A COMMON EXTENSION OF ARHANGEL'SKII'S THEOREM AND THE HAJNAL-JUHÁSZ INEQUALITY

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Abstract. We present a bound for the weak Lindelöf number of the G_δ-modification of a Hausdorff space which implies various known cardinal inequalities, including the following two fundamental results in the theory of cardinal invariants in topology: $|X| \leq 2^{L(X)\chi(X)}$ (Arhangel'skiĭ) and $|X| \leq 2^{c(X)\chi(X)}$ (Hajnal-Juhász). This solves a question that goes back to Bell, Ginsburg and Woods [6] and is mentioned in Hodel's survey on Arhangel'skiĭ's Theorem [15]. In contrast to previous attempts we do not need any separation axiom beyond T_2.

1. Introduction

Two of the milestones in the theory of cardinal invariants in topology are the following inequalities:

Theorem 1. *(Arhangel'skiĭ, 1969)* [2] *If X is a T_2 space, then $|X| \leq 2^{L(X)\chi(X)}$.*

Theorem 2. *(Hajnal-Juhász, 1967)* [13] *If X is a T_2 space, then $|X| \leq 2^{c(X)\chi(X)}$.*

Here $\chi(X)$ denotes the *character* of X, $c(X)$ denotes the *cellularity* of X, that is the supremum of the cardinalities of the pairwise disjoint collection of non-empty open subsets of X and $L(X)$ denotes the *Lindelöf degree* of X, that is the smallest cardinal κ such that every open cover of X has a subcover of size at most κ.

The intrinsic difference between the cellularity and the Lindelöf degree makes it non-trivial to find a common extension of the two previous inequalities. The first attempt was done in 1978 by Bell, Ginsburg and Woods [6], who used the notion of weak Lindelöf degree. The weak Lindelöf degree of X ($wL(X)$) is defined as the least cardinal κ such

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that every open cover of X has a $\leq \kappa$-sized subcollection whose union is dense in X. Clearly, $wL(X) \leq L(X)$ and we also have $wL(X) \leq c(X)$, since every open cover without $< \kappa$-sized dense subcollections can be refined to a κ-sized pairwise disjoint family of non-empty open sets by an easy transfinite induction. Unfortunately, the Bell-Ginsburg-Woods result needs a separation axiom which is much stronger than Hausdorff.

Theorem 3. [6] *If X is a normal space, then $|X| \leq 2^{wL(X)\chi(X)}$.*

It is still unknown whether this inequality is true for regular spaces, but in [6] it was shown that it may fail for Hausdorff spaces. Indeed, the authors constructed Hausdorff non-regular first-countable weakly Lindelöf spaces of arbitrarily large cardinality.

Arhangel'skiĭ [3] got closer to obtaining a common generalization of these two fundamental results by introducing a relative version of the weak Lindelöf degree, namely the cardinal invariant $wL_c(X)$, i.e. the least cardinal κ such that for any closed set F and any family of open sets $\mathcal{U}$ satisfying $F \subseteq \bigcup \mathcal{U}$ there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $F \subseteq \overline{\bigcup \mathcal{V}}$.

Theorem 4. [3] *If X is a regular space, then $|X| \leq 2^{wL_c(X)\chi(X)}$.*

O. Alas [1] showed that the previous inequality continues to hold for Urysohn spaces, but it is still open whether it's true for Hausdorff spaces.

In [4] Arhangel'skii made another step ahead by introducing the notion of strict quasi-Lindelöf degree, which allowed him to give a common refinement of *the countable case* of his 1969 theorem and the Hajnal-Juhász inequality. He defined a space X to be *strict quasi-Lindelöf* if for every closed subset F of X, for every open cover $\mathcal{U}$ of F and for every countable decomposition $\{\mathcal{U}_n : n < \omega\}$ of $\mathcal{U}$ there are countable subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$, for every $n < \omega$ such that $F \subset \bigcup \{\overline{\bigcup \mathcal{V}_n} : n < \omega\}$. It is easy to see that every Lindelöf space is strict quasi-Lindelöf and every ccc space is strict-quasi Lindelöf. Arhangel'skii proved that every strict quasi-Lindelöf first-countable space has cardinality at most continuum.

