\} \cup \{\langle B_\sigma^0, B_\sigma^1 \rangle: \sigma \in \omega\},$$

and consider the space  $Y_\nu = \langle \omega, \tau_{\mathbb{B}'_\nu} \rangle$ . Clearly  $Y_\nu$  is homeomorphic to  $\mathbb{Q}$ .

Let

$$\mathbb{A}_\nu = \{\langle A_i: i < \omega \rangle: A_i \subset F_i, A_i \text{ is nowhere dense in } Y_\nu\}. \quad (15)$$

If  $A = \langle A_i: i < \omega \rangle$  and  $A' = \langle A'_i: i < \omega \rangle$  are from  $\mathbb{A}_\nu$  let  $A \prec A'$  iff  $A_i \subseteq A'_i$  for each  $i < \omega$ .

Since  $F_i$  is dense in  $Y_\nu$ , it follows that  $F_i$  is also homeomorphic to  $\mathbb{Q}$  for all  $i \in \omega$ . So  $Z$ , the topological sum of the spaces  $\{F_i: i < \omega\}$ , is homeomorphic to the disjoint union of  $\omega$  copies of  $\mathbb{Q}$  which is homeomorphic to  $\mathbb{Q}$ .

Fixing a homeomorphism  $h : \mathbb{Q} \rightarrow Z$ , define the map  $\varphi : \text{nwd}_{\mathbb{Q}} \rightarrow \mathbb{A}_{\nu}$  as follows:  $\varphi(A) = \langle \varphi'' A \cap F_i : i < \omega \rangle$ . Since  $\varphi$  is an isomorphism between  $\langle \text{nwd}_{\mathbb{Q}}, \subset \rangle$  and  $\langle \mathbb{A}_{\nu}, \prec \rangle$ , we have

$$\text{cof}(\mathbb{A}_{\nu}, \prec) = \text{cof}(\text{nwd}_{\mathbb{Q}}).$$

Fremlin (see [3, Theorem 1.6]) proved that

$$\text{cof}(\mathcal{M}) = \text{cof}(\text{nwd}_{\mathbb{Q}}), \quad (16)$$

where  $\text{nwd}_{\mathbb{Q}}$  is the family of nowhere dense subsets of  $\mathbb{Q}$ . So we have

$$\text{cof}(\mathbb{A}_{\nu}, \prec) = \text{cof}(\text{nwd}_{\mathbb{Q}}) = \text{cof}(\mathcal{M}) = \omega_1. \quad (17)$$

Let  $\{A_{\alpha} : \alpha \in I_{\nu}\}$  enumerate a cofinal subset of  $\mathbb{A}_{\nu}$ . Write  $A_{\alpha} = \langle A_{\alpha}^i : i < \omega \rangle$ .

We have to show that  $(\phi_{\nu+1})$  holds.

Assume, indirectly, that for some  $j < \omega$  and  $\varepsilon \in \text{Fn}(\omega_1, 2; \omega)$  we have

$$F_j \cap \mathbb{B}_{\nu+1}[\varepsilon] = \emptyset.$$

Fix  $\sigma \leq \nu$  with  $\nu \in I_{\sigma}$ .

Let  $\eta = \varepsilon \upharpoonright \sigma$ . Since  $\langle A_{\nu}^i : i < \omega \rangle \in \mathbb{A}_{\sigma}$ , the set  $A_{\nu}^j$  was nowhere dense in  $Y_{\sigma}$ , i.e. there is  $\eta' \in \text{Fn}(\sigma, 2; \omega)$ ,  $\eta' \supseteq \eta$ , such that

$$\mathbb{B}'_{\sigma}[\eta'] \cap A_{\nu}^j = \emptyset. \quad (18)$$

But

$$\mathbb{B}'_{\sigma}[\eta'] = \mathbb{B}_{\nu}[\eta'] = \mathbb{B}_{\nu+1}[\eta'], \quad (19)$$

so

$$(F^j \setminus A_{\nu}^j) \cap \mathbb{B}_{\nu}[\eta' \cup \varepsilon] = F^j \cap \mathbb{B}_{\nu}[\eta' \cup \varepsilon] \neq \emptyset \quad (20)$$

by  $(\phi_{\nu})$ . Thus

$$F^j \cap \mathbb{B}_{\nu+1}[\varepsilon] \supset F^j \cap \mathbb{B}_{\nu+1}[\eta' \cup \varepsilon] \supset (F^j \setminus A_{\nu}^j) \cap \mathbb{B}_{\nu}[\eta' \cup \varepsilon] \neq \emptyset,$$

a contradiction; the relation in the middle follows from the fact that  $(F^j \setminus A_{\nu}^j) \cap C_{\nu}^i = (F^j \setminus A_{\nu}^j) \cap B_{\nu}^i$ .

Finally we show that the sequence  $\langle F^j : j < \omega \rangle$  witnesses that  $X_{\mathbb{C}}$  is not *NWD*-separable. Assume that  $E_i \subset F^i$  is nowhere dense; being nowhere dense is witnessed by a dense open set which, in turn, is the countable union of basic open sets. Thus there is  $\sigma < \omega_1$  such that  $E_i$  is nowhere dense in  $Y_{\sigma}$  for all  $i < \omega$ . Then there is  $\nu \in I_{\sigma}$  such that  $E_i \subset A_{\nu}^i$  for  $i < \omega$ .

Then  $\bigcup \{E_i : i \in \omega\} \subset \bigcup \{A_{\nu}^i : i \in \omega\} \subset C_{\nu}^0$ , i.e.  $\bigcup \{E_i : i \in \omega\}$  is not dense because it does not intersect  $C_{\nu}^1$ .  $\square$

Fig. 2 summarizes the (trivial) implications between separation properties we considered in this section. The labels of the arrows indicate the examples showing that the implications cannot be reversed.

We will get further consistency results in the next section, however the following question remained open.

**Question 4.13.** Is there a *d*-separable, *NWD*-separable, non-*D*-separable space in ZFC?



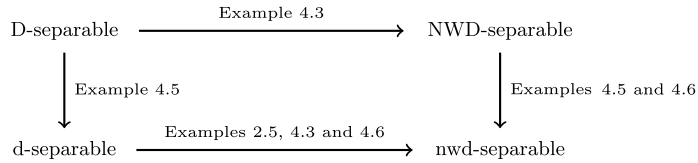


Fig. 2. Separation results in ZFC.

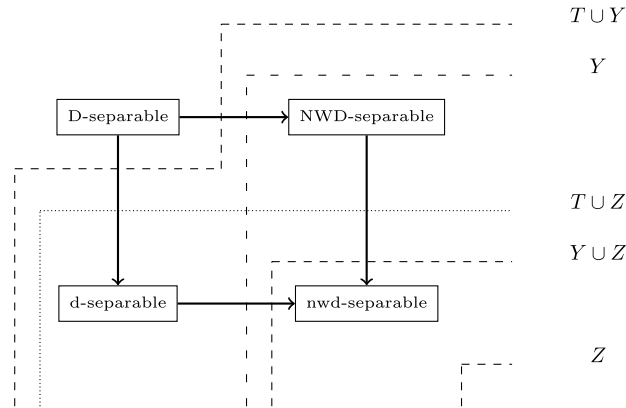


Fig. 3. Implications and counterexamples.

## 5. In the class of first-countable spaces – forcing counterexamples

In [4] Barman and Dow proved that every separable Fréchet space is  $M$ -separable. In [12] G. Gruenhage and M. Sakai observed that separable Fréchet spaces are  $R$ -separable. The aim of this section is showing that no theorem of this kind can be proved in the context of  $d$ -separability ( $nwd$ -separability) and  $D$ -separability ( $NWD$ -separability), not even if one replaces Fréchet with first-countable.

First, let us start with a lemma.

