

Some notes about distribution frame multipliers

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Abstract Inspired by a recent work about distribution frames, the definition of multiplier operator is extended in the rigged Hilbert spaces setting and a study of its main properties is carried on. In particular, conditions for the density of domain and boundedness are given. The case of Riesz distribution bases is examined in order to develop a symbolic calculus.

Key words: distributions; rigged Hilbert spaces; frames; multipliers

1 Introduction

Bessel multipliers were introduced by Balazs in [8] and they became objects of several works [9, 40, 41, 42, 43, 44, 45]. Multipliers have been studied also in particular cases ([15, 21, 30]) and found applications in physics, signal processing, acoustics and mathematics. To define them we need to recall some notions (see [16, 31]).

Let \mathcal{H} be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$. A sequence $\phi = \{\phi_n\}_{n \in \mathbb{N}}$ is a *Bessel sequence* of \mathcal{H} with upper bound $B > 0$ if

$$\sum_{n \in \mathbb{N}} |\langle f | \phi_n \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

A sequence $\phi = \{\phi_n\}_{n \in \mathbb{N}}$ is a (*discrete*) *frame* if there exist $A, B > 0$ such that

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$$A\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f | \phi_n \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Let $\phi = \{\phi_n\}_{n \in \mathbb{N}}, \psi = \{\psi_n\}_{n \in \mathbb{N}}$ be two sequences of \mathcal{H} and $m : \mathbb{N} \rightarrow \mathbb{C}$. The operator $M_{m, \phi, \psi}$ defined by

$$M_{m, \phi, \psi} f = \sum_{n \in \mathbb{N}} m_n \langle f | \psi_n \rangle \phi_n,$$

and with domain the subspace of $f \in \mathcal{H}$ such that $\sum_{n \in \mathbb{N}} m_n \langle f | \psi_n \rangle \phi_n$ is convergent, is called the *multiplier* of ϕ, ψ with symbol m . When ϕ, ψ are Bessel (frame) sequences and m is a bounded sequence, then $M_{m, \phi, \psi}$ is defined on \mathcal{H} , bounded, and it is called a *Bessel (frame) multiplier*.

Independently in [1] and in [33], the notion of continuous frame was introduced as generalization of discrete frame and later in [10] the correspondent notion of Bessel continuous (frame) multiplier was formulated, which we now recall. The setting involves a measure space (X, μ) with positive measure μ . A map $F : x \in X \rightarrow F_x \in \mathcal{H}$ is called a *continuous frame* with respect to (X, μ) if

1. F is weakly measurable, i.e. $f \mapsto \langle f | F_x \rangle$ is μ -measurable for every $f \in \mathcal{H}$;
2. there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \int_X |\langle f | F_x \rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1)$$

A weakly measurable map $F : x \in X \rightarrow F_x \in \mathcal{H}$ is called a *Bessel continuous map* with respect to (X, μ) if the second inequality in (1) holds. Let $F, G : X \rightarrow \mathcal{H}$ be Bessel continuous maps and $m \in L^\infty(X, \mu)$. An operator $M_{m, F, G}$ can be weakly defined by

$$\langle M_{m, F, G} f | g \rangle = \int_X m(x) \langle f | F_x \rangle \langle G_x | g \rangle d\mu, \quad f, g \in \mathcal{H}.$$

This operator is called the *Bessel continuous multiplier* of F, G with symbol m (and *continuous frame multiplier* if F, G are in addition continuous frames). Among continuous frame multipliers one can find time-frequency localization operators [19, 20] (also called short-time Fourier transform multipliers) and Calderón-Toeplitz operators [36, 37].

Recently, a notion of frame (and related topics such as bases, Bessel maps, Riesz bases and Riesz-Fischer maps) in space of distributions is appeared in [46, 48], involving a *rigged Hilbert space*, or *Gel'fand triplet*, i.e. a triple $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$, where $\mathcal{D}[t]$ is a dense subspace of \mathcal{H} endowed with a locally convex topology t , stronger than the one induced by the Hilbert norm and $\mathcal{D}^\times[t^\times]$ is the conjugate dual of $\mathcal{D}[t]$ with the strong dual topology t^\times . If $\mathcal{D}[t]$ is reflexive, then the inclusions are dense and continuous. Analogous concepts in rigged Hilbert spaces and for the discrete case have been considered also in [14]. The aim of this paper is then to give a correspondent notion of multipliers of distribution maps. A preliminary, but very confined, study about distribution multipliers actually was given in [46].

In contrast with a large part of the current literature, we will not pay attention to bounded multipliers only. To give an example of the importance of unbounded multipliers, we mention [4, 5] where they were used as tools to define non-selfadjoint hamiltonians. Sufficient conditions for operators to be written as multipliers with a fixed sequence (or map) have been given in [12, 13, 25].

The paper is organized as follows. We start by recalling some preliminaries in Section 2. Then, in Section 3, we give the definitions of distribution multipliers. They can actually be formulated in two different ways, i.e. as operators from \mathcal{D} to \mathcal{D}^\times or as operators on \mathcal{H} . Questions about density of domain and closedness of unbounded multipliers are discussed in Section 4, while Riesz distribution multipliers are studied and some results about symbolic calculus is obtained in Section 5.

2 Preliminary definitions and facts

Throughout the paper \mathcal{H} indicates a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$. We denote by $D(T)$ and $R(T)$ the domain and the range of an operator $T : D(T) \subset \mathcal{H} \rightarrow \mathcal{H}$. If T is densely defined, then we write T^* for its adjoint.

A sequence $\phi = \{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is called *total* if $\langle f | \phi_n \rangle = 0$ for every $n \in \mathbb{N}$ implies that $f = 0$. In particular, discrete frames are total sequences. A *Riesz basis* ϕ is a total sequence satisfying for some $A, B > 0$

$$A \sum_{n \in \mathbb{N}} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{N}} c_n \phi_n \right\|^2 \leq B \sum_{n \in \mathbb{N}} |c_n|^2, \quad \forall \{c_n\} \in \ell_2(\mathbb{N}),$$

where $\ell_2(\mathbb{N})$ is the usual space of square integrable complex sequences.

