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2 **Analytical Solution of the Richards Equation under Gravity-Driven Infiltration**
3 **and Constant Rainfall Intensity**

4
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9
10 **Abstract**

11
12 In the field of soil hydrology, the Richards equation is commonly used to model water flow in
13 unsaturated soils. The high nonlinearity of the Richards equation makes it very challenging to solve
14 analytically, for situations that are meaningful in practical applications. In this paper, an exact and
15 simple analytical solution of the Richards equation under gravity-driven infiltration and constant
16 rainfall intensity is derived. First, the solution is presented under Torricelli's law, which mimics the
17 soil hydraulic conductivity function and also describes the emptying or filling process of non-linear
18 water reservoirs. Then, following a similar approach, the solution is extended to the Brooks and Corey
19 soil hydraulic conductivity function, which is generally considered to describe soil hydrological
20 characteristics well. The approach followed in this study is a simple hydraulic approach, thus the
21 derived solutions are not affected by uncertainty as long as the hypothesis of the gravity-driven
22 infiltration is satisfied for the selected soils. A comparison with the solution numerically derived by
23 the Richards equation where the gravity-driven assumption is relaxed, is performed and discussed.
24 Interestingly, the comparison showed that the suggested solution delimits the solution domain of the
25 Richards equation.

26

27 **1. Introduction**

28

29 Infiltration is the process where water enters into the soil surface as rainfall and then vertically
30 moves through the soil profile under the influence of gravity (Brutsaert 2005; Smith et al. 2012).
31 When this movement takes place in the absence of gravity forces, it is considered a “sorption process”
32 (Assouline 2013). Infiltration is a key component of the hydrologic cycle, partitioning rainfall into
33 surface runoff and that flowing into the soil. Knowledge of the infiltration process makes it possible
34 to determine the runoff hydrograph, soil moisture patterns and groundwater recharge, because of
35 infiltration’s interaction with the other components of the hydrologic cycle. The infiltration process
36 is affected by various factors, including soil hydrological properties, rainfall intensity, antecedent soil
37 moisture content (Tramblay et al. 2010, Baïamonte and Singh 2016; 2017), and by additional factors
38 that could be related to biological activities in the soil column (Singh 1996; Assouline 2013). Because
39 of the fundamental role that it plays in applied hydrology and soil science, the infiltration process has
40 been widely studied, and a multitude of models have been introduced and compared (Brutsaert 2005;
41 Singh 2010; Cui and Zhu 2018; Vand et al. 2018).

42 The vertical movement of water into porous media is theoretically represented by the Richards
43 equation (Richards, 1931) and theoretically based are all the simplified models that derive by the
44 Richards equation, although under simplified assumptions (Philip 1957a; 1969; Green and Ampt
45 1911). As an example, the Green-Ampt infiltration model (Green and Ampt 1911) was derived from
46 the Richards equation under ponding conditions and by assuming a delta function diffusivity
47 (Parlange, 1971). Indeed Philip (1957b) showed that this formulation of the infiltration process may
48 be derived from the governing diffusion equation of Richards (1931), if the diffusivity is taken to be
49 a delta function of nonzero value only at a relative soil saturation equal to 1. The delta function
50 formulation is, of course, only an approximation of the rapid rise in diffusivity near saturation and it
51 has been widely discussed in the hydrologic literature (Milly 1985, Parlange 1975, Govindaraju et al.

1992, Russo and Dagan 1993). Other infiltration models are empirical (Kostiakov 1932; Horton 1938; Holtan 1961; Overton 1964, Soil Conservation Service – Curve Number, *SCS – CN* 2004; Singh and Yu 1990), thus they could be applied under the conditions for which the models were derived.

An important aspect is related to the model's application scale. For example, the *SCS-CN* model, which does not definitely serve as an infiltration model, was developed for runoff estimation at the relatively large basin scale, whereas the classic GA infiltration model (Green and Ampt 1911) could be applicable in principle at the point spatial scale. However, because of their simplicity and the few parameters required, simplified infiltration models have also been applied at spatial scales much larger than the point scale, providing satisfactory results (James et al. 1992; Grimaldi et al. 2013). Moreover, models that were derived from empirical and theoretical formulations have also been compared (Rawls and Brakensiek 1986; Ficklin and Zhang 2013; Sihag et al. 2017). Recently, Baiamonte (2019a) used the *SCS-CN* and Green and Ampt models jointly in order to analytically establish connecting relationships that allow the positive features of both models to be taken into account in applications. Baiamonte (2019a) showed that the suggested procedure made it is possible to move from the empirical model to the theoretical one and vice versa, according to the derived equations linking the relative parameters.

Hooshyar and Wang (2016) provided a physical basis for the *SCS-CN* method and its proportionality hypothesis from the infiltration excess runoff generation perspective. To achieve this purpose, Hooshyar and Wang (2016) derived an analytical solution of the Richards equation for ponded infiltration in shallow water table environments and under specific boundary conditions.

Although, simplified empirical and theoretical formulations describing infiltration have been widely used and tested, the limits of both have also been noted (Triadis and Broadbridge 2012; Barlett et al. 2016). Thus, infiltration models purely based on the Richards equation have also been widely considered, since they can be applied in any kind of soil, regardless of the simplified assumptions taken into account and can also be extended to cases that consider root water uptake according to the root depth and/or the water content (Broadbridge et al. 2017). The latter is another important issue,

78 since it makes it possible to study the role of soil's physical ability to store and provide water to plant
79 roots for both forest and agricultural soils (Cullotta et al. 2016).

80 Many efforts have been made in order to derive exact solutions of the Richards equations in a
81 linearized form and under steady-state conditions (Raats 1976; Batu 1978; White and Broadbridge
82 1988) as well as under time-dependent conditions (Broadbridge and White 1988; Sander et al. 1988).
83 Recently, exact solutions were introduced of the Richards equation with nonlinear plant-root
84 extraction, as a function of the soil water content (Broadbridge et al. 2017).

85 When the sorption component is neglected, the infiltration process is dominated by gravity and
86 modeled by capacitance models, also known as tank or bucket models (Manabe 1969, Struthers et al
87 2006; Romano et al. 2011), which are commonly employed in hydrological prediction because of
88 their conceptual simplicity and their ability to describe fundamental constraints on hydrological
89 behavior. For example, they might be pertinent when the prediction of groundwater recharge is not
90 the primary modeling objective. The latter was observed by Kim et al. (1996), whose framework
91 reasonably agreed with the Richards equation's numerical solution, for a wide range of soils and
92 climates, although it included the important simplifying assumption of considering water
93 redistribution to be instantaneous within the unsaturated zone, and thus only considered the total
94 storage in the soil water reservoir. In fact, the fundamental limitation of the capacitance approaches
95 lay in their inability to replicate time delays between input fluxes to the soil system and the generation
96 of output fluxes. However, the ability of capacitance models to replicate internal travel times may be
97 improved through the discretization of the soil profile into multiple tanks connected in series
98 (Struthers et al 2006).

99 Several studies (Milly, 1994; Farmer et al., 2003) have shown that capacitance models can be
100 efficiently employed for predicting some components of the soil water budget, as they provide results
101 similar to those observed in the field or to those simulated by models that are much more detailed in
102 their description of soil water dynamics, and in soils with specific climatic regimes, and also include

103 accurate descriptions of plant ecology (Guswa et al., 2002). This probably explains why the
104 capacitance approach is still applied, investigated and interpreted (Romano et al., (2011).

105 In this paper, an exact and simple to use analytical solution is derived of the Richards equation
106 under gravity-driven infiltration and constant rainfall intensity. First, the Richards equation solution
107 is presented under Torricelli's law, which mimics the soil hydraulic conductivity function and
108 interestingly describes the emptying or filling process of a non-linear water reservoir. Then, following
109 a similar approach, the solution is extended to the Brooks and Corey soil hydraulic conductivity
110 function (Brooks and Corey 1964), which has been tested and used in similar contexts (Severino et
111 al. 2005, Severino et al. 2006; Nasta et al. 2013; Xing et al. 2018).

112 Although the plant-root extraction component was not considered in this study, the easy-to-
113 apply derived approach, could also be used to take into account the water uptake by plant roots in
114 different dimensions (Vrugt et al. 2001) and in terms of the complexity and degree of
115 mechanical/biological principles explicitly accounted for (de Willigen et al. 2012; Javaux et al. 2013).
116 Because of gravity-driven infiltration was assumed, the procedure is better suited to be extended to
117 the case in which water root uptake is modeled by the soil water content alone, since the instantaneous
118 redistribution limitation was assumed, which is typical of the capacitance approach.

119 The paper is organized as follows. Following this introduction, section 2 discusses the solution
120 of the Richards equation under Torricelli's law. The solution is extended to the Brooks and Corey
121 soil hydraulic conductivity function in section 3. Applications for both cases are performed in section
122 2 and 3. In section 4, a comparison between the present solution and a numerical one derived by the
123 Richards equation is performed and discussed. The paper is concluded in section 5.

124

125 **2. Gravity driven infiltration under Torricelli's law**

126

127 Consider an open cylindrical tank of soil and water of unit cross sectional area, A , unit height
128 $Z = 1$ m, and radius R , as displayed in **Fig. 1**. A small circular hole (much smaller than that in the

figure) with radius r is made at the bottom of the tank. A stationary and uniformly distributed rainfall intensity, i , is applied over the tank. For a homogenous and isotropic soil, the tank water level mimics the actual volumetric soil water content, $\Theta = (\theta - \theta_{res})/\Delta\theta$, which is normalized with respect to the effective porosity, $\Delta\theta = \theta_s - \theta_{res}$, where θ ($\text{cm}^3 \text{ cm}^{-3}$) is the actual volumetric soil water content, and θ_s ($\text{cm}^3 \text{ cm}^{-3}$) and θ_{res} ($\text{cm}^3 \text{ cm}^{-3}$) are the saturated and the residual volumetric water content, respectively. Thus, $\Theta = 0$ and $\Theta = 1$ (**Fig. 1**) indicate that the soil is completely dry or saturated, respectively. Moreover, the outflow rate corresponds to the water infiltration rate at the bottom of the soil layer.

For one-dimensional infiltration, the Richards equation is written:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial h} \frac{\partial h}{\partial t} = \frac{\partial}{\partial z} \left[K(h) \left(\frac{\partial h}{\partial z} + 1 \right) \right] \quad (1)$$

where t is the time, $K(h)$ (m s^{-1}) is the soil hydraulic conductivity as a function of the soil matric potential h (m) and z (m) is the elevation.

Under gravity-driven infiltration, i.e. by assuming that the matric potential gradient is negligible compared to the gravitational potential gradient, we can write (Severino et al. 2005; Baiaumont et al. 2019):

$$\frac{\partial}{\partial z}(z + h) \cong 1 \quad \Rightarrow \quad \frac{\partial h}{\partial z} \equiv \frac{dh}{d\theta} \frac{d\theta}{dz} \ll 1 \quad (2)$$

In other words, it is assumed that the distribution of the water content along the vertical may generate lower/larger humidity gradients, so that when adopting a gravity-dominated model, the key assumption is that the slope $dh/d\theta$ of the retention curve, $h(\theta)$, is low enough (irrespective of the order of magnitude of $\partial\theta/\partial z$). Therefore, neglecting the matric gradient (compared with that of the

gravitational one) is purely a retention property and unrelated to the permeability. Under the
aforementioned assumptions, Eq. (1) reduces to the simple 1D continuity equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial K(h)}{\partial z} \quad (3a)$$

that according to the sketch displayed in **Fig. 1** and considering the inlet rainfall intensity, i , becomes:

$$AZ d\theta = (iA - Q)dt \quad \Rightarrow \quad Z\theta_s \frac{d\Theta}{dt} = \left(i - \frac{Q}{A} \right) \quad (3b)$$

where Q ($\text{m}^3 \text{s}^{-1}$) is the outflow discharge at the bottom of the soil layer and, by neglecting the residual
water content, $\Theta = \theta/\theta_s$ is the saturation degree, as previously defined. Eq. (3b) denotes the continuity
equation, establishing that during a rainfall event with constant intensity, i , the water volume exiting
the jet in time dt is equal to the water volume removed from the tank in the same time dt .