**Lemma 5.1.** *Suppose that we fixed some ideal  $\mathbb{A}_X$  for each space  $X$  as in Theorem 3.14. If  $X = \bigcup_{n \in \omega} A_n$  for some  $A_n \in \mathbb{A}_X$  then  $\text{MA}_{\pi(X)}(\text{countable})$  implies that  $X$  is  $\mathbb{A}$ -separable.*

**Proof.** We fix a  $\pi$ -base  $\mathcal{B}$  of  $X$  of size  $\pi(X)$  and a sequence of dense sets  $D_n \subseteq X$ . Let us define  $D_B = \{p \in \mathbb{C} : \exists n \in \text{dom}(p) : D_n \cap A_{p(n)} \cap B \neq \emptyset\}$  for  $B \in \mathcal{B}$  where  $\mathbb{C}$  denotes the Cohen poset; note that each  $D_B$  is dense in  $\mathbb{C}$ . Consider a filter  $G \subseteq \mathbb{C}$  which is generic to  $\{D_B : B \in \mathcal{B}\}$ ; this exists by  $\text{MA}_{\pi(X)}(\text{countable})$ . Let  $g = \bigcup G$  and define  $E_n = D_n \cap A_{g(n)} \in \mathbb{A}_X$ .

We claim that  $\bigcup_{n \in \omega} E_n$  is dense in  $X$ . It suffices to show that for every  $B \in \mathcal{B}$  there is  $n \in \omega$  such that  $B \cap E_n \neq \emptyset$ . As  $G \cap D_B \neq \emptyset$ , there is  $n \in \omega$  such that  $A_{g(n)} \cap D_n \cap B \neq \emptyset$ ; that is  $E_n \cap B \neq \emptyset$ .  $\square$

An uncountable space  $X$  is *Luzin* iff every nowhere dense subset of  $X$  is countable. We continue by our main theorem:

**Theorem 5.2.** (See Fig. 3.) *It is consistent that there is a left-separated in type  $\omega_1$ , first-countable, 0-dimensional Hausdorff space of size  $\omega_1$  such that  $X$  has a partition  $X = Z \cup T \cup Y$  into dense uncountable subspaces such that*

- (1)  $T$  is  $D$ -separable;
- (2)  $Y$  is  $NWD$ -separable, but not  $d$ -separable;
- (3)  $Z$  is *Luzin*, so it is not  $nwd$ -separable.

Moreover,

- (4)  $T \cup Z$  is  $d$ -separable but not  $NWD$ -separable;
- (5)  $Y \cup Z$  is  $nwd$ -separable, but not  $d$ -separable and not  $NWD$ -separable;
- (6)  $T \cup Y$  is  $d$ -separable,  $NWD$ -separable, but not  $D$ -separable.

**Proof.** First we show that (4)–(6) follow automatically from (1)–(3):

- (4)  $T$  is a dense,  $D$ -separable subspace of  $T \cup Z$ , so  $T \cup Z$  is  $d$ -separable.  $Z$  is a dense, not  $nwd$ -separable subspace of  $T \cup Z$ , so  $T \cup Z$  is not  $NWD$ -separable.
- (5)  $Y$  is a dense,  $NWD$ -separable subspace of  $Y \cup Z$ , so  $Y \cup Z$  is  $nwd$ -separable.  $Z$  is a dense, not  $nwd$ -separable subspace of  $Y \cup Z$ , so  $Y \cup Z$  is not  $NWD$ -separable.  
Assume that  $\{F_n: n \in \omega\}$  are discrete subspaces of  $Y \cup Z$ , and let  $F = \bigcup \{F_n: n \in \omega\}$ . Then  $F_n \cap Z$  is discrete, hence nowhere dense and so countable. Thus  $F \cap Z$  is countable. Since  $X$  is left-separated in type  $\omega_1$ , it follows that  $F \cap Z$  is nowhere dense. But  $Y$  is not  $d$ -separable, so  $X \neq \overline{F \cap Z}$ . Thus  $X \neq \overline{F}$ , i.e.  $Y \cup Z$  is not  $d$ -separable.
- (6)  $T \cup Y$  is the union of two  $NWD$ -separable spaces, so it is  $NWD$ -separable by Corollary 3.17.  $T$  is a dense,  $D$ -separable subspace of  $T \cup Y$ , so  $T \cup Y$  is  $d$ -separable.  $Y$  is a dense, not  $d$ -separable subspace of  $Y \cup Z$ , so  $Y \cup Z$  is not  $D$ -separable.

Now we define a poset  $\mathbb{Q}$  which has property  $K$ , thus c.c.c., and forces a left-separated, first-countable, 0-dimensional Hausdorff topology on the set  $X = \omega_1 \times (\omega + 1)$  such that  $X$  has a partition  $X = Z \cup T \cup Y$  into dense uncountable subspaces with the following properties:

- (A)  $T$  is  $\sigma$ -discrete,
- (B)  $Y$  is  $\sigma$ -nowhere dense and  $s(Y) = \omega$ , i.e. every discrete subset of  $Y$  is countable,
- (C)  $Z$  is Luzin.

We will do this in such way, that a condition  $p \in \mathbb{Q}$  will be a finite approximation of a countable neighborhood base.

For  $i < 2$  let  $I_i = \{n \in \omega: n \equiv i \pmod{2}\}$ , and let  $I_2 = \{\omega\}$ . The underlying set of  $T$ ,  $Y$  and  $Z$  will be  $\omega_1 \times I_0$ ,  $\omega_1 \times I_1$  and  $\omega_1 \times I_2$ , respectively.

We will use the following notations: if  $x = (\alpha, k) \in X$  let

$$Q_x = \begin{cases} [\alpha, \omega_1) \times [k+1, \omega] \cup \{x\} & \text{if } k \in I_0, \\ [\alpha, \omega_1) \times [k, \omega] & \text{if } k \in I_1, \\ [\alpha, \omega_1) \times [0, \omega] & \text{if } k \in I_2, \end{cases} \quad (21)$$

see Fig. 4.

Let  $\mathbb{Q}$  consist of the following conditions

$$p = \langle I^p, n^p, \langle U^p(x, j): x \in I^p, j < n^p \rangle \rangle$$

such that

- (Q-a)  $I^p \in [\omega_1 \times (\omega + 1)]^{<\omega}$  and  $n^p \in \omega$ ,
- (Q-b)  $x \in U^p(x, j) \subset I^p \cap Q_x$  for all  $x \in I^p$  and  $j < n^p$ .

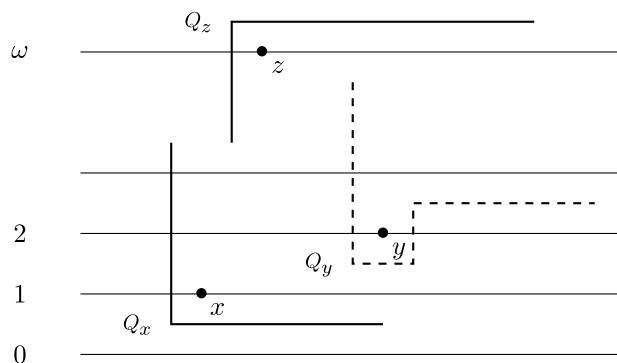


Fig. 4. Basic neighborhoods.

If  $p, q \in \mathbb{Q}$  let  $q \leq p$  iff

- (Q-i)  $I^p \subseteq I^q$ ,  $n^p \leq n^q$  and  $U^p(x, j) \subseteq U^q(x, j)$  for all  $x \in I^p$  and  $j < n^p$ ,
- (Q-ii)  $U^p(x, j) \subseteq U^p(y, k)$  implies  $U^q(x, j) \subseteq U^q(y, k)$  for all  $x, y \in I^p$  and  $j, k < n^p$ ,
- (Q-iii)  $U^p(x, j) \cap U^p(y, k) = \emptyset$  implies  $U^q(x, j) \cap U^q(y, k) = \emptyset$  for all  $x, y \in I^p$  and  $j, k < n^p$ .