However, Arhangel'skii's approach cannot be extended to higher cardinals. Indeed, it's not even clear whether $|X| \leq 2^{\chi(X)}$ is true for every strict quasi-Lindelöf space X. This inspired us to introduce the following cardinal invariants:

Definition 5.

- *The* piecewise weak Lindelöf degree of X *($pwL(X)$) is defined as the minimum cardinal κ such that for every open cover $\mathcal{U}$ of*

X and every decomposition $\{\mathcal{U}_i : i \in I\}$ *of* $\mathcal{U}$*, there are* $\leq \kappa$*-sized families* $\mathcal{V}_i \subset \mathcal{U}_i$*, for every* $i \in I$ *such that* $X \subset \bigcup\{\overline{\bigcup \mathcal{V}_i} : i \in I\}$*.*

- *The* piecewise weak Lindelöf degree for closed sets of X *(*$pwL_c(X)$*) is defined as the minimum cardinal* κ *such that for every closed set* $F \subset X$*, for every open family* $\mathcal{U}$ *covering* F *and for every decomposition* $\{\mathcal{U}_i : i \in I\}$ *of* $\mathcal{U}$*, there are* $\leq \kappa$*-sized subfamilies* $\mathcal{V}_i \subset \mathcal{U}_i$ *such that* $F \subset \bigcup\{\overline{\bigcup \mathcal{V}_i} : i \in I\}$*.*

As a corollary to our main result, we will obtain the following bound, which is the desired common extension of Arhangel'skii's Theorem and the Hajnal-Juhász inequality.

Theorem 6. *For every Hausdorff space* X*,* $|X| \leq 2^{pwL_c(X)\cdot\chi(X)}$*.*

For undefined notions we refer to [11]. Our notation regarding cardinal functions mostly follows [14]. To state our proofs in the most elegant and compact way we use the language of elementary submodels, which is well presented in [10].

2. A cardinal bound for the G_δ-modification

The following proposition collects a few simple general facts about the piecewise weak Lindelöf number which will be helpful in the proof of the main theorem.

Proposition 7. *For any space* X *we have:*

1. $pwL(X) \leq pwL_c(X)$*.*
2. $pwL_c(X) \leq L(X)$*.*
3. $pwL_c(X) \leq c(X)$*.*
4. *If* X *is* T_3 *then* $wL_c(X) \leq pwL(X)$*.*

Proof. The first two items are trivial. To prove the third item, let F be a closed subset of X and $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \in I\}$ be an open collection satisfying $F \subseteq \bigcup \mathcal{V}$. Suppose $c(X) \leq \kappa$. For every $i \in I$ let $\mathcal{C}_i$ be a maximal collection of pairwise disjoint non-empty open subsets of X such that for each $C \in \mathcal{C}_i$ there is some $V_C \in \mathcal{V}_i$ with $C \subseteq V_C$. By letting $\mathcal{W}_i = \{V_C : C \in \mathcal{C}_i\}$, the maximality of $\mathcal{C}_i$ implies that $\bigcup \mathcal{V}_i \subseteq \overline{\bigcup \mathcal{W}_i}$ and so $F \subseteq \bigcup\{\overline{\cup \mathcal{W}_i} : i \in I\}$. Since $|\mathcal{W}_i| \leq |\mathcal{C}_i| \leq \kappa$, we have $pwL_c(X) \leq \kappa$.