The outflow discharge per unit tank cross-sectional area, Q/A , is denoted as f (m/s), which
mimics the soil infiltration rate and can be expressed according to Torricelli's law:

$$f = \frac{Q}{A} = c_c \frac{A_0}{A} (2g\Theta)^{1/2} = c_c r_*^2 (2g\Theta)^{1/2} \cong K_s \Theta^{1/2} \quad (4a)$$

where g is gravity's acceleration, A_0 is the hole cross-sectional area, r_* is the ratio between the circular
hole radius r and that of the water tank, R , and c_c is the contraction coefficient that is close to the
unity, since R is much larger than r , and K_s (m s^{-1}), which equals $r_*^2 (2g)^{1/2}$, mimics the saturated
hydraulic conductivity, i.e. the maximum value of the soil infiltration rate, f , for $c_c = 1$ and $\Theta = 1$.

176 According to Eq. (4a), the soil infiltration rate normalized with respect to the saturated soil
 177 hydraulic conductivity, f_* , can be expressed as:

$$179 \quad f_* = \frac{f}{K_s} \cong \Theta^{1/2} \quad (4b)$$

180
 181 With ρ denoting the rainfall intensity, i , normalized with respect to K_s (m/s), and assuming a
 182 unit height ($Z = 1000$ mm), and substituting Eq. (4b) in Eq. (3b), yields the dimensionless differential
 183 flow equation:

$$185 \quad \frac{d\Theta}{d\tau} = \rho - \Theta^{1/2} \quad (5)$$

186
 187 where τ is the time, t , normalized with respect to the time, t_c (s), representing a characteristic time of
 188 the infiltration process:

$$190 \quad t_c = \frac{Z \theta_s}{K_s} = \frac{Z \theta_s}{r_*^2 (2g)^{1/2}} \quad (6)$$

191
 192 From a physical point of view, t_c could also be expressed as the travel time required for the
 193 water particle to cross the entire soil layer of unit length, under saturated and steady state conditions.
 194 Of course the more r_* increases the less the travel time will be.

195 Soil characteristics are assumed to be spatially constant and according to the diagram
 196 displayed in **Fig. 1**, runoff is generated through a fast or slow mechanism of runoff generation
 197 (Baiamonte et al. 2015; Baiamonte and Agnese, 2016; Baiamonte, 2019b), depending on whether the
 198 normalized rainfall intensity exceeds or falls behind the unity, i.e. when $i > K_s$ or $i < K_s$, respectively
 199 (Baiamonte, 2016).

The differential Eq. (5) could be integrated, by fixing the initial condition:

$$\Theta = \Theta_0 \quad \text{for} \quad \tau = \tau_0 \quad (7)$$

on the assumption that during the close interval $[\tau_0, \tau]$ the normalized infiltration rate (Eq. 4b) is constant:

$$\int_{\tau_0}^{\tau} d\tau = \int_{\Theta_0}^{\Theta} \frac{d\Theta}{\rho - \Theta^{1/2}} \quad (8)$$

Two different cases can be distinguished:

i) $\rho \neq 0$. In this case, for $\tau_0 = 0$, the integration of Eq. (8) gives an explicit τ solution:

$$\tau = 2 \left(\rho \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - \sqrt{\Theta}} \right| - (\sqrt{\Theta} - \sqrt{\Theta_0}) \right) \quad (9a)$$

that can also be written in terms of f^* by using Eq. (4b):

$$\tau = 2 \left(\rho \log \left| \frac{\rho - f_{*0}}{\rho - f_*} \right| - (f_* - f_{*0}) \right) \quad (9b)$$

where f_{*0} is the antecedent infiltration rate normalized with respect to K_s .

Eq. (9a) shows that for $\rho \leq 1$, the time to equilibrium, τ_{eq} , i.e. the time needed to achieve the steady state condition of the infiltration process, cannot be achieved in a finite time, so that the water

221 content asymptotically attains an equilibrium value, Θ_∞ , whereas for $\rho > 1$, τ_{eq} is achieved in a finite
 222 time at the soil saturation condition, which can be derived by imposing $\Theta = \Theta_{eq} = 1$:

223

$$224 \quad \Theta_\infty = \rho^2 \quad \rho \leq 1 \quad \tau_{eq} \sim \infty \quad (10a)$$

$$225 \quad \tau_{eq} = 2 \left(\rho \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - 1} \right| - (1 - \sqrt{\Theta_0}) \right) \quad \rho > 1 \quad \Theta_{eq} = 1 \quad (10b)$$

226

227 ii) $\rho = 0$. This case describes the recession of the infiltration process, starting at the end of the rainfall
 228 at the time, τ_r , i.e. $\tau_0 = \tau_r$ and $\Theta_0 = \Theta_r$, with Θ_r denoting the normalized water content at the end of
 229 the rainfall period. Thus, Eq. (8) can be rewritten as:

230

$$231 \quad \int_{\tau_r}^{\tau} d\tau = \int_{\Theta}^{\Theta_r} \frac{d\Theta}{\Theta^{1/2}} \quad (11)$$

232

233 which once integrated provides the following explicit solution in τ or in Θ , respectively:

234

$$235 \quad \tau = \tau_r + 2 \left(\Theta_r^{1/2} - \Theta^{1/2} \right) \quad (12a)$$

$$236 \quad \Theta = \left(\Theta_r^{1/2} - \frac{\tau - \tau_r}{2} \right)^2 \quad (12b)$$

237

238 Eq. (12a) allows to the time required for the zero flux condition, τ_{zf} , i.e. for $\Theta = 0$, to be determined:

239

$$240 \quad \tau_{zf} = \tau_r + 2 \Theta_r^{1/2} \quad (13)$$

241

242 For the case $\rho < 1$, the time to equilibrium can also be established as the time for which the
 243 normalized infiltration rate achieves a high fraction of the asymptotic value. The latter, using Eqs.
 244 (4b) and (10a), matches ρ , thus τ_{eq} can be expressed as:

$$245 \quad \tau_{eq} = \min \{ \tau : f_* \geq \beta \rho \} \quad \rho \leq 1 \quad (14a)$$

246
 247 in which β could be chosen as equal to 0.95. Imposing Eq. (14a) in Eq. (9a) yields:
 248

$$249 \quad \tau_{eq} = 2 \left(\rho \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho(1-\beta)} \right| - (\beta \rho - \sqrt{\Theta_0}) \right) \quad \rho \leq 1 \quad (14b)$$

250
 251 The assumption made to derive Eq. (14b) agrees with Saghafian and Julien (1995), who observed that
 252 the equilibrium time on pervious watersheds is seldom attained, because the infiltration rate could
 253 reach saturated hydraulic conductivity after a time that is much longer than the duration of the rainfall.
 254 Accordingly, Eq. (14b) denotes a time to virtual equilibrium that may serve as a measure of the
 255 response time of pervious watersheds subject to uniform or distributed rainstorms (Baiaamonte and
 256 Singh, 2016), although it provides an approximated rather than an exact time.

257 For both cases $\rho > 1$ (Eq. 10b) and $\rho \leq 1$ (Eq. 14b), the dependence of τ_{eq} on ρ shows the non-
 258 linearity of the studied process. For $\rho \leq 1$ and $\rho > 1$ and under zero antecedent soil moisture conditions
 259 (ASMC), $\Theta_0 = 0$, Eq. (14b) and Eq. (10b) can be rewritten, respectively:

$$261 \quad \tau_{eq} = 2 \left(\rho \log \left| \frac{1}{1-\beta} \right| - \beta \rho \right) \quad \rho \leq 1 \quad \Theta_{eq} = (\beta \rho)^2 \quad (15a)$$

$$262 \quad \tau_{eq} = 2 \left(\rho \log \left| \frac{\rho}{\rho-1} \right| - 1 \right) \quad \rho > 1 \quad \Theta_{eq} = 1 \quad (15b)$$

263

264 For $\Theta_0 = 0$ and for different ρ values, the normalized water content, Θ , calculated by Eq. (9a),
 265 versus τ is plotted in **Fig. 2**, which makes it possible to confirm that for $\rho > 1$, equilibrium is achieved
 266 in a finite time, τ_{eq} (Eq. 15b), whereas for $\rho \leq 1$, Θ asymptotically attains the equilibrium conditions
 267 $\Theta_\infty = \rho^2$. For $\rho \leq 1$ and by assuming $\beta = 0.95$, **Fig. 2** also illustrates the virtual time to equilibrium
 268 calculated by Eq. (15a), corresponding to $\Theta_{eq} = (\beta \rho)^2$.

269 For the case $\rho > 1$, the time to equilibrium can also be seen as the time to ponding (Broadbridge
 270 and White 1987), i.e. the time that is required for the runoff generation process. In fact, starting from
 271 this condition, the infiltration rate achieves its maximum value, K_s , and under a constant rainfall with
 272 indefinite duration, the overland flow occurs with a constant rainfall excess, $i - K_s$. Contrarily, for ρ
 273 ≤ 1 rainfall excess cannot be generated.

274 In order to investigate the influence of the AMSC on the time to equilibrium, for different Θ_0
 275 values and for both $\rho \leq 1$ and $\rho > 1$, **Fig. 3a** and **Fig. 3b** plot τ_{eq} versus ρ , respectively, showing that
 276 for an increasing ρ , for $\rho \leq 1$, τ_{eq} increases, whereas for $\rho > 1$, τ_{eq} decreases. Moreover, for any ρ at
 277 an increasing Θ_0 , as expected, τ_{eq} decreases. However, the influence of Θ_0 on τ_{eq} seems to be much
 278 more important for $\rho \leq 1$ than for $\rho > 1$.

279 From a practical point of view, it is interesting to derive the infiltration coefficient, δ_i , which
 280 is denoted here as the ratio between the infiltration volume, I , at the time t , and that of the rainfall.
 281 Towards this aim, for time $\tau \leq \tau_r$ (with $\tau_r = t_r/t_c$), the infiltration volume at the bottom of the soil layer,
 282 I_* , normalized with respect to the infiltration volume, $K_s t_c$, can be expressed according to:

$$284 \quad I_* = f_* \tau - \int_{f_{*0}}^{f_*} \tau(f_*) df_* \quad (16a)$$

285

286 The infiltration volume, $K_s t_c$, which normalized the infiltration volume, I , represents, as t_c , a
 287 characteristic value, i.e. the volume that infiltrates in saturated condition and in the time $t = t_c$,
 288 Substituting Eq. (9b) in Eq. (16a) yields:

289

$$290 \quad I_* = f_* \tau - 2\rho \int_{f_{*0}}^{f_*} \log \left| \frac{\rho - f_{*0}}{\rho - f_*} \right| df_* + 2 \int_{f_{*0}}^{f_*} (f_* - f_{*0}) df \quad (16b)$$

291

292 that once integrated becomes:

293

$$294 \quad I_* = 2\rho^2 \log \left| \frac{\rho - f_{*0}}{\rho - f_*} \right| - 2\rho (f_* - f_{*0}) - (f_*^2 - f_{*0}^2) \quad (16c)$$

295

296 By using Eq. (4b), Eq. (16c) can be rewritten in terms of Θ :

297

$$298 \quad I_* = 2\rho^2 \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - \sqrt{\Theta}} \right| - 2\rho (\sqrt{\Theta} - \sqrt{\Theta_0}) - (\Theta - \Theta_0) \quad (17)$$

299

300 For any time $\tau > \tau_r$, Eq. (16c) needs to be modified by putting f_* equal to the infiltration rate
 301 at the end of the rainfall, f_{*r} , and also by considering the additional infiltration volume, during the
 302 falling limb, from the end of the rainfall to the time $\tau > \tau_r$, which can be expressed by the following
 303 integral on the right, according to Eq. (12b) and Eq. (4b):

304

$$305 \quad I_* = 2\rho^2 \log \left| \frac{\rho - f_{*0}}{\rho - f_{*r}} \right| - 2\rho (f_{*r} - f_{*0}) - (f_{*r}^2 - f_{*0}^2) + \int_{\tau_r}^{\tau} \left(f_{*r} - \frac{\tau - \tau_r}{2} \right) d\tau \quad (18a)$$