If  $G$  is a generic filter in  $\mathbb{Q}$  then let

$$U^G(x, j) = \bigcup \{U^p(x, j) : p \in G, x \in I^p, j < n^p\} \quad (22)$$

for any  $x \in X$  and  $j < \omega$ . Let  $B^G(x) = \{U^G(x, j) : j < \omega\}$  for  $x \in X$ .

**Lemma 5.3.**  $\bigcup \{B^G(x) : x \in X\}$  forms a base for a Hausdorff, 0-dimensional topology  $\tau^G$  on  $X$ , such that  $B^G(x)$  is a countable neighborhood base for the point  $x \in X$ .  $X$  is left-separated in type  $\omega_1$ .

**Proof.** The statement follows from standard density arguments.  $\square$

Let  $E_n = \omega_1 \times \{n\}$  for  $n \leq \omega$ . The next lemma follows from easy density arguments as well:

**Lemma 5.4.**

- (a) The subspace  $E_n$  is discrete in  $X$  for  $n \in I_0$ ; hence  $T$  is  $\sigma$ -discrete.
- (b) The subspace  $E_n$  is nowhere dense in  $Y$  for  $n \in I_1$ ; hence  $Y$  is  $\sigma$ -nowhere dense.
- (c)  $\omega_1 \times \{\omega\}$  is dense in  $X$ .
- (d) If  $N \subset \omega$  is infinite then  $\omega_1 \times N$  is dense in  $X$ .

Denote  $\pi$  the projection from  $\omega_1 \times (\omega + 1)$  onto  $\omega_1$ , i.e.  $\pi(\langle \alpha, n \rangle) = \alpha$ .

**Definition 5.5.** We say that the conditions  $p$  and  $q$  are *twins* iff  $n^p = n^q$ ,  $|\pi[I^p]| = |\pi[I^q]|$  and denoting by  $\sigma$  the unique  $<$ -preserving bijection between  $\pi[I^p]$  and  $\pi[I^q]$  we have

- (1)  $\pi[I^p] \cap \pi[I^q] < \pi[I^p] \setminus \pi[I^q] < \pi[I^q] \setminus \pi[I^p]$ ,
- (2) using the notation  $\sigma^*(\langle \alpha, n \rangle) = \langle \sigma(\alpha), n \rangle$ ,
  - (i)  $I^q = \sigma^*[I^p]$ ,
  - (ii)  $U^q(\sigma^*(x), i) = \sigma^*[U^p(x, i)]$  for  $x \in I^p$  and  $i < n^p$ .

We say that  $\sigma^*$  is the *twin function* from  $p$  to  $q$ .

The following lemma is rather technical although it will be essential in finishing our proof. We encourage the reader to skip the proof of [Lemma 5.6](#) at first read and see its applications in what follows.

**Lemma 5.6.** Assume that  $p$  and  $q$  are twin conditions,  $n^p = n^q = n$ , and  $\sigma^*$  is the twin function from  $p$  to  $q$ .

(i) If  $u \in I^p \cap (\omega_1 \times I_1)$ , then there is a condition  $r \leq p, q$  such that

$$\sigma^*(u) \in \bigcap_{\ell < n^p} U^r(u, \ell). \quad (23)$$

(ii) If

- (a)  $r \leq p$ ,  $I^r = I^p$ ,  $n^r = n + 1$ ,  $U^r(x, i) = U^p(x, i)$  for  $i < n$  and  $U^r(x, n) = \{x\}$  for each  $x \in I^r$ ,
  - (b)  $s \leq r$  such that  $\pi[I^s] < \pi[I^q \setminus I^p]$ ,
  - (c)  $u \in (I^p \setminus I^q) \cap (\omega_1 \times I_2)$ , and  $v \in U^s(u, n)$ ,
- then there is a condition  $t \leq s, q$  such that

$$\sigma^*(u) \in \bigcap_{i < n^s} U^t(v, i). \quad (24)$$

**Proof.** (i) Define  $r = \langle I^r, n^r, \langle U^r(x, j): x \in I^r, j < n^r \rangle \rangle$  as follows:

- $I^r = I^p \cup I^q$ ,  $n^r = n^p = n^q$ .
- If  $x = (\alpha, n) \in I^r$ , let

$$U^r(x, j) = \begin{cases} U^p(x, j) \cup U^q(x, j) & \text{if } x \in I^p \cap I^q, \\ U^p(x, j) \cup \{\sigma^*(u)\} & \text{if } x \in I^p \setminus I^q \text{ and } u \in U^p(x, j), \\ U^p(x, j) & \text{if } x \in I^p \setminus I^q \text{ and } u \notin U^p(x, j), \\ U^q(x, j) & \text{if } x \in I^q \setminus I^p. \end{cases}$$

It is clear that  $r$  satisfies (Q-a). If (Q-b) fails then  $U^r(x, j) \not\subset Q_x$  for some  $x \in I^r$ . Since  $p$  and  $q$  are twins, the only possibility is that  $U^r(x, j) = U^p(x, j) \cup \{\sigma^*(u)\}$  and  $\sigma^*(u) \notin Q_x$ . But  $u \in Q_x$ , so the only possibility is that  $x = u$ . But  $u \in \omega_1 \times I_1$ , so  $\sigma^*(u) \in Q_u$ . So  $r$  satisfies (Q-b) as well. Hence  $r \in \mathbb{Q}$ .

To show  $r \leq p, q$ , first remark that (Q-i) is trivial by the construction.

(Q-ii) can be easily seen to hold since  $U^p(x, j) \subset U^p(y, k)$  iff  $U^q(\sigma^*(x), j) \subset U^q(\sigma^*(y), k)$  as  $p$  and  $q$  are twins.

To check (Q-iii) assume first that  $x, y \in I^p$  and  $U^p(x, j) \cap U^p(y, k) = \emptyset$ . We can assume that  $u \notin U^p(x, j)$  and so  $\sigma^*(u) \notin U^r(x, j)$ . Thus

$$U^r(x, j) \cap U^r(y, k) \subset (U^p(x, j) \cup U^q(\sigma^*(x), j)) \cap (U^p(y, k) \cup U^q(\sigma^*(y), k)). \quad (25)$$

But  $p$  and  $q$  are twins, and so  $U^p(x, j) \cap U^p(y, k) = \emptyset$  implies

$$(U^p(x, j) \cup U^q(\sigma^*(x), j)) \cap (U^p(y, k) \cup U^q(\sigma^*(y), k)) = \emptyset \quad (26)$$

as well.

Thus  $r$  and  $p$  satisfy (Q-iii), and so  $r \leq p$ .

Now let  $x, y \in U^q$  such that  $U^q(x, j) \cap U^q(y, k) = \emptyset$ .

Pick  $x', y' \in I^p$  with  $\sigma^*(x') = x$  and  $\sigma^*(y') = y$ .

Assume that  $\sigma^*(u) \notin U^q(x, j)$ . Then  $\sigma^*(u) \notin U^r(x, j)$ . Thus

$$U^r(x, j) \cap U^r(y, k) \subset (U^q(x, j) \cup U^p(x', j)) \cap (U^q(y, k) \cup U^p(y', k)). \quad (27)$$

But  $p$  and  $q$  are twins, and so  $U^q(x, j) \cap U^q(y, k) = \emptyset$  implies

$$U^r(x, j) \cap U^r(y, k) \subset (U^q(x, j) \cup U^p(x', j)) \cap (U^q(y, k) \cup U^p(y', k)) = \emptyset \quad (28)$$

as well.

Thus  $r$  and  $q$  satisfy (Q-iii), and so  $r \leq q$ .

So we proved  $r \leq p, q$ .

Finally  $\sigma^*(u) \in U^r(u, \ell)$  for  $\ell < n^p$  is clear from the construction. This proves Lemma 5.6(i).