Let $\mathcal{D}[t]$ be a dense subspace of \mathcal{H} endowed with a locally convex topology t , stronger than the topology induced by the Hilbert norm. The vector space of all continuous conjugate linear functionals on $\mathcal{D}[t]$ (the conjugate dual of $\mathcal{D}[t]$) is denoted by $\mathcal{D}^\times[t^\times]$, and is endowed with the *strong dual topology* t^\times , defined by the seminorms

$$q_{\mathcal{M}}(F) = \sup_{g \in \mathcal{M}} |\langle F | g \rangle|, \quad F \in \mathcal{D}^\times,$$

where \mathcal{M} is a bounded subsets of $\mathcal{D}[t]$. With a well-known identification procedure (see [32]), \mathcal{H} is considered as subspace of $\mathcal{D}^\times[t^\times]$. The triplet

$$\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times],$$

is called *rigged Hilbert space* or *Gel'fand triplet* [26, 27]. If $\mathcal{D}[t]$ is reflexive \mathcal{H} is continuously and densely embedded in $\mathcal{D}^\times[t^\times]$. Denoting by \hookrightarrow the continuous and dense embedding, the triple is also denoted by

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times].$$

In this way, the sesquilinear form $B(\cdot, \cdot)$ which puts \mathcal{D} and \mathcal{D}^\times in duality extends the inner product of \mathcal{H} ; i.e. $B(\xi, \eta) = \langle \xi | \eta \rangle$, for every $\xi, \eta \in \mathcal{D}$: we adopt the symbol $\langle \cdot | \cdot \rangle$ for both of them.

Let us denote by $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$ ([2]). If $\mathcal{D}[t]$ is reflexive, it is possible introduce an involution $X \mapsto X^\dagger$ in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ by the identity:

$$\langle X^\dagger \eta | \xi \rangle = \overline{\langle X \xi | \eta \rangle}, \quad \forall \xi, \eta \in \mathcal{D}.$$

Hence, in this case, $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is a † -invariant vector space.

If $\mathcal{D}[t]$ is a smooth space (e.g., Fréchet and reflexive), then $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is a quasi * -algebra over $\mathcal{L}^\dagger(\mathcal{D})$ (Definition 2.1.9 of [2]).

We also denote by $\mathcal{L}(\mathcal{D})$ the algebra of all continuous linear operators $Y : \mathcal{D}[t] \rightarrow \mathcal{D}[t]$ and by $\mathcal{L}(\mathcal{D}^\times)$ the algebra of all continuous linear operators $Z : \mathcal{D}^\times[t^\times] \rightarrow \mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is reflexive, for every $Y \in \mathcal{L}(\mathcal{D})$ there exists a unique operator $Y^\times \in \mathcal{L}(\mathcal{D}^\times)$, the *adjoint* of Y , such that

$$\langle F | Y g \rangle = \langle Y^\times F | g \rangle, \quad \forall F \in \mathcal{D}^\times, g \in \mathcal{D}.$$

In similar way an operator $Z \in \mathcal{L}(\mathcal{D}^\times)$ has an adjoint $Z^\times \in \mathcal{L}(\mathcal{D})$ such that $(Z^\times)^\times = Z$. We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the algebra of all closable operators A in \mathcal{H} such that $D(A) = \mathcal{D}$, $D(A^*) \supseteq \mathcal{D}$, and A, A^* leave \mathcal{D} invariant. With the involution $A \mapsto A^* \upharpoonright_{\mathcal{D}} = A^\dagger$, $\mathcal{L}^\dagger(\mathcal{D})$ is a * -algebra.

In this paper (X, μ) denotes a measure space with a σ -finite positive measure μ . We recall that a measurable set $A \subseteq X$ is called an *atom* if $\mu(A) > 0$ and for every $B \subseteq A$ we have either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure space (X, μ) is called *atomic* if there exists a partition $\{A_n\}_{n \in \mathbb{N}}$ of X consisting of atoms and sets of measure zero. We write $L^1(X, \mu)$, $L^2(X, \mu)$ and $L^\infty(X, \mu)$ for the usual spaces of (classes of) measurable functions. Moreover, $\|m\|_\infty$ denotes the essential supremum of $m \in L^\infty(X, \mu)$. For simplicity, we write $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ when we assume the Lebesgue measure. The Fourier transform of $f \in L^1(\mathbb{R})$ is defined as $\hat{f}(\gamma) = \int_{\mathbb{R}} f(x) e^{-2\pi i \gamma x} dx$ and it extends to a unitary operator of $L^2(\mathbb{R})$ in a standard way. In this paper we consider weakly measurable maps: given a measure space (X, μ) with μ a σ -finite positive measure, $\omega : x \in X \rightarrow \omega_x \in \mathcal{D}^\times$ is a *weakly measurable map* if, for every $f \in \mathcal{D}$, the complex valued function $x \mapsto \langle f | \omega_x \rangle$ is μ -measurable. If not otherwise specified, throughout the paper we will work with a fixed rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ with $\mathcal{D}[t]$ reflexive and a measure space (X, μ) as described before. We start by recalling simple definitions about weakly measurable maps. Since the form which puts \mathcal{D} and \mathcal{D}^\times in conjugate duality is an extension of the inner product of \mathcal{H} , we write $\langle f | \omega_x \rangle$ for $\overline{\langle \omega_x | f \rangle}$, $f \in \mathcal{D}$.

Definition 1 ([46, Definition 2.2]) Let $\omega : x \in X \rightarrow \omega_x \in \mathcal{D}^\times$ be a weakly measurable map, then:

1. ω is *total* if, $f \in \mathcal{D}$ and $\langle f | \omega_x \rangle = 0$ μ -a.e. $x \in X$ implies $f = 0$;
2. ω is μ -*independent* if the unique μ -measurable function $\xi : X \rightarrow \mathbb{C}$ such that: $\int_X \xi(x) \langle g | \omega_x \rangle d\mu = 0$, for every $g \in \mathcal{D}$, is $\xi(x) = 0$ μ -a.e.

Definition 2 ([46, Definition 3.2]) A weakly measurable map ω is a *Bessel distribution map* (briefly: Bessel map) if for every $f \in \mathcal{D}$, $\int_X |\langle f | \omega_x \rangle|^2 d\mu < \infty$.

It is convenient to consider $\mathcal{D}[t]$ as a Fréchet space because of the following proposition.

Proposition 1 ([46, Proposition 3.1]) Let $\mathcal{D}[t]$ be a Fréchet space and $\omega : x \in X \rightarrow \omega_x \in \mathcal{D}^\times$ a weakly measurable map. The following statements are equivalent.

1. ω is a Bessel map.
2. There exists a continuous seminorm p on $\mathcal{D}[t]$ such that

$$\left(\int_X |\langle f | \omega_x \rangle|^2 d\mu \right)^{1/2} \leq p(f), \quad \forall f \in \mathcal{D}.$$

(iii) For every bounded subset \mathcal{M} of \mathcal{D} there exists $C_{\mathcal{M}} > 0$ such that

$$\sup_{f \in \mathcal{M}} \left| \int_X \xi(x) \langle \omega_x | f \rangle d\mu \right| \leq C_{\mathcal{M}} \|\xi\|_2, \quad \forall \xi \in L^2(X, \mu).$$

As a consequence of the previous proposition, we have [46]:

- the conjugate linear functional on \mathcal{D} :

$$\Lambda_\omega^\xi := \int_X \xi(x) \omega_x d\mu$$

is defined in weak sense, and is continuous, i.e. $\Lambda_\omega^\xi \in \mathcal{D}^\times[t^\times]$;

- the *synthesis operator* $T_\omega : L^2(X, \mu) \rightarrow \mathcal{D}^\times[t^\times]$ defined by $T_\omega : \xi \mapsto \Lambda_\omega^\xi$ is continuous;
- the *analysis operator* $T_\omega^\times : \mathcal{D}[t] \rightarrow L^2(X, \mu)$ defined by $(T_\omega^\times f)(x) = \langle f | \omega_x \rangle$ is continuous;
- the *frame operator* $S_\omega : \mathcal{D}[t] \rightarrow \mathcal{D}^\times[t^\times]$, $S_\omega := T_\omega T_\omega^\times$ is continuous, i.e. $S_\omega \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$.

