

Complex Rating Curves for Sharp Crested Orifices and Rectangular or Triangular Weirs under Unsteady Flow Conditions

Giorgio Baiamonte, M.ASCE

Department of Agricultural, Food and Forest Sciences (SAAF), University of Palermo, viale delle
Scienze, Bldg. 4, Palermo 90128, Italy.

Abstract

The importance of discharge measurements is fully acknowledged in many research fields, mostly in hydrology. Numerous measurement devices and various overflow structures have been proposed for discharge measurements; however, their use is based on calibrated simple stage discharge relationships that may provide important errors when unsteady flow conditions occur. This issue is quite common, because of rainfall and runoff temporal variability that inhibits the achievement of the steady state. Although this issue was already experimentally investigated, it seems that a physically-based line of approach has not been attempted before. In this paper, unsteady stage discharge relationships on theoretical base are derived for the most common shapes of weirs. According to previous experimental investigations, a transposition of the simple (steady-state) rating curve occurs, with higher input discharges than those under steady-state, when the water level rises, and with lower input discharges, when the water level falls. The corresponding time to equilibrium is also studied and two different applications are performed. Finally, when the outflow process is unsteady, an error analysis quantitatively shows that using simple rating curves may determine high errors in discharge measurements, which decrease more and more at disappearing of the transposition of the rating curve, because of the attaining of the steady state. Triangular weirs result more sensitive to the unsteadiness than rectangular weirs, which in turn result more sensitive than the sharp crested orifices. The

suggested procedure could be relevant to improve the accuracy in discharge estimate, as it could be demonstrated by comparing the discharges derived by the proposed procedure and the ones provided by the application of hydrodynamic and rainfall-runoff models.

Key-Words: rating curves, unsteady state, stage-discharge relationships, sharp crested orifice, triangular weir, rectangular weir.

Introduction

Flow measurement is a key element in water resource management. Knowledge of flow rates allows quantifying deliveries to water users. Importantly, flow measurements allow to carried out a large amount of experimental activities that are focused on understanding fundamental hydrological processes (Bagarello and Ferro 2010; Camporese et al. 2014; Romano et al. 2018).

In general, rating curves are commonly used to calculate or predict a variable as the discharge, which is difficult to measure continuously, from another variable that is much easier to determine. To convert water height or stage into a discharge, hydrographers must establish a relationship between them by making frequent direct discharge measurements at stream gauging stations (Marg and Khas 1999).

The rating curve depends on the hydraulic characteristics of the stream channel and floodplain, and vary over time at almost every gauged station. There might be sensitive changes to a stream channel, such as the growth of aquatic vegetation in the summer, frequent shifting of a sand-bed stream bottom, catastrophic changes due to floods, or man-made changes (Merkley 2008), which might require only minor or temporary adjustments to streamflow records, or could require a complete modification of the rating curve.

50 Artificial control sections and overflow structures have been developed in order to stabilizing
51 the relationship between stage and discharge, thus partially overcoming the aforementioned issues.
52 In this context, many different flow measurement flumes and weirs have been designed and tested,
53 but only a few are commonly used in practice today. Both flumes and weirs are widespread and both
54 present drawbacks and advantages.

55 Flumes consist of a narrowed canal section with a particular, well-defined shape and partially-
56 constricted throat section. The flume geometry is often designed to cause the flow regime to pass
57 through critical depth, providing a means for determining the rate of flow from a single (upstream)
58 water depth measurement. The advantage of flumes over weirs is the small drop in water level (head
59 loss), and so flumes can be used in relatively shallow canals with flat grades.

60 However, boxes equipped by weirs can be useful where flow is above ground and the flow of
61 water is piped. Weirs are sharp-crested, easy to construct and can measure the discharge accurately
62 when correctly installed and the data interpreted well. It is important that the water level downstream
63 is always below the weir crest; otherwise, the discharge estimate could be incorrect.

64 Ever since the development of the Parshall flume (Parshall 1926), continuous improvements
65 have been made to reduce cost, increase accuracy and performance, and reduce head losses in open
66 channel flumes (Samani and Magallanez 2000; Baiamonte and Ferro 2007; Vatankhah 2010;
67 Vatankhah 2019). With respect to the previous Venturi flume, the Parshall flume is characterized by
68 reducing the inlet converging angle to reduce flow separation, lengthening the throat section to a
69 better development of the flow and to improve energy recovery. Most importantly, Parshall flume
70 introduces a drop through the throat, and the subsequent partial recovery in the discharge section.

71 Later, at Utah State University, Skogerboe et al. (1967) developed the cutthroat flume, with a
72 zero length in the throat section, which makes construction simple (Heiner et al. 2011). When the
73 throat base width of a trapezoidal flume is zero, these are called “V-notch flumes”. Similar to the V-
74 notch weirs, V-notch flumes are most commonly used for measuring water with a small head due to

75 a more rapid change of head with change in discharge. In general, flumes are particularly suitable
76 when the structure must have a minimal impact on existing flow conditions and water surface
77 elevations.

78 A disadvantage of flumes over weirs is that they are relatively expensive and they cannot easily
79 be combined with other structures, whereas that is possible with weirs. The crest of a measurement
80 weir is usually regular and perpendicular to the direction of flow. If this is not the case, special
81 calibrations must be made to develop a stage discharge relationship.

82 For both flumes and weirs, a wide and intensive use of dimensional analysis was performed,
83 having the great advantage to detect the dimensionless groups governing the particular shape device
84 to be studied. Therefore, a numerous shapes of flumes and weirs have been investigated (Carollo et
85 al. 2016; Bijankhan and Ferro 2017), usually providing excellent fitting of the rating curves,
86 according to the numerical coefficients but physical meaningless, which appear in the power
87 equations, as a result of the dimensional analysis. However, the latter is a sensitive issue, since actual
88 theoretical rating curves could differ from simple power equations that dimensional analysis imposes,
89 and the empirical coefficients can be applied when the experimental calibrating conditions are
90 rigorously reproduced in the field.

91 Dimensional analysis was also widely used to derive the discharge coefficient, usually
92 depending on the upstream level (Bagheri and Heidarpour 2010; Bautista-Capetillo et al. 2014),
93 which characterizes the stage discharge relationship. Recently, by using the dimensional analysis,
94 Vatankhah and Khamisabadi (2019) derived the discharge coefficient of sharp-crested weir with
95 power-law sides that reduces to the widely used rectangular, parabolic, and triangular weirs, where
96 empirical data are required to determine the numerical coefficient of power-laws. Similarly, Izadinia
97 and Heidarpour (2016) derived a discharge coefficient equation for a circular-crested side weir in
98 rectangular channels.

99 However, theoretical based derivation of the discharge coefficient was also made in the past.
100 For example, for rectangular weirs, Henderson (1966) reported a relationship between the discharge
101 coefficient, C_d , the contraction coefficient and the upstream kinetic head, and evidenced a logical
102 dependence of C_d by the ratio between the upstream water level, h , and the weir width, L , which was
103 verified by using the experimental data of Rehbock (1929), $C_d = 0.611 + 0.08 h/L$. Indeed, for high
104 L , the outflow process is two-dimensional and C_d reduces to 0.611, which also matches the value $\pi/(2$
105 $+\pi)$ obtained by Kirchhoff, (1869) who gave a theoretical minimum jet contraction coefficient.

106 Starting from theoretical considerations, Ramamurthy et al. (1987) applied the momentum
107 principle to estimate the discharge coefficient for sharp-crested weirs and then derived pressure and
108 momentum coefficients experimentally.

109 The stage discharge relationships governing flumes and weirs are not always simple, namely
110 discharge does not depend upon stage only (Henderson 1966; Neely and Bingham 1986; Fread 2007).
111 A complex rating curve should be considered when additional variables such as the slope of the
112 energy line or the rate of change of stage occur. Therefore, unsteady flow conditions have effects on
113 the one-way nature of rating curve. Marg and Khas (1999) suggested that when the simple rating
114 curve is affected by the variation in the local acceleration due to unsteady flow, the plotted points
115 could be annotated with the corresponding rate of change of slope with respect to time. A smooth
116 curve (steady state curve) passing through those points having the least values of rate of change of
117 stage is drawn first. In this way, all those points having positive values of rate of change of stage are
118 towards the right side of the curve and those with negative values are towards the left of it.

119 Le Coz (2012) reported that non-stationarity could determine important errors in the deriving
120 discharge by the simple stage discharge relationship. Chanson and Wang (2013) used the relationship
121 between flow discharge and upstream water elevation in the integral form of the continuity equation
122 but to derive the discharge coefficient of a large V-notch weir. Bortoni et al. (2017) also used unsteady
123 flow conditions for calibration of weirs. Recently, Ghasemzadeh et al. (2020) experimentally have

124 found that the deviation between steady and unsteady flow rates determines a looped rating curves,
125 which depends on the weir type and on the hydrograph gradient (i.e. the temporal depth variation),
126 especially when the latter changes rapidly. Ghasemzadeh et al. (2020) have also proposed calibrated
127 relationships, where two empirical coefficients account for the so called “Hydrograph Equivalent
128 Slope” defined as the geometric mean of the absolute slope value of the water head above the weir
129 crest, between the rising and the falling limb, which incorporates the effect of flow unsteadiness
130 (Baiamonte 2020a).

131 The performance of water measurement structures and devices in unsteady flow condition has
132 rarely been studied and the impacts of the flow dynamics on the estimated discharge still need further
133 attention and study. Heiner et al. (2011) presented an interesting table, assessing the accuracies of a
134 wide variety of flow measurement devices currently in service. In addition to the visual inspections,
135 actual flow measurements were taken and compared with the theoretical flows provided from each
136 device. Two-thirds of all the devices that were tested indicated flow measurements that exceeded
137 manufacturer design specifications, with 37% of those over estimating the discharge through the
138 structure. It was determined that errors in flow measurement were caused by a variety of reasons
139 (vegetation/debris in entrance, bypassing flow, incorrect geometry, etc.) and emphasized the
140 increasing importance of accurately measuring water-flow rates and volumes. Even Steinbakk et al
141 (2016) investigated the effect of rating curve and sample uncertainty on design flood estimation. As
142 observed by McMillan et al. (2010), in order to quantify the errors affecting hydrological models and
143 improve predictions, we must explicitly recognize errors in discharge measurements used to calibrate
144 a rainfall-runoff model, caused by stage–discharge rating curve uncertainty.

145 The importance of unsteadiness on rating curves was pointed out well by Muste and Lee
146 (2013), who stated that *the steady flow assumption is not applicable during propagation of storm-*
147 *generated waves hence the question on the validity of the steady rating curves during unsteady flow*
148 *is of both scientific and practical interest.* Moreover, scarce experimental evidence and analytical

149 inferences substantiate that during unsteady flows the relationship between some of the variables is
150 not unique leading to looped rating curves (also labeled hysteresis), thus this issue needs to be further
151 investigated. Kiang et al. (2018) when compared methods for streamflow uncertainty estimation,
152 among the multiple sources of uncertainty that impact the formulation of rating curves and hence
153 streamflow estimation, also mentioned the flow unsteadiness. Kiang et al. (2018) highlighted that
154 currently only the GesDyn method (Morlot et al. 2014) explicitly models time variation. However,
155 GesDyn is based on an approximated and not easy to apply procedure that updates the rating curve
156 after each new gauging, and uses variographic analysis to represent changes in uncertainty caused by
157 down weighting of previous rating curves with time (Kiang et al. 2018).

158 In conclusion, understanding causes and characteristics of discharge estimation uncertainty,
159 as well as how it can best be estimated, is an important research task.

160 Recently, Baiamonte (2020b) has theoretically derived unsteady stage discharge relationships
161 for a sharp crested orifice made at the bottom or at the side of a water tank, which is devoted to collect
162 the discharge to be estimated. As the weirs, the orifice, which needs to be accurately designed, makes
163 it possible to reduce the volume that common closed tank requires, also accounting for unsteadiness
164 during the outflow process. Experimental measurements demonstrated the reliability of the proposed
165 procedure.

166 In this context, the objective of this work is to simplify the already available complex rating
167 curves for sharp crested orifices, and to extend the discharge estimate procedure to the most common
168 shapes of weirs, rectangular and triangular, which was poorly attempted or never attempted before
169 on theoretical base, thus avoiding the need to achieve the steady state condition that weir and flume
170 methods commonly require.

171 It will be shown that, under unsteady flow conditions, this procedure makes it possible to
172 measure a constant inlet runoff discharge, by measuring a water level variation into the tank and the
173 time that is spent for the occurrence of the same water level variation. The proposed solutions are not

174 affected by the orifice or weirs geometry neither by the discharge coefficient, since they are presented
175 in dimensionless terms, which make results more general and more useful for their ability to contract,
176 or make more succinct, the functional form of physical relationships.

177 The paper is organized as follows. First, in the following “theoretical background” section,
178 for tanks equipped by crested orifice or weirs, the unsteady outflow process is analyzed in
179 dimensionless terms and the already introduced differential flow continuity equation is briefly
180 summarized and extended to the considered weirs. The solutions corresponding to the rising and to
181 the falling water level are presented, and the time to equilibrium, indicating the time required for the
182 achievement of the steady state conditions, is derived and tested. Second, a simple procedure to
183 determine the input discharge under unsteady state conditions is suggested. In this latter section, the
184 discharge magnitude corresponding to these devices operating under unsteadiness is also analyzed
185 and an errors analysis quantitatively described the effect on discharge estimate of assuming steady
186 state the outflow when it is not. The paper is concluded in the last section.

187

188 Theoretical background

189

190 **Fig. 1** illustrates an open rectangular tank, with sides L (m) and B (m), base area, A (m²), and
191 height, Z (m). The same **Fig. 1** is used to define the geometric parameters characterizing i) a circular
192 sharp-crested orifice (C) with diameter d (m) and cross-sectional area, a (m²), made at the bottom or
193 at a side of the tank, ii) a rectangular weir (R), with width L (m) and height H (m), and iii) a triangular
194 weir (T), with slope side $\tan \theta$.