(ii) Define  $t = \langle I^t, n^t, \langle U^t(x, j): x \in I^t, j < n^t \rangle \rangle$  as follows. For  $x \in I^s$  and  $j < n^s$  let

$$V(x, j) = \bigcup \{U^q(z, \ell): z \in I^p \cap I^q, \ell < n^p, U^s(z, \ell) \subset U^s(x, j)\}, \quad (29)$$

and

$$W(x, j) = \begin{cases} \{\sigma^*(u)\} & \text{if } U^s(v, i) \subset U^s(x, j) \text{ for some } i < n^s, \\ \emptyset & \text{otherwise.} \end{cases} \quad (30)$$

Let

- (a)  $I^t = I^s \cup I^q, n^t = n^s$ .
- (b) For  $x \in I^t$  and  $j < n^t$  let

$$U^t(x, j) = \begin{cases} U^s(x, j) \cup V(x, j) \cup W(x, j) & \text{if } x \in I^s, \\ U^q(x, j) & \text{if } x \in I^q \setminus I^s \text{ and } j < n^q, \\ \{x\} & \text{if } x \in I^q \setminus I^s \text{ and } n^q \leq j < n^t. \end{cases} \quad (31)$$

Clearly  $t$  satisfies (Q-a).

Assume on the contrary that  $w \in U^t(x, j) \setminus Q_x$  witnesses that (Q-b) fails. Since  $\sigma^*(u) \in E_\omega$ , we have  $w \neq \sigma^*(u)$ .

So  $w \in V(x, j) \setminus U^s(x, j) \subset V(x, j) \setminus I^p$ . Pick  $w' \in I^p \setminus I^q$  with  $\sigma^*(w') = w$ . There is  $z \in I^p \cap I^q$  and  $\ell < n^p$  such that  $w \in U^q(z, \ell)$  and  $U^s(z, \ell) \subset U^s(x, j)$ . Thus  $x \notin I^p \setminus I^q$ , and so  $x \neq w'$ . Thus  $w' \in U^p(z, \ell) \subseteq U^s(z, \ell) \subseteq U^s(x, j) \subset Q_x$  implies  $w = \sigma^*(w') \in Q_x$ . This contradiction shows that (Q-b) must hold. Thus, we proved that  $t \in \mathbb{Q}$ .

We will check that  $t \leq s, q$ . (Q-i) is trivial.

(Q-ii) holds for  $t \leq s$ , because the construction is “monotone” in (29)–(31). (Q-ii) also holds for  $t \leq q$  because if  $U^q(x, j) \subset U^q(y, k)$  then it is not possible that  $x \in I^p \cap I^q$  and  $y \in I^q \setminus I^p$ , so we can use that the construction is “monotone”.

Now we check (Q-iii) for  $t \leq s$ . So let  $U^s(x, j) \cap U^s(y, k) = \emptyset$ , and assume on the contrary that  $U^t(x, j) \cap U^t(y, k) \neq \emptyset$ . Since  $p$  and  $q$  are twins, we have

$$(U^s(x, j) \cup V(x, j)) \cap (U^s(y, k) \cup V(y, k)) = \emptyset. \quad (32)$$

Indeed, assume that  $w \in (U^s(x, j) \cup V(x, j)) \cap (U^s(y, k) \cup V(y, k))$ . Since  $V(x, j) \cap I^s \subset U^s(x, j)$ , we can assume  $w \in I^q \setminus I^p$ , i.e.  $w \in V(x, j) \cap V(y, k)$ . Then  $\sigma^{*-1}(w) \in U^s(x, j) \cap U^s(y, k)$ . A contradiction.

So, by (32),  $U^t(x, j) \cap U^t(y, k) \neq \emptyset$  implies  $\sigma^*(u) \in U^t(x, j) \cap U^t(y, k)$ . Since  $u, v \notin U^s(x, j) \cap U^s(y, k)$ , we can assume that

$$\sigma^*(u) \in W(x, j) \quad \text{and} \quad \sigma^*(u) \in V(y, k), \quad (33)$$

so

$$U^s(v, i) \subset U^s(x, j) \quad \text{and} \quad u \in U^s(z, \ell) \subset U^s(y, k), \quad (34)$$

for some  $i < n^s$ ,  $z \in I^p \cap I^q$  and  $\ell < n^p$ .

So  $U^s(z, \ell) \cap U^s(u, n) \neq \emptyset$ . Thus  $s \leq r$  implies  $U^r(z, \ell) \cap U^r(u, n) \neq \emptyset$ , that is  $u \in U^r(z, \ell)$ . But  $U^r(u, n) = \{u\}$ , so  $U^r(u, n) \subset U^r(z, \ell)$ . Thus  $U^s(u, n) \subset U^s(z, \ell)$ , and so  $v \in U^s(u, n) \subset U^s(z, \ell) \subset U^s(y, k)$ . Thus  $v \in U^s(x, j) \cap U^s(y, k)$ . A contradiction, thus  $U^t(x, j) \cap U^t(y, k) = \emptyset$ .

Finally we check (Q-iii) for  $t \leq q$ . So let  $U^q(x, j) \cap U^q(y, k) = \emptyset$ .

We should distinguish three cases as follows.

If  $x, y \in I^p \cap I^q$ , then  $x, y \in I^s$ . As  $U^p(x, j) \cap U^p(y, k) = \emptyset$  and  $s \leq p$ , we have that  $U^s(x, j) \cap U^s(y, k) = \emptyset$ . We have just verified that (Q-iii) holds in this case.

If  $x, y \in I^q \setminus I^p$ , then  $U^t(x, j) = U^q(x, j)$  and  $U^t(y, k) = U^q(y, k)$ , so (Q-iii) is trivial.

Finally let  $x \in I^p \cap I^q$  and  $y \in I^q \setminus I^p$ . Then

$$U^t(x, j) \cap U^t(y, k) = (U^s(x, j) \cup V(x, j) \cup W(x, j)) \cap U^q(y, k). \quad (35)$$

Let  $y' = \sigma^{*-1}(y) \in I^p \setminus I^q$ . Then  $U^p(x, j) \cap U^p(y', k) = \emptyset$ , and so  $U^s(x, j) \cap U^s(y', k) = \emptyset$ . Since  $U^q(y, k) \cap I^s \subset U^p(y', k) \subset U^s(y', k)$ , we have

$$U^s(x, j) \cap U^q(y, k) = \emptyset. \quad (36)$$

If  $U^s(z, \ell) \subset U^s(x, j)$ , then  $U^s(x, j) \cap U^s(y', k) = \emptyset$  implies that  $U^s(z, \ell) \cap U^s(y', k) = \emptyset$ , and so  $U^p(z, \ell) \cap U^p(y', k) = \emptyset$ . Thus  $U^q(z, \ell) \cap U^q(y, k) = \emptyset$ . Hence

$$V(x, j) \cap U^q(y, k) = \emptyset. \quad (37)$$

Assume that  $W(x, j) = \{\sigma^*(u)\}$ . Then  $U^s(v, i) \subset U^s(x, j)$  for some  $i < n^s$ . Thus  $U^s(u, n) \cap U^s(x, j) \neq \emptyset$ , so  $U^r(u, n) \cap U^r(x, j) \neq \emptyset$ , so  $u \in U^p(x, j)$ . Thus  $u \notin U^p(y', k)$  and so  $\sigma^*(u) \notin U^q(y, k)$ .

So  $U^t(x, j) \cap U^t(y, k) = \emptyset$ .

Thus  $t \leq s, q$  and  $\sigma^*(u) \in \bigcap_{i < n^s} U^t(v, i)$  holds as well.  $\square$

**Lemma 5.7.**  $\mathbb{Q}$  has property  $K$ .

**Proof.** If  $\langle p_\alpha: \alpha < \omega_1 \rangle \subset \mathbb{Q}$ , then by standard  $\Delta$ -system arguments we can find an uncountable  $I \subset \omega_1$  such that  $p_\alpha$  and  $p_\beta$  are twins whenever  $\alpha < \beta \in I$ . So  $p_\alpha$  and  $p_\beta$  are compatible by Lemma 5.6.  $\square$

**Lemma 5.8.** If  $m \in I_1$  then  $E_m$  does not contain any uncountable discrete subspace; in particular,  $s(Y) = \omega$ .