We will often work with a special class of Bessel maps which is defined as follows.

Definition 3 ([46, Definition 3.2]) A Bessel distribution map ω is called *bounded Bessel map* if there exists $B > 0$ such that

$$\int_X |\langle \omega_x | f \rangle|^2 d\mu \leq B \|f\|^2, \quad \forall f \in \mathcal{D}.$$

If ω is a bounded Bessel map, with a limit procedure, we have [46]:

- Λ_ω^ξ is bounded in $\mathcal{D}(\|\cdot\|)$, then it has a bounded extension $\tilde{\Lambda}_\omega^\xi$ to \mathcal{H} ;
- the synthesis operator T_ω has range in \mathcal{H} , it is bounded and $\|T_\omega\| \leq \sqrt{B}$;

- the (Hilbert adjoint) operator $T_\omega^* : \mathcal{H} \rightarrow L^2(X, \mu)$ extends T_ω^\times and $T_\omega^* f = \eta$, where η is the limit in $L^2(X, \mu)$ of the functions $T_\omega^\times f_n : x \mapsto \langle f_n | \omega_x \rangle$, where $\{f_n\}$ is a sequence in \mathcal{D} converging to f (we will also denote the function $T_\omega^* f$ by $x \mapsto \langle f | \tilde{\omega}_x \rangle$ for $f \in \mathcal{H}$, i.e. we consider $\tilde{\omega}_x$ as an ‘extension’ of the linear functional ω_x);
- the operator $\hat{S}_\omega = T_\omega T_\omega^*$ is bounded and it is an extension of S_ω .

Now it is time to recall a notion of frames in the distribution context (Definition 3.6 of [46]).

Definition 4 ([46, Definition 3.6]) Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, with $\mathcal{D}[t]$ a reflexive space and ω a weakly measurable map. We say that ω is a *distribution frame* if there exist $A, B > 0$ (called *frame bounds*) such that

$$A\|f\|^2 \leq \int_X |\langle f | \omega_x \rangle|^2 d\mu \leq B\|f\|^2, \quad \forall f \in \mathcal{D}.$$

Moreover, we say that

- ω is a *tight* distribution frame if we can choose $A = B$ as frame bounds of ω ;
- ω is a *Parseval* distribution frame if $A = B = 1$ are frame bounds of ω .

If ω is a distribution frame, then the frame operator \hat{S}_ω satisfies the inequalities

$$A\|f\| \leq \|\hat{S}_\omega f\| \leq B\|f\|, \quad \forall f \in \mathcal{H}.$$

Since \hat{S}_ω is symmetric, this implies that \hat{S}_ω has a bounded inverse \hat{S}_ω^{-1} everywhere defined in \mathcal{H} . A Parseval distribution frame satisfies Definition 4 with $S_\omega = I_{\mathcal{D}}$, the identity operator of \mathcal{D} , and $\hat{S}_\omega = I_{\mathcal{H}}$, the identity operator of \mathcal{H} .

Example 1 ([46, Example 3.18]) Let us consider the rigged Hilbert space

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing C^∞ -functions on \mathbb{R} and the conjugate dual $\mathcal{S}^\times(\mathbb{R})$ is the space of tempered distributions. Let δ be the weakly measurable map $\delta : x \in \mathbb{R} \rightarrow \delta_x \in \mathcal{S}^\times(\mathbb{R})$, where δ_x stands for the δ distribution centered at x . As known, δ_x acts in the following way $\langle \delta_x | \phi \rangle = \overline{\phi(x)}$, for every $\phi \in \mathcal{S}(\mathbb{R})$. Then one trivially has

$$\int_{\mathbb{R}} |\langle \phi | \delta_x \rangle|^2 dx = \int_{\mathbb{R}} |\phi(x)|^2 dx = \|\phi\|^2, \quad \forall \phi \in \mathcal{S}(\mathbb{R}),$$

where $\|\phi\|^2$ is the norm in $L^2(\mathbb{R})$; hence, δ is a Parseval frame.

We note that this example is based on the measure space $(X, \mu) = (\mathbb{R}, \lambda)$, where λ is the Lebesgue measure on \mathbb{R} , and of course it is not a continuous frame. On the contrary, the definition of distribution frame reduces to that of discrete frame when (X, μ) is the set \mathbb{N} with the counting measure γ . Indeed we have the following.

Proposition 2 Let $\omega : n \in \mathbb{N} \rightarrow \omega_n \in \mathcal{D}^\times$ be a bounded Bessel distribution (resp., distribution frame) on $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. Then $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ and $\{\omega_n\}_{n \in \mathbb{N}}$ is a Bessel sequence (resp., frame) of \mathcal{H} .

Proof Assume that $\{\omega_n\}_{n \in \mathbb{N}}$ is a bounded Bessel distribution map and fix $m \in \mathbb{N}$. The linear functional $f \mapsto \langle f | \omega_m \rangle$ for $f \in \mathcal{D}$ is bounded with respect to the norm of \mathcal{H} , because

$$|\langle f | \omega_m \rangle|^2 \leq \sum_{n \in \mathbb{N}} |\langle f | \omega_n \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{D}. \quad (2)$$

This means that $\omega_m \in \mathcal{H}$. A standard argument ([16, Lemma 5.1.9]) shows that (2) extends for each $f \in \mathcal{H}$, i.e. $\{\omega_n\}_{n \in \mathbb{N}}$ is a Bessel sequence.

If $\{\omega_n\}_{n \in \mathbb{N}}$ is a distribution frame, then the conclusion follows in a similar way. \square

Definition 5 Let ω, θ be distribution frames. We say that θ is a dual frame of ω if

$$\langle f | g \rangle = \int_X \langle f | \theta_x \rangle \langle \omega_x | g \rangle d\mu, \quad \forall f, g \in \mathcal{D}.$$

To formulate the following result, we recall that there exists a unique operator $R_\omega \in \mathcal{L}(\mathcal{D})$ such that $S_\omega R_\omega f = f$ for every $f \in \mathcal{D}$ ([46, Lemma 3.8]).