195 A stationary and uniformly distributed water intensity, r (m s⁻¹) corresponding to an input
196 discharge Q_{in} (m³ s⁻¹), is applied over the tank (**Fig. 1**). Under steady-state conditions, the well-known
197 C, R and T rating curves describe the relationship between the output discharge, Q_o (m³/s), and the

198 water level, h , above the crested weirs or above the orifice. Thus, for steady state, the output discharge
199 Q_0 , equals the input discharge Q_{in} , which can be measured by the simple rating curves.

200 The water level, h , could be measured either by a submersible pressure transducer coupled
201 with a data logger, or by an ultrasonic probe (**Fig. 1**). The latter can be installed at the top of a wide
202 piezometer connected to the water tank, in order to measure the height to which a column of the water
203 rises against gravity, without the disturbance effects at the open water surface caused by the incoming
204 discharge into the tank.

205 Denoting y (m) the water level measured with respect to the base of the tank and p (m) the
206 crest elevation, the water level with respect to p equals:

207

$$208 \quad h = y - p \quad (1a)$$

209

210 Of course for C, when the crested orifice is placed at the bottom of the tank $p = 0$ (**Fig. 1**) and $h = y$.

211 **Table 1** reports the geometric characteristics of C, R and T, together with some reasonable
212 operating range of the same parameters that should be satisfied. As example, for R and T, the crest
213 elevation, p , and the width, L , should be high enough to neglect the kinetic head, which is not
214 considered in the rating curve, and partially avoids the dependence of the discharge coefficient by the
215 upward kinetic head; whereas, for C, h should be high enough in order to avoid vortices formation
216 towards the hole.

217 The maximum depth of the water level is denoted as H (m), and depends on Z and p , $H = Z -$
218 p (**Fig. 1**). The water level can also be expressed in dimensionless terms as $h_* = h/H$:

219

$$220 \quad h_* = y_* - p_* \quad (1b)$$

221

222 where $y^* = y/H$ and $p^* = p/H$. Thus, $h^* = 1$ and $h^* = 0$ indicate that the tank is completely full (maximum
223 outflow discharge) or that no outflow discharge occurs, respectively.

224 As observed above, under steady-state conditions, C, R and T rating curves make it possible
225 to derive the output discharge, Q_0 (m³/s), which equals the input discharge (i.e. runoff rate), Q_{in} (m³/s),
226 according to power equations, to the aforementioned geometry parameters, and to the different
227 exponent x of the water level, h , above the crested orifice or weir. As well-known (Henderson 1966),
228 in the rating curves, $x = 1/2$ for circular holes (Swamee and Swamee 2010), $x = 3/2$ for rectangular
229 weirs (Aydin et al. 2011), and $x = 5/2$ for triangular weirs (Bautista-Capetillo et al. 2014):

230

$$231 \quad Q_0^C = C_d a \sqrt{2g} h^{1/2} \quad (2a)$$

$$232 \quad Q_0^R = C_d \frac{2}{3} L \sqrt{2g} h^{3/2} \quad (2b)$$

$$233 \quad Q_0^T = C_d \frac{8}{15} \tan \theta \sqrt{2g} h^{5/2} \quad (2c)$$

234

235 where Q_0 , according to the different superscripts C, R, and T, is the steady-state discharge, C_d is the
236 discharge coefficient, a (m²) is the cross-sectional area of the crested orifice (C), L (m) is the width
237 of the rectangular weir (R), $\tan \theta$ is the slope side of the triangular weir (T) (**Fig. 1**) and g (m/s²) is
238 the acceleration due to gravity.

239 The outlet discharge per unit surface tank base, A , is denoted as v (m/s):

240

$$241 \quad v^C = C_d \frac{a}{A} \sqrt{2g} h^{1/2} \quad (3a)$$

$$242 \quad v^R = C_d \frac{2}{3} \frac{L}{A} \sqrt{2g} h^{3/2} \quad (3b)$$

$$243 \quad v^T = C_d \frac{8}{15} \frac{\tan \theta}{A} \sqrt{2g} h^{5/2} \quad (3c)$$

244

245 Of course, the maximum value of Q_0 , $Q_{0,max}$, is achieved when the water level h equals its
 246 maximum value, H :

247

$$248 \quad Q_{0,max}^C = C_d a \sqrt{2g} H^{1/2} \quad (4a)$$

$$249 \quad Q_{0,max}^R = C_d \frac{2}{3} L \sqrt{2g} H^{3/2} \quad (4b)$$

$$250 \quad Q_{0,max}^T = C_d \frac{8}{15} \tan \theta \sqrt{2g} H^{5/2} \quad (4c)$$

251

252 For C, R and T, the maximum value of v , v_{max} , can be analogously defined:

253

$$254 \quad v_{max}^C = \frac{Q_{0,max}^C}{A} = C_d \frac{a}{A} \sqrt{2g} H^{1/2} \quad (5a)$$

$$255 \quad v_{max}^R = \frac{Q_{0,max}^R}{A} = C_d \frac{2}{3} \frac{L}{A} \sqrt{2g} H^{3/2} \quad (5b)$$

$$256 \quad v_{max}^T = \frac{Q_{0,max}^T}{A} = C_d \frac{8}{15} \frac{\tan \theta}{A} \sqrt{2g} H^{5/2} \quad (5c)$$

257

258 Under steady-state conditions, the simple rating curves can also be jointly written for C, R and
 259 T, by normalizing Q_0 (Eq. 2a-c) with respect to $Q_{0,max}$ (Eq. 4a-c), and h with respect to H , denoted as
 260 as h^* (Eq. 1b):

261

$$262 \quad Q_{*0} = \frac{Q_0^C}{Q_{0,max}^C} = \frac{Q_0^R}{Q_{0,max}^R} = \frac{Q_0^T}{Q_{0,max}^T} = h_*^x \quad (6)$$

263

264 Now, let define a characteristic time of the water tank, t_c (s), as the travel time required by the
 265 water particle to cross the entire depth, $H = Z - p$, under uniform motion conditions, for $v = v_{max}$ (Eqs.
 266 5a-c):

267

$$t_c = \frac{H}{v_{max}} \quad (7)$$

269

270 Equation (7) shows that t_c also corresponds to the minimum travel time that the water particle,
 271 starting from the top of the tank, requires to cross H , therefore, the more v_{max} increases the less the
 272 travel time, t_c , will be.

273 Under unsteady-state conditions, meaning that a new input discharge $Q_{in} \neq Q_0$ occurs, the one
 274 dimensional differential flow equation can be written:

275

$$A dh = (Q_{in} - Q_0)dt \quad (8)$$

277

278 Eq. (8) states that for a constant input discharge, Q_{in} , the water volume flowing into the tank
 279 in time dt , $Q_{in} dt$, is equal to that released by the outlet, $Q_0 dt$, added to that stored in the tank, in the
 280 same time dt , $A dh$.

281 Once the characteristic time t_c has been defined, by dividing Eq. (8) with respect to t_c , the
 282 differential flow equation can be rewritten:

283

$$\frac{A}{t_c} dh = (Q_{in} - Q_0) d \frac{t}{t_c} \quad (9)$$

285

286 By denoting t/t_c as the normalized time, τ , and inserting t_c (Eq. 7) into the left side of Eq. (9), yields:

287

$$\frac{A v_{max}}{H} dh = (Q_{in} - Q_0) d\tau \quad (10)$$

289

290 Now, by using the normalized water level $h^* = h/H$, let define ρ the runoff excess intensity, r
 291 (m s^{-1}), that is Q_{in}/A , normalized with respect to v_{max} (Baiaumont 2020b):

292

293
$$\rho = \frac{r}{v_{max}} = \frac{Q_{in}}{A v_{max}} \quad (11)$$

294

295 According to Eq. (11), Eq. (10) can be rewritten:

296

297
$$dh_* = \left(\rho - \frac{Q_0}{A v_{max}} \right) d\tau \quad (12)$$

298

299 Considering that $Q_0/(A v_{max})$ is the normalized output discharge expressed by Eq. (6), Eq. (12)
300 can be further simplified as:

301

302
$$\frac{dh_*}{d\tau} = \rho - h_*^x \quad (13)$$

303

304 where of course under unsteady state conditions $\rho \neq h_*^x = Q_{*0}$.

305 In the following three sections, for C, R and T, first, the rising limb, the time to equilibrium
306 and the falling limb of the unsteady outflow process is theoretically derived and a Ψ function allowing
307 the mentioned derivation is introduced. Then, for two practical cases, the reliability of the suggested
308 solutions is shown. However, from a practical point of view, the following three sections help
309 understanding the fundamentals of the unsteady outflow process, but they are not directly required to
310 determine the input discharge under unsteady state. The latter procedure is described in a later section
311 and starts from the differential equation of the outflow process (Eq. 13), once the aforementioned Ψ
312 function was been introduced, as it is shown in the following sections.

313

314 **Rising limb**

315

316 For the case C ($x = 1/2$) and $\rho > 0$ an analytical solution of Eq. (13), i.e. the rising limb, was
 317 already derived (Baiamonte 2020b), providing the dimensionless unsteady-state relationship linking
 318 the normalized water levels, h^* and h_{*0} , corresponding to the normalized actual time, τ , and to the
 319 antecedent time, τ_0 , respectively, for a given ρ :

320

$$321 \quad \tau = \tau_0 + 2 \left(\rho \log \left| \frac{\rho - \sqrt{h_{*0}}}{\rho - \sqrt{h_*}} \right| - (\sqrt{h_*} - \sqrt{h_{*0}}) \right) \quad (14)$$

322

323 For $p = 0$, that is when the hole is made at the bottom of the tank, Eq. (14) was also experimentally
 324 tested, providing satisfactory results (Baiamonte 2020b). Eq. (14), which is explicit in τ , can be solved
 325 with respect to h^* in an implicit form:

326

$$327 \quad h_* = \left(\rho \log \left| \frac{\rho - \sqrt{h_{*0}}}{\rho - \sqrt{h_*}} \right| + \sqrt{h_{*0}} - \left(\frac{\tau - \tau_0}{2} \right) \right)^2 \quad (15)$$

328

329 Eq. (15) can be also made explicit, which could be useful in practice (Baiamonte 2020a), by using
 330 the Lambert function, W_0 (Olver et al. 2010) that was also applied in other different contexts (Serrano
 331 2001; Parlange et al. 2002):

332

$$333 \quad h_* = \rho^2 \left[1 + W_0 \left(\frac{\sqrt{h_{*0}} - \rho}{\rho} \exp \left(\frac{2\sqrt{h_{*0}} - (\tau - \tau_0)}{2\rho} - 1 \right) \right) \right]^2 \quad (16)$$

334

335 For the cases R and T ($x = 3/2$ and $x = 5/2$, respectively), by fixing the initial condition $h = h_0$
 336 for $t = t_0$, or in dimensionless terms:

337

$$338 \quad h_* = h_{*0} \quad \tau = \tau_0 \quad (17)$$

339

340 the differential equation of the unsteady outflow process (Eq. 13) needs considering, under the
341 reasonable assumption that during the short observed interval $[\tau_0, \tau]$, the runoff discharge Q_{in} , and the
342 corresponding ρ , are time-invariant:

343

344
$$\int_{\tau_0}^{\tau} d\tau = \int_{h_{*0}}^{h_*} \frac{dh_*}{\rho - h_*^x} \quad (18)$$

345

346 For any x , a general solution of Eq. (18) was already derived in order to provide in a very
347 different context, an analytical solution for the Richards equation, under gravity-driven infiltration
348 and constant rainfall intensity to be applied to the soil (Baiamonte 2020c), which was also treated as
349 a non-linear reservoir.

350 The aforementioned approach has been reformulated here, according to a different
351 characteristic time of the water tank and to different input and output dimensionless groups. Indeed,
352 there are similarities between the two processes (outflow process from a tank and water content
353 variably into the soil). When the unsteady differential equation (Eq. 13) is applied to the soils, the
354 known input was the rainfall intensity and the soil water content, mimed by the water tank level, has
355 to be determined (Baiamonte 2020d; Zhang et al 2020). In the tank case study of the present work,
356 the variation of the water level is the input and the incoming discharge, Q_{in} , is the output.

357 Moreover, in Eq. (13) for the soils (Baiamonte 2020c), according to the exponent x , which is
358 a dimensionless conductivity shape parameter (Nasta et al., 2012), in place of the normalized water
359 level, appears the Brooks and Corey hydraulic conductivity (Brooks and Corey 1964), which is
360 commonly used to model soil water fluxes and to assess soil physical quality (Severino and Coppola
361 2012; Cullotta et al. 2016). The solution of Eq. (18) applied to the soils in Baiamonte (2020c) is
362 rewritten for the water tank here:

363

$$\tau = \tau_0 + \frac{1}{\rho} \left[h_* {}_2F_1 \left(1, \frac{1}{x}, 1 + \frac{1}{x}, \frac{h_*^x}{\rho} \right) - h_{*0} {}_2F_1 \left(1, \frac{1}{x}, 1 + \frac{1}{x}, \frac{h_{*0}^x}{\rho} \right) \right] \quad (19)$$

365

366 where ${}_2F_1$ is the hypergeometric function ${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$ (Abramowitz and Stegun 1972,
367 p. 556).

368 For R and T weirs, contrarily to the soils in which x is linked to the pore connectivity index
369 and the retention curve shape parameter and may assume various values, x is imposed ($x = 3/2$ for R,
370 and $x = 5/2$ for T), and solutions according to arctangent and logarithm functions, much easier to be
371 used than hypergeometric function, can be derived, as it is shown in the following.

372 For R and T weirs, by assuming $\tau_0 = h_{*0} = 0$ as initial conditions, after simple but long algebra,
373 it was found that integration of Eq. (18) can be expressed by a Ψ function that only depends on the η
374 variable, which is defined by the root square of the ratio between the normalized water level, h_* , and
375 its asymptotic value, $h_{*\infty} = \rho^{1/x}$ (Baiamonte 2020b).