**Proof.** Assume that  $p \Vdash \dot{A} = \{\dot{x}_\zeta: \zeta < \omega_1\} \in [E_m]^{\omega_1}$  is discrete". For each  $\zeta < \omega_1$  pick  $p_\zeta$ ,  $k_\zeta$  and  $x_\zeta$  such that

$$p_\zeta \Vdash \dot{x}_\zeta = x_\zeta \text{ and } U^G(x_\zeta, k_\zeta) \cap \dot{A} = \{x_\zeta\}. \quad (38)$$

We can assume that the elements  $\{x_\zeta: \zeta < \omega_1\}$  are pairwise different,  $x_\zeta \in I^{p_\zeta}$  and  $k_\zeta = k < n^{p_\zeta}$ .

By standard  $\Delta$ -system arguments we can find  $\zeta < \xi < \omega_1$  such that  $p_\zeta$  and  $p_\xi$  are twins, and  $\sigma^*(x_\zeta) = x_\xi$ , where  $\sigma^*$  is the twin function.

Then, by Lemma 5.6 part (i), there is a  $q \leq p_\zeta, p_\xi$  such that  $\sigma^*(x_\zeta) = x_\xi \in \bigcap_{\ell < n^{p_\zeta}} U^q(x_\zeta, \ell)$  and so

$$q \Vdash \{x_\zeta, x_\xi\} \subset U^G(x_\zeta, k) \cap \dot{A}. \quad (39)$$

This contradicts the choice of the neighborhoods which finishes the proof.  $\square$

**Lemma 5.9.** *Every uncountable subset  $A$  of  $Z$  is somewhere dense in  $X$ . In particular,  $Z$  is a Luzin subspace of  $X$ .*

**Proof.** Assume  $p \Vdash \dot{A} = \{\dot{a}_\zeta: \zeta < \omega_1\} \subset E_\omega$ .

Pick conditions  $\{p_\zeta: \zeta < \omega_1\}$ , and ordinals  $\{\alpha_\zeta: \zeta < \omega_1\} \subset \omega_1$  such that  $p_\zeta \Vdash \dot{a}_\zeta = \langle \alpha_\zeta, \omega \rangle$ . We can assume that:

- (i) If  $\zeta < \xi < \omega_1$  then  $p_\zeta$  and  $p_\xi$  are twins, so  $n^{p_\zeta} = n$ .
- (ii)  $a_\zeta \in I^{p_\zeta} \setminus I^{p_\xi}$  for  $\xi \neq \zeta$ .
- (iii)  $\sigma_{\zeta, \xi}^*(a_\zeta) = a_\xi$ , where  $\sigma_{\zeta, \xi}^*$  is the twin function from  $p_\zeta$  to  $p_\xi$ .

Let  $r \leq p_0$ ,  $I^r = I^{p_0}$ ,  $n^r = n + 1$ ,  $U^r(x, i) = U^{p_0}(x, i)$  for  $i < n$  and  $U^r(x, n) = \{x\}$  for  $x \in I^r$ .

**Claim 5.9.1.**  $r \Vdash \dot{A}$  is dense in  $U^G(a_0, n)$ .

Indeed, assume that  $s \leq r$  such that  $s \Vdash v \in U^G(a_0, n)$ , i.e.  $v \in U^s(a_0, n)$ . Pick  $\xi < \omega_1$  such that  $\pi[I^s] < \pi[I^{p_\xi} \setminus I^{p_0}]$ .

Then, by Lemma 5.6, there is a condition  $t \leq s, p_\xi$  such that

$$a_\xi \in \bigcap_{i < n^s} U^t(v, i). \quad (40)$$

Thus

$$t \Vdash \dot{A} \cap \bigcap_{i < n^s} U^G(v, i) \neq \emptyset. \quad (41)$$

Since  $s$  and  $v$  were arbitrary, we proved the claim, and so does the lemma.  $\square$

Now let  $\mathbb{P} = \mathbb{Q} \times \mathbb{C}_{\omega_2}$ , where  $\mathbb{C}_{\omega_2}$  is the standard poset adding  $\omega_2$  many Cohen reals.

Let  $G = G_0 \times G_1$  be a generic filter in  $\mathbb{P}$ , such that  $G_0$  is generic in  $\mathbb{Q}$ . Consider the space  $X^{G_0} = (\omega_1 \times (\omega + 1), \tau^{G_0})$ . Since  $V[G] = V[G_1][G_0]$ , it follows that  $X^{G_0}$  and the corresponding  $T$ ,  $Y$  and  $Z$  satisfy (A)–(C). However  $V[G] = V[G_0][G_1]$  as well, so  $\text{MA}_{\omega_1}(\text{countable})$  also holds.

We claim that  $T$  is  $D$ -separable; indeed,  $T$  is  $\sigma$ -discrete,  $w(T) = \omega_1$  and  $\text{MA}_{\omega_1}(\text{countable})$  holds hence Lemma 5.1 implies that

(D)  $T$  is  $D$ -separable.

Similarly, since  $Y$  is  $\sigma$ -nowhere dense,  $w(Y) = \omega_1$  and  $\text{MA}_{\omega_1}(\text{countable})$  holds, Lemma 5.1 implies that

(E)  $Y$  is  $NWD$ -separable.

Finally observe that (A)–(E) imply (1)–(3). This finishes the proof of the theorem.  $\square$



## 6. Monotonically normal spaces – positive results

Barman and Dow's aforementioned result suggests that convergence properties have some influence on selective versions of separability. In this section our first aim is to prove that *nwd*-separability and *D*-separability are equivalent in the class of monotonically normal spaces. This result exploits a weak convergence property which is satisfied by all monotonically normal spaces.

**Definition 6.1.** ([9]) A space  $X$  is called *discretely generated* (*nowhere densely generated*) if for every set  $A \subset X$  and every point  $x \in \bar{A}$  there is a discrete (nowhere dense)  $D \subset A$  such that  $x \in \bar{D}$ .

The property of being discretely generated (nowhere densely generated) is called *discrete tightness* (*nowhere dense tightness*) by Bella and Malykhin in [6]. Of course every crowded discretely generated space is nowhere densely generated, but the converse doesn't hold, as the following example shows.

**Example 6.2.** There is a nowhere densely generated space which is not discretely generated.

**Proof.** Let  $X = \Sigma_\omega(2^{\omega_1})$ , i.e. the set of all countably supported functions in  $2^{\omega_1}$  with the countably supported box product topology, and  $Y$  be any countable non-discretely generated space (for example, a countable maximal space). We claim that  $X \times Y$  is the desired example.

**Claim 6.2.1.** *Every meager set is nowhere dense in  $X$ .*

**Proof.** Recall that  $X$  has a base  $\mathcal{B}$  so that every countable decreasing sequence of sets from  $\mathcal{B}$  has non-empty interior (see Lemma 4.1); now the claim clearly follows.  $\square$

**Claim 6.2.2.** *The space  $X$  is discretely generated.*

**Proof.** Note that the character of a point  $x \in X$  is equal to  $\aleph_1$ . Let  $A \subset X$  be a non-closed set and  $x \in \bar{A} \setminus A$ . Since  $X$  is a  $P$ -space we can fix a decreasing local base  $\{U_\alpha: \alpha < \aleph_1\}$  at  $x$ . For every  $\alpha < \aleph_1$  pick  $x_\alpha \in U_\alpha \cap A$ . Then  $S = \{x_\alpha: \alpha < \aleph_1\}$  converges to  $x$ . If  $S$  had another accumulation point  $y \neq x$ , then, since  $X$  is a  $P$ -space, every neighborhood of  $y$  should hit  $S$  into uncountably many points. But that contradicts convergence. So  $S$  is a discrete set such that  $x \in \bar{S}$ .  $\square$

The space  $X \times Y$  is not discretely generated because it contains a homeomorphic copy of  $Y$ . Let  $A \subset X \times Y$  and  $(x, y) \in \bar{A}$ .