Proposition 3 ([46, Proposition 3.10]) Let ω be a distribution frame with frame bounds A and B . Then the map $\theta : X \rightarrow \mathcal{D}^\times$ defined by $\theta_x := R_\omega^\times \omega_x$ for $x \in X$, is a distribution frame with bounds B^{-1} and A^{-1} and it is a dual frame of ω .

The map θ in Proposition 3 is called the *canonical dual frame* of ω .

Definition 6 ([46, Definition 2.3]) Let $\mathcal{D}[t]$ be a locally convex space, \mathcal{D}^\times its conjugate dual and $\omega : x \in X \rightarrow \omega_x \in \mathcal{D}^\times$ a weakly measurable map. Then ω is a *distribution basis* for \mathcal{D} if, for every $f \in \mathcal{D}$, there exists a *unique* μ -measurable function ξ_f such that:

$$\langle f | g \rangle = \int_X \xi_f(x) \langle \omega_x | g \rangle d\mu, \quad \forall f, g \in \mathcal{D}$$

and, for every $x \in X$, the linear functional $f \in \mathcal{D} \rightarrow \xi_f(x) \in \mathbb{C}$ is continuous in $\mathcal{D}[t]$.

Given a distribution basis ω we can simply write in weak sense

$$f = \int_X \xi_f(x) \omega_x d\mu, \quad \forall f \in \mathcal{D}.$$

Moreover, ω is μ -independent. Since $f \in \mathcal{D} \rightarrow \xi_f(x)$ continuously, there exists a unique weakly μ -measurable map $\theta : X \rightarrow \mathcal{D}^\times$ such that: $\xi_f(x) = \langle f | \theta_x \rangle$ for every $f \in \mathcal{D}$. We call θ *dual map* of ω . If θ is μ -independent, then it is a distribution basis too.

The next two notions are the counterparts of orthonormal and Riesz bases of the discrete context, which are particular cases of Definition 6.

Definition 7 ([46]) A weakly measurable map $\omega : X \rightarrow \mathcal{D}^\times$ is a *Riesz distribution basis* if it is a μ -independent distribution frame.

A weakly measurable map $\zeta : X \rightarrow \mathcal{D}^\times$ is *Gel'fand distribution basis* if it is a μ -independent Parseval distribution frame.

In a way similar to the discrete case, we can give some equivalent conditions for a Bessel distribution map to be a Riesz distribution basis.

Proposition 4 ([46, Proposition 3.19]) Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and let $\omega : x \in X \rightarrow \omega_x \in \mathcal{D}^\times$ be a Bessel distribution map. Then the following statements are equivalent.

1. ω is a Riesz distribution basis;
2. if ζ is a Gel'fand distribution basis, then the operator W defined, for $f \in \mathcal{H}$, by

$$f = \int_X \xi_f(x) \zeta_x d\mu \mapsto Wf = \int_X \xi_f(x) \omega_x d\mu$$

is continuous and has bounded inverse;

3. the synthesis operator T_ω is a topological isomorphism of $L^2(X, \mu)$ onto \mathcal{H} ;
4. ω is total and there exist $A, B > 0$ such that

$$A\|\xi\|_2^2 \leq \left\| \int_X \xi(x) \omega_x d\mu \right\|^2 \leq B\|\xi\|_2^2, \quad \forall \xi \in L^2(X, \mu).$$

If ω is a Riesz distribution basis with frame bounds A, B , then ω possesses a unique dual frame θ (so the canonical dual frame) which is also a Riesz distribution basis with frame bounds B^{-1} and A^{-1} (see [46, Proposition 3.20]). In particular, a Gel'fand distribution basis ζ coincides with its dual basis.

Example 2 ([46, Example 3.18]) Let us come back to Example 1. The distribution frame $\delta : x \in \mathbb{R} \mapsto \delta_x \in \mathcal{S}^\times(\mathbb{R})$ is clearly λ -independent, then δ is a Gel'fand distribution frame.

In Proposition 2 we made a consideration about the case $(X, \mu) = (\mathbb{N}, \gamma)$ and discrete frames. Now we compare distribution frames with continuous frames. There is indeed a remarkable difference about the possibility to define ‘Riesz maps’. To explain the difference in details, we recall that a family $\{F_x\}_{x \in X} \subset \mathcal{H}$ is a Riesz continuous map if one of the following statements holds (see [3, 24])

1. $\{F_x\}_{x \in X}$ is a continuous frame and the operator $C_F : \mathcal{H} \rightarrow L^2(X, \mu)$ defined by $(C_F f)(x) = \langle f | F_x \rangle$ is surjective;
2. $\{F_x\}_{x \in X}$ is a continuous frame and μ -linearly independent, i.e. if $c : X \rightarrow \mathbb{C}$ and $\int_X c(x) \langle f | F_x \rangle d\mu(x) = 0$ for all $f \in \mathcal{H}$, then $c(x) = 0$ μ -almost everywhere.

It was proven in [35, Corollary 4.3] and in [39, Theorem 9], that Riesz continuous maps can be defined only if the space (X, μ) is atomic. In contrast, Riesz distribution

maps can be defined with non-atomic measure spaces (X, μ) (as we have seen in Example 1). Note that the formulations of Riesz continuous map and Riesz distribution map are totally analogue.

3 Distribution multipliers

Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space. In the distribution context, there is more than one way to define multipliers. The first way we describe consists of operators acting on $\mathcal{D}[t]$ with values in $\mathcal{D}^\times[t^\times]$.

We suppose that $\mathcal{D}[t]$ is a reflexive Fréchet space and (X, μ) is a measure space with μ a σ -finite positive measure. Let $\omega, \theta : X \rightarrow \mathcal{D}^\times$ be two weakly measurable Bessel maps and $m \in L^\infty(X, \mu)$. Then the sesquilinear form

$$\Omega_{m, \omega, \theta}(f, g) := \int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu$$

is defined for all $f, g \in \mathcal{D}$. By Proposition 1(ii) we have

$$|\Omega_{m, \omega, \theta}(f, g)| \leq \|m\|_\infty \|\langle f | \omega_x \rangle\|_2 \|\langle \theta_x | g \rangle\|_2 \leq \|m\|_\infty p(f)p(g)$$

for all $f, g \in \mathcal{D}$. This means that $\Omega_{m, \omega, \theta}$ is jointly continuous on $\mathcal{D}[t]$ and then there exists an operator $\mathcal{M}_{m, \omega, \theta} \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$, such that

$$\langle \mathcal{M}_{m, \omega, \theta} f | g \rangle = \Omega_{m, \omega, \theta}(f, g), \quad \forall f, g \in \mathcal{D}.$$

For brevity we write

$$\mathcal{M}_{m, \omega, \theta} f = \int_X m(x) \langle f | \omega_x \rangle \theta_x d\mu, \quad \forall f \in \mathcal{D}$$

and we call $\mathcal{M}_{m, \omega, \theta}$ the *outer* distribution multiplier of ω and θ with symbol m .