376 In particular, for R weirs ($x = 3/2$), $\eta = h_*/\rho^{1/3}$, and integration of Eq. (18) yields:

377

$$\tau = \frac{\Psi(\eta)}{\rho^{1/3}} \quad (20a)$$

379

380 where the Ψ function equals:

381

$$\Psi(\eta) = \frac{1}{3\sqrt{3}} \left[\pi - 6 \arctan \left(\frac{2\eta+1}{\sqrt{3}} \right) + \sqrt{3} \log \left(\frac{\eta^2+\eta+1}{(\eta-1)^2} \right) \right] \quad (20b)$$

383

384 Whereas, for T weirs ($x = 5/2$), $\eta = h_*/\rho^{2/5}$, and integration of Eq. (18) yields:

385

$$\tau = \frac{\Psi(\eta)}{\rho^{3/5}} \quad (21a)$$

387

388 where the Ψ function equals:

389

$$390 \quad \Psi(\eta) = \frac{1}{5} (\Psi_{T1} + \Psi_{T2}) \quad (21b)$$

391

392 Similarly to the R case, for T weirs, the Ψ function also depends on an arctangent and a logarithmic
 393 function according to Ψ_{T1} and Ψ_{T2} , defined as:

394

$$395 \quad \Psi_{T1}(\eta) = 5^{1/4} \left[\frac{\pi}{\sqrt{5(\sqrt{5}+2)}} - a_1 \arctan\left(\frac{\sqrt{5}-(1+4\eta)}{5^{1/4}a_2}\right) \right. \\ \left. - a_2 \arctan\left(\frac{\sqrt{5}+(1+4\eta)}{5^{1/4}a_1}\right) \right] \quad (21c)$$

396 with $a_1 = \sqrt{2(\sqrt{5}-1)}$ and $a_2 = \sqrt{2(\sqrt{5}+1)}$ and

$$397 \quad \Psi_{T2}(\eta) = \log \frac{(2+\eta(1+2\eta-\sqrt{5}))^{(\sqrt{5}+1)/2}}{2(\eta-1)^2(2+\eta(1+2\eta+\sqrt{5}))^{(\sqrt{5}-1)/2}} \quad (21d)$$

398

399 Inspection of the logarithmic function, for R weirs (Eq. 20b) and for T weirs (Eq. 21d),
 400 highlights that both functions attain an asymptotic value for $\eta \sim 1$. Indeed, for $\rho < 1$, the
 401 aforementioned η variable attains the unity when the h^* limiting condition, $h^*_{\infty} = \rho^{1/x}$, is achieved,
 402 and can be expressed for both R and T weirs as a function of x :

403

$$404 \quad \eta = \sqrt{\frac{h_*}{h_{*\infty}}} = \sqrt{\frac{h_*}{\rho^{1/x}}} \quad (22)$$

405

406 Interestingly, Eq. (22) also matches the limiting occurrence for circular crested orifices (Eq.
 407 14) already studied, that is when $x = 1/2$, where it was shown that $h^*_{\infty} = \rho^2$ (Baiamonte, 2020b).

408 Importantly, η variable is in the range]0-1], which provides relevant implications in deriving
409 unsteady rating curves, as will be shown later.

410 For C, R and T, **Table 2** reports the asymptotic h^* relationship, h^*_{∞} (column 4), and the
411 corresponding relationships of the η variable (column 8), thus according to the different x .

412 In order to make comparable unsteady rating curves derived for R and T with the one already
413 known for C, Eq. (14) can be similarly rewritten by assuming $\tau_0 = h^*_{\infty} = 0$ as initial conditions, and
414 by denoting the corresponding Ψ function:

415

416
$$\tau = \rho \Psi(\eta) \quad (23a)$$

417

418 where for the C case the Ψ function equals (see Eq. 14):

419

420
$$\Psi(\eta) = 2 \left(\log \left| \frac{1}{1-\eta} \right| - \eta \right) \quad (23b)$$

421

422 and η equals (see **Table 2**):

423

424
$$\eta = \sqrt{\frac{h^*}{\rho^{1/x}}} = \frac{\sqrt{h^*}}{\rho} \quad (23c)$$

425

426 Following this finding, in conclusion for C, R and T, the rising limb of the dimensionless
427 unsteady-state discharge relationship, expressed by Eq. (23a), Eq. (20a) and Eq. (21a), respectively,
428 can be jointly written by a general equation, which is valid for any of the considered x values:

429

430
$$\tau = \frac{\Psi(\eta)}{\rho^{1-\frac{1}{x}}} \quad (24)$$

431

432 where the Ψ function needs to be expressed by Eq. (23b), Eq. (20b) and Eq. (21b) for the C, R and
433 T case, respectively, with η expressed by Eq. (22).

434 For C, R and T, **Fig. 2** plots the Ψ function obtained by using Eq. (23b), Eq. (20b), Eq. (21b),
435 respectively, as a function of η , showing that the Ψ function, which only depends on η , is able to
436 consolidate the unsteady rating curve behavior of both the rectangular and triangular crested weirs,
437 and that of the sharp crested orifice. Of course, the Ψ function monotonically increases at increasing
438 η , showing that for any ρ value, at increasing of the normalized water level, larger and larger times
439 are required.

440

441 **Time to equilibrium**

442

443 For the case $\rho < 1$, the normalized time to equilibrium, τ_{eq} , cannot be attained rigorously in a
444 finite time. As example for C, Eq. (14) shows that for a fixed $\rho < 1$, at increasing h^* , when $h^* \sim \rho^2$
445 (less than one), the logarithm denominator attains zero. As a consequence, the logarithm diverges and
446 so the time to equilibrium. The latter occurrence can also be observed by inspection of the logarithm
447 function appearing in the Ψ functions (Eqs. 23b, 20b and 21d) when $\eta = 1$. However, for $\rho < 1$ τ_{eq}
448 could be as well defined as the time for which the normalized water level achieves a high fraction of
449 the asymptotic value, ρ^2 . Accordingly, for the C case, τ_{eq} can be expressed as:

450

$$451 \quad \tau_{eq} = \min\{\tau: h_* \geq (\beta \rho)^2\} \quad \rho < 1 \quad (25a)$$

452

453 in which β is a value close to the unity and could be chosen as equal to 0.95. For C, this occurrence
454 was already highlighted (Baiamonte, 2020b), observing that Eq. (14) for $\rho = \sqrt{h^*}$ shows a limiting

condition, so that the normalized water depth asymptotically attains an equilibrium value, $h_{*\infty} = \rho^2$, and according to Eq. (25a) and for C, the normalized water level at the equilibrium, $h_{*eq} = (\beta \rho)^2$, was already defined. Also the normalized water level at equilibrium, h_{eq} , can be generalized for C, R and T, according to:

$$h_{*eq} = (\beta \rho)^{1/x} \quad (25b)$$

For C, R and T, the time to equilibrium can also be expressed in terms of η , by imposing Eq. (25b) in Eq. (23c), and denoting η_{eq} , the η variable corresponding to the equilibrium:

$$\eta_{eq} = \sqrt{\frac{h_*}{h_{*\infty}}} = \sqrt{\frac{h_*}{\rho^{1/x}}} = \frac{(\beta \rho)^{1/(2x)}}{\rho} \quad (25c)$$

For the different x values, in column 3 and 9 of **Table 2**, h_{*eq} and η_{eq} relationships are reported that once inserted into Eq. (23b), Eq. (20b), Eq. (21b), provide the corresponding Ψ_{eq} functions, which make it possible to calculate τ_{eq} by Eq. (24), for C, R and T, respectively:

$$\Psi_{eq}(\beta) = 2 \left(\log \left| \frac{1}{1-\beta} \right| - \beta \right) \quad (26a)$$

$$\Psi_{eq}(\beta) = \frac{1}{3\sqrt{3}} \left[\pi - 6 \arctan \left(\frac{2\beta^{1/3}+1}{\sqrt{3}} \right) + \sqrt{3} \log \left(\frac{\beta^{2/3}+\beta^{1/3}+1}{(\beta^{1/3}-1)^2} \right) \right] \quad (26b)$$

$$\Psi_{T1,eq}(\beta) = 5^{1/4} \left[\frac{\pi}{\sqrt{5(\sqrt{5}+2)}} - a_1 \arctan \left(\frac{\sqrt{5}-(1+4\beta^{1/5})}{5^{1/4} a_2} \right) - a_2 \arctan \left(\frac{\sqrt{5}+(1+4\beta^{1/5})}{5^{1/4} a_1} \right) \right] \quad (26c)$$

$$\Psi_{T2,eq}(\beta) = \log \frac{(2+\beta^{1/5}(1+2\beta^{1/5}-\sqrt{5}))^{(\sqrt{5}+1)/2}}{2(\beta^{1/5}-1)^2(2+\beta^{1/5}(1+2\beta^{1/5}+\sqrt{5}))^{(\sqrt{5}-1)/2}} \quad (26d)$$

477

478 The assumption made to derive the time to equilibrium when $\rho < 1$ (as example for C in Eq.
 479 25a) was considered in different contexts where temporal limiting conditions also occurred. For
 480 example, Saghafian and Julien (1995) observed that the time to equilibrium on pervious watersheds
 481 is seldom attained, because the infiltration rate could reach saturated hydraulic conductivity after a
 482 time that is much longer than the duration of the rainfall. For hillslopes of complex topography,
 483 Agnese et al. (2007) defined a time of a virtual equilibrium, for a non-linear overland flow storage
 484 model that may be used as a measure of the response time of a hillslope subject to uniform or
 485 distributed rainstorms, although it provides an approximated rather than an exact time (Baiamonte
 486 and Singh 2016).

487 For C, R and T, and for $\rho < 1$, **Fig. 3a** graphs the time to equilibrium, τ_{eq} , calculated by Eqs.
 488 (26a-b), as a function of ρ . It is interesting to observe that at increasing ρ , for R and T, τ_{eq} decreases
 489 whereas for C increases. Moreover, there are characteristics ρ values (cross symbols) for which τ_{eq}
 490 values are equal, for R and T, $\rho = 0.43$ and $\tau_{eq} = 3.286$, for C and T, $\rho = 0.635$ and $\tau_{eq} = 2.599$, and
 491 for C and R, $\rho = 0.687$ and $\tau_{eq} = 2.811$.

492 Contrarily to $\rho < 1$, when $\rho > 1$ the time to equilibrium of the water level rising limb can be
 493 derived in a finite time (Baiamonte 2020b), since at increasing h^* , the logarithm denominator does
 494 not diverge, meaning that h^* can attain the unity, and requires imposing $h^* = 1$ in the η variable (Eq.
 495 22), which yields:

496

$$\eta_{eq} = \sqrt{\frac{h^*}{\rho^{1/x}}} = \frac{1}{\rho^{1/(2x)}} \quad (27)$$

498

499 For the different x values, in column 10 of **Table 2**, the corresponding η_{eq} relationships for
500 the case $\rho > 1$ are also reported. The corresponding Ψ_{eq} function can be derived by Eq. (23b), for C,
501 by Eq. (20b), for R, and by Eq. (21c and 21d), for T, by using the η_{eq} relationships, in a simpler form
502 than the case $\rho < 1$, directly as a function of ρ , $\Psi_{eq}(\rho)$, respectively:

503

$$504 \quad \Psi_{eq}(\rho) = 2 \left(\log \left| \frac{1}{1-1/\rho} \right| - \frac{1}{\rho} \right) \quad (28a)$$

$$505 \quad \Psi_{eq}(\rho) = \frac{1}{3\sqrt{3}} \left[\pi - 6 \arctan \left(\frac{2+\rho^{1/3}}{\sqrt{3}\rho^{1/3}} \right) + \sqrt{3} \log \left(\frac{\rho^{2/3}+\rho^{1/3}+1}{(\rho^{1/3}-1)^2} \right) \right] \quad (28b)$$

$$506 \quad \Psi_{eq,T1}(\rho) = 5^{1/4} \left[\frac{\pi}{\sqrt{5}(\sqrt{5}+2)} - a_1 \arctan \left(\frac{\sqrt{5}-(1+4/\rho^{1/5})}{5^{1/4}a_2} \right) \right. \\ \left. - a_2 \arctan \left(\frac{\sqrt{5}+(1+4/\rho^{1/5})}{5^{1/4}a_1} \right) \right] \quad (28c)$$

$$507 \quad \Psi_{eq,T2}(\rho) = \log \frac{\left(2 + \frac{2}{\rho^{2/5}} + \frac{1-\sqrt{5}}{\rho^{1/5}} \right)^{(\sqrt{5}+1)/2}}{2 \left(\frac{1}{\rho^{1/5}} - 1 \right)^2 \left(2 + \frac{2}{\rho^{2/5}} + \frac{1+\sqrt{5}}{\rho^{1/5}} \right)^{(\sqrt{5}-1)/2}} \quad (28d)$$

508

509 By using again Eq. (24), Eqs. (28) makes it possible to calculate τ_{eq} also for $\rho > 1$. Contrarily
510 to $\rho < 1$ (**Fig. 3a**), for $\rho > 1$, **Fig. 3b** shows that τ_{eq} exhibits the same decreasing relationship versus
511 ρ , for C, R and T. However, for any ρ values, greater times are required for the achievement of the
512 equilibrium condition, for C with respect to R, and for R with respect to T. Both **Fig. 3a** and **Fig. 3b**
513 indicate the τ_{eq} values (black dots) corresponding to the application that will be discussed later.

514

515 **Falling limb**

516

517 This case describes the recession of the water level, when the input discharge is over, and it is
518 described by Eq. (18), by imposing $\rho = 0$:

519

$$\int_{\tau_0}^{\tau} d\tau = - \int_{h_{*0}}^{h_*} \frac{dh_*}{h_*^x} = \int_{h_*}^{h_{*0}} \frac{dh_*}{h_*^x} \quad (29)$$

521

522 For any x value, by assuming as initial conditions, $\tau_0 = \tau_{eq}$ and $h_{*0} = h_{*eq}$, integration of Eq.
 523 (29) provides:

524

$$\tau = \tau_{eq} + \frac{h_{*eq}^{1-x} - h_*^{1-x}}{1-x} \quad (30)$$

526

527 For the C case ($x = 1/2$) and for $h_* = \tau_{eq} = 0$, if using Eq. (5a) and Eq. (7), Eq. (30) matches
 528 the well-known time to emptying of a water reservoir. Eq. (30) also shows that the time that needs
 529 for the achievement of the zero flux conditions, τ_{zf} , is finite only for $x < 1$, that is for the C case,
 530 whereas for R and T cases ($x > 1$), the zero flux condition is asymptotic. Eq. (30) will be used to
 531 describe the falling limb of the water level in the following section.