Let  $\{y_n: n < \omega\}$  be an enumeration of the set  $\pi_Y(A)$  and set  $P_n = \{z \in X: (z, y_n) \in A\}$ . Moreover define  $B \subset Y$  to be the set

$$B = \{y_n: x \in \overline{P_n}\}.$$

**Claim 6.2.3.** *The point  $y$  is in the closure of  $B$ .*

**Proof.** Suppose that this is not the case and let  $V$  be a neighborhood of  $y$  which misses  $B$ . Let  $S \subset \omega$  be the set such that  $V \cap \pi_Y(A) = \{y_n: n \in S\}$ . For every  $n \in S$  we have that  $x \notin \overline{P_n}$ , and thus we can find an open neighborhood  $U_n$  of  $x$  such that  $U_n \cap P_n = \emptyset$ . But then  $(\bigcap_{n < \omega} U_n) \times V$  is a neighborhood of  $(x, y)$  which misses  $A$  and this is a contradiction.  $\square$

Let  $T \subset \omega$  such that  $B = \{y_n: n \in T\}$ . For every  $n \in T$  we have that  $x \in \overline{P_n}$ , so, by Claim 6.2.2, there is a discrete  $D_n \subset P_n$  such that  $x \in \overline{D_n}$ . Now since  $X$  is dense-in-itself and  $D_n$  is nowhere dense, by Claim 6.2.2

we have that  $\bigcup_{n \in T} D_n$  is nowhere dense. Thus the set  $N := \bigcup_{n \in T} D_n \times \{y_n\} \subset A$  is also nowhere dense and it is easy to see that  $(x, y) \in \overline{N}$ . This proves that  $X \times Y$  is nowhere densely generated.  $\square$

These convergence-type properties are very useful in our context and this is apparent from the following fact.

**Fact 6.3.** Every separable discretely generated (nowhere densely generated) space is  $D$ -separable ( $NWD$ -separable).

Of course, we would be happier to obtain a relationship between  $d$ -separability and  $D$ -separability, but unfortunately, we already saw that there can be even first-countable,  $d$ -separable spaces which are not  $D$ -separable, so there is no way to simply replace separability with  $d$ -separability in Fact 6.3. Another approach would be to try and strengthen discrete generability to something more suitable to our purposes. This amounts to nothing more than replacing points with discrete sets:

**Definition 6.4.** A space is *discretely discretely generated* (in short, DDG) if for every set  $A \subset X$  and every discrete set  $D \subset \overline{A}$  there is a discrete set  $E \subset A$  such that  $D \subset \overline{E}$ .

**Fact 6.5.** ([2]) Every discretely discretely generated  $d$ -separable space is  $D$ -separable.

The authors of [2] proved that every monotone normal space is DDG; hence monotone normal,  $d$ -separable spaces are  $D$ -separable. We need the following closely related result:

**Lemma 6.6.** Let  $X$  be a monotonically normal space,  $A \subset X$  be a dense set and  $N \subset X$  be nowhere dense. Then there is a discrete set  $D \subset A$  such that  $N \subset \overline{D}$ .

**Proof.** Let  $H : X \times \tau_X \rightarrow \tau_X$  witness the monotone normality of  $X$  and define  $H^2 : X \times \tau_X \rightarrow \tau_X$  by  $H^2(x, U) = H(x, H(x, U))$ . Let  $\mathcal{U}$  be a maximal system of pairs  $\langle x, U \rangle \in A \times \tau_X$  such that

- (1)  $x \in U \subset X \setminus N$ ,
- (2) if  $\langle x, U \rangle \neq \langle x', U' \rangle \in \mathcal{U}$  then  $H^2(x, U) \cap H^2(x', U') = \emptyset$ .

Let  $D = \{x : \langle x, U \rangle \in \mathcal{U}\}$ . Clearly  $D \subset A$  is discrete.

We will show that  $N \subset \overline{D}$ . Let  $y \in N$ . Assume on the contrary that  $y \in W \in \tau_X$  with  $W \cap D = \emptyset$ .

If  $\langle x, U \rangle \in \mathcal{U}$ , then  $y \notin U \subset X \setminus N$  and  $x \notin W$  so

$$H(y, W) \cap H(x, U) = \emptyset. \quad (42)$$

Let  $V \subset H(y, W) \cap X \setminus N$  be open and pick  $z \in A \cap V$ . Then  $z \notin H(x, U)$  and  $x \notin V$ . Thus

$$H(z, V) \cap H^2(x, U) = \emptyset. \quad (43)$$

Thus  $\mathcal{U}$  was not maximal because  $\mathcal{U} \cup \{\langle z, V \rangle\}$  also satisfies (1) and (2). A contradiction.  $\square$

**Theorem 6.7.** Every monotonically normal,  $nwd$ -separable space is  $D$ -separable.

**Proof.** Let  $X$  be a monotonically normal space with a  $\sigma$ -nowhere dense set  $D = \bigcup_{n < \omega} N_n$ . Fix a sequence of dense sets  $\{D_n : n \in \omega\}$  as well. Let us apply Lemma 6.6 to pick discrete sets  $E_n \subset D_n$  such that  $N_n \subset \overline{E_n}$ . Then  $\bigcup_{n < \omega} E_n$  is a dense subset of  $X$  and this witnesses that  $X$  is  $D$ -separable.  $\square$

The following theorem can be derived from Theorem 6.7 and Theorem 28 of [7]. We offer an alternative proof based on Mary Ellen Rudin's famous result that every compact monotonically normal space is the continuous image of a compact linearly ordered space.

**Theorem 6.8.** *Every compact, monotonically normal,  $nwd$ -separable space has a  $\sigma$ -disjoint  $\pi$ -base.*

**Proof.** Let us remark that  $X$  does not have isolated points because it is  $nwd$ -separable.

Assume first that  $X$  is a GO-space, i.e., it is a subspace of an ordered space  $Y$ .

Let  $\{N_n: n \in \omega\}$  be a family of nowhere dense subsets of  $X$  such that  $\bigcup\{N_n: n \in \omega\}$  is dense. We can assume  $N_0 \subset N_1 \subset \dots$ .

For each  $n \in \omega$  consider  $Y \setminus \overline{N_n}$ , and let  $\mathcal{U}_n$  be the natural partition of  $Y \setminus \overline{N_n}$  into maximal convex sets. Let  $\mathcal{V}_n = \{U \cap X: U \in \mathcal{U}_n\}$ .

We claim that  $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$  is a  $\pi$ -base.

Indeed, let  $(y, y')$  be an open interval with  $X \cap (y, y') \neq \emptyset$ . Since  $X$  is dense-in-itself, we can find  $x_0, x_1, x_2 \in X$  with  $y < x_0 < x_1 < x_2 < y'$ . Then  $(y, x_1) \cap X \neq \emptyset \neq (x_1, y') \cap X$ . So there is  $n$  such that  $(y, x_1) \cap N_n \neq \emptyset \neq (x_1, y') \cap N_n$ . Pick  $x'_0 \in (y, x_1) \cap N_n$  and  $x'_2 \in (x_1, y') \cap N_n$ . Since  $X \cap (x'_0, x'_2) \neq \emptyset$  and  $N_n$  is nowhere dense, we can find  $x'_1 \in X \cap (x'_0, x'_2) \setminus \overline{N_1}$ . Pick  $U \in \mathcal{U}_n$  with  $x'_1 \in U$ . Then  $x'_0, x'_2 \notin U$ , so  $U \subset (x'_0, x'_2) \subset (y, y')$ . Thus  $\emptyset \neq U \cap X \subset X \cap (y, y')$ .

Now let  $X$  be arbitrary. Then, by Rudin's theorem,  $X$  is the continuous image of a compact, ordered space  $Y$ ,  $f: Y \rightarrow X$ . Then there is a closed subspace  $Z$  of  $Y$  such that map  $g = f \upharpoonright Z$  is irreducible. Then  $Z$  is a GO-space, and it does not have isolated points because  $g$  is irreducible, and  $X$  is dense-in-itself.