306

307 The integration of Eq. (18a) and using of Eq. (12a) yields:

308

$$309 \quad I_* = 2\rho^2 \log \left| \frac{\rho - f_{*0}}{\rho - f_{*r}} \right| - 2\rho(f_{*r} - f_{*0}) - (f_{*r}^2 - f_{*0}^2) \quad (18b)$$

310

311 As for $\tau \leq \tau_r$ (Eq. 17), Eq. (18b) can be rewritten as a function of Θ , by using Eq. (4b):

312

$$313 \quad I_* = 2\rho^2 \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - \sqrt{\Theta_r}} \right| - 2\rho(\sqrt{\Theta_r} - \sqrt{\Theta_0}) - (\Theta - \Theta_0) \quad (19)$$

314

315 Thus, for both $\tau \leq \tau_r$ and $\tau > \tau_r$, the infiltration coefficient, δ_i , referring to the bottom of the
 316 soil layer, can be derived in terms of f^* or Θ by dividing Eq. (16c) or Eq. (18b) and Eq. (17) or Eq.
 317 (19), respectively, with respect to the rainfall volume $V_* = \rho \tau$, which is normalized, as I_* , with respect
 318 to the characteristic volume $K_s t_c$. In the following, δ_i is written with respect to Θ for both cases;

319

320

$$321 \quad \delta_i = \frac{I_*}{V_*} = \frac{1}{\rho \tau} \left[2\rho^2 \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - \sqrt{\Theta}} \right| - 2\rho(\sqrt{\Theta} - \sqrt{\Theta_0}) - (\Theta - \Theta_0) \right] \quad \text{for } \tau \leq \tau_r \quad (20a)$$

$$322 \quad \delta_i = \frac{I_*}{V_{*r}} = \frac{1}{\rho \tau_r} \left[2\rho^2 \log \left| \frac{\rho - \sqrt{\Theta_0}}{\rho - \sqrt{\Theta_r}} \right| - 2\rho(\sqrt{\Theta_r} - \sqrt{\Theta_0}) - (\Theta - \Theta_0) \right] \quad \text{for } \tau > \tau_r \quad (20b)$$

323

324 where in Eq. (20b), V_{*r} is the normalized rainfall volume at the end of the rainfall. Checking the water
 325 balance requires that the rainfall volume, V_r , equals $I + Z(\theta - \theta_0)$ whereas, in dimensionless terms, it
 326 requires $\rho \tau = I_* + (\Theta - \Theta_0)$, i.e. $1 = \delta_i + (\Theta - \Theta_0)/(\rho \tau)$.

Three practical cases, reported in **Table 1**, were considered in order to verify the consistence of Eq. (20a) and Eq. (20b). The first two cases refer to the normalized rainfall intensity $\rho > 1$ and $\Theta_0 = 0$ and $\Theta_0 > 0$, whereas the third case refers to $\rho < 1$ and $\Theta_0 = 0$. **Fig. 4** shows the temporal variation of the normalized water content, Θ (**Fig. 4a**), the normalized infiltration rate, f^* (**Fig. 4b**), and the infiltration coefficient, δ_i (**Fig. 4c**). In the same figures the characteristic times τ_0 , τ_r , τ_{zf} , and the corresponding normalized water content and infiltration coefficient are also plotted (**Table 1**). The accuracy of the derived solutions is tested by the water balance of the inflowing and outflowing volumes, corresponding to the condition $\delta_i = 1$ at the end of the $\Theta(\tau)$ falling limb, when the initial conditions are re-established, so that when $\Theta = \Theta_0$, $\rho\tau = I^*$ and $\delta_i = I^* / (\rho\tau) = 1$.

3. Gravity driven infiltration under the Brooks and Corey hydraulic conductivity function

Most of the results described in the previous section, although formulated for a simplified hydrologic soil behavior, meet and extend the already known fundamentals of hydraulics (De Marchi 1954; Faber 1995) that govern the process of emptying or filling a water reservoir where there is a given inlet and outlet. However, the solutions derived in section 2 cannot be applied to actually describe the soil hydraulic conductivity function, which is far from that described by Torricelli's law.

In order to extend the previous approach to the soil infiltration process, Brooks and Corey's (1964) soil hydraulic conductivity function was considered here:

$$K(\Theta) = K_s \Theta^{1/c} \quad (21)$$

where $c = \lambda / (2 + (2 + l) \lambda)$ is the pore connectivity index, which is a function of the pore size distribution index, λ , and of the tortuosity factor, l , which are characteristics of the soil hydraulic functions (Assouline 2005) and sometimes are considered to be merely empirical coefficients

352 affecting the hydraulic function (Šimůnek et al. 2011). For example, Nasta et al. (2013) derived a
353 simple closed-form equation that predicts the saturated hydraulic conductivity, which uses a
354 tortuosity-connectivity calibration parameter to fit the measured K_s .

355 The comparison between Eq. (21) and Eq. (4b) shows that Eq. (20) fits Torricelli's law for c
356 $= 2$, whereas according to the Brooks and Corey hydraulic conductivity function, c values less than 2
357 should be considered. Therefore, the hydraulic conductivity function and the characteristic time of
358 the infiltration process previously introduced (Eq. 4b and Eq. 6, respectively) need to be extended to
359 any c value according to:

360

361
$$K(\Theta) = K_s \Theta^{1/c} = r_*^2 (2g)^{1/c} \Theta^{1/c} \quad (22a)$$

362
$$t_c = \frac{Z \theta_s}{K_s} = \frac{Z \theta_s}{r_*^2 (2g)^{1/c}} \quad (22b)$$

363

364 where the r_* value that matches the saturated hydraulic conductivity of the Brooks and Corey model,
365 can be derived as:

366

367
$$r_* = \sqrt{\frac{K_s}{(2g)^{1/c}}} \quad (23)$$

368

369 This consideration allows us to use a well-consolidated hydraulic approach, which is not
370 affected by uncertainties, to describe the infiltration process under the assumption of gravity-driven
371 infiltration. In other words, according to Eq. (23), the ratio between the radius of the circular hole and
372 that of the water tank, r_* , can be associated to the saturated hydraulic conductivity, the high spatial
373 variability of which complicates its estimation (Leij et al. 2004; Fan et al. 2016; Baiamonte et al.
374 2017; 2019), and the described approach can thus be extended to describe soil behavior.

375 In order to detect the r^* value range that needs to be considered for common soils, by assuming $l = 1$,
 376 the dataset reported in **Table 2** was taken into account (Brakensiek et al. 1981), showing that for soils,
 377 the c value is actually much less than 2. **Fig. 5a** shows the relationship between r^* and K_s for the c
 378 values in the range 0.06 – 0.15, which cover the c values reported in **Table 2**. As expected, at
 379 increasing K_s , r^* also increases, since the larger the hole radius describing the soil behavior is, the
 380 higher K_s will be. **Fig. 5a** also plots the r^* - K_s relationship derived for a c value higher than $c = 0.15$
 381 ($c = 0.2$), which is typical of highly permeable soils, such as sandy soils (Matlan et al. 2014;
 382 Baiamonte et al. 2019), which could better fit the gravity-driven infiltration assumption. For the same
 383 c values, in **Fig. 5b** the Brooks and Corey hydraulic conductivity function is reported. For
 384 comparison, the hydraulic conductivity function for $c = 2$, according to Torricelli's law is also
 385 illustrated, indicating that $c = 2$ provides normalized infiltration rates that are much higher than those
 386 of the soils. Accounting for Eq. (22a) rather than Eq. (4b) to describe the hydraulic conductivity
 387 function, complicates the approach followed in section 2 to derive analytical solutions. In fact, the
 388 differential equation and the corresponding integral form reported in Eq. (5) and in Eq. (8) need to be
 389 rewritten according to any $c < 2$:

390

$$391 \quad \frac{d\Theta}{d\tau} = \rho - \Theta^{1/c} \quad (24a)$$

$$392 \quad \tau = \int_{\Theta_0}^{\Theta} \frac{1}{\rho - \Theta^{1/c}} d\Theta \quad (24b)$$

393

394 As for $c = 2$, also for $c < 2$, two different cases can be distinguished:

395 i) $\rho \neq 0$. The integration of (24b) gives an explicit τ solution:

396

$$397 \quad \tau = \frac{1}{\rho} \left(\Theta {}_2F_1 \left(1, c, 1+c, \frac{\Theta^{1/c}}{\rho} \right) - \Theta_0 {}_2F_1 \left(1, c, 1+c, \frac{\Theta_0^{1/c}}{\rho} \right) \right) \quad (25)$$

398

399 where ${}_2F_1$ is the hypergeometric function ${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}$ (Abramovitz 1972, p. 556).

400 Eq. (25) shows that, for $c = 2$, it matches with Eq. (9b), demonstrating that Eq. (9b) is a
 401 particular case of Eq. (25). The latter could be observed by putting $c = 2$ in Eq. (25), and considering
 402 that, for $c = 2$, the hypergeometric function ${}_2F_1(.,.)$ can be expressed according to (Abramovitz 1972,
 403 p. 556, 15.1.4):

404

$$405 \quad {}_2F_1\left(1, 2, 3, \frac{\Theta^{1/2}}{\rho}\right) = \frac{2\rho}{\Theta} \left(\rho \log\left(\frac{\rho}{\rho - \Theta^{1/2}}\right) - \Theta^{1/2} \right) \quad (26)$$

406

407 For $\Theta_0 = 0$ and $c < 2$, Eq. (10a) and Eq. (10b) were rewritten:

408

$$409 \quad \Theta_{eq} = \rho^c \quad \rho \leq 1 \quad \tau_{eq} \sim \infty \quad (27a)$$

$$410 \quad \tau_{eq} = \frac{1}{\rho} \left({}_2F_1\left(1, c, 1 + c, \frac{1}{\rho}\right) \right) \quad \rho > 1 \quad \Theta_{eq} = 1 \quad (27b)$$

411

412 ii) for $\rho = 0$, as for $c = 2$, integration of Eq. (24b) gives an explicit τ solution that can be written in
 413 terms of f^* or Θ :

414

$$415 \quad \tau = \tau_r + \frac{c}{c-1} \left(f_{*r}^{c-1} - f_*^{c-1} \right) \quad (28a)$$

$$416 \quad \tau = \tau_r + \frac{c}{c-1} \left(\Theta_r^{1-1/c} - \Theta^{1-1/c} \right) \quad (28b)$$

417

418 Since the approach for $c < 2$ is similar to that already described, only the final equations are
 419 reported in the following. In particular, Eq. (17) and Eq. (19) become, respectively:

420

$$421 \quad I_* = \frac{1}{\rho} \left[\rho \tau \Theta^{1/c} + (\Theta^{1/c} - \Theta_0^{1/c}) \Theta_0 {}_2F_1 \left(1, c, 1+c, \frac{\Theta_0^{1/c}}{\rho} \right) + \right. \\ \left. - \frac{1}{1+c} \left(\Theta^{1+1/c} {}_2F_1 \left(1, c, 2+c, \frac{\Theta^{1/c}}{\rho} \right) - \Theta_0^{1+1/c} {}_2F_1 \left(1, c, 2+c, \frac{\Theta_0^{1/c}}{\rho} \right) \right) \right] \quad \text{for } \tau \leq \tau_r \quad (29a)$$

$$422 \quad I_* = \frac{1}{\rho} \left[\rho \tau_r \Theta_r^{1/c} + (\Theta_r^{1/c} - \Theta_0^{1/c}) \Theta_0 {}_2F_1 \left(1, c, 1+c, \frac{\Theta_0^{1/c}}{\rho} \right) + \right. \\ \left. - \frac{1}{1+c} \left(\Theta_r^{1+1/c} {}_2F_1 \left(1, c, 2+c, \frac{\Theta_r^{1/c}}{\rho} \right) - \Theta_0^{1+1/c} {}_2F_1 \left(1, c, 2+c, \frac{\Theta_0^{1/c}}{\rho} \right) \right) \right] + I_{*fall} \quad \text{for } \tau > \tau_r \quad (29b)$$

423

424 In Eq. (29b), I_{*fall} denotes the infiltration volume during the falling limb, which in the case $c = 2$ was
 425 represented by the second integral on the right side in Eq. (18a) that was then incorporated into Eq.
 426 (18b).