As in [8], we have, if T_ω^\times and T_θ are the analysis and synthesis operators of ω and θ , respectively, $D_m : L^2(X, \mu) \rightarrow L^2(X, \mu)$ is the multiplication by m defined by $D_m f(x) := m(x)f(x)$, then $\mathcal{M}_{m, \omega, \theta} = T_\theta D_m T_\omega^\times$. Moreover, $\mathcal{M}_{m, \omega, \theta}^\dagger = \mathcal{M}_{\bar{m}, \theta, \omega}$.

Now we move to the second way to define multipliers, namely operators acting on \mathcal{H} . Again we consider a rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$, a measure space (X, μ) with μ a σ -finite positive measure, but the choice of ω, θ and m is more general. Indeed, let ω, θ be two weakly measurable maps and $m : X \rightarrow \mathbb{C}$ a μ -measurable function. Let us define the subspace $D(\mathcal{M}_{m, \omega, \theta})$ of $f \in \mathcal{D}$ such that the integral

$$\int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu$$

is convergent for all $g \in \mathcal{D}$ and the linear functional

$$g \mapsto \int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu \quad (3)$$

is bounded with respect to the norm of \mathcal{H} . An operator $M_{m,\omega,\theta} : D(M_{m,\omega,\theta}) \rightarrow \mathcal{H}$ can be defined as follows: for every $f \in D(M_{m,\omega,\theta})$, $M_{m,\omega,\theta}f$ is the unique element in \mathcal{H} associated to the functional (3) by the Riesz lemma, i.e.

$$\langle M_{m,\omega,\theta}f | g \rangle = \int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu, \quad \forall f \in D(M_{m,\omega,\theta}), g \in \mathcal{D}.$$

For shortness, we write $M_{m,\omega,\theta}f = \int_X m(x) \langle f | \omega_x \rangle \theta_x$ and call $M_{m,\omega,\theta}$ the *distribution multiplier* of ω and θ with *symbol* m . If ω and θ are Bessel distribution maps (resp. Gel'fand bases, Riesz distribution bases, distribution frames), then $M_{m,\omega,\theta}$ is called a *Bessel distribution* (resp. *Gel'fand distribution*, *Riesz distribution*, *distribution frame*) multiplier.

In the language of representation of sesquilinear forms [34, 38], we can say that $M_{m,\omega,\theta}$ is the operator associated to $\Omega_{m,\omega,\theta}$. In the discrete setting, operators associated to sesquilinear forms induced by sequences have been studied in [5, 17, 18].

We first pay attention to distribution multipliers defined (resp. bounded) on \mathcal{D} .

Proposition 5 *Let $m \in L^\infty(X, \mu)$.*

1. *If ω is a Bessel distribution map and θ is a bounded Bessel distribution map, then $M_{m,\omega,\theta}$ is a well-defined operator $M_{m,\omega,\theta} : \mathcal{D} \rightarrow \mathcal{H}$.*
2. *If ω and θ are bounded Bessel distribution maps with bound B_ω, B_θ , respectively, then $M_{m,\omega,\theta}$ is bounded and it extends to a bounded operator $\hat{M}_{m,\omega,\theta}$ on \mathcal{H} with norm $\|\hat{M}_{m,\omega,\theta}\| \leq \sqrt{B_\omega B_\theta} \|m\|_\infty$.*
3. *If ω, θ are bounded Bessel maps, then $\hat{M}_{m,\omega,\theta} = T_\theta D_m T_\omega^*$ and $M_{m,\omega,\theta}^* = \hat{M}_{\bar{m},\omega,\theta}$.*

The proof is an adaptation of [10, Lemma 3.3] and it is omitted. However, we want to make the following remark about Proposition 5.

Remark 1 Let $m \in L^\infty(X, \mu)$ and ω, θ Bessel distribution maps, then $M_{m,\omega,\theta}$ is not necessarily bounded in the norm of \mathcal{H} . Indeed let us consider $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$ and $\omega : \mathbb{R} \rightarrow \mathcal{S}^\times(\mathbb{R})$ defined by $\omega_x = x\delta_x$, i.e. the distributions $\langle f | x\delta_x \rangle = xf(x)$ for $f \in \mathcal{S}(\mathbb{R})$. Then ω is a Bessel distribution map since $\int_{\mathbb{R}} |\langle f | x\delta_x \rangle|^2 dx = \int_{\mathbb{R}} |xf(x)|^2 dx$ is finite for all $f \in \mathcal{S}(\mathbb{R})$. Let θ be the distribution frame given by $\theta_x = \delta_x$ and $m(x) = 1$ for $x \in \mathbb{R}$, then $M_{m,\omega,\theta}$ is defined on $f \in \mathcal{S}(\mathbb{R})$ and $(M_{m,\omega,\theta}f)(x) = xf(x)$ for $x \in \mathbb{R}$. Clearly, $M_{m,\omega,\theta}$ is not bounded.

Multipliers with a bounded inverse defined on the whole space have a special interest, since they lead to reconstruction formulas, as shown in the discrete case in [9, 44, 43]. In the following result we show how reconstruction formulas can be found by a distribution multiplier having a right or left inverse.

Theorem 1 Let $\omega, \theta : X \rightarrow \mathcal{D}^\times$ be weakly measurable maps and $m : X \rightarrow \mathbb{C}$ such that the distribution multiplier $M_{m,\omega,\theta}$ is defined on \mathcal{D} .

1. If there exists $J \in \mathcal{L}(\mathcal{D})$ such that $M_{m,\omega,\theta}Jf = f$ for every $f \in \mathcal{D}$, then the weakly measurable map $\rho : X \rightarrow \mathcal{H}$ defined by $\rho_x = J^\dagger(\overline{m(x)}\omega_x)$ satisfies

$$\langle f|g \rangle = \int_X \langle f|\rho_x \rangle \langle \theta_x|g \rangle d\mu, \quad \forall f, g \in \mathcal{D}.$$

2. If there exists $K \in \mathcal{L}(\mathcal{D}^\times)$ such that $KM_{m,\omega,\theta}f = f$ for every $f \in \mathcal{D}$ the weakly measurable map $\tau : X \rightarrow \mathcal{H}$ defined by $\tau_x = K(m(x)\theta_x)$ satisfies

$$\langle f|g \rangle = \int_X \langle f|\omega_x \rangle \langle \tau_x|g \rangle d\mu, \quad \forall f, g \in \mathcal{D}.$$