To prove the fourth item assume X is a regular space and let κ be a cardinal such that $pwL(X) \leq \kappa$. Let F be a closed subset of X and $\mathcal{U}$ be an open cover of F. If $\mathcal{U}$ covers X we're done. Otherwise use regularity to choose, for every $p \in X \setminus \bigcup \mathcal{U}$ an open set U_p such that $p \in U_p$ and $F \cap \overline{U}_p = \emptyset$. Note that $\mathcal{U} \cup \{U_p : p \in X \setminus F\}$ is an open cover of X, so by $pwL(X) \leq \kappa$, there is a κ-sized subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X \subset \overline{\bigcup \mathcal{V}} \cup \bigcup \{\overline{U_p} : p \in X \setminus F\}$. Hence $F \subset \overline{\bigcup \mathcal{V}}$ and we are done. □

Corollary 8. *If X is a regular space then $|X| \leq 2^{pwL(X) \cdot \chi(X)}$.*

Proof. Combine Proposition 7, (4) and Arhangel'skii's result that $|X| \leq 2^{wL_c(X) \cdot \chi(X)}$ for every regular space X. □

We state our main theorem in terms of the G_κ-*modification* of a space. Let κ be a cardinal number. By X_κ we denote the topology on X generated by κ-sized intersections of open sets of X. We call X_κ, the G_κ-*modification of* X; in case $\kappa = \omega$ we speak of the G_δ-modification of X and we often use the symbol X_δ instead. This construction has been extensively studied in the literature; various authors have tried to bound the cardinal functions of X_κ in terms of their values on X (see, for example [8], [12], [16], [17], [18]) and results of this kind have found applications to other topics in topology, like the estimation of the cardinality of compact homogeneous spaces (see [5], [8], [9] and [18]).

By X^c_κ we denote the topology on X generated by G^c_κ-sets, that is those subsets G of X such that there is a family $\{U_\alpha : \alpha < \kappa\}$ of open sets with $G = \bigcap \{U_\alpha : \alpha < \kappa\} = \bigcap \{\overline{U_\alpha} : \alpha < \kappa\}$. In general, the topology of X^c_κ is coarser than the G_κ-modification of X, but if X is a regular space then $X^c_\kappa = X_\kappa$.

Theorem 9. *Let X be a Hausdorff space such that $t(X) \cdot pwL_c(X) \leq \kappa$ and X has a dense set of points of character $\leq \kappa$. Then $wL(X^c_\kappa) \leq 2^\kappa$.*

Proof. Let $\mathcal{F}$ be a cover of X by G^c_κ-sets. Let θ be a large enough regular cardinal and M be a κ-closed elementary submodel of $H(\theta)$ such that $|M| = 2^\kappa$ and M contains everything we need (that is, $X, \mathcal{F} \in M$, $\kappa + 1 \subset M$ etc...).

For every $F \in \mathcal{F}$ choose open sets $\{U_\alpha : \alpha < \kappa\}$ such that $F = \bigcap \{U_\alpha : \alpha < \kappa\} = \bigcap \{\overline{U_\alpha} : \alpha < \kappa\}$.

Claim 1. $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim 1. Let $x \in \overline{X \cap M}$. Since $\mathcal{F}$ is a cover of X we can find a set $F \in \mathcal{F}$ such that $x \in F$. Moreover, using $t(X) \leq \kappa$, we can find a κ-sized subset S of $X \cap M$ such that $x \in \overline{S}$. Note that $x \in \overline{U_\alpha \cap S}$, for every $\alpha < \kappa$. Moreover, by κ-closedness of M, the set $U_\alpha \cap S$ belongs to M. Set $B = \bigcap \{\overline{U_\alpha \cap S} : \alpha < \kappa\}$. Note that $x \in B \subset F$ and $B \in M$. Therefore $H(\theta) \models (\exists G \in \mathcal{F})(x \in B \subset G)$ and all the free variables in the previous formula belong to M. Therefore, by elementarity we also have that $M \models (\exists G \in \mathcal{F})(x \in B \subset G)$ and hence there exists a set $G \in \mathcal{F} \cap M$ such that $x \in G$, which is what we wanted to prove.

△

Claim 2. $\mathcal{F} \cap M$ has dense union in X.

Proof of Claim 2. Suppose by contradiction that $X \nsubseteq \overline{\bigcup(\mathcal{F} \cap M)}$. Then we can fix a point $p \in X \setminus \overline{\bigcup(\mathcal{F} \cap M)}$ such that $\chi(p, X) \leq \kappa$. Let $\{V_\alpha : \alpha < \kappa\}$ be a local base at p.