532

533 **Testing the solutions of the rising limb, time to equilibrium and falling limb**

534

535 According to the suggested procedure, applications were performed with the aim to describe
 536 the filling and the emptying processes occurring in C, R and T, under a constant input discharge, but
 537 unsteady state conditions, so that Eq. (2a-c) cannot be applied provided the steady state condition is
 538 achieved.

539 Two applications were carried out, one for $\rho = 0.8 (< 1)$, for which time to equilibrium, τ_{eq} , is
 540 defined by the β fraction, assumed 0.95, and the other for $\rho = 2 (> 1)$. For C, R and T, and for both
 541 the selected ρ values, **Table 3** reports the characteristics normalized times and water levels and, for
 542 R and T, the asymptotic time to zero flux. For $\rho = 2$, **Table 3** also reports the asymptotic normalized
 543 water level, $h_{*\infty} = 4, 1.59$ and 1.32 , for C, R and T, respectively, indicating that water levels higher

544 than the maximum H should be achieved. However, this latter occurrence is meaningless by a physical
545 point of view, since for $\rho > 1$, the time equilibrium condition is defined when $h^* = 1$, by the finite
546 time τ_{eq} (column 8 of **Table 3**).

547 Once the characteristics water levels and times are known, in order to graph the rising limb of
548 the water level, by assuming for both $\rho = 0.8$ and $\rho = 2$ that the constant input discharge flows into
549 the tank until $\tau = \tau_{eq}$, by varying h^* and η (Eq. 22), the explicit in τ Eq. (24) was applied. Of course,
550 for C, R and T, the Ψ function expressed by Eq. (23b), Eq. (20b) and Eq. (21b), respectively, was
551 considered.

552 Results are illustrated in **Fig. 4** that shows the asymptotic achievement of the steady state for
553 $\rho < 1$ (**Fig. 4a**), the virtually assumed at τ_{eq} (**Table 3**), and those ones achieved for $\rho = 2$ in a finite
554 time at $h^* = 1$ (**Fig. 4b**). For C, R and T and for both ρ values, τ_{eq} is also indicated in **Fig. 3**, testing
555 the consistence of the τ_{eq} relationships described in the previous section.

556 By varying h^* , **Figs. 4a-b** also graph the falling limb of the normalized water level obtained
557 by using Eq. (30) for which it can be observed that, for C, the zero flux condition is established at τ
558 $= \tau_{eq} + \tau_{zf}$ (**Table 3**), whereas for R and T is asymptotically attained.

559 The reliability of the presented procedure could be also checked in terms of water volume
560 balance, since it could be verified that during the normalized time τ , the water volume flowing into
561 the tank equals to the amount of that released by the orifice or by weir outlet, and that stored into the
562 tank. As a simpler consequence, at the end of the falling limb (asymptotic for R and T), since the
563 normalized water level, h^* , goes back to that of the initial condition ($h^*_{i0} = 0$), one can verify that the
564 volume stored into the tank should attains to zero, and the inlet volume, V^*_{in} , should balance the outlet
565 volume, V^*_{out} .

566 Towards this aim, the cumulated water volume flowing into the tank, V^*_{in} , linearly increases
567 during the time and, in dimensionless terms, equals $V^*_{in} = \rho \tau$, where the slope ρ of the straight line
568 of course differs for the selected two cases (**Fig. 5a** and **Fig.5b**). On purpose, **Fig. 5ab** were equally

569 scaled in order to succinctly observe the differences determined by the two ρ values, and by the
 570 cumulated outlet volume, as it will be discussed. **Fig. 5ab** also reports the pairs $(\tau_{eq}, V_{*,in})$
 571 corresponding to the input volume at τ_{eq} (black dots).

572 Similarly to $V_{*,in}$, the dimensionless outlet volume, $V_{*,out}$, can be numerically calculated by the
 573 rising and the falling limbs illustrated in **Fig. 4ab**:

574

$$575 \quad V_{*,out} = 0.5 \sum_{i=1}^n (Q_{*0,i} + Q_{*0,i-1}) (\tau_i - \tau_{i-1}) \quad (31)$$

576

577 where n is the number of intervals in which the water level variation, h_{*eq} , for $\rho = 0.8$ (Eq. 25b) and
 578 for $\rho = 2$ ($h_{*eq} = 1$), was subdivided, and $Q_{*0,i}$ and $Q_{*0,i-1}$ are the outflow discharges, calculated for the
 579 different x by Eq. (4) at the time τ_i and τ_{i-1} , respectively.

580 For C, R and T and for both ρ values, **Fig. 5ab** show the cumulated outflow volume flowing out
 581 from the tank, $V_{*,out}$, calculated by Eq. (31). As expected, it can be observed that for C, the zero flux
 582 condition (cross symbols) is achieved in a finite time, $\tau_{eq} + \tau_{zf}$ (**Table 3**), when $V_{*,out} = V_{*,in}$.

583 Contrarily, for R and T, $V_{*,out}$ asymptotically attains to $V_{*\infty}$, which is represented by the dashed
 584 lines corresponding, as for C, to $V_{*,in}$ values (black dots) reported in the input volume straight lines,
 585 validating the mass balance. White dots reported for C, R and T, in the V_{*} relationships refer to the
 586 volume drained at τ_{eq} from the outlet and indicate the inflection points of those curves, when the
 587 falling limb starts.

588 A comparison between the overall results obtained for $\rho = 0.8$ (**Fig. 5a**) and for $\rho = 2$ (**Fig. 5b**)
 589 highlights that for $\rho = 0.8$, the equilibrium condition requires larger times than $\rho = 2$. As a
 590 consequence, for $\rho = 0.8$, much higher volumes than for $\rho = 2$ drain from the outlet.

591 Discrepancies between V_{*out} and V_{*in} reflect the errors in using Eqs. (4), by assuming the flow
592 conditions steady state, under the actual unsteady conditions, which appear much higher for triangular
593 and rectangular weirs than for the sharp crested orifice.

594

595 **Determining the input discharge under unsteady state**

596

597 As described in Baiamonte (2020b), for a circular crested orifice (C), made at the bottom or
598 at the side of a water tank, the input discharge can be determined by the water level variation and by
599 the time that is spent for it. To this aim, in light of the Ψ function described in **Fig. 2**, a different
600 formulation is presented here, providing simpler input discharge calculus than in Baiamonte (2020b),
601 and a more powerful graphical illustration, which can also be applied for R and T.

602 With reference to the normalized discharge, ρ , to be determined by discrete times and water
603 levels measurements, knowledge of the Ψ function makes it possible to express the normalized input
604 discharge, ρ , by the following implicit equation derived by Eq. (24):

605

$$606 \quad \rho = \left(\frac{\Psi(\eta) - \Psi(\eta_0)}{\tau - \tau_0} \right)^{x/(x-1)} \quad (32)$$

607

608 where the initial conditions of the corresponding variables h^*_0 , h_0 , η_0 , τ_0 , Ψ_0 , were introduced. Eq.
609 (32), which is implicit in ρ and depends on depths and times, can be used to derive the input discharge
610 ρ , for any x .

611 In order to describe the application of Eq. (32), for an already investigated small tank (27.5
612 cm x 37.8 cm x 19.3 cm) where a circular crested orifice ($x = 1/2$, $C_d = 0.671$, $d = 11.1$ mm) was made
613 at its bottom, for an unsteady outflow process, a set of ten runs was considered (Baiamonte 2020b).

614 A vertical baffle, separating the tank fraction where the discharge inflows, from that where
615 the water levels are measured, was inserted into the tank to limit the disturbance effect next to the
616 water level measurements (cf. Fig. 7 in Baiamonte 2020b), thus mitigating the water surface
617 fluctuations and the noise generated by the inflow discharge.

618 Similarly, the device could work in the field, where the discharge by a plot or by a small basin
619 needs to be measured. Runoff could be intercepted by a gutter placed along the lower end of the
620 plot/small basin, and collected into the storage tank system. Two examples of common tanks, with
621 an arrangement similar to that above described for field's applications are reported in Fig. 2 of
622 Baiamonte (2020b).

623 **Table 4** reports the data corresponding to the ten runs, t_0 , t , h_0 , h , times and water level
624 variations, together with the dimensionless parameters involved in the procedure. A constant input
625 discharge $Q_{in,m} = 0.119$ l/s was applied to the tank and the water level gently raised, starting from
626 different initial conditions and, for a step $h - h_0 = 0.5$ cm, spent different time intervals (**Table 4**).

627 First, the temporal scale of the unsteady outflow process, t_c , has to be established by Eq. (7)
628 and Eq. (5b) that gave, for $v_{max}^C = 1.21$ cm/s (Eq. 5a), $t_c = 158.8$ s. Knowledge of t_c made it possible
629 to calculate τ and τ_0 (**Table 4**).

630 Because of Eq. (32) is implicit in ρ , a starting ρ point must be provided, which allows
631 determining η and η_0 (Eq. 22) and the corresponding first attempt values of the Ψ function (Eq. 23b).
632 The actual ρ values have to be obtained by iterations until they match those calculated by Eq. (23b).

633 **Table 4** reports the actual ρ values and the corresponding calculated discharges, $Q_{in} = \rho A$
634 v_{max}^C , which are very close to the constant input discharge, $Q_{in,m}$, validating the suggested procedure.
635 This can also be checked by the corresponding relative errors, RE ($|RE|$ in the range 1-5%, **Table 4**),
636 calculated with respect to the input discharge value ($Q_{in,m} = 0.119$ l/s).

637 A slight dependence of the discharge coefficient, C_d , by the water level, which was assumed
638 as a constant (0.671), was detected. However, as it can be observed by **Table 4**, RE a bit increases at

639 increasing the water level into the tank, indicating that accounting of this well-known C_d dependence
640 would further improve the discharge estimate, in truth already satisfactory. The latter issue could be
641 addressed by considering the formulations already available in the literature (Bautista-Capetillo et al.
642 2014; Vatankhah and Khamisabadi 2019; Swamee and Swamee 2010) or, more properly, by
643 calibrating an *ad hoc* formulation under steady-state condition, which account for the hydraulic head
644 and for orifice/weir geometry, to be considered in the discharge estimate, as well in t_c (for $h = H$).

645 As it will be described in the next section, experimental calibration is preferred because of the
646 various factors that may affect the discharge coefficient, mainly the hydraulic head and the shape of
647 the crested orifices and weirs.

648 **Table 4** also reports the calculated input discharge under the assumption of a steady state
649 condition, $Q_{in,ss}$ (Eq. 2a), which one would have calculated if ignoring the rising water level ($h_0 = h$),
650 and the corresponding very high relative errors, RE_{ss} ($|RE_{ss}|$ in the range 26-49%).

651 Clearly, a similar procedure could be followed for R and T, but accurately selecting the
652 corresponding x value and the Ψ function. It should be noted that, for C, in the case that the hole is
653 made at the side of the water tank, rather than at the bottom, as considered above, discharge estimate
654 could start only when the hole is fully submerged, so that a water level high enough is established
655 above the hole. The latter option is easier to apply because it limits the drop in elevation, as for the
656 hole made at the bottom of the tank, and also allows designating a dead volume at the tank bottom,
657 where solid volume, if present, could take place.

658 The simplicity of Eq. (32), which is easier to apply than the only valid for C solution reported
659 in Baiamonte (2020b), also lays in the shifted limiting condition of the h^* variable, $h^*_{\infty} = \rho^{1/x}$, to that
660 of the introduced η variable. Indeed, the η variable of the Ψ function has to be searched in its domain
661 $]0,1]$, thus relaxing the h^*_{∞} limiting condition that, for the C case, was described by a yellow 3D
662 vertical surface in Baiamonte (2020b, cf. Fig. 10).

663 In practice, for calculating the input discharge, if continuous h^* versus τ function would be
664 known, the input discharge for any x would be possible to be determined by its derivative, according
665 to the continuity equation (Eq. 13). Since this is not the case, because of the discrete nature of water
666 level data, Eq. (32) needs to be applied. However, with reference to a graphical illustration of the
667 input discharge, the continuity equation (Eq. 13) can be written by using again the Ψ function (Eq.
668 24), displayed in **Fig. 2**, and by the η variable, if we express h^* by Eq. (22):

$$670 \quad h_* = \rho^{1/x} \eta^2 \quad (33)$$

672 Substituting Eq. (33) and Eq. (24) into the continuity equation (Eq. 13), provides, respectively:

$$674 \quad \rho^{1/x} \frac{d\eta^2}{d\tau} = \rho - h_*^x \quad (34)$$

$$675 \quad \rho \frac{d\eta^2}{d\Psi} = \rho - h_*^x \quad (35)$$

677 Comparison between Eq. (35) and Eq. (13) indicates that the normalized water level rate can
678 be expressed as the reciprocal of the Ψ function gradient multiplied by ρ :

$$680 \quad \frac{dh_*}{d\tau} = \rho \frac{d\eta^2}{d\Psi} \quad (36)$$

682 Now, substituting again h^* (Eq. 33) into the right side of Eq. (35) provides the Ψ function
683 gradient with respect to η^2 as a function of η :

$$685 \quad \frac{d\Psi}{d\eta^2} = \frac{1}{1-\eta^{2x}} \quad (37)$$

687 For the three considered x values, it was verified that Eq. (37) matches the derivative of the Ψ
688 functions obtained for $x = 1/2$ (Eq. 23b), $x = 3/2$ (Eq. 20b) and $x = 5/2$ (Eq. 21b). As an example, for
689 the simplest Ψ function corresponding to the C case (Eq. 23b), the reliability of Eq. (37) is shown
690 here. Indeed, the derivative of Eq. (23b) with respect to η^2 can be written:

$$692 \quad \frac{\Psi(\eta)}{d\eta^2} = \frac{d}{d\eta^2} \left(2 \log \left| \frac{1}{1-\sqrt{\eta^2}} \right| - 2\sqrt{\eta^2} \right) \quad (38)$$

693
694 that can be easily expressed as:

$$695 \quad \frac{\Psi(\eta)}{d\eta^2} = \frac{2}{2(1-\sqrt{\eta^2})\sqrt{\eta^2}} - \frac{1}{\sqrt{\eta^2}} = \frac{1}{1-\sqrt{\eta^2}} \quad (39)$$

696
697
698 Eq. (39), once simplified, verifies Eq. (37) if, for this C case, $x = 1/2$ is imposed into it. For different
699 x , the Ψ function derivatives are reported in **Table 2** (last column). Therefore, the input discharge can
700 also be derived by substituting Eq. (39) into the continuity equation expressed by Eq. (35):

$$702 \quad \rho(1 - \eta^{2x}) = \rho - h_*^x \quad (40)$$

703
704 In conclusion, both Eq. (32) and Eq. (40) can be used to determine the input discharge under
705 transient conditions for C, for R as well for T. However, Eq. (32) has to be used when discrete water
706 level and times measurements are available, whereas for a graphical illustration, Eq. (40) is more
707 suitable than Eq. (32), since the water level rate is incorporated into it by the Ψ function derivative,
708 expressed by Eq. (39). It is noteworthy that Eq. (40) also matches Eq. (22), indicating that also the
709 introduced η variable, although not explicitly depending on the time, intrinsically accounts for the
710 water level rate and thus by the time.