427 For the case $c < 2$, I_{*fall} requires a separated derivation, since it needs to be recalculated
 428 according to:

429

$$430 \quad I_{*fall} = \int_{\tau}^{\tau_r} \tau (f_*) d\tau - (f_{*r} - f_*) \tau_r + (\tau - \tau_r) f_* \quad (30a)$$

431

432 Substituting Eq. (28a) into Eq. (30a) yields:

433

$$434 \quad I_{*fall} = \int_{\tau}^{\tau_r} \left(\tau_r + \frac{c}{c-1} (f_{*r}^{c-1} - f_*^{c-1}) \right) d\tau - (f_{*r} - f_*) \tau_r + (\tau - \tau_r) f_* \quad (30b)$$

435

436 Eq. (30b) can be integrated, providing the following solution:

437

$$I_{*fall} = -\frac{f_*^c (c f_{*r}^c + (c-1)f_{*r}\tau_r) - f_{*r} (f_{*r}^c + (c-1)(f_{*r}^c + f_{*r}\tau_r))}{f_{*r}(c-1)} - (f_{*r} - f_*)\tau_r + (\tau - \tau_r)f_* \quad (30c)$$

that can be simplified in the following forms in terms of f_* or Θ :

$$I_{*fall} = \frac{f_*^c - c f_* f_{*r}^{c-1}}{c-1} + f_{*r}^c + (\tau - \tau_r)f_* \quad \text{for } \tau > \tau_r \quad (30d)$$

$$I_{*fall} = \frac{\Theta - c \Theta^{1/c} \Theta_r^{1-1/c}}{c-1} + \Theta_r + (\tau - \tau_r)\Theta^{1/c} \quad \text{for } \tau > \tau_r \quad (30e)$$

Therefore, also for the case $c < 2$, and for $\tau \leq \tau_r$ and for $\tau > \tau_r$, the infiltration coefficient, δ_i , referred to the bottom soil layer, can be derived by dividing Eq. (29a) and Eq. (29b) with respect to the normalized rainfall volume, V_* , and by using Eq. (30e):

$$\delta_i = \frac{I_*}{V_{*r}} = \frac{I_*}{\tau \rho} \quad \text{for } \tau \leq \tau_r \quad (31a)$$

$$\delta_i = \frac{I_*}{V_{*r}} = \frac{I_{*r}}{\tau_r \rho} + \frac{1}{\tau \rho} \left(\frac{\Theta - c \Theta^{1/c} \Theta_r^{1-1/c}}{c-1} + \Theta_r + (\tau - \tau_r)\Theta^{1/c} \right) \quad \text{for } \tau > \tau_r \quad (31b)$$

First, an application was made for a fixed c value (0.134), and by varying the normalized rainfall intensity, ρ , for $\rho > 1$ with $\Theta_0 = 0$ and $\Theta_0 > 0$, and for $\rho < 1$ with $\Theta_0 = 0$ (**Table 3**). For these cases, **Fig. 6** shows the time-varying normalized water content, Θ (**Fig. 6a**), the normalized infiltration rate, f_* (**Fig. 6a**), and the infiltration coefficient, δ_i (**Fig. 6c**). The same figures also show the characteristic times τ_0 , the normalized duration of rainfall, τ_r , and the corresponding normalized water content and infiltration coefficient (**Table 3**).

As for $c = 2$, the accuracy of the derived solutions (Eq. 31a and Eq. 31b) is tested by the water balance of the inflowing and outflowing volumes corresponding to the condition that the infiltration

coefficient, δ_i , equals the unity at the end of the falling limb, $\Theta(\tau)$, when the initial condition is re-established. **Fig. 6a** shows that when $\Theta_0 = 0.2$ (#1), the saturated condition is achieved quicker than the case $\Theta_0 = 0$ (**Table 3**) and the corresponding infiltration volume also increases quicker (**Fig. 6a**). Similar expected results are obtained by comparing, for $\Theta_0 = 0$, the graphs for different ρ values ($\rho < 1$, #1 and $\rho > 1$ #2, #4).

Expected results were also obtained by varying c values in the range $0.06 - 0.2$, and by fixing the normalized rainfall intensity ($\rho = 3 > 1$, **Fig. 7**, **Table 4** and $\rho = 0.6 < 1$, **Fig. 8**, **Table 5**). In both **Fig. 7** and **Fig. 8**, it is interesting to observe that the $\Theta(\tau)$ rising limb is poorly affected by the c value, whereas the $\Theta(\tau)$ falling limb, which requires much higher times than the rising limb, is strongly affected by c . Of course, as c increases, the $\Theta(\tau)$ falling limb becomes steeper, according to the shape of the hydraulic conductivity function displayed in **Fig. 5b**.

No differences were observed in the $\Theta(\tau)$ falling limbs between **Fig. 7** and **Fig. 8**, since it is not influenced by the rainfall intensity but only by the soil's hydrological characteristics, whereas the $\Theta(\tau)$ rising limb, and the corresponding infiltration coefficient, appear steeper for $\rho = 3$ (**Fig. 7**) than for $\rho = 0.6$ (**Fig. 8**).

Finally, the effect of the c parameter on the time to equilibrium, τ_{eq} , was investigated by varying both the AMSC, Θ_0 , and the normalized rainfall intensity, separately for $\rho \leq 1$ and $\rho > 1$. Towards this aim, for any Θ_0 , by using Eq. (25) and by imposing the normalized water content at the equilibrium, Θ_{eq} , Eq. (15a) and Eq. (15b) were rewritten:

$$\tau_{eq} = \frac{1}{\rho} \left((\beta \rho)^c {}_2F_1(1, c, 1+c, \beta) - \Theta_0 {}_2F_1\left(1, c, 1+c, \frac{\Theta_0^{1/c}}{\rho}\right) \right) \quad \rho \leq 1 \quad \Theta_{eq} = (\beta \rho)^c \quad (32a)$$

$$\tau_{eq} = \frac{1}{\rho} \left({}_2F_1\left(1, c, 1+c, \frac{1}{\rho}\right) - \Theta_0 {}_2F_1\left(1, c, 1+c, \frac{\Theta_0^{1/c}}{\rho}\right) \right) \quad \rho > 1 \quad \Theta_{eq} = 1 \quad (32b)$$

For $c = 0.06, 0.08, 0.1, 0.15$ and 0.2 , **Fig. 9** plots the time to equilibrium versus Θ_0 and ρ , for $\rho \leq 1$ (**Fig. 9a**) and for $\rho > 1$ (**Fig. 9b**). In any case, according to what was expected, τ_{eq} decreases with an increasing ρ and Θ_0 . Interestingly, the influence of c on τ_{eq} only appears high for low ρ values, for both $\rho \leq 1$ (**Fig. 9a**, $0.01 < \rho < 0.07$) and for $\rho > 1$ (**Fig. 9b**, $1 < \rho < 2$). Moreover, for a fixed Θ_0 and in the mentioned ρ ranges, for $\rho > 1$, at decreasing c , τ_{eq} decreases, in agreement with the corresponding hydraulic conductivity functions graphed in **Fig. 5b**, whereas for $\rho \leq 1$ the opposite occurs. The latter results could be explained by the saturated hydraulic conductivity that is higher than the rainfall intensity ($\rho < 1$), and does not reflect the trend of the infiltration rate with c , plotted in **Fig. 5b**, which was normalized with respect to K_s .

Experimental validation might be appropriate to test the suggested procedure, however discrepancies between this theory and any experimental results should be ascribed to the experimental conditions, which wouldn't be able to reproduce the gravity-driven assumption made in the present work. Because it is based on analytical solutions, it should provide exact results.

4. Comparison between the present solution and the Richards equation's numerical solution

This section contains a comparison between the present solution (PS), and the more accurate, but numerically demanding solution derived by the Richards equation (RCH), where the assumption of the gravity-driven infiltration is relaxed (Eq. 2).

RCH solutions were obtained by using the HYDRUS-1D software application (Šimůnek et al. 2011). The PS and RCH comparison was actually non-homogeneous, since the two procedures require a different number of parameters, differ in the temporal variation of the volumetric water content profile, and in the water redistribution dynamic within the unsaturated zone, which was

507 assumed to be instantaneous in the PS. However, the comparison revealed interesting results, as will
508 be shown.

509 The RCH was applied by using the Brooks and Corey hydraulic conductivity function, which
510 was already considered in the PS. Furthermore, the corresponding water retention curve, which was
511 ignored in PS, was considered:

512

$$513 \quad \Theta = \begin{cases} |\alpha h|^{-\lambda} & h < -\frac{1}{\alpha} \\ 1 & h \geq -\frac{1}{\alpha} \end{cases} \quad (33)$$

514

515 where h is the matric water potential and α is the reciprocal of the air-entry pressure head (m^{-1}).

516 For a set of soil hydrological characteristics, corresponding to a highly permeable porous
517 media, reported in **Table 6** (run #1), the RCH application provided the water content profiles plotted
518 in **Figure 10a**, for different rainfall durations, starting from $t = 0$, where an initial uniform water
519 content profile was assumed ($\theta_0 = 0.1$) until the end of the rainfall ($t_r = 10$ h). As can be observed in
520 **Figure 10a**, for the time period $t \leq 4$ h, the rainfall volume is stored in the soil layer $Z = 1$ m depth,
521 whereas for $t > 4$ h, an infiltration process at the bottom of the soil layer (for the set free drainage
522 boundary conditions) occurred.

523 **Figure 11** illustrates the corresponding cumulative infiltration volume storage into the soil
524 layer, V_{st} , and the infiltration volume at the bottom of the soil layer, I , which made it possible to check
525 that their temporal variation was consistence with the water content profiles described in **Figure 10a**.
526 Moreover, the water balance was also checked, giving a rainfall volume, V_r (RCH), which was equal
527 to $I + V_{st}$ (**Figure 11**).

528 Under gravity-driven infiltration, the cumulative infiltration volume, I (mm), was obtained by
529 applying Eq. (29a), and considering that the latter referred to the dimensionless variables, $f^* = f/K_s$
530 and $\tau, = t/t_c$, thus:

531

532
$$I = I_* K_s t_c = I_* K_s \left(\frac{Z \theta_s}{K_s} \right) = Z \theta_s I_* \quad (34)$$

533

534 where Z equals 1,000 mm to obtain volumes that are also in (mm). The volume storage into the soil
535 layer during rainfall was obtained by calculating the volumetric water content with Eq. (25) and then
536 the corresponding integral storage volume, V_{st} :

537

538
$$V_{st} = Z(\theta - \theta_0) = Z \theta_s (\Theta - \Theta_0) \quad (35)$$

539

540 The uniform water content profiles corresponding to the storage volume, under instantaneous
541 soil moisture distribution throughout the tank, so that the newly infiltrated water is immediately
542 available for efflux, are illustrated in **Figure 10b**. In **Figure 11**, the cumulative infiltration volume,
543 I , and the cumulative storage volume, V_{st} , are compared with those corresponding to the Richards
544 equation application. The water balance was also checked in this case (PS), and yielded V_r (PS) = I
545 + V_{st} (**Figure 11**).

546 Interestingly, results showed the influence of assuming the matric potential gradient to be
547 negligible with respect to the gravitational potential gradient in the infiltration process dynamic. In
548 particular, when gravity-driven infiltration is assumed, infiltration at the bottom of the soil layer
549 increases more quickly than in the RCH and the storage volume increase more slowly, in agreement
550 with what expected.

551 **Table 6** also reports the time to equilibrium, $t_{eq} = \tau_{eq} t_c$, calculated for $\rho < 1$ by Eq. (32a),
552 which needed to achieve $f^* = 0.95 \rho$ (and $\Theta = \Theta_{eq}$, see **Table 6**) and was equal to 15.85 h.

553 Similar considerations can be done for **Figure 12** and **Figure 13**, where for another set of soil
554 hydrological characteristics **Table 6** (run #2), the RCH and PS water content profiles (**Figure 12a**
555 and **Figure 12b**, respectively) and the corresponding cumulative volumes (**Figure 13**) are plotted.