Proof (1) Let $J \in \mathcal{L}(\mathcal{D})$ as in the statement. Then

$$\langle f|g \rangle = \langle M_{m,\omega,\theta}Jf|g \rangle = \int_X m(x) \langle Jf|\omega_x \rangle \langle \theta_x|g \rangle d\mu = \int_X \langle f|J^\dagger(\overline{m(x)}\omega_x) \rangle \langle \theta_x|g \rangle d\mu,$$

for all $f, g \in \mathcal{D}$. (2) Let $K \in \mathcal{L}(\mathcal{D}^\times)$ as in the statement. Then

$$\begin{aligned} \langle f|g \rangle &= \langle KM_{m,\omega,\theta}f|g \rangle = \langle M_{m,\omega,\theta}f|K^\dagger g \rangle = \int_X m(x) \langle f|\omega_x \rangle \langle \theta_x|K^\dagger g \rangle d\mu \\ &= \int_X \langle f|\omega_x \rangle \langle K(m(x)\theta_x)|g \rangle d\mu, \end{aligned}$$

for all $f, g \in \mathcal{D}$. □

Example 3 Let us consider again the rigged Hilbert space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$ and $\omega = \theta$ the distribution frames defined by $\omega_x = \theta_x = \delta_x$ for every $x \in \mathbb{R}$ and $m \in C^\infty(\mathbb{R})$ a function such that $0 < \inf_{x \in \mathbb{R}} |m(x)| \leq \sup_{x \in \mathbb{R}} |m(x)| < \infty$. The multiplier $M_{m,\omega,\theta}$ is of course defined on $\mathcal{S}(\mathbb{R})$ and $M_{m,\omega,\theta}f = mf$. Clearly, the operator $J : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ defined by $Jf = m^{-1}f$ belongs to $\mathcal{L}(\mathcal{S}(\mathbb{R}))$ and the operator $K : \mathcal{S}^\times(\mathbb{R}) \rightarrow \mathcal{S}^\times(\mathbb{R})$ defined by $KF = m^{-1}F$ belongs to $\mathcal{L}(\mathcal{S}^\times(\mathbb{R}))$. Moreover, $KM_{m,\omega,\theta}f = M_{m,\omega,\theta}Jf = f$ for every $f \in \mathcal{S}(\mathbb{R})$. The reconstruction formulas in Theorem 1 hold in particular with $\rho_x = \tau_x = \delta_x$.

In Section 4 we will give conditions for a Riesz multiplier to be invertible with bounded inverse.

4 Unbounded distribution multipliers

In Proposition 5 we gave a condition for a distribution multiplier to be bounded. Not only bounded multipliers are interesting, of course; we refer to [4, 5] where unbounded discrete multipliers have been studied in the context of non-selfadjoint

hamiltonians. So in this section we want to analyze some aspects of unbounded distribution multipliers.

Assuming that $\mathcal{D}[t]$ is a reflexive Fréchet space, ω, θ are Bessel distribution maps and $m \in L^\infty(X, \mu)$, we easily see that $M_{m,\omega,\theta}$ is actually a restriction of $\mathcal{M}_{m,\omega,\theta}$. Indeed, $D(M_{m,\omega,\theta}) = \{f \in \mathcal{D} : \mathcal{M}_{m,\omega,\theta} f \in \mathcal{H}\}$ and $M_{m,\omega,\theta} = \mathcal{M}_{m,\omega,\theta}|_{D(M_{m,\omega,\theta})}$. This fact leads, for instance, to the following conclusions.

Proposition 6 *Let $\mathcal{D}[t]$ be a reflexive Fréchet space. Let ω, θ be Bessel distribution maps and $m \in L^\infty(X, \mu)$. If $\mathcal{M}_{m,\omega,\theta} : \mathcal{D} \rightarrow \mathcal{D}^\times$ is bijective with a bounded inverse, then*

1. $M_{m,\omega,\theta} : D(M_{m,\omega,\theta}) \rightarrow \mathcal{H}$ is bijective, densely defined and has a bounded inverse (consequently $M_{m,\omega,\theta}$ is closed);
2. $M_{m,\omega,\theta}^* = M_{\bar{m},\theta,\omega}$.

Proof 1. That $M_{m,\omega,\theta} : D(M_{m,\omega,\theta}) \rightarrow \mathcal{H}$ is bijective follows easily since $M_{m,\omega,\theta}$ is a restriction of $\mathcal{M}_{m,\omega,\theta}$. Since the inclusions $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ are continuous

- there exists a continuous seminorm p on $\mathcal{D}[t]$ and $\|f\| \leq p(f)$ for all $f \in \mathcal{D}$;
- for all continuous seminorms q on $\mathcal{D}^\times[t^\times]$ there exists $\alpha_q > 0$ such that $q(f) \leq \alpha_q \|f\|$ for all $f \in \mathcal{D}$.

By hypothesis $\mathcal{M}_{m,\omega,\theta}^{-1} : \mathcal{D}^\times \rightarrow \mathcal{D}$ is bounded, so there exists a continuous semi-norm q on $\mathcal{D}^\times[t^\times]$ such that for all continuous semi-norms p on $\mathcal{D}[t]$ we have $p(M_{m,\omega,\theta}^{-1} F) \leq q(F)$ for all $F \in \mathcal{D}^\times[t^\times]$. Hence, for all $h \in \mathcal{H}$

$$\|M_{m,\omega,\theta}^{-1} h\| \leq p(M_{m,\omega,\theta}^{-1} h) = p(\mathcal{M}_{m,\omega,\theta}^{-1} h) \leq q(h) \leq \alpha_q \|h\|.$$

Thus $M_{m,\omega,\theta}$ has a bounded inverse. The continuity of $\mathcal{M}_{m,\omega,\theta}^{-1}$ implies also that $D(M_{m,\omega,\theta})$ is dense, because $D(M_{m,\omega,\theta})$ is the inverse image of \mathcal{H} which is dense in $\mathcal{D}^\times[t^\times]$.

2. Clearly, $M_{m,\omega,\theta}^*$ is bijective and $M_{\bar{m},\theta,\omega} \subseteq M_{m,\omega,\theta}^*$. The conclusion of point (1) holds also for $M_{\bar{m},\theta,\omega}$ since $M_{\bar{m},\theta,\omega}$ coincides with a restriction of $\mathcal{M}_{\bar{m},\theta,\omega}^\dagger = \mathcal{M}_{\bar{m},\theta,\omega}$. Thus we can conclude that $M_{\bar{m},\theta,\omega} = M_{m,\omega,\theta}^*$. \square

Even though $M_{m,\omega,\theta}$ is not necessarily bounded, Proposition 6 makes use of boundedness of m . This may be a strong hypothesis, thus we now look for less restrictive assumptions to ensure that $M_{m,\omega,\theta}$ is densely defined.