For every $F \in \mathcal{F} \cap M$, let $\{U_\alpha(F) : \alpha < \kappa\} \in M$ be a sequence of open sets such that $F = \bigcap\{U_\alpha(F) : \alpha < \kappa\} = \bigcap\{\overline{U_\alpha(F)} : \alpha < \kappa\}$. Note that $\{U_\alpha(F) : \alpha < \kappa\} \subset M$. Let $\mathcal{C} = \{U_\alpha(F) : F \in \mathcal{F} \cap M, \alpha < \kappa\}$. Note that $\mathcal{C}$ is an open cover of $\overline{X \cap M}$ and $\mathcal{C} \subset M$.

For every $x \in \overline{X \cap M}$, we can choose, using Claim 1, a set $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$. Since $p \notin F_x$, there is $\alpha < \kappa$ such that $p \notin \overline{U_\alpha(F_x)}$. Hence we can find an ordinal $\beta_x < \kappa$ such that $V_{\beta_x} \cap U_\alpha(F_x) = \emptyset$. This shows that $\mathcal{U} = \{U \in \mathcal{C} : (\exists \beta < \kappa)(U \cap V_\beta = \emptyset)\}$ is an open cover of $\overline{X \cap M}$. Let $\mathcal{U}_\alpha = \{U \in \mathcal{U} : U \cap V_\alpha = \emptyset\}$. Then $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is a decomposition of $\mathcal{U}$ and hence we can find a κ-sized family $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ for every $\alpha < \kappa$ such that $\overline{X \cap M} \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$. Note that by κ-closedness of M the sequence $\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$ belongs to M and hence the previous formula implies that:

$$M \models X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$$

So, by elementarity:

$$H(\theta) \models X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$$

But that is a contradiction, because $p \notin \overline{\bigcup \mathcal{V}_\alpha}$, for every $\alpha < \kappa$.

△

Since $|\mathcal{F} \cap M| \leq 2^\kappa$, Claim 2 proves that $wL(X^c_\kappa) \leq 2^\kappa$, as we wanted.

□

As a first consequence, we derive the desired common extension of Arhangel'skii's Theorem and the Hajnal-Juhász inequality.

Recall that the *closed pseudocharacter of the point x in X* ($\psi_c(x, X)$) is defined as the minimum cardinal κ such that there is a κ-sized family $\{U_\alpha : \alpha < \kappa\}$ of open neighbourhoods of x with $\bigcap\{\overline{U_\alpha} : \alpha < \kappa\} = \{x\}$. The closed pseudocharacter of X ($\psi_c(X)$) is then defined as $\psi_c(X) = \sup\{\psi_c(x, X) : x \in X\}$.

Corollary 10. *Let X be a Hausdorff space. Then $|X| \leq 2^{pwL_c(X) \cdot \chi(X)}$.*

Proof. It suffices to note that in a Hausdorff space $\psi_c(X)\cdot t(X) \leq \chi(X)$ and hence if κ is a cardinal such that $\chi(X) \leq \kappa$ then X^c_κ is a discrete set. Thus $wL(X^c_\kappa) \leq 2^\kappa$ if and only if $|X| = |X^c_\kappa| \leq 2^\kappa$. □

Remark. Corollary 10 is a *strict* improvement of both Arhangel'skii's Theorem and the Hajnal-Juhász inequality. Indeed, if S is the Sorgenfrey line and $A([0,1])$ the Aleksandroff duplicate of the unit interval, then the space $X = (S \times S) \oplus A([0,1])$ is first countable, $pwL_c(X) = \aleph_0$ and $L(X) = c(X) = \mathfrak{c}$.

Recall that a space is initially κ-compact if every open cover of cardinality $\leq \kappa$ has a finite subcover (for $\kappa = \omega$ we obtain the usual notion of countable compactness). The following Lemma essentially says that if X is an initially κ-compact spaces such that $wL_c(X) \leq \kappa$, then it satisfies the definition of $pwL_c(X) \leq \kappa$ when restricted to decompositions of cardinality at most κ.