556 For this case (run #2), rainfall duration was shorter than that in run #1 ($t_r = 8$ h, **Table 6**),
557 whereas soil hydrological parameters were similarly set. The tortuosity parameter and the saturated
558 hydraulic conductivity were set lower and higher, respectively, which resulted in a quicker response
559 of the infiltration process than run #1, for both RCH and PS, which is reasonable. Of course, for this
560 case, the time to equilibrium was lower than that for run #1 (**Table 6**).

561 Although the suggested solution is applied under the restricted conditions of assuming that
562 infiltration is gravity-driven and that the soil moisture is distributed evenly and instantaneously
563 throughout the tank, it also sheds light on an important issue for the general RCH model, since it
564 detects the RCH operation domain, delimiting the boundary outside which the RCH model cannot
565 find solutions.

566 In **Figure 13**, the light and dark gray areas indicate the Richards solutions domains, for the
567 storage and the infiltration volumes, respectively. In fact, the lower boundary of the light gray area
568 shows the minimum storage volume, whereas the top boundary of the dark gray area corresponds to
569 the maximum storage volume.

570 Following this simple to apply approach, may even open doors and lead to deriving RCH
571 solutions under the assumption, opposite to that considered in the present study, that the gravitational
572 gradient is negligible compared to the matric potential gradient, further delimiting the Richards
573 solution domain described here (**Figure 13**). In fact, it is expected that under the latter condition, new
574 upper and lower bounds for the storage and infiltrating volume, should emerge.

575

576 **5. Conclusions**

577

578 The most important points covered in this paper are summarized as follows:

- 579 1. An analytical solution of the Richards equation under gravity-driven infiltration and constant
580 rainfall intensity is suggested.

2. The approach followed is a hydraulic approach that is first introduced under Torricelli's law, which mimics the soil hydraulic conductivity function and which could be used to describe the emptying or filling process of a non-linear water reservoir.
3. The same approach is applied to the Brooks and Corey hydraulic conductivity function, which has been recognized as describing the soil hydraulic conductivity function well, especially under gravity-driven infiltration.
4. A relationship linking the saturated hydraulic conductivity to the hole radius characterizing the outlet of the non-linear reservoir is presented.
5. Time to equilibrium for rainfall intensity both exceeding and non-exceeding the saturated hydraulic conductivity is derived, which shows how important the roles are of the antecedent soil moisture condition and of the rainfall intensity.
6. The reliability of the derived solutions is demonstrated by the positive verification of the water volume balance, according to a number of applications under both Torricelli's law and Brooks and Corey's hydraulic conductivity functions.
7. A comparison was performed and discussed between the present solution and the more accurate, but numerically demanding, solution derived by the Richards equation where the gravity-driven assumption is considered. Interestingly, the comparison showed that the suggested solutions delimit the solutions domain of the Richards equation.
8. Finally, the fundamental limitation of this capacitance approach should be noted, which relates to its inability to replicate time delays between input fluxes to the soil system and the generation of output fluxes.

Further research on the issues explored in this paper is suggested:

1. The water uptake by plant roots was not considered in the suggested procedure, however this simple approach could also be extended to address this issue, since it could make it possible to study the role of soil's physical quality in storing and providing water to roots for both forest and agricultural soils.

2. RCH solutions under the opposite assumption, to that considered in the present study, i.e. that the gravitational gradient is negligible compared to the matric potential gradient, could further delimit the Richards solution domain described here. In fact, it is expected that under the latter condition, new upper and lower bounds for the storage and infiltration volume, should emerge.
3. Although the suggested procedure is not affected by uncertainties, experimental verification would be advisable in order to investigate the suitability of the gravity-driven assumption for selected soils.

Data Availability Section

All data, models, and code generated or used during the study appear in the published article.

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799

800 **Notation**

801

802 *The following symbols are used in this paper:*

803

804 A = base area of the tank $[L^2]$

805 A_0 = hole cross-sectional area $[L^2]$

806 c = pore connectivity index

807 c_c = contraction coefficient

808	f	=	<i>soil infiltration rate</i>	$[LT^{-1}]$
809	f_0	=	<i>infiltration capacity at the time t_0</i>	$[LT^{-1}]$
810	f_r	=	<i>infiltration capacity at the end of the rainfall</i>	$[LT^{-1}]$
811	f^*	=	<i>soil infiltration rate normalized with respect to K_s</i>	
812	f^*_{*r}	=	<i>normalized infiltration capacity at the time τ_r</i>	
813	f^*_{*0}	=	<i>normalized infiltration capacity at the antecedent condition</i>	
814	g	=	<i>acceleration due to gravity</i>	$[LT^{-2}]$
815	h	=	<i>soil matric potential</i>	$[L]$
816	i	=	<i>rainfall intensity</i>	$[LT^{-1}]$
817	I	=	<i>infiltration volume</i>	$[L]$
818	I^*	=	<i>infiltration volume normalized with respect to the volume $K_s t_c$</i>	
819	I^*_{fall}	=	<i>falling limb infiltration volume normalized with respect to the volume $K_s t_c$</i>	
820	K	=	<i>soil hydraulic conductivity</i>	$[L T^{-1}]$
821	K_s	=	<i>saturated hydraulic conductivity</i>	$[L T^{-1}]$
822	l	=	<i>tortuosity factor</i>	
823	Q	=	<i>outflow discharge by the hole at the bottom of the tank</i>	$[L^3 T^{-1}]$
824	r^*	=	<i>ratio between the circular hole radius r and that of the water tank, R</i>	
825	r	=	<i>hole radius at the bottom of the tank</i>	$[L]$
826	R	=	<i>base radius of the tank</i>	$[L]$
827	t	=	<i>time</i>	$[T]$
828	t_0	=	<i>time at the antecedent condition</i>	$[T]$
829	t_c	=	<i>temporal scale of the infiltration process</i>	$[T]$
830	t_{eq}	=	<i>time to equilibrium</i>	$[T]$

831	t_r	=	<i>duration of rainfall</i>	[T]
832	V_{st}	=	<i>storage water volume into the soil layer</i>	[L]
833	V_r	=	<i>rainfall volume at the end of rainfall with duration t_r</i>	[L]
834	V_{*r}	=	$(\rho \tau_r)$ <i>rainfall volume at the end of rainfall normalized with respect to the volume $K_s t_c$</i>	
835	z	=	<i>elevation</i>	[L]
836	Z	=	<i>depth of the soil layer</i>	[L]
837	α	=	<i>reciprocal of the air-entry pressure head</i>	[L ⁻¹]
838	β	=	<i>correction factor of the asymptotic equilibrium condition ($f^* = \beta \rho$, for $\rho < 1$)</i>	
839	δ_i	=	<i>infiltration coefficient</i>	
840	δ_r	=	<i>infiltration coefficient at the end of the rainfall</i>	
841	$\Delta\theta$	=	<i>effective porosity</i>	[LT ⁻¹]
842	λ	=	<i>pore size distribution index</i>	
843	θ	=	<i>volumetric water content</i>	[L ³ L ⁻³]
844	θ_r	=	<i>volumetric water content at the end of the rainfall</i>	[L ³ L ⁻³]
845	θ_{res}	=	<i>residual volumetric water content</i>	[L ³ L ⁻³]
846	θ_s	=	<i>saturated volumetric water content</i>	[L ³ L ⁻³]
847	θ_0	=	<i>antecedent volumetric water content</i>	[L ³ L ⁻³]
848	Θ	=	<i>normalized volumetric water content</i>	
849	Θ_0	=	<i>antecedent normalized volumetric water content</i>	
850	Θ_{eq}	=	<i>normalized volumetric water content at the equilibrium condition</i>	
851	Θ_r	=	<i>normalized volumetric water content at the end of the rainfall</i>	
852	Θ_∞	=	<i>asymptotic normalized volumetric water content</i>	

- 853
- ρ
- =
- rainfall intensity normalized with respect to K_s*
- 854
- τ
- =
- time normalized with respect to t_c*
- 855
- τ_0
- =
- normalized time at the antecedent condition*
- 856
- τ_{eq}
- =
- time to equilibrium normalized with respect to t_c*
- 857
- τ_r
- =
- duration of rainfall normalized with respect to t_c*
- 858
- τ_{zf}
- =
- time normalized with respect to t_c for the zero-flux condition*
- 859
-
- 860

861 Table 1 – Rainfall and soils parameters considered in the applications illustrated in Figure 4

#	c	i (mm/h)	K_s (mm/h)	K_s (m/s)	ρ	r^*	Θ_0	f^*_0	τ_r	Θ_r	f^*_r	δ_r	τ_{zf}
1		30	25	6.944E-06	1.2	1.252E-03	0.2	0.447	0.763	0.620	0.787	0.541	2.338
2	2	30	25	6.944E-06	1.2	1.252E-03	0	0.000	0.987	0.620	0.787	0.477	2.562
3		10	11.1	3.086E-06	0.9	8.347E-04	0	0.000	0.919	0.400	0.632	0.516	2.184

862
863
864 Table 2 – Saturated hydraulic conductivity values, pore size distribution index, λ , and pore
865 connectivity index, c .

Textural class	K_s (m/s)	K_s (mm/h)	λ	c
Sand	4.20E-05	151.13	0.546	0.150
Loamy Sand	1.10E-04	395.49	0.449	0.134
Sandy Loam	2.63E-05	94.64	0.378	0.121
Loam	9.05E-06	32.59	0.246	0.090
Silt Loam	2.28E-06	8.21	0.207	0.079
Sand Clay Loam	6.86E-06	24.68	0.345	0.114
Clay Loam	5.33E-06	19.18	0.259	0.093
Silty Clay Loam	1.57E-06	5.65	0.164	0.066
Silty Clay	3.27E-06	11.79	0.186	0.073
Clay	1.65E-06	5.93	0.187	0.073

866 Source: Data from Brakensiek et al. (1981).

867
868 Table 3 – Rainfall and soils parameters considered in the applications illustrated in Figure 6

#	c	i (mm/h)	K_s (mm/h)	K_s (m/s)	ρ	r^*	Θ_0	f^*_0	τ_r	Θ_r	f^*_r	δ_r
1		60	30.0	8.33E-06	2	4.38E-08	0.2	6.15E-06	0.442	1	1	0.095
2	0.134	60	30.0	8.33E-06	2	4.38E-08	0	0	0.542	1	1	0.077
3		10	16.7	4.63E-06	0.6	3.27E-08	0	0	0.834	0.500	5.70E-03	0.001
4		30	25.0	6.94E-06	1.2	4.00E-08	0	0	0.501	0.600	2.22E-02	0.002

869
870
871 Table 4 – Rainfall and soils parameters considered in the applications illustrated in Figure 7

#	c	i (mm/h)	K_s (mm/h)	K_s (m/s)	ρ	r^*	Θ_0	f^*_0	τ_r	Θ_r	f^*_r	δ_r
1	0.060	20	6.7	1.852E-06	3	2.298E-14	0	0	0.341	1	1	0.023
2	0.080	20	6.7	1.852E-06	3	1.134E-11	0	0	0.343	1	1	0.029
3	0.100	20	6.7	1.852E-06	3	4.681E-10	0	0	0.343	1	1	0.036
4	0.134	20	6.7	1.852E-06	3	2.043E-08	0	0	0.350	1	1	0.046
5	0.150	20	6.7	1.852E-06	3	6.680E-08	0	0	0.351	1	1	0.051
6	0.200	20	6.7	1.852E-06	3	7.981E-07	0	0	0.356	1	1	0.064

872

873

874 Table 5 – Rainfall and soils parameters considered in the applications illustrated in Figure 8

#	c	i (mm/h)	K_s (mm/h)	K_s (m/s)	ρ	r^*	Θ_0	f^*_0	τ_r	Θ_r	f^*_r	δ_r
1	0.060	20	33.3	9.259E-06	0.6	5.138E-14	0	0	1.586	0.921	0.255	0.031
2	0.080	20	33.3	9.259E-06	0.6	2.535E-11	0	0	1.605	0.912	0.316	0.053
3	0.100	20	33.3	9.259E-06	0.6	1.047E-09	0	0	1.633	0.903	0.359	0.078
4	0.134	20	33.3	9.259E-06	0.6	4.569E-08	0	0	1.690	0.887	0.409	0.124
5	0.150	20	33.3	9.259E-06	0.6	1.494E-07	0	0	1.713	0.880	0.426	0.144
6	0.200	20	33.3	9.259E-06	0.6	1.785E-06	0	0	1.806	0.858	0.464	0.208

875

876

877 Table 6 – Rainfall data and soil hydrological parameters considered for the
878 comparison between the present solution (PS) and that derived by the Richards
879 equation (RCH).