711 In order to show a graphical illustration accounting for unsteadiness, it needs to consider that
 712 the sign of the Ψ function derivative could be positive or negative, depending if the constant input
 713 discharge has been measured during a filling or an emptying process of the water tank. Thus, for the
 714 filling process, assuming in Eq. (40) the positive sign, provides:

715

$$716 \quad \rho = \frac{h_*^x}{(1-(1-\eta^{2x}))} = \frac{h_*^x}{\eta^{2x}} \quad (41a)$$

717

718 Whereas, under an emptying process, assuming the negative sign of the Ψ function derivative
 719 (Eq. 37) into Eq. (40), yields:

720

$$721 \quad \rho = \frac{h_*^x}{(1+(1-\eta^{2x}))} = \frac{h_*^x}{2-\eta^{2x}} \quad (41b)$$

722

723 In the following, Eq. (41a) and Eq. (41b) will be usefully applied to show the influence of
 724 assuming steady state an actually unsteady outflow process, when measuring a constant input
 725 discharge ρ .

726 In **Fig. 6ac**, for the three considered cases (C, R and T), by varying η (0-1), Eq. (41a) for the
 727 filling process, and Eq. (41b) for the emptying process, are plotted. Interestingly, it can be observed
 728 that for the C case, the empirical data carried out by Baiamonte (2020b) during a filling process fits
 729 the theoretical surface. Moreover, **Fig. 6ac** illustrates the steady state condition ($\rho = h_*^x$), indicating
 730 that for the filling process, the transposition of the rising and of the falling limbs occurred with higher
 731 input discharges (i.e. lower output discharges) in the filling process, and with lower input discharges
 732 (i.e. higher output discharges) in the emptying process, which effect increases more and more at
 733 decreasing η . Comparison between C, R and T indicates that, with respect to steady state conditions,

734 the underestimate and overestimate of the unsteady discharge increases by moving from C to R and
735 from R to T.

736 For the filling process, this latter occurrence can be better observed in log scales in **Fig. 7a**,
737 where the corresponding surfaces were grouped for C, R and T, providing regular planes that can be
738 clearly distinguished. Whereas, for the emptying process (**Fig. 7b**), by using normal axes, it can be
739 observed that the C, R and T surfaces intersect each other. Intersections likely derive by the falling
740 limb behavior of the normalized water level, already discussed for **Fig. 4a** and **Fig. 4b**, although in
741 that case, $\rho = 0$ was assumed. Intersections of the surfaces can be easily recognized by considering
742 the different overlapping colors that can be distinguish in **Fig. 7a**.

743 For the filling process, **Fig. 7a** also plots the horizontal lines for $\rho = 0.8$ (dashed lines) and for
744 $\rho = 2$ (solid lines), considered in the application of **Fig. 4** and **Fig. 5**, which of course lay in the
745 corresponding planes.

746 An investigation on the transposition of the rating curves due to the unsteadiness, as already
747 observed in similar analysis, although carried out by numerical or empirical point of view, was also
748 performed. For C, R and T, and for three different η values ($\eta = 0.98, 0.95$ and 0.9), **Fig. 8** plots the
749 relationships between the unsteady state discharge during a filling and an emptying process, ρ_{rise} and
750 ρ_{fall} , and that of the steady state discharge, Q^*_{θ} , versus the normalized water level, h^* .

751 As it was already recognized in the past, but also recently (Henderson 1966; Neely and
752 Bingham 1986; Ghasemzadeh et al 2020), and according to **Fig. 6** and **Fig. 7**, the transposition of the
753 rising and of the falling limbs occurred for the filling process as well for the emptying process, which
754 effect increases more and more at decreasing η . These results are a logical consequence of the
755 imposed continuity equation (Eq. 13). For example, it is obvious that when water level gradients are
756 positive, the input discharge, ρ_{rise} , has to exceed the output discharge.

757

758 **Discharge magnitude, tank height and error analysis**

759

760 Another important aspect is the tank height required by this device, which is linked to the
761 excavation costs that are needed for the drop at the outlet of an experimental plot or of a small basin.
762 Of course, costs are determined by the discharge magnitude, and this is why this device is better aimed
763 to small/moderate discharges by small plots or basins, than to high discharges. However, it should be
764 noted that the tank height strictly necessary for the discharge estimate is H (**Fig. 1**) and does not
765 include the quantity p displayed in **Fig. 1**. The latter p is required to only assure the negligibility of
766 the inflow kinetic head, which was ignored in the present procedure.

767 In light of this consideration, an analysis of the discharge magnitude associated with tank
768 height was performed for two reasonable H values ($H = 0.5$ and $H = 1$ m), by varying the parameters
769 characterizing the geometry of the circular crested orifice (d), and the rectangular (L) and the
770 triangular (θ) weirs. For this purpose, Eq. (41a) and Eq. (41b) make also possible to detect the
771 discharge magnitude associated with tank height that this device requires.

772 According to different orifice/weirs geometries, **Fig. 9a-c**, for $H = 0.5$ m, and **Fig. 9d-f**, for H
773 $= 1$ m, show the input discharge versus the dimensionless parameters, η , derived by Eq. (41a), for a
774 rising water level, and by Eq. (41b), for a falling water level. Of course, for $\eta = 1$ (steady state), both
775 formulations provide the same input discharge that could be also derived by Eqs. 2 (**Fig. 9**). The
776 geometry of the crested orifice and that of the weirs highly influence the discharge magnitude that
777 increases at increasing d , L and θ , and also H , as expected. Moreover, **Fig. 9** shows that higher
778 discharges could be measured during the rising process than during the falling process and that
779 discharges higher than those displayed in **Fig. 9** should require higher d , L and θ , but also higher H
780 values, probably increasing too much excavation costs. Therefore, a cost-benefit analysis should be
781 performed in light of the aforementioned considerations, in order to select the most convenient choice.

782 It should be noted that the discharge coefficient, C_d , which in the application described in **Fig.**
783 **9** was assumed independent of the water level and equal to 0.6, is a crucial point of the suggested

784 procedure. In fact, although the presented dimensionless solutions are not affected by C_d , C_d needs to
785 be known for discharge estimate in dimensional terms, as in **Fig. 9**. However, with respect to other
786 methodologies, only one physical meaningful parameters needs to be calibrated.

787 The discharge coefficient depends by the crested orifice and weirs geometry, thus it may vary
788 from case to case, and it needs determining by a separate procedure. From a practical point of view,
789 C_d could be determined by calibration, minimizing the errors between measured and calculated
790 discharges, i) according to the suggested procedure, i.e. under unsteady state, or ii) under steady state
791 by taking into account its dependency from the water level, if it exists. For example for C, the first
792 option was considered in Baiamonte (2020b), and results showed that C_d was slightly influenced by
793 the water level values.

794 Finally, an error analysis was carried out, by considering that the maximum relative errors for
795 the rising and the falling conditions can be calculated as displayed in **Fig. 8i**, and considering that
796 $Q_{*0,max}$ and ρ_{fall} only depend on η , as illustrated in **Fig. 10a** and **Fig. 10b**, respectively. Therefore,
797 relative errors were differently expressed for the rising and falling conditions, RE_{rise} and RE_{fall} ,
798 according to:

799

$$800 \quad RE_{rise} = \frac{Q_{*0}-1}{1} \quad (42a)$$

$$801 \quad RE_{fall} = \frac{1-\rho_{fall}}{\rho_{fall}} \quad (42b)$$

802

803 where in Eq. (42a), the unity is the maximum normalized discharge for the unsteady state rising limb,
804 ρ_{rise} (**Fig. 8i**), whereas, in Eq. (42b), the unity is the maximum normalized discharge under steady
805 state, Q_{*0} .

806 For C, R and T, **Fig. 10c** graphs the relative errors given by Eq. (42a) and Eq. (42b) that are
807 made when computing the input discharge, by assuming a steady state flow condition that is unsteady.
808 Of course, no errors occur for $\eta = 1$, and it seems that T is much more sensitive to the unsteadiness

809 than R, which in turn is more sensitive than C. As expected, following **Fig. 6**, **Fig. 7** and **Fig. 8**, RE s
810 are negative for the rising condition of the normalized water level, whereas they are positive for the
811 falling condition.

812 Interestingly, **Fig. 10c** also plots RE calculated by Eq. (42a) for a number of h^* values
813 corresponding to the rising condition considered in the two applications made in **Fig. 4a** (for $\rho = 0.8$)
814 and in **Fig. 4b** (for $\rho = 2$), which perfectly match the error analysis performed. Of course, no errors
815 comparison between the falling limb of **Fig. 4a** and **Fig. 4b** was made, since they refer to the case ρ
816 $= 0$, and thus no errors occurred.

817 In **Fig. 10c**, it is also interesting to observe that, for C, RE magnitude is similar to that (RE_{ss})
818 referred to the experimental runs ($\langle \rho \rangle = 0.92$), discussed above (**Table 4**). These results indicate that
819 **Fig. 10c** is of general validity and can be used to correct the steady state discharge rating curves,
820 when the steadiness is not achieved, for C, R and as well for T.

821 Due to the high errors that could be made (**Fig. 10c**), the approach presented in this work
822 suggests that if the simple steady-state rating curves one wants to use, checking of the steady state
823 condition is highly recommended. Moreover, calculating input discharges under unsteadiness could
824 be performed with less uncertainties if the suggested procedure is applied.

825

826 **Conclusions**

827

828 In this work, complex rating curves of sharp crested orifices, of rectangular and triangular
829 weirs for unsteady conditions are presented. According to previous experimental investigations, a
830 transposition of the simple rating curve occurs, with higher input discharges than those by simple
831 rating curve, when the water level rises, and with lower input discharges, when the water level falls.
832 The corresponding time to equilibrium was also studied and two very different applications were
833 performed.

834 An error analysis quantitatively showed that using simple rating curves when the outflow
835 process is unsteady, may determine relevant errors in discharge measurements, which decrease more
836 and more at disappearing of the loop of the rating curve, because of the attaining of the steady state.
837 Triangular weirs result more sensitive to the unsteadiness than rectangular weirs, which in turn result
838 more sensitive than the sharp crested orifices.

839 The suggested procedure could help researchers improve the accuracy in discharge
840 measurements, with expected improvements that should arise by the comparison between discharges
841 obtained by the suggested procedure with those derived by the application of hydrodynamic and
842 rainfall-runoff models. Moreover, it is expected that this procedure, which was already tested for
843 sharp crested orifice, could work well also for rectangular and triangular weirs and, if implemented
844 in a data-logger software application, could provide directly discharges data by water levels measured
845 by probes.

846 As a further remark, similar considerations could be made about the uncertainties of simple
847 rating curves used under unsteadiness, for the large amount of steady state discharge relationships
848 derived according to the Buckingham theorem and dimensional analysis, and for a multiplicity of
849 gauge geometries and shapes introduced in the scientific literature.

850

851 **Data Availability Section**

852

853 All data, models, and code generated or used during the study appear in the submitted article.

854

855 **List of symbols**

856

857 a = cross-sectional area of the orifice/weir $[L^2]$.

858 A = base area of the tank $[L^2]$.

859	B	= base side of the tank	$[L]$.
860	C	= circular crested orifice.	
861	C_d	= discharge coefficient.	
862	d	= diameter of the circular crested orifice	$[L]$.
863	g	= acceleration due to gravity	$[L\ T^{-2}]$.
864	h	= water level above the crested orifice/weir	$[L]$.
865	h_0	= tank water level at the antecedent condition ($t = t_0$)	$[L]$.
866	h^*	= tank water level normalized with respect to H .	
867	h^*_{eq}	= normalized water level corresponding to the time to equilibrium, τ_{eq} .	
868	$h^*_{t_0}$	= tank water level at the antecedent condition normalized with respect to H .	
869	h^*_{∞}	= asymptotic value of the normalized water level.	
870	H	= maximum depth of the water level	$[L]$.
871	L	= base side of the tank	$[L]$.
872	n	= number of intervals in which the water level variation, $h^* - h^*_{t_0}$, was subdivided.	
873	p	= crest elevation	$[L]$.
874	p^*	= normalized crest elevation.	
875	Q_{in}	= runoff input discharge	$[L^3\ T^{-1}]$.
876	$Q_{in,m}$	= measured input discharge	$[L^3\ T^{-1}]$.
877	$Q_{in,ss}$	= input discharge calculated by assuming a steady-state condition	$[L^3\ T^{-1}]$.
878	Q_0	= outflow discharge from the tank	$[L^3\ T^{-1}]$.
879	Q_0^C	= outflow discharge from the circular crested orifice	$[L^3\ T^{-1}]$.
880	Q_0^R	= outflow discharge from the rectangular weir	$[L^3\ T^{-1}]$.
881	Q_0^T	= outflow discharge from the triangular weir	$[L^3\ T^{-1}]$.
882	$Q_{0,max}$	= maximum value of Q_0	$[L^3\ T^{-1}]$.
883	$Q^*_{t_0}$	= outflow discharge from the tank normalized with respect $Q_{0,max}$.	