Lemma 11. *Let X be an initially κ-compact space such that $wL_c(X) \leq \kappa$ and F be a closed subset of X. If $\mathcal{U}$ is an open cover of F and $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is a κ-sized decomposition of $\mathcal{U}$, then there are κ-sized subfamilies $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ such that $F \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$*

Proof. Let $U_\alpha = \bigcup \mathcal{U}_\alpha$. Then $\{U_\alpha : \alpha < \kappa\}$ is an open cover of F of cardinality κ, so by initial κ-compactness there is a finite subset S of κ such that $F \subset \{U_\alpha : \alpha \in S\}$. Let now $\mathcal{W} = \bigcup\{\mathcal{U}_\alpha : \alpha \in S\}$. We then have $F \subset \bigcup \mathcal{W}$ and hence by $wL_c(X) \leq \kappa$ we can find a κ-sized subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $F \subset \overline{\bigcup \mathcal{W}'}$. Set now $\mathcal{V}_\alpha = \{W \in \mathcal{W}' : W \in \mathcal{U}_\alpha\}$. Then $|\mathcal{V}_\alpha| \leq \kappa$ and $F \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$, as we wanted. □

Noticing that in the proof of Theorem 9 we only needed to apply the definition of $pwL_c(X) \leq \kappa$ to decompositions of cardinality κ, Theorem 9 and Lemma 11 imply the following corollaries.

Corollary 12. [8] *Let X be an initially κ-compact space containing a dense set of points of character $\leq \kappa$ and such that $wL_c(X) \cdot t(X) \leq \kappa$. Then $wL(X^c_\kappa) \leq 2^\kappa$.*

Corollary 13. *(Alas,* [1]*) Let X be an initially κ-compact space with a dense set of points of character κ, such that $wL_c(X) \cdot t(X) \cdot \psi_c(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

3. Open Questions

Corollary 8 can be slightly improved by replacing regularity with the Urysohn separation property (that is, every pair of distinct points can be separated by disjoint closed neighbourhoods). Indeed, in a similar way as in the proof of Proposition 7 (4) it can be shown that if X is Urysohn then $wL_\theta(X) \leq pwL(X)$, where $wL_\theta(X)$ is the weak Lindelöf number for *θ-closed sets* (see [7]). Moreover, $|X| \leq 2^{wL_\theta(X)\cdot\chi(X)}$ for every Urysohn space X. However it's not clear whether regularity can be weakened to the Hausdorff separation property. That motivates the next question.

Question 3.1. *Is the inequality $|X| \leq 2^{pwL(X)\cdot\chi(X)}$ true for every Hausdorff space X?*

Moreover, we were not able to find an example which distinguishes countable piecewise weak Lindelöf number for closed sets from the strict quasi-Lindelöf property.

Question 3.2. *Is there a strict quasi-Lindelöf space X such that $pwL_c(X) > \aleph_0$?*

Finally, Arhangel'skii's notion of a strict quasi-Lindelöf space suggests a natural cardinal invariant. Define the strict quasi-Lindelöf number of X ($sqL(X)$) to be the least cardinal number κ, such that for every closed subset F of X, for every open cover $\mathcal{U}$ of F and for every *κ-sized* decomposition $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of $\mathcal{U}$ there are κ-sized subfamilies $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ such that $X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$. Obviously $sqL(X) \leq pwL_c(X)$. It's not at all clear from our argument whether the piecewise weak-Lindelöf number for closed sets can be replaced with the strict quasi-Lindelöf number in Corollary 10.

Question 3.3. *Let X be a Hausdorff space. Is it true that $|X| \leq 2^{sqL(X)\cdot\chi(X)}$?*

Even the following special case of the above question seems to be open.

Question 3.4. *Let X be a strict quasi-Lindelöf space. Is it true that $|X| \leq 2^{\chi(X)}$?*

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