Parameters	Unit	Run #	
		1	2
Rainfall intensity and duration, and normalized rainfall	i (mm/h)	10	10
	t_r (h)	10	8
	$\rho = i / K_s$	0.005	0.00333
Volumetric water contents ($\text{cm}^3 \text{cm}^{-3}$) and normalized water contents	θ_0	0.1	0.1
	θ_{res}	0	0
	θ_s	0.437	0.437
	Θ_0	0.229	0.229
	Θ_{res}	0	0
	Θ_s	1	1
	$\Theta_{\text{eq}} = (0.95 r)^c$	0.403	0.343
Parameters of the Brooks and Corey hydraulic conductivity function	λ	0.592	0.592
	l	0.5	0
	$2/\lambda + l + 2$	5.878	5.378
	$1/c$	5.878	5.378
	c	0.17	0.186
	$Ks \text{ (mm/h)} \Theta$	2000	3000
Reciprocal of the air-entry pressure head, α (cm^{-1})		0.138	0.138
Time scale of the infiltration process, t_c (h) (Eq. 21b)		0.2185	0.1457
Time to equilibrium t_{eq} (h) (Eq. 32a)		15.85	12.49

880

881

882 **Figure captions**

883

884 Figure 1 – Hydraulic sketch reproducing the infiltration process under gravity-driven infiltration

885

886 Figure 2 – Temporal variation of the normalized water content, for different ρ values. Time to
887 equilibrium, τ_{eq} , for $\rho > 1$, and for $\rho \leq 1$, and for the latter case, the corresponding asymptotic Θ
888 value, Θ_∞ , are also illustrated.

889

890 Figure 3 – Time to equilibrium, τ_{eq} , a) for $\rho \leq 1$ and b) for $\rho > 1$, versus the normalized rainfall
891 intensity, for different antecedent moisture conditions, Θ_θ .

892

893 Figure 4 – Temporal variation a) of the normalized water content, Θ , of the normalized infiltration
894 rate, f^* , and c) of the infiltration coefficient, δ_i , for the application parameters reported in Table 1.
895 The characteristic times, τ_θ , τ_r , τ_{zf} , and the corresponding normalized water content and infiltration
896 coefficient are also plotted.

897

898 Figure 5 – a) Relationship between r^* and K_s for different c values (Table 2). For the same c values,
899 b) illustrates the corresponding hydraulic conductivity function according to Brooks and Corey. b)
900 also plots the Torricelli's law for $c = 2$ (dashed line).

901

902 Figure 6 – Temporal variation a) of the normalized water content, Θ , of the normalized infiltration
903 rate, f^* , and c) of the infiltration coefficient, δ_i , for the application parameters reported in Table 3. The
904 characteristic times, τ_θ , τ_r , and the corresponding normalized water content and infiltration coefficient
905 are also plotted.

906

907

908 Figure 7 – Temporal variation a) of the normalized water content, Θ , of the normalized infiltration
909 rate, f^* , and c) of the infiltration coefficient, δ_i , for the application parameters reported in Table 4. The
910 characteristic times, τ_θ , τ_r , and the corresponding normalized water content and infiltration coefficient
911 are also plotted.

912

913 Figure 8 – Temporal variation a) of the normalized water content, Θ , of the normalized infiltration
914 rate, f^* , and c) of the infiltration coefficient, δ_i , for the application parameters reported in Table 5. The
915 characteristic times, τ_θ , τ_r , and the corresponding normalized water content and infiltration coefficient
916 are also plotted.

917

918 Figure 9 – For different c values, time to equilibrium, τ_{eq} , versus the normalized rainfall intensity, ρ ,
919 and versus the antecedent water content, Θ_θ , a) for $\rho \leq 1$ and b) for $\rho > 1$.

920

921 Figure 10 – Water content profiles corresponding to the set of parameters reported in Table 6 (run
922 #1), a) for the Richards equation and b) for the present solution.

923

924 Figure 11 – For the set of parameters reported in Table 6 (run #1), cumulative rainfall volume, V_r ,
925 cumulative storage volume, V_{st} , and infiltration volume at the bottom of the soil layer, V_i .

926

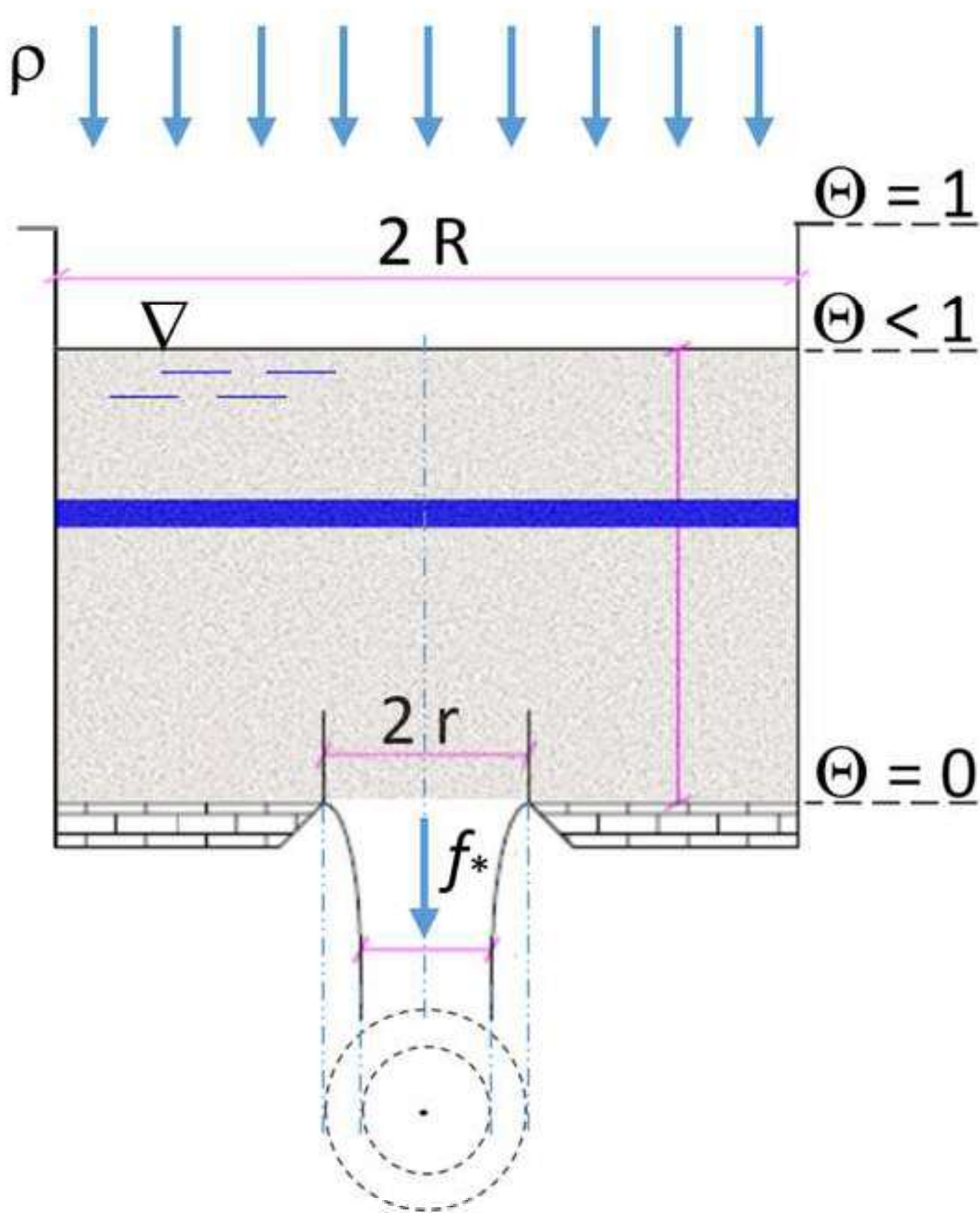
927 Figure 12 – Water content profiles corresponding to the set of parameters reported in Table 6 (run
928 #2), a) for the Richards equation and b) for the present solution.

929

930 Figure 13 – For the set of parameters reported in Table 6 (run #2), cumulative rainfall volume, V_r ,
931 cumulative storage volume, V_{st} , and infiltration volume at the bottom of the soil layer, V_i . Light and

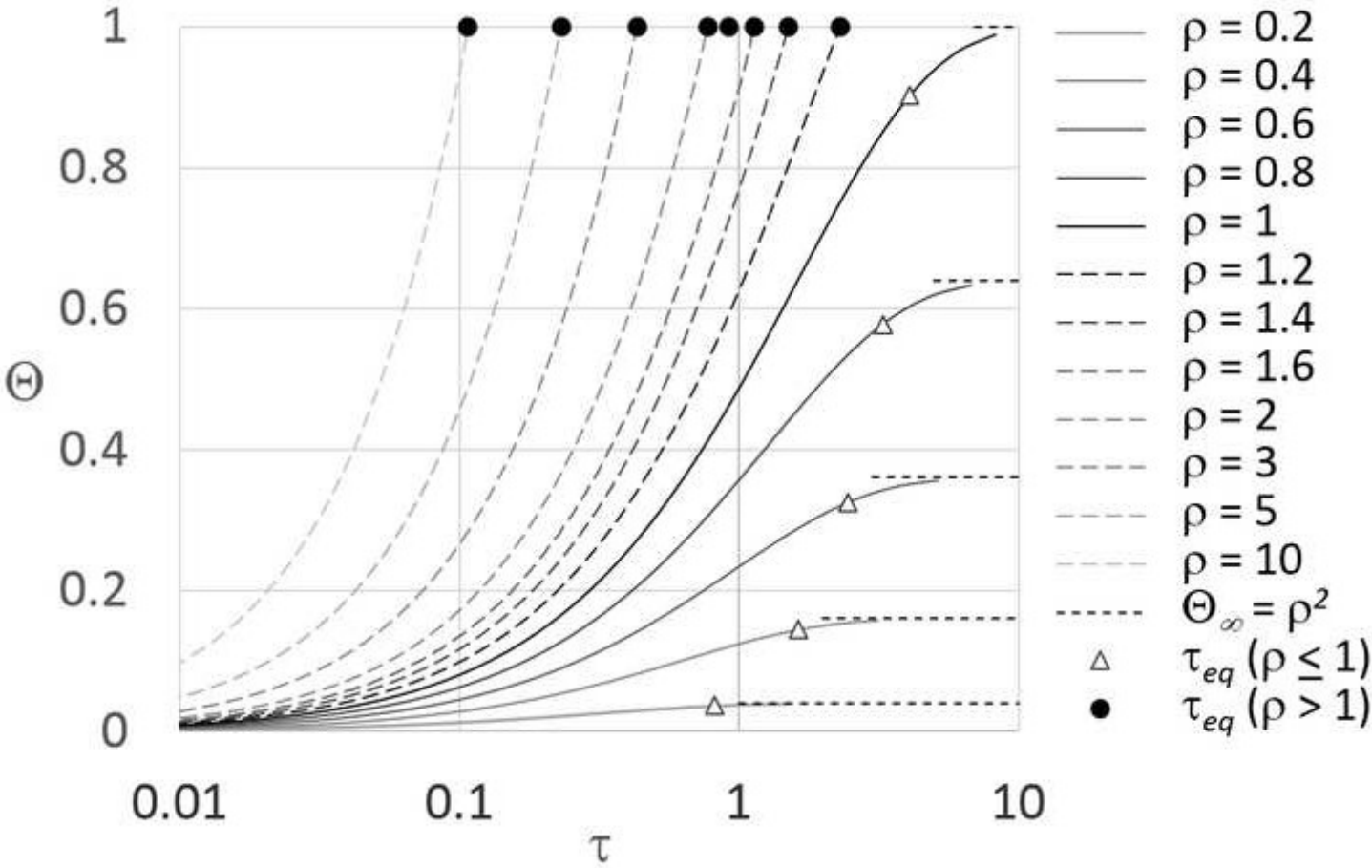
932 dark gray areas indicate the Richards solutions domain of the storage and infiltration volumes,
933 respectively.