So  $Z$  has a  $\sigma$ -disjoint  $\pi$ -base  $\mathcal{U}$ .

We claim that

$$\mathcal{V} = \{X \setminus g[Z \setminus U]: U \in \mathcal{U}\} \quad (44)$$

is a  $\sigma$ -disjoint  $\pi$ -base of  $X$ .

First observe that if  $U \in \mathcal{U}$ , then  $X \setminus g[Z \setminus U] \neq \emptyset$ , i.e.  $X \neq g[Z \setminus U]$ , because  $g$  is irreducible. Thus  $\emptyset \notin \mathcal{V}$ .

To check that  $\mathcal{V}$  is a  $\pi$ -base, pick an arbitrary non-empty set  $V \subset X$ . Then there is  $U \in \mathcal{U}$  with  $U \subset g^{-1}V$ . Then  $X \setminus g[Z \setminus U] \subset V$ .

Finally we show that  $\mathcal{V}$  is  $\sigma$ -disjoint. Since  $\mathcal{U}$  was  $\sigma$ -disjoint, it is enough to show that  $U \cap U' = \emptyset$  implies  $(X \setminus f[Z \setminus U]) \cap (Z \setminus f[X \setminus U']) = \emptyset$ . Indeed, assume that  $(X \setminus f[Z \setminus U]) \cap (Z \setminus f[X \setminus U']) \neq \emptyset$ . Pick  $x \in X \setminus (f[Z \setminus U] \cup f[Z \setminus U'])$ . Fix  $z \in Z$  with  $g(z) = x$ . Then  $z \in U \cap U'$ , i.e.  $U \cap U' \neq \emptyset$ .  $\square$

Finally, we turn our attention to a particularly interesting space:

$$\sigma(2^{\omega_1}) = \{x \in 2^{\omega_1}: |x^{-1}\{1\}| < \omega\}$$

with the Tychonoff product topology and to the question whether this space is  $D$ -separable. First, note that  $\sigma(2^{\omega_1})$  is  $\sigma$ -discrete and hence  $d$ -separable. A natural approach would be then to prove that  $\sigma(2^{\omega_1})$  is DDG. However, we will show that it is independent of ZFC whether  $\sigma(2^{\omega_1})$  is DDG. More precisely, we prove that

**Theorem 6.9.** *If  $\text{MA}_{\aleph_1}$  holds then  $\sigma(2^{\omega_1})$  is DDG.*

and

**Theorem 6.10.** *If  $\diamond$  holds then  $\sigma(2^{\omega_1})$  is not DDG.*

If  $a \in [\omega_1]^{<\omega}$ , then we denote the characteristic function of  $a$  by  $\chi_a$ . The map  $a \mapsto \chi_a$  is a bijection between  $[\omega_1]^{<\omega}$  and  $\sigma(2^{\omega_1})$ . For  $A \subset [\omega_1]^{<\omega}$  write  $\chi[A] = \{\chi_a: a \in A\}$ .

Let

$$\tau = \{A \subset [\omega_1]^{<\omega} : \chi[A] \text{ is open in } \sigma(2^{\omega_1})\}.$$

Instead of  $\sigma(2^{\omega_1})$  we will consider a homeomorphic copy of that space: the space  $\mathcal{X} = \langle [\omega_1]^{<\omega}, \tau \rangle$ .

For  $x, y \in [\omega_1]^{<\omega}$  with  $x \cap y = \emptyset$ , let

$$U(x, y) = \{z \in [\omega_1]^{<\omega} : x \subset z \wedge y \cap z = \emptyset\}.$$

If  $a \in [\omega_1]^{<\omega}$ , then the family

$$\{U(a, b) : b \in [\omega_1 \setminus a]^{<\omega}\}$$

is a neighborhood base of  $a$  in  $\mathcal{X}$ .

**Lemma 6.11.** *Let  $a \in [\omega_1]^{<\omega}$  and  $B \subset [\omega_1 \setminus a]^{<\omega}$ . Then  $a \in B'$  iff  $B$  contains an infinite  $\Delta$ -system with kernel  $a$ .*

**Proof.** Assume first that  $a \in B'$ . Choose  $b_0, b_1, \dots$  from  $B \setminus \{a\}$  such that

$$b_n \in B \cap U(a, (b_0 \cup \dots \cup b_{n-1}) \setminus a).$$

Since  $a \in B'$  we can construct such a sequence, and observe that  $\{b_0, b_1, \dots\}$  is an infinite  $\Delta$ -system with kernel  $a$ .

Assume now that  $B$  is an infinite  $\Delta$ -system with kernel  $a$ . If  $U(a, c)$  is a neighborhood of  $a$ , then we can pick  $b \in B \setminus \{a\}$  with  $c \cap b = \emptyset$ , and then  $b \in U(a, c)$ . So  $a \in B'$ .  $\square$

Since  $E \subset A'$ , by the previous lemma, for each  $e \in E$  we can fix an infinite  $\Delta$ -system  $A_e \subset A \setminus \{e\}$  with kernel  $e$ .

To prove [Theorem 6.9](#) we need the following lemma.

**Lemma 6.12** ( $\text{MA}_{\aleph_1}$ ). *If  $E \subset \sigma(2^{\omega_1})$  is discrete,  $A \subset \sigma(2^{\omega_1})$ , and  $E \subset A'$ , then there is a discrete  $D_1 \subset A$  with  $E \subset D_1'$ .*

**Proof.** For  $e \in E$  pick  $z(e) \in [\omega_1 \setminus e]^{<\omega}$  with  $E \cap U(e, z(e)) = \{e\}$ . Since  $e \in A'$ , there is an infinite  $\Delta$ -system  $A_e \subset U(e, z(e)) \cap A$  with kernel  $e$ .

Define  $\mathcal{P} = \langle P, \leq \rangle$  as follows. Let

$$P = \{p \mid p \in [A \times E]^{<\omega} \wedge \forall \langle a, e \rangle \in p \ a \in A_e \wedge \forall \langle a, e \rangle \neq \langle a', e' \rangle \in p \ a \notin U(a', z(e'))\}. \quad (45)$$

Let  $p \leq q$  iff  $p \supseteq q$ .

**Claim 6.12.1.**  $\mathcal{P}$  satisfies c.c.c.

**Proof.** Assume  $\{p_\alpha : \alpha < \omega_1\} \subset P$ .

Let  $S_\alpha = \text{supp}(p_\alpha) = \bigcup_{\langle a, e \rangle \in p_\alpha} (a \cup z(e))$ . There are  $\alpha < \beta < \omega_1$  and a bijection  $\sigma : S_\alpha \rightarrow S_\beta$  such that  $\sigma \upharpoonright A_\alpha \cap A_\beta = \text{id}$ , and

$$p_\beta = \{\langle \sigma[a], \sigma[e] \rangle : \langle a, e \rangle \in p_\alpha\}. \quad (46)$$

Then  $p_\alpha \cup p_\beta \in P$ . Indeed, if  $\langle a, e \rangle \in p_\alpha \setminus p_\beta$  and  $\langle a', e' \rangle \in p_\beta \setminus p_\alpha$ , then  $a' \notin S_\alpha \cap S_\beta$ , so  $a \notin U(a', \emptyset)$ .  $\square$

For  $e \in E$  and  $n \in \omega$  let

$$\mathcal{D}_{e,n} = \{p \in P: |\{a: \langle a, e \rangle \in p\}| \geq n\}.$$

**Claim 6.12.2.**  $\mathcal{D}_{e,n}$  is dense.

**Proof.** It is enough to show that if  $p \in P$ , then there is  $a \in A \cap U(e, z)$  such that  $p \cup \{\langle a, e \rangle\} \in P$  for any  $z \in [\omega_1]^{<\omega}$  with  $e \cap z = \emptyset$ .