In the discrete context it is very easy to prove that a multiplier $M_{m,\phi,\psi}$ of a Hilbert space \mathcal{H} , where $\phi = \{\phi_n\}_{n \in \mathbb{N}}$ is a Riesz basis and $\psi = \{\psi_n\}_{n \in \mathbb{N}}$ a sequence of \mathcal{H} , is densely defined whatever the symbol $m = \{m_n\}_{n \in \mathbb{N}}$ is. Indeed, there exist a unique total sequence (in particular a Riesz basis) $\tilde{\phi} = \{\tilde{\phi}_n\}$ biorthogonal to ϕ , i.e. $\langle \tilde{\phi}_m | \phi_n \rangle = \delta_{m,n}$ (the Kronecker symbol) for all $m, n \in \mathbb{N}$. Thus $D(M_{m,\phi,\psi})$ is dense, because it contains $\tilde{\phi}$.

On the contrary, when $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ is a rigged Hilbert space and $\omega : X \rightarrow \mathcal{D}^\times$ is a Riesz distribution basis, then we may not find a function $\rho : X \rightarrow \mathcal{D}$ which is biorthogonal to ω in the sense that

$$\langle \rho_y | \omega_x \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Indeed, let us consider the Riesz distribution basis given by the Dirac deltas $\omega_x = \delta_x$, $x \in \mathbb{R}$, on the rigged Hilbert space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\times$ (Example 1). Then there is no $x \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$ such that $f(x) := \langle f | \delta_x \rangle = 1$ and $f(z) := \langle f | \delta_z \rangle = 0$ for all $z \neq x$. Thus we have to manage a new problem in order to study densely defined distribution multipliers.

Taking again the example of $\omega_x = \delta_x$, $x \in \mathbb{R}$, on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\times$, we note that for any symbol $m : \mathbb{R} \rightarrow \mathbb{C}$ and for $\theta = \omega$ the multiplier $M_{m,\omega,\theta}$ is densely defined. We will give the proof in Theorem 2 in a more general context. Here we confine ourselves to give the following remark. Note that we say that a subset V of a Hilbert space \mathcal{H} is *total* if $\langle f | h \rangle = 0$ for all $f \in V$ implies $h = 0$ (then the linear span of V is dense in \mathcal{H}).

Remark 2 Let λ the Lebesgue measure on \mathbb{R} . Let $\alpha : \mathbb{R} \rightarrow \mathbb{C}$ be a positive λ -measurable function and define $V_\alpha := \{f \in C_0^\infty(\mathbb{R}) : |f(x)| \leq \alpha(x), x \in \mathbb{R}\}$. We prove that the subset V_α is total in $L^2(\mathbb{R})$ dividing the proof into three steps.

First of all, α is locally bounded away from zero in a.e. $x \in \mathbb{R}$, i.e. $B^c := \mathbb{R} \setminus B$ is measurable with measure zero, where

$$B = \{x \in \mathbb{R} : \text{there exists an interval } U_x \subset \mathbb{R} \text{ of } x \text{ such that } \text{essinf}_{y \in U_x} \alpha(y) > 0\}.$$

Indeed, fix $n \in \mathbb{N}$. For every $x \in B^c$, there exists a measurable set $U_{n,x}$ containing x such that $\text{essinf}_{y \in U_{n,x}} \alpha(y) \leq \frac{1}{n}$. Then $B^c \subseteq \bigcup_{x \in B^c} U_{n,x} \subseteq \{y \in \mathbb{R} : \alpha(y) \leq \frac{1}{n}\}$, thus the outer measure $\lambda_o(B^c) \leq \lambda(\{y \in \mathbb{R} : \alpha(y) \leq \frac{1}{n}\}) \rightarrow 0$ for $n \rightarrow \infty$. Hence B^c is measurable with measure zero.

Now, if $x \in B$ and $m_x = \text{essinf}_{y \in U_x} \alpha(y) > 0$, then $m_x \chi_I \in V_\alpha$ where χ_I is the characteristic function of any interval $I \subset U_x$. Taking into account that B^c has measure zero, we conclude that there is a subset V total in $L^2(\mathbb{R})$ such that every $f \in V$ satisfies $f(x) = m \chi_I$ (for some $m > 0$ and an interval I) and $0 \leq f(x) \leq \alpha(x)$ for every $x \in \mathbb{R}$.

Finally, every function $f \in V$ can be approximated with C_0^∞ functions $\{f_k\}$ with $0 \leq f_k \leq f$. Hence, V_α is total.

The reason of talking about the example $\omega_x = \delta_x$, is that it suggests to consider variations of the condition of biorthogonality as stated in the next definitions. For a better comparison, we rewrite a property of V_α (with $\alpha : \mathbb{R} \rightarrow \mathbb{C}$ a positive measurable function) in the following way: for every $f \in V_\alpha$ there exists a bounded subset $X_f \subset \mathbb{R}$ and $|\langle f | \delta_x \rangle| \leq \alpha(x)$ for $x \in X_f$ and $\langle f | \delta_x \rangle = 0$ for $x \notin X_f$.

Definition 8 Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and $\omega : X \rightarrow \mathcal{D}^\times$ a weakly measurable function. We say that

1. ω is *pseudo-orthogonal* if there exists a subset $V \subset \mathcal{D}$ total in \mathcal{H} such that for every $f \in V$ there exists a measurable subset $X_f \subset X$ with $\mu(X_f) < \infty$, $\sup_{x \in X_f} |\langle f | \omega_x \rangle| < \infty$ for $x \in X_f$ and $\langle f | \omega_x \rangle = 0$ for $x \notin X_f$;

2. ω is *hyper-orthogonal* if for every positive measurable function $\alpha : X \rightarrow \mathbb{C}$ there exists a subset $V_\alpha \subset \mathcal{D}$ total in \mathcal{H} such that for every $f \in V_\alpha$ there exists a measurable subset $X_f \subset X$ with $\mu(X_f) < \infty$, $|\langle f | \omega_x \rangle| \leq \alpha(x)$ for $x \in X_f$ and $\langle f | \omega_x \rangle = 0$ for $x \notin X_f$.

Note that these definitions are covered by a Riesz (discrete) basis $\{\phi_n\}_{n \in \mathbb{N}}$ (more generally by a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ having a total biorthogonal sequence $\{\psi_n\}_{n \in \mathbb{N}}$, i.e. $\langle \phi_n | \psi_m \rangle = \delta_{n,m}$). Furthermore, if ω is hyper-orthogonal then it is also pseudo-orthogonal. We are now able to formulate results about the density of domains of distribution multipliers. We denote by $L^2_{loc}(X, \mu)$ the space of measurable functions f on X such that $f \in L^2(U)$ for every bounded measurable subset $U \subseteq X$.