Figure_1



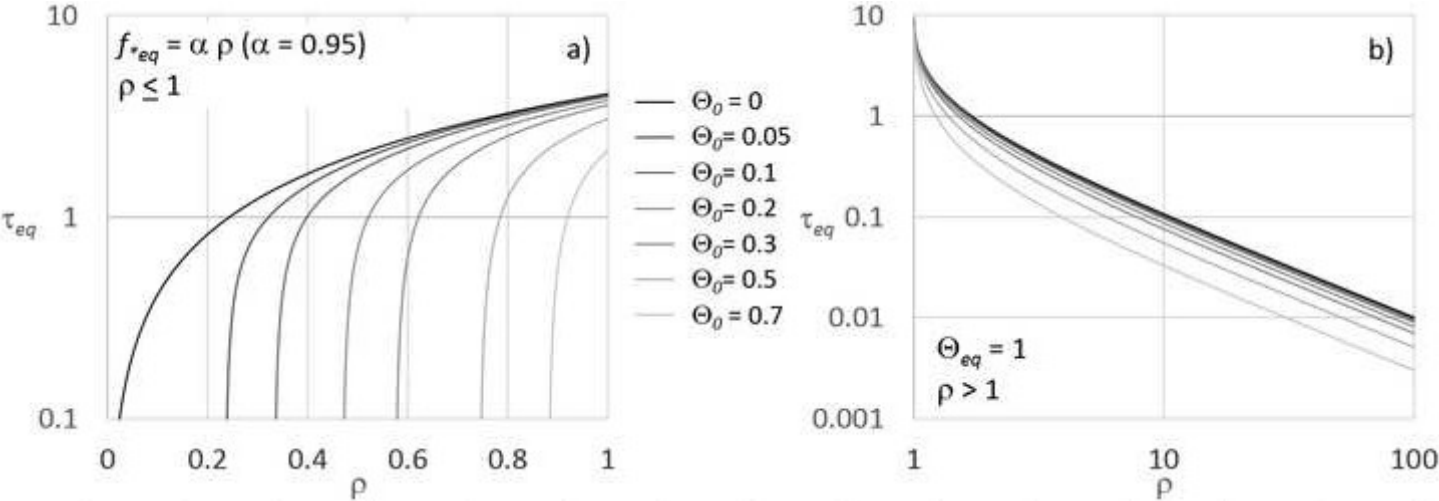
Figure_2

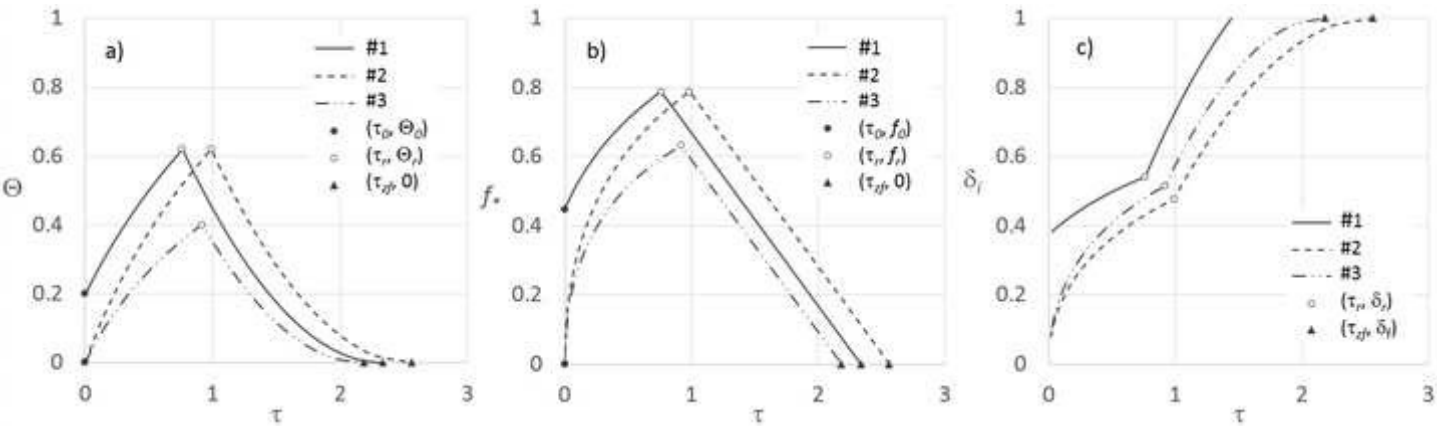
[Click here to access/download;Figure;Figure_2.tif](#)



Figure_3ab

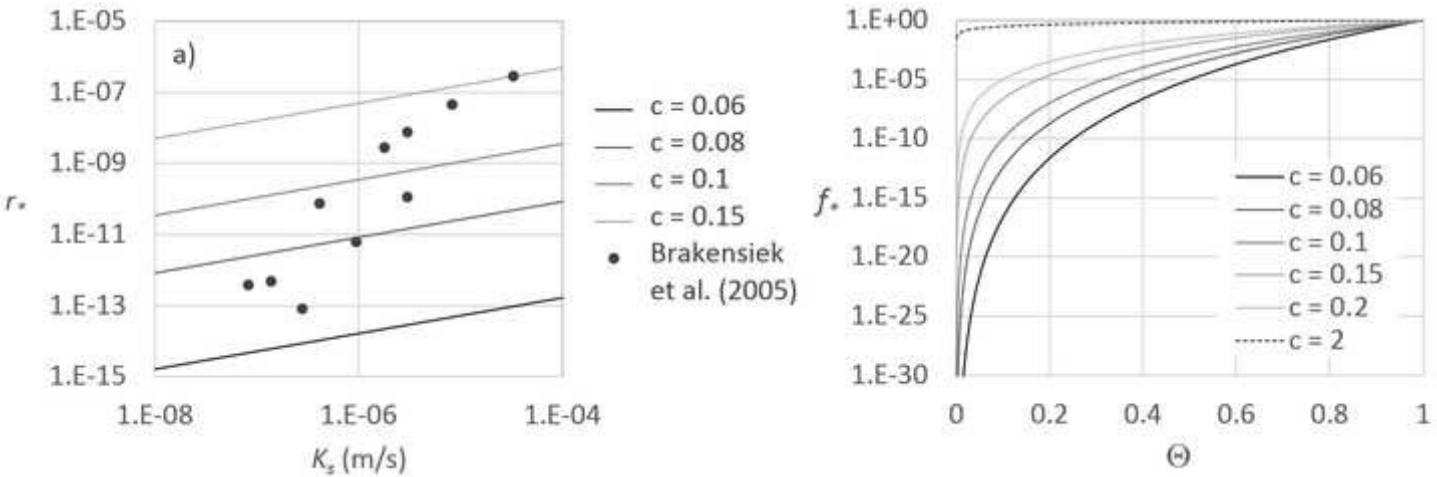
[Click here to access/download;Figure;Figure_3ab.tif](#)





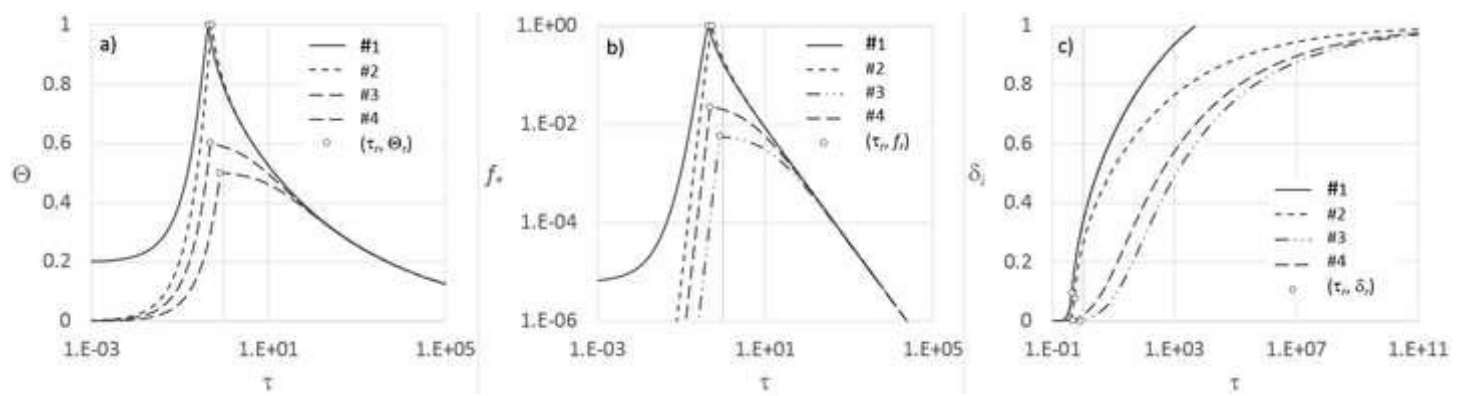
Figure_5ab

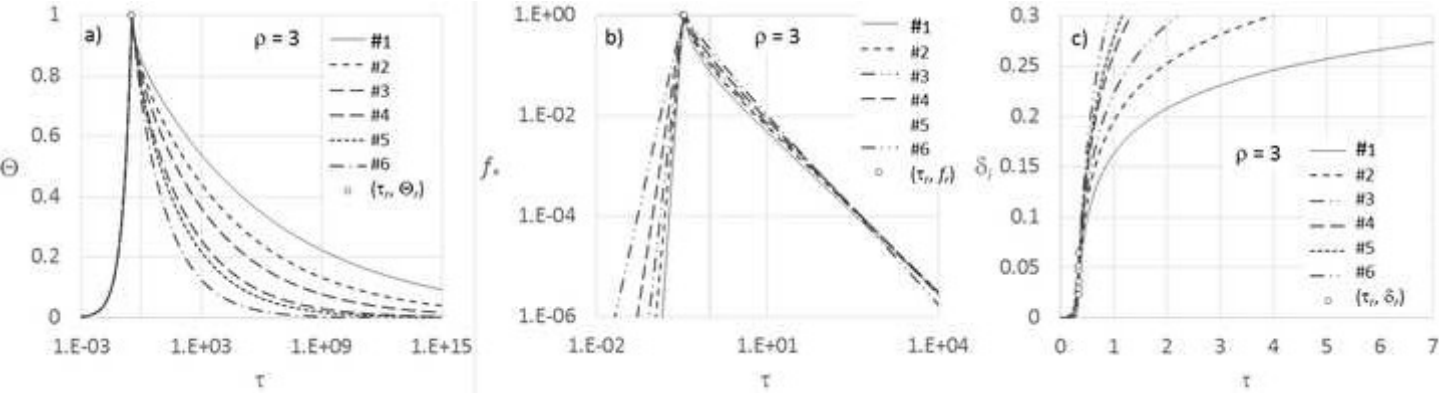
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Figure_6abc