884	r	= runoff intensity	$[L\ T^{-1}]$.
885	R	= rectangular weir.	
886	RE_{fall}	= input discharge relative error for the falling condition.	
887	RE_{rise}	= input discharge relative error for the rising condition.	
888	t	= time	$[T]$.
889	t_c	= characteristic time of the emptying tank process	$[T]$.
890	T	= triangular weir.	
891	t_0	= time at $h = h_0$	$[T]$.
892	v	= outlet discharge per unit surface tank base	$[L\ T^{-1}]$.
893	v_{max}	= maximum value of v	$[L\ T^{-1}]$.
894	v^C	= outlet discharge per unit surface tank base from the crested orifice	$[L\ T^{-1}]$.
895	v^R	= outlet discharge per unit surface tank base from the rectangular weir	$[L\ T^{-1}]$.
896	v^T	= outlet discharge per unit surface tank base from the triangular weir	$[L\ T^{-1}]$.
897	V_{*in}	= normalized inlet water volume ($V_{*in} = \rho\ \tau$).	
898	V_{*out}	= normalized outlet water volume (Eq. 31).	
899	$V_{*\infty}$	= asymptotic value of V_{*out} (for R and T weirs).	
900	x	= water level exponent of the rating curve.	
901	y	= water level measured with respect to the base of the tank	$[L]$.
902	y^*	= normalized water level measured with respect to the base of the tank.	
903	Z	= height of the tank	$[L]$.
904	β	= correction factor of asymptotic time of equilibrium (close to one)	
905	η	= square root of the ratio $h^*/h_{*\infty}$.	
906	η_{eq}	= value of η corresponding to the equilibrium condition.	
907	θ	= side angle of the triangular weir	
908	ρ	= runoff intensity normalized with respect to v_{max} .	

909 ρ_{fall} = runoff intensity normalized with respect to v_{max} . for the falling condition
 910 ρ_{rise} = runoff intensity normalized with respect to v_{max} . for the rising condition
 911 ρ = runoff intensity normalized with respect to v_{max} .
 912 τ = time normalized with respect to t_c .
 913 τ_{eq} = time to equilibrium normalized with respect to t_c .
 914 τ_{zf} = time normalized with respect to t_c corresponding to the zero flux condition
 915 τ_0 = time normalized with respect to t_c corresponding to the initial water level
 916 Ψ = Ψ function that makes it possible to compact τ relationships. It only depends on η .
 917 Ψ_{eq} = value of the Ψ function corresponding to the equilibrium condition.

918
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Table 1 – Geometric characteristics of C, R and T, together with the operating range of some parameters.

Crested orifice or weirs	Symbol	Geometric parameters	Operating range
Circular	C	Hole diameter, d	$h > \alpha p, \alpha = 1.2-1.3, p \geq 0$
Rectangular	R	Width, L	$p > \alpha h, \alpha = 1.2-1.3$
triangular	T	Vertical angle, θ	$p > \alpha h, \alpha = 1.2-1.3$

Table 2 – For C, R and T, characteristic relationships of the main variables.

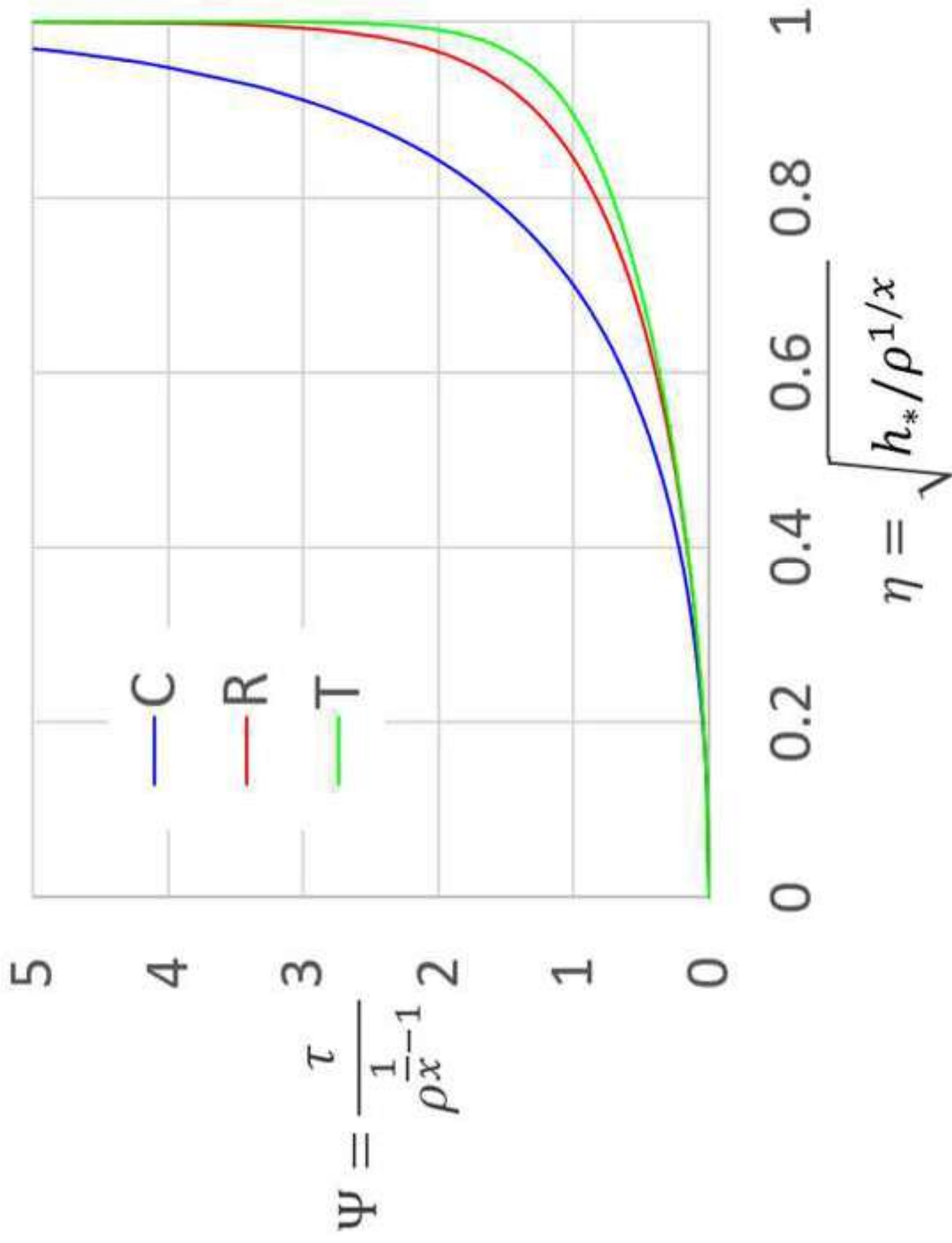
Orifice or weirs	x	$h_{eq} = (\beta \rho)^{1/x}$	$h^*_{\infty} = \rho^{1/x}$	h^*_{∞} / ρ	$\sqrt{h^*_{\infty}} = \rho^{1/(2x)}$	h^* / h^*_{∞}	$\eta = \sqrt{h^* / \rho^{1/(2x)}} = \sqrt{h^* / h^*_{\infty}}$	$\eta_{eq} (\rho < 1)$	$\eta_{eq} (\rho > 1)$	$d\Psi/d\eta^2$
C	1/2	$(0.95 \rho)^2$	ρ^2	ρ	ρ	h^* / ρ^2	$\sqrt{h^* / \rho}$	α	$1/\rho$	$(1-\eta)^{-1/2}$
R	3/2	$(0.95 \rho)^{2/3}$	$\rho^{2/3}$	$\rho^{1/3}$	$\rho^{1/3}$	$h^* / \rho^{2/3}$	$\sqrt{h^* / \rho^{1/3}}$	$\alpha^{1/3}$	$1/\rho^{1/3}$	$(1-\eta)^{-3/2}$
T	5/2	$(0.95 \rho)^{2/5}$	$\rho^{2/5}$	$\rho^{3/5}$	$\rho^{1/5}$	$h^* / \rho^{2/5}$	$\sqrt{h^* / \rho^{1/5}}$	$\alpha^{1/5}$	$1/\rho^{1/5}$	$(1-\eta)^{-5/2}$

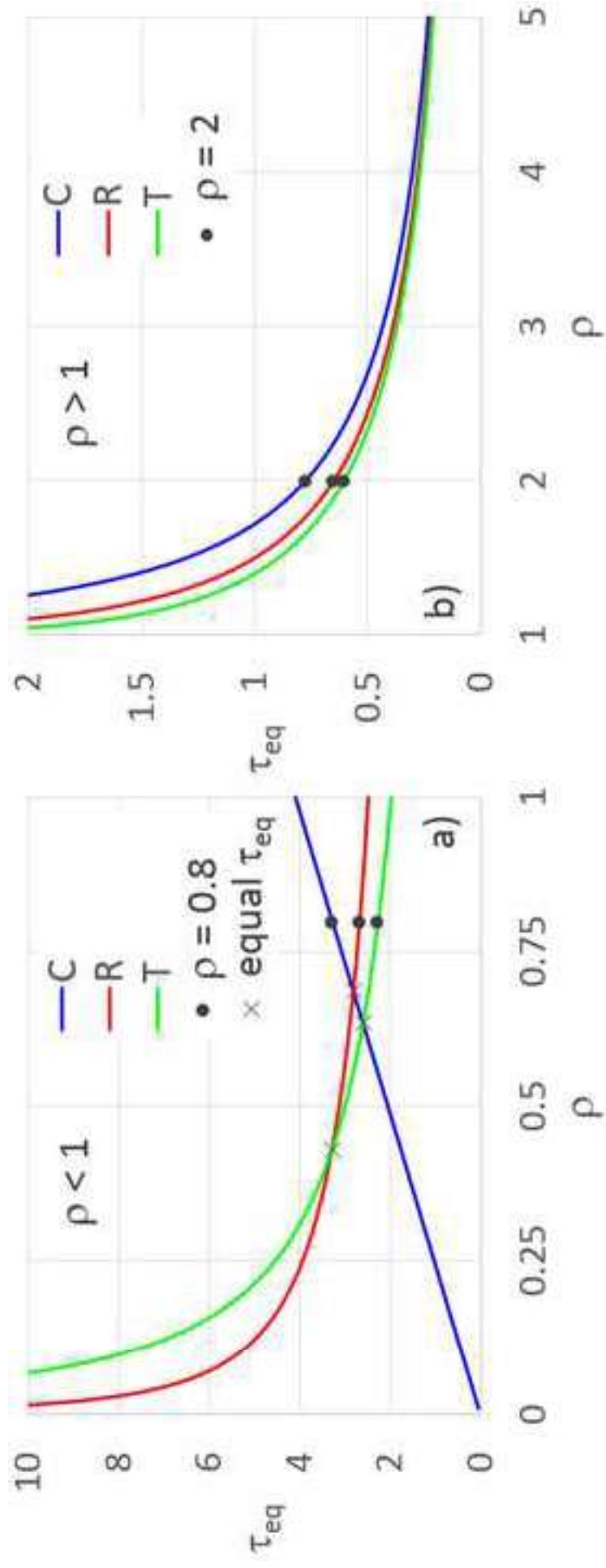
Table 3 – For C, R and T, characteristic values of the main variables, for the applications performed for $\rho < 1$ ($\rho = 0.8$) and for $\rho > 1$ ($\rho = 2$).

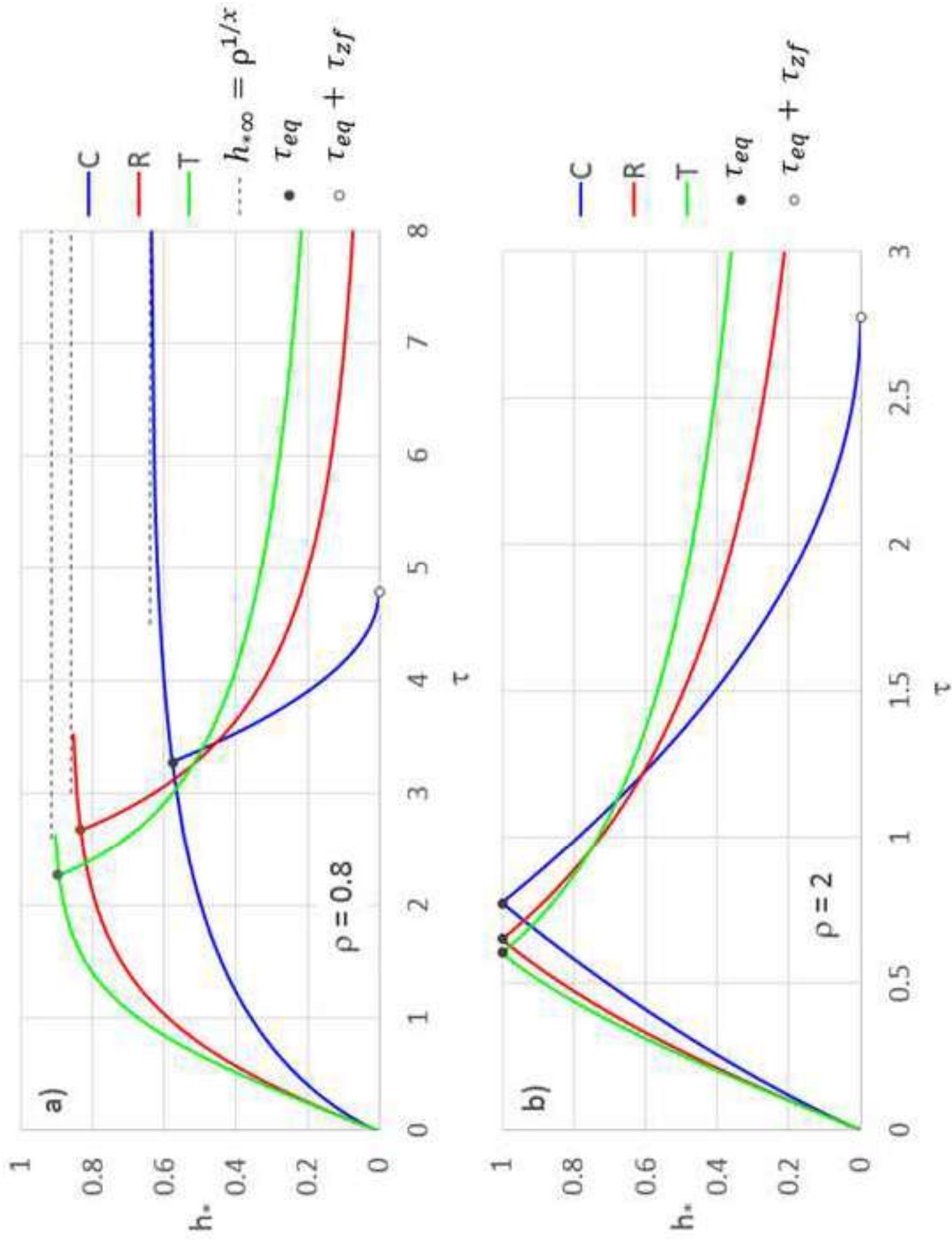
Orifice or weirs	x	$\rho = 0.8$					$\rho = 2$				
		$\tau_r = \tau_{eq}$	h^*_{eq}	τ_{zf}	$\tau_{eq} + \tau_{zf}$	h^*_{∞}	$\tau_r = \tau_{eq}$	h^*_{eq}	τ_{zf}	$\tau_{eq} + \tau_{zf}$	h^*_{∞}
C	1/2	3.2732	0.5776	1.520	4.7932	0.6400	0.7726	1	2.000	2.7754	4.000
R	3/2	2.6713	0.8328	∞	∞	0.8618	0.6503	1	∞	∞	1.587
T	5/2	2.2633	0.8960	∞	∞	0.9146	0.6045	1	∞	∞	1.320