Since  $A_e$  is an infinite  $\Delta$ -system with kernel  $e$ , there is  $a \in A_e$  such that

$$(*) \quad (\text{supp}(p) \cup z) \cap a = e.$$

We claim that this  $a$  works.

Indeed, if  $\langle a', e' \rangle \in p$ , then  $a' \not\supseteq a$ , and so  $a' \notin U(a, \emptyset) \supset U(a, z(e))$ . On the other hand, assume on the contrary that  $a \in U(a', z(e')) \subset U(e', z(e'))$ . Then  $e \in U(a', z(e'))$  by (\*). Thus  $e = e'$  and  $a, a' \in A_e$  so  $a \notin U(a', \emptyset)$ . A contradiction.  $\square$

By Claim 6.12.2, the sets  $\mathcal{D}_{e,n}$  are dense. Let

$$\mathbb{D} = \{D_{e,n}: e \in E, n \in \omega\}.$$

By Claim 6.12.1,  $\text{MA}_{\aleph_1}$  implies that there is a  $\mathbb{D}$ -generic filter  $\mathcal{G} \subset P$ . For  $e \in E$  let

$$F_e = \left\{a \in A: \langle a, e \rangle \in \bigcup \mathcal{G}\right\},$$

and

$$D_1 = \bigcup \{F_e: e \in E\}.$$

Then  $D_1$  is discrete, because if  $\langle a, e \rangle \in \bigcup \mathcal{G}$ , then  $D_1 \cap U(a, z(e)) = \{a\}$  by the construction of the poset. Moreover, for  $e \in E$  the set  $F_e$  is infinite by the genericity of  $\mathcal{G}$ , and  $F_e$  is a  $\Delta$ -system with kernel  $e$  by the construction of the poset. So  $e \in F'_e \subset D'_1$ .

Thus we proved the lemma.  $\square$

We now finish with the proof of the first theorem:

**Proof of Theorem 6.9.** Let  $A \subset \sigma(2^{\omega_1})$ , and let  $E \subset \overline{A}$  be a discrete set. We need to find a discrete set  $D \subset A$  with  $E \subset \overline{D}$ .

To start with let  $E_0 = \{e \in E \cap A: e \notin A'\}$ .

Then  $E_0$  is discrete, moreover  $E_0 \cap \overline{A \setminus E_0} = \emptyset$ . Let  $A_1 = A \setminus \overline{E_0}$  and  $E_1 = E \setminus \overline{E_0}$ . Then  $E_1 \subset A'_1$ , so, by Lemma 6.12, there is a discrete set  $D_1 \subset A_1$  with  $E_1 \subset D'_1$ . Then

- (I)  $E_0 \cap \overline{D_1} = \emptyset$  because  $E_0 \cap A' = \emptyset$ , and
- (II)  $\overline{E_0} \cap D_1 = \emptyset$  because  $D_1 \subset A_1 = A \setminus \overline{E_0}$ .

Thus  $D = E_0 \cup D_1$  is the required set.  $\square$

**Proof of Theorem 6.10.** Consider the discrete subspace  $D = [\omega_1]^1$  of  $\sigma(2^{\omega_1})$ . Using  $\diamond$  we will construct  $A \subset [\omega_1]^{\geq 2}$  such that  $D \subset A'$ , but there is no discrete  $\mathcal{E} \subset A$  with  $D \subset \mathcal{E}'$ .

Fix a  $\diamond$ -sequence  $\langle \mathcal{C}_\alpha: \alpha < \omega_1 \rangle$  which guesses subsets of  $[\omega_1]^{<\omega}$ , i.e.

$$\forall \mathcal{C} \subset [\omega_1]^{<\omega} \quad \{\alpha: \mathcal{C} \cap [\alpha]^{<\omega} = \mathcal{C}_\alpha\} \text{ is stationary.} \quad (47)$$

We construct a continuous sequence  $\langle \mathcal{A}_\beta: \omega \leq \beta \leq \omega_1 \rangle$  such that

- (a)  $\mathcal{A}_\beta \subset \{b \in [\beta]^{<\omega}: |b| \geq 2\}$ ,
- (b)  $\{\{\alpha\}: \alpha < \beta\} \subset \mathcal{A}'_\beta$ ,
- (c) if  $b \in \mathcal{A}_\beta$  and  $|b \cap \nu| \geq 2$  for some  $\nu < \beta$ , then  $b \cap \nu \in \mathcal{A}_\nu$ ,

as follows. Let

$$\mathcal{A}_\omega = \{x \in [\omega]^{<\omega}: |x| \geq 2\}. \quad (48)$$

If  $\beta$  is limit, let

$$\mathcal{A}_\beta = \bigcup \{\mathcal{A}_\zeta: \alpha < \beta\}. \quad (49)$$

Assume that  $\beta = \alpha + 1$ .

If  $\mathcal{C}_\alpha \not\subset \mathcal{A}_\alpha$  or there is  $\nu < \alpha$  such that  $\{\nu\}$  is not an accumulation point of  $\mathcal{C}_\alpha$ , then let

$$\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{\{\alpha, 2n, 2n+1\}: n < \omega\}. \quad (50)$$

Assume now that  $\mathcal{C}_\alpha \subset \mathcal{A}_\alpha$  and

$$\{\{\nu\}: \nu < \alpha\} \subset \mathcal{C}'_\alpha. \quad (51)$$

Then for each  $\nu < \alpha$  there is a  $\Delta$ -system in  $\mathcal{C}_\alpha$  with kernel  $\{\nu\}$ . So there are infinitely many pairwise disjoint elements  $\{b_n^\alpha: n < \omega\}$  in  $\mathcal{C}_\alpha$ . Let

$$\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{\{\alpha\} \cup b_n^\alpha: n < \omega\}. \quad (52)$$

It is clear that (a)–(c) hold.

Let  $\mathcal{A} = \mathcal{A}_{\omega_1}$ . Then  $\{\{\alpha\}: \alpha < \omega_1\} \subset \mathcal{A}'$  by (b). Assume on the contrary that there is a discrete set  $\mathcal{E} \subset \mathcal{A}$  with  $\{\{\alpha\}: \alpha < \omega_1\} \subset \mathcal{E}'$ . For each  $e \in \mathcal{E}$  fix neighborhood  $U(e, z(e))$  of  $e$  with  $U(e, z(e)) \cap \mathcal{E} = \{e\}$ .

Then there is  $\alpha < \omega_1$  such that

$$\{\{\nu\}: \nu < \alpha\} \subset (\mathcal{E} \cap [\alpha]^{<\omega})', \quad z(e) \subset \alpha \quad \text{for } e \in \mathcal{E} \cap [\alpha]^{<\omega}, \quad \text{and} \quad \mathcal{C}_\alpha = \mathcal{E} \cap [\alpha]^{<\omega}.$$

Pick  $b \in \mathcal{E} \cap U(\{\alpha\}, \emptyset)$ . Since  $\{\{\nu\}: \nu < \alpha\} \subset \mathcal{C}'_\alpha$ , we have  $b \cap (\alpha + 1) = \{\alpha\} \cup b_n^\alpha$  for some  $n < \omega$ . Since  $b_n^\alpha \in \mathcal{C}_\alpha \subset \mathcal{E}$ , we also have

$$U(b_n^\alpha, z(b_n^\alpha)) \cap \mathcal{E} = \{b_n^\alpha\}.$$

But  $b_n^\alpha \cup z(b_n^\alpha) \subset \alpha$  and  $b \cap \alpha = b_n^\alpha$ , so

$$b \in U(b_n^\alpha, z(b_n^\alpha)). \quad (53)$$

A contradiction.  $\square$

The following remains unsolved:

**Question 6.13.** Is it provable in ZFC that  $\sigma(2^{\omega_1})$  is  $D$ -separable? Does  $\diamond$  imply that  $\sigma(2^{\omega_1})$  is not  $D$ -separable?

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