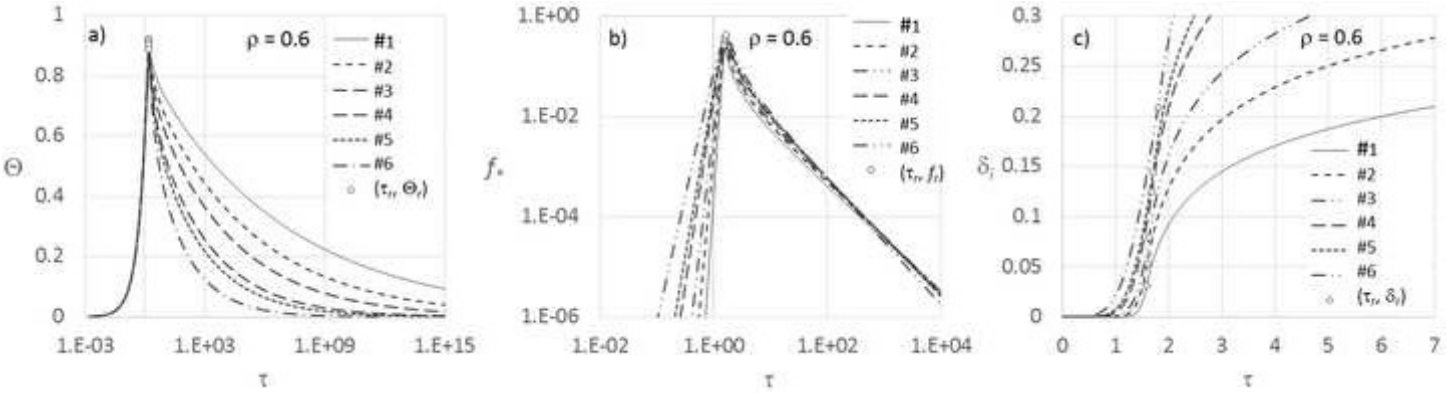
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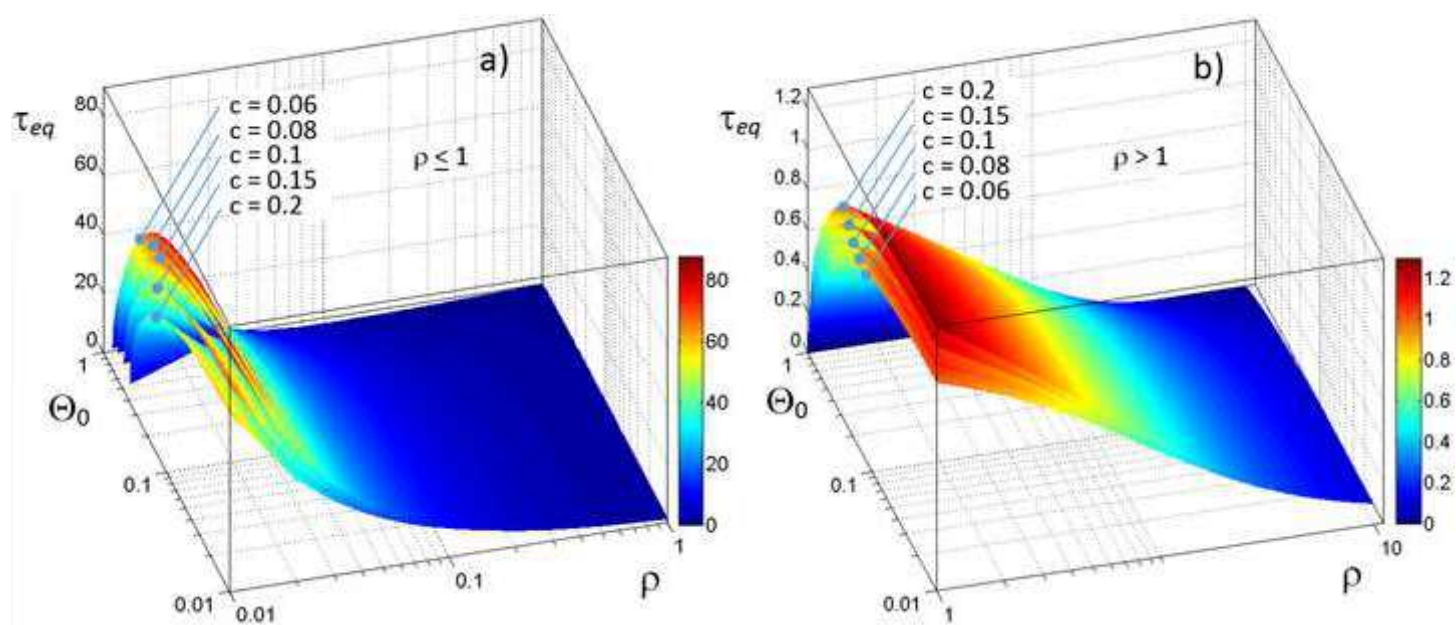
Figure_8abc

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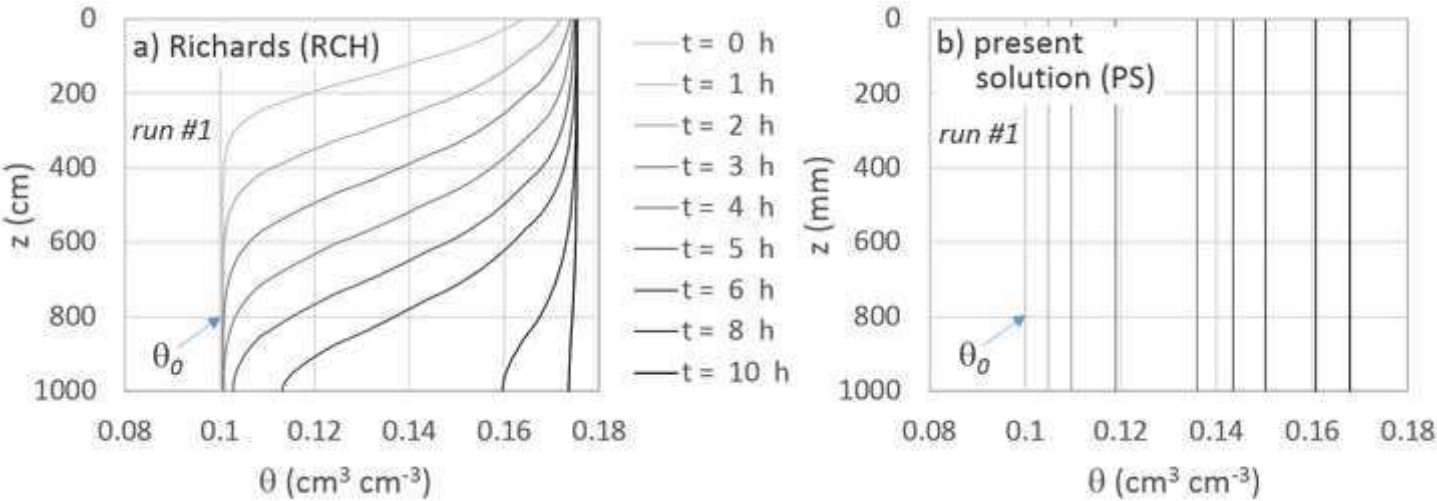
Figure_9ab

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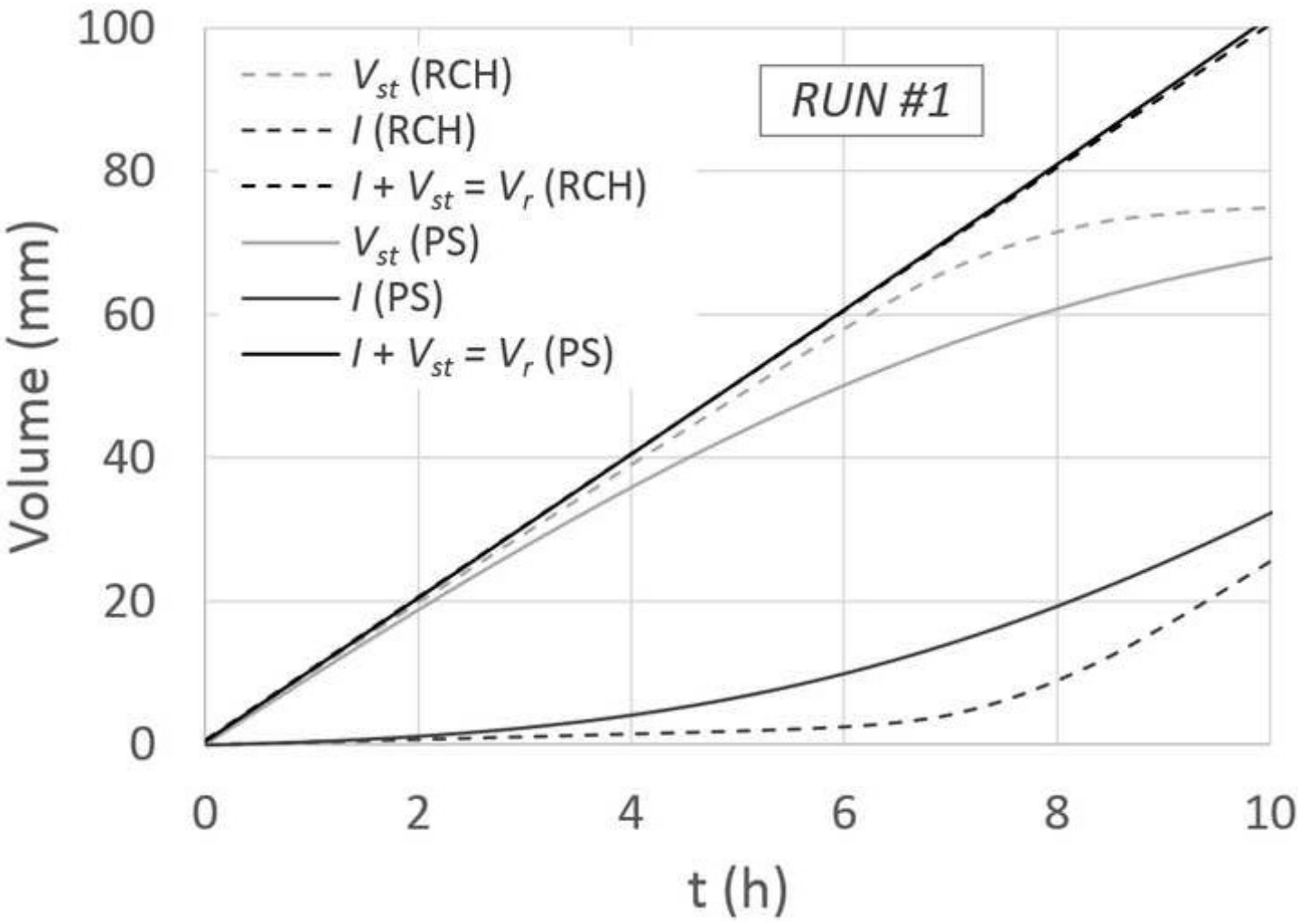
Figure_10ab

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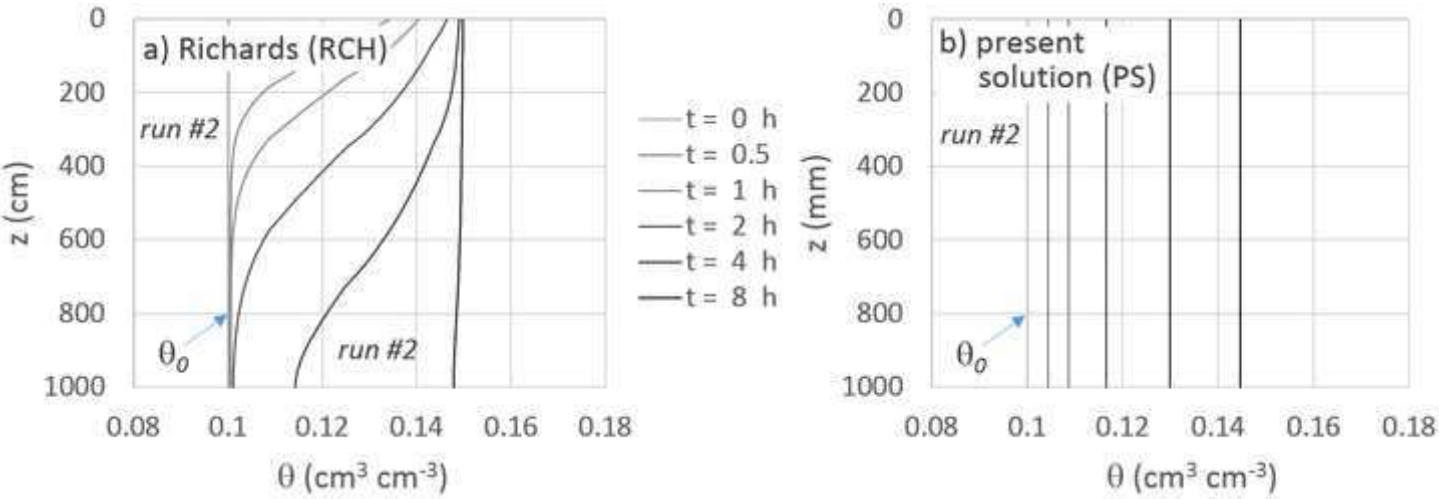
Figure_11

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Figure_12ab

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Figure_13

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