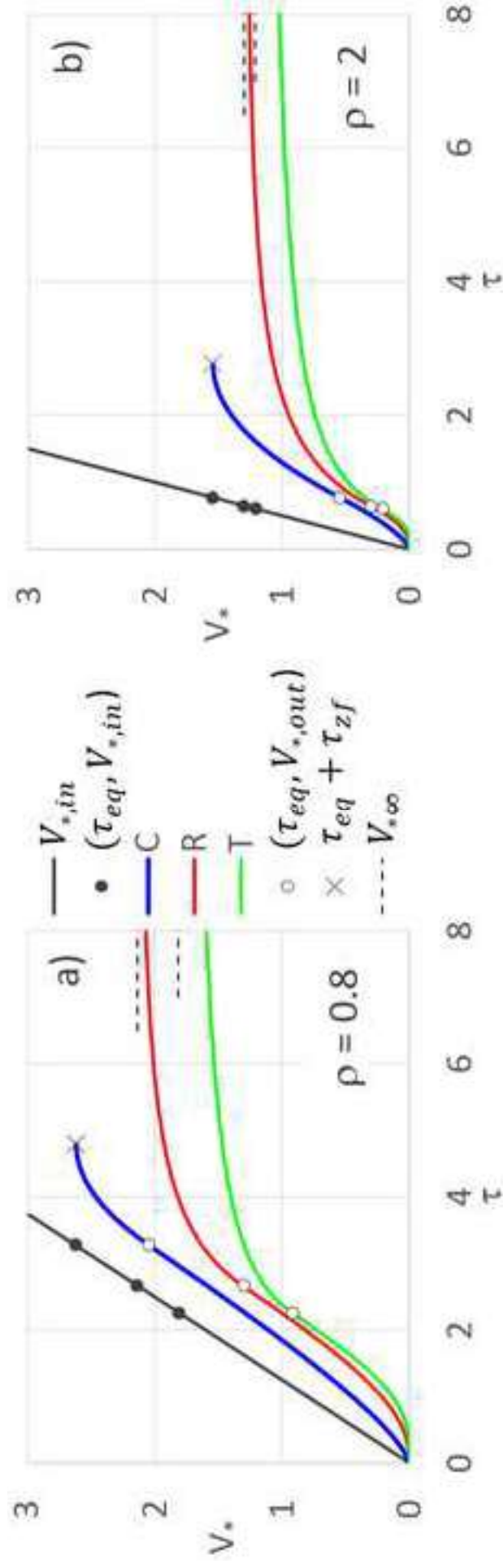
Table 4 – For a set of ten experimental runs considered in Baiamonte (2020b), data and dimensionless parameters that need to calculate the input discharge Q_{in} . The discharge by assuming a steady-state condition, $Q_{in,ss}$, and the relative errors are also reported.

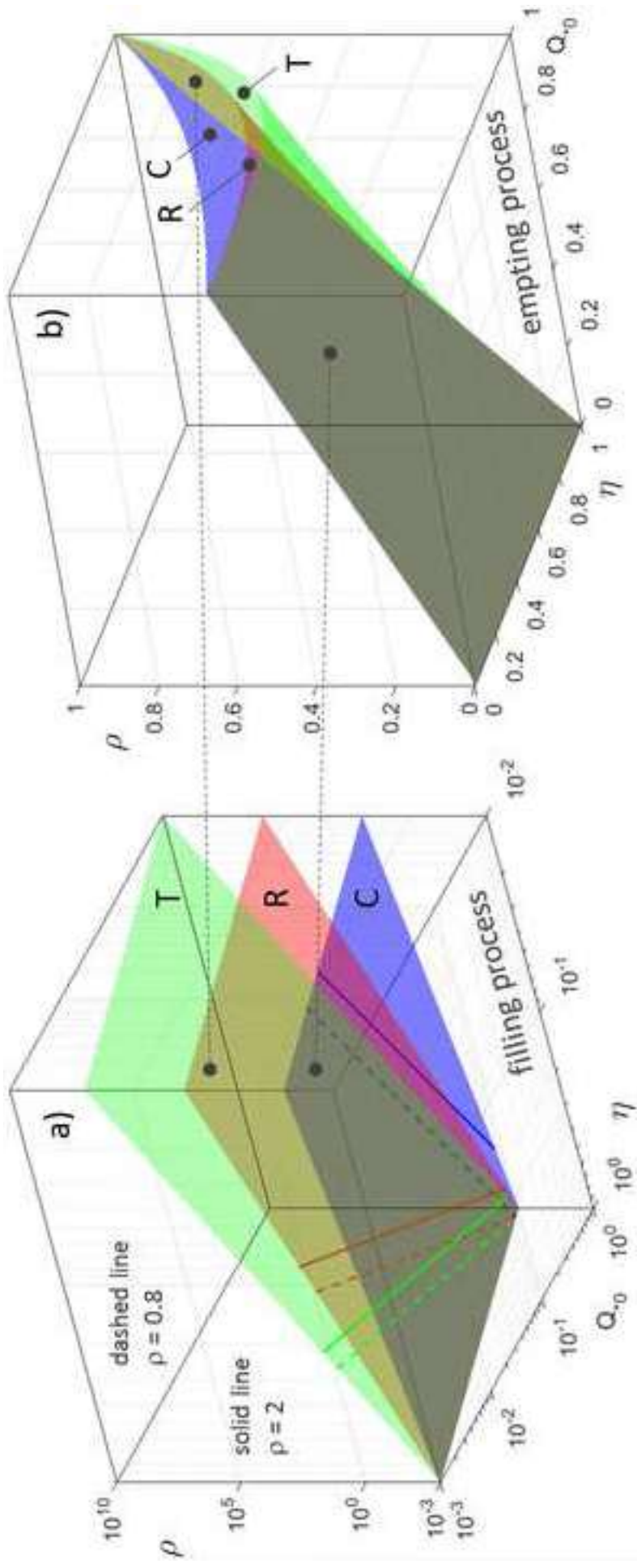
run #	experimental data						dimensionless parameter										Q_{in} (l/s)	RE	$Q_{in,ss}$ (l/s)	RE_{ss}
	t_0 (s)	t (s)	$t - t_0$ (s)	h_0 (cm)	h (cm)	$h - h_0$ (cm)	τ	τ_0	h^*_{*0}	h^*	ρ	η_0	η	Ψ_0	Ψ					
1	10	20	10	4.5	5	0.5	0.13	0.06	0.23	0.26	0.91	0.53	0.56	0.45	0.52	0.115	-4%	0.061	-49%	
2	20	31	11	5	5.5	0.5	0.20	0.13	0.26	0.28	0.90	0.57	0.60	0.54	0.62	0.113	-5%	0.064	-46%	
3	44	56	12	6	6.5	0.5	0.35	0.28	0.31	0.34	0.91	0.61	0.64	0.67	0.75	0.115	-4%	0.070	-41%	
4	56	69	13	6.5	7	0.5	0.43	0.35	0.34	0.36	0.91	0.64	0.66	0.76	0.85	0.115	-4%	0.073	-39%	
5	69	82	13	7	7.5	0.5	0.52	0.43	0.36	0.39	0.93	0.65	0.67	0.79	0.88	0.117	-2%	0.076	-36%	
6	82	96	14	7.5	8	0.5	0.60	0.52	0.39	0.41	0.93	0.67	0.69	0.89	0.98	0.117	-2%	0.079	-34%	
7	96	109	13	8	8.5	0.5	0.69	0.60	0.41	0.44	0.97	0.66	0.68	0.85	0.94	0.123	3%	0.081	-32%	
8	109	126	17	8.5	9	0.5	0.79	0.69	0.44	0.47	0.92	0.72	0.75	1.13	1.25	0.116	-3%	0.084	-30%	
9	126	143	17	9	9.5	0.5	0.90	0.79	0.47	0.49	0.93	0.73	0.75	1.16	1.28	0.118	-1%	0.086	-28%	
10	143	161	18	9.5	10	0.5	1.01	0.90	0.49	0.52	0.94	0.75	0.77	1.25	1.37	0.119	-1%	0.089	-26%	

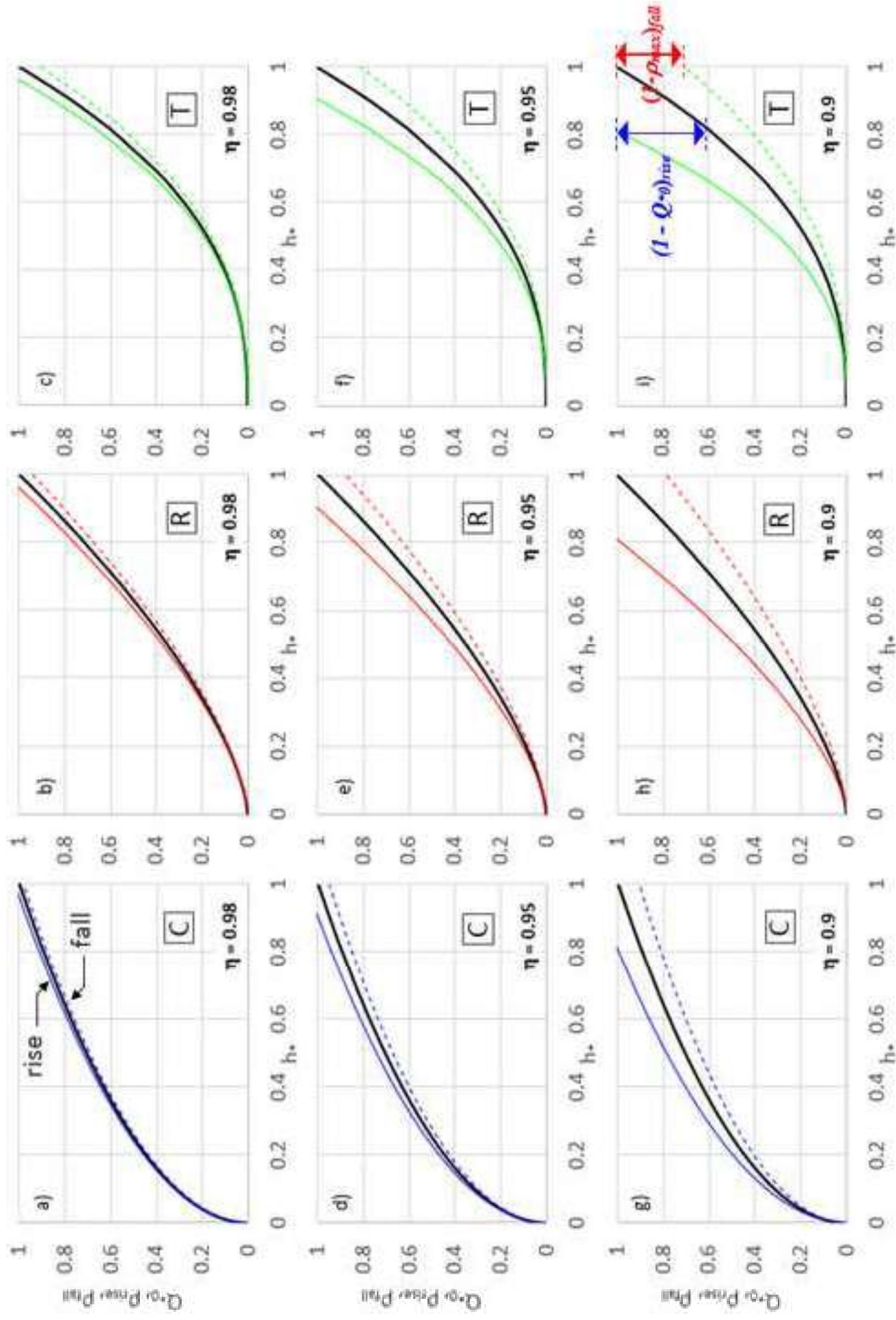


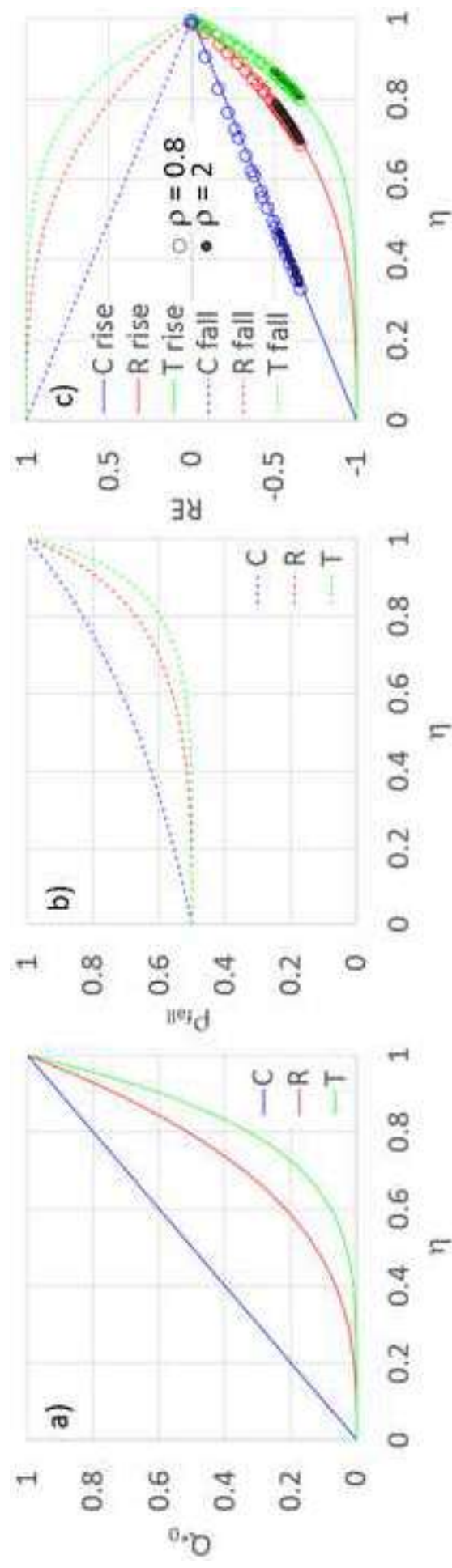


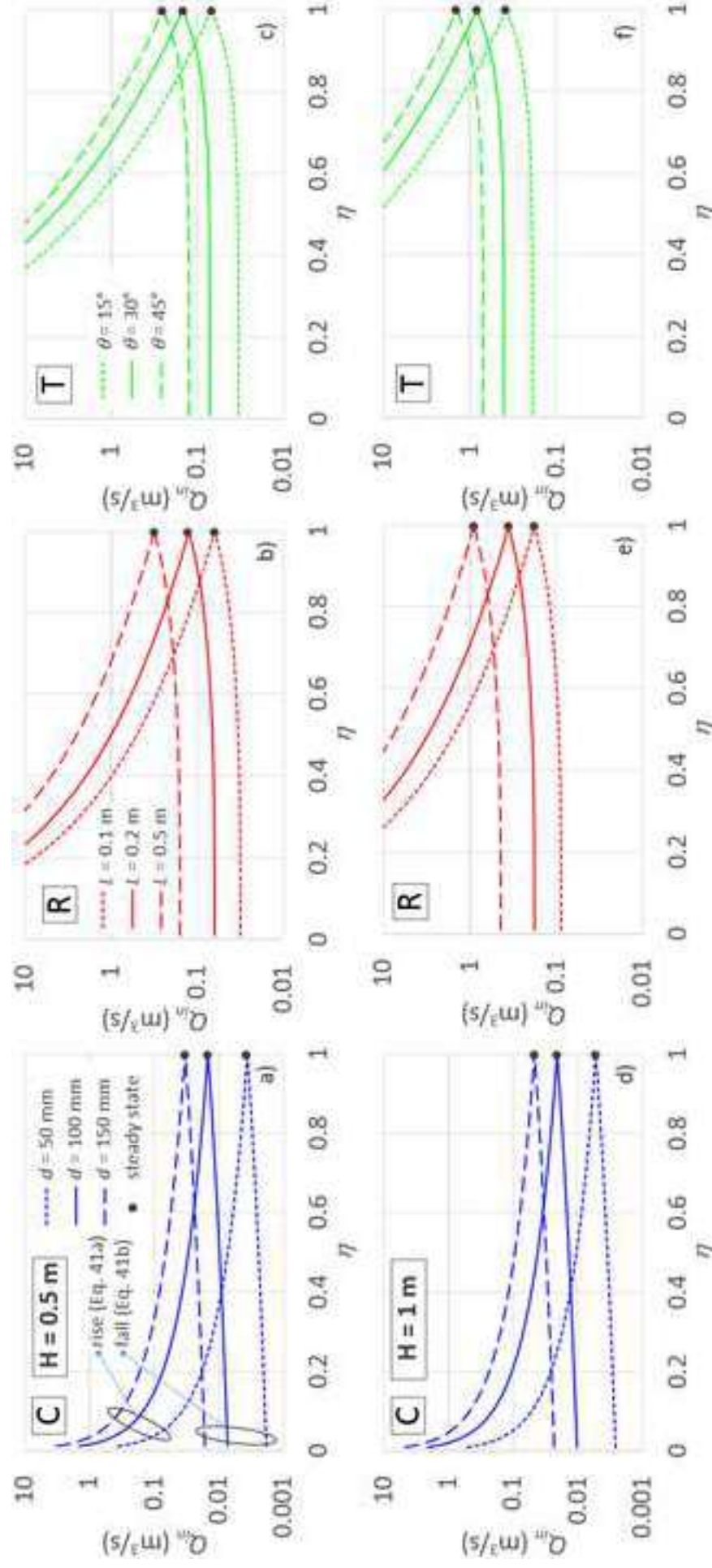


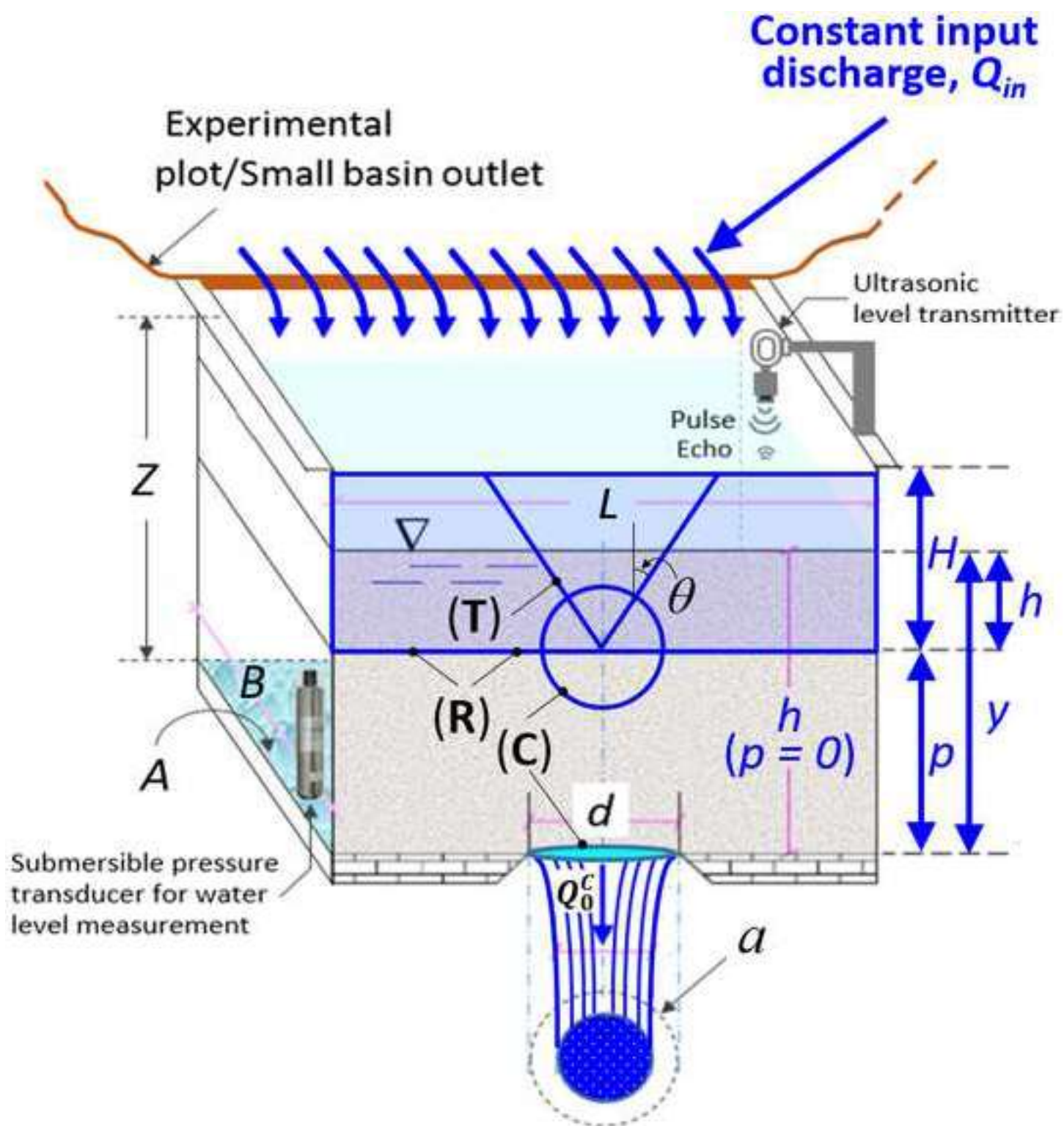












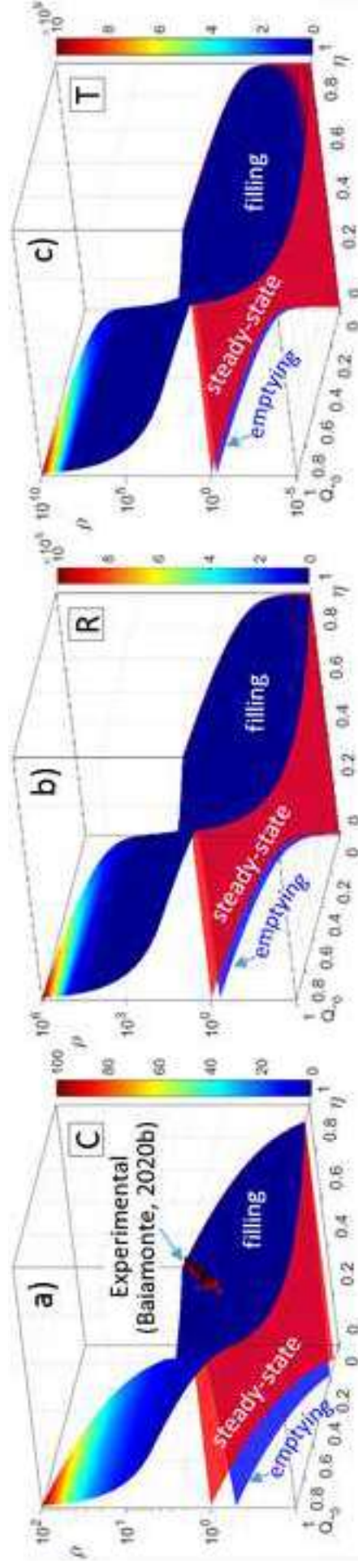


Figure Captions

Fig. 1 – Open rectangular tank, with sides L (m) and B (m), base area, A (m²), and height, Z (m), illustrating the geometric and hydraulic characteristics of i) a circular sharp-crested orifice (C) made at the bottom or at a side of the tank, ii) a rectangular weir (R), and iii) a triangular weir (T).

Fig. 2 – Relationships between the Ψ functions derived for C, R and T versus η parameter.

Fig. 3 – For C, R and T, time to equilibrium versus ρ , a) for $\rho < 1$, and b) for $\rho > 1$. Black dots refer to the application illustrated in Fig. 4. Cross symbols indicate equal τ_{eq} , for C-R and C-T, and for R-T.

Fig. 4 – For C, R and T, and a) for $\rho = 0.8$ and b) for $\rho = 2$, rising and falling limbs of the normalized water level, $h^* = h/H$, versus the normalized time $\tau = t/t_c$. The time to equilibrium asymptotic (for $\rho = 0.8$) and finite (for $\rho = 2$) is indicated. For C, the zero flux condition occurs in a finite time ($\tau_{eq} + \tau_{zf}$), whereas for R and T is asymptotic.

Fig. 5 – For C, R and T, cumulative and normalized inlet volume, $V^* = \rho \tau$, and corresponding outlet volume, V^*_{out} . According to Fig. 4, for C, the zero flux condition occurs in a finite time ($\tau_{eq} + \tau_{zf}$), whereas for R and T is asymptotic. Black dots indicate the inlet volume at $\tau = \tau_{eq}$, which is balanced by the outlet volume, when the zero flux conditions are achieved.

Fig. 6 – For a) C, b) R and c) T, comparison between the input discharge for unsteady conditions, ρ , during a filling and emptying process, and for steady steady-state, Q^*_{s0} , for different η . For C, experimental measurements carried out during a filling process (Baiamonte, 2020a) are also plotted.

Fig. 7 – For C, R and T, comparison between the input discharge for unsteady conditions, ρ , and for steady-state discharge, Q^*_{s0} , for different η , during a) a filling process and b) an emptying process, where different colors indicate the overlapping order according to a). For C, R and T, and for the filling process (a), the horizontal lines corresponding to $\rho = 0.8$ (dashed lines) and to $\rho = 2$ (solid lines), considered in the application of Fig. 4 and Fig. 5, are also plotted.

Fig. 8 – For C, R and T, and for different η values ($\eta = 0.98, 0.95$ and 0.9), runoff rate curves for a steady state (solid bold lines), for a filling unsteady state (solid lines), and for an emptying unsteady state (dashed lines).

Fig. 9 – For C, R and T, for $C_d = 0.6$, and for two H values, (a, b, c, for $H = 0.5$ and c, d, f for $H = 1$ m), relationships between the input discharge, R , versus η parameter (Eq. 22), by varying d (for C), by varying L (for R), and by varying θ (for T). For a-f), the rising and the falling conditions of the water level are indicated in Fig. 9a, and black dots refer to the steady-state condition.

Fig. 10 – For C, R and T, relationships between a) the input discharge of the rising unsteady state, ρ_{rise} , and b) the input discharge of the falling unsteady state, ρ_{fall} , versus η . In c), the corresponding absolute relative errors, defined in Fig. 8i, are reported. Circles refer to the application of Fig. 4 and Fig. 5.