Theorem 2 *Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, $\omega : X \rightarrow \mathcal{D}^\times$ a weakly measurable function, $\theta : X \rightarrow \mathcal{D}^\times$ a bounded Bessel distribution map with Bessel bound B_θ and $m : X \rightarrow \mathbb{C}$ a μ -measurable function.*

1. *If ω is pseudo-orthogonal and $m \in L^2_{loc}(X, \mu)$, then the distribution multiplier $M_{m, \omega, \theta}$ is densely defined.*
2. *If ω is a bounded Bessel distribution map and hyper-orthogonal, then the distribution multiplier $M_{m, \omega, \theta}$ is densely defined.*

Proof 1. Let V and X_f be as in Definition 8(1). For $f \in V, g \in \mathcal{D}$ we have by Cauchy-Schwarz inequality that

$$\left| \int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu \right| \leq C \int_{X_f} |m(x)| |\langle \theta_x | g \rangle| d\mu \leq C B_\theta^{\frac{1}{2}} \|m\|_{L^2(X_f)} \|g\|,$$

where $C = \sup_{x \in X_f} |\langle f | \omega_x \rangle|$. Thus $f \in D(M_{m, \omega, \theta})$, and consequently $D(M_{m, \omega, \theta})$ is dense because V is total.

2. The proof is divided into three parts. If $m \in L^\infty(X, \mu)$, then the conclusion follows by Proposition 5 since $D(M_{m, \omega, \theta}) = \mathcal{D}$. If $|m(x)| \geq 1$ a.e., then we take $\alpha(x) = |m(x)|^{-1}$ a.e. in Definition 8(2). Thus there exists $V_\alpha \subset \mathcal{D}$ total in \mathcal{H} such that for every $f \in V_\alpha$ there exists a compact subset $X_f \subset X$ and

$$\left| \int_X m(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu \right| \leq \int_{X_f} |m(x)| |m(x)|^{-1} |\langle \theta_x | g \rangle| d\mu \leq \mu(X_f)^{\frac{1}{2}} B_\theta^{\frac{1}{2}} \|g\|,$$

for every $g \in \mathcal{D}$ (the last inequality is due to Cauchy-Schwarz inequality). This means that $D(M_{m, \omega, \theta})$ is dense. Finally, let m be a generic measurable function. Then it is possible to write m as sum of two measurable functions $m = m_1 + m_2$ such that $m_1 \in L^\infty(X)$ and $|m_2| \geq 1$. Then $M_{m_1, \omega, \theta}$ is well-defined on \mathcal{D} and $M_{m_2, \omega, \theta}$ is defined on a subspace of \mathcal{D} dense in \mathcal{H} . As consequence, also $M_{m, \omega, \theta} = M_{m_1, \omega, \theta} + M_{m_2, \omega, \theta}$ is densely defined. \square

We show other examples of weakly measurable maps satisfying Definition 8.

Example 4 For $x \in \mathbb{R}$ consider the function ω_x defined by $\omega_x(y) := e^{-2\pi i x y}$ for $y \in \mathbb{R}$. Then ω_x is a distribution on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and in Example 3.17 of [46] it

was proved that ω is a distribution frame. Choosing $V = \{f \in L^2(\mathbb{R}) : \widehat{f} \in C_0^\infty(\mathbb{R})\}$, where \widehat{f} denotes the Fourier transform of f , in Definition 8, we conclude that ω is pseudo-orthogonal.

Now let $\alpha : \mathbb{R} \rightarrow \mathbb{C}$ be a positive measurable function and define $V_\alpha = \{f \in L^2(\mathbb{R}) : \widehat{f} \in C_0^\infty(\mathbb{R}) \text{ and } |\widehat{f}(x)| \leq \alpha(x), x \in \mathbb{R}\}$. By the considerations in Remark 2 and since the Fourier transform is unitary in $L^2(\mathbb{R})$, the set V_α is total, i.e. ω is also hyper-orthogonal.

Example 5 Let $\mathcal{D}[t]$ be a dense subspace of $L^2(\mathbb{R})$ endowed with a locally convex topology t , stronger than the topology of $L^2(\mathbb{R})$, and such that $C_0^\infty(\mathbb{R}) \subset \mathcal{D}$.

Let $g \in L^2(\mathbb{R})$ have support in a bounded interval I . Define $\omega_x(t) = g(t - x)$ for $t \in \mathbb{R}$. Then the weakly measurable map $\omega : \mathbb{R} \rightarrow L^2(\mathbb{R})$ is pseudo-orthogonal (again one can take $V = C_0^\infty(\mathbb{R})$).

We conclude this section by turning the attention to a sufficient condition for a distribution multiplier to be closable.

Proposition 7 *Let $\omega, \theta : X \rightarrow \mathcal{D}^\times$ be weakly measurable functions and $m : X \rightarrow \mathbb{C}$ be a μ -measurable function. If $M_{\overline{m}, \theta, \omega}$ is densely defined (in particular if ω, θ are bounded Bessel distribution maps and θ is hyper-orthogonal), then $M_{m, \omega, \theta}$ is closable.*

Proof For $f \in D(M_{m, \omega, \theta})$ and $g \in D(M_{\overline{m}, \theta, \omega})$ we have $\langle M_{m, \omega, \theta} f | g \rangle = \langle f | M_{\overline{m}, \theta, \omega} g \rangle$. This means that $M_{m, \omega, \theta} \subseteq (M_{\overline{m}, \theta, \omega})^*$, i.e. $M_{m, \omega, \theta}$ is closable. \square

5 Riesz distribution multipliers

In this section, we examine the case where ω and θ are Riesz distribution bases, reconsidering the Examples 4.1 and 4.2 of [46]. Let ω and θ be distribution Riesz bases, $\check{\omega}$ and $\check{\theta}$ their extensions to \mathcal{H} with the limit procedure described just after Definition 3, and a μ -measurable function $m : X \rightarrow \mathbb{C}$ such that the integral:

$$\int_X m(x) \langle f | \check{\omega}_x \rangle \langle \check{\theta}_x | g \rangle d\mu$$

is convergent for all $f, g \in \mathcal{H}$. Since θ is a Riesz basis, the analysis operator T_θ^* is a topological isomorphism of \mathcal{H} onto $L^2(X, \mu)$, then $T_\theta^*(\mathcal{H}) = L^2(X, \mu)$. It follows that $m(x) \langle f | \check{\omega}_x \rangle \in L^2(X, \mu)$ for all $f \in \mathcal{H}$. Furthermore, the operator $M_{m, \omega, \theta} : \mathcal{D} \rightarrow \mathcal{H}$:

$$M_{m, \omega, \theta} f = \int_X m(x) \langle f | \omega_x \rangle \theta_x d\mu, \quad \forall f \in \mathcal{D},$$

is well-defined. Analogously, for all $g \in \mathcal{H}$ one has $m(x) \langle g | \check{\theta}_x \rangle \in L^2(X, \mu)$ and the operator $M_{m, \omega, \theta}^\dagger : \mathcal{D} \rightarrow \mathcal{H}$:

$$M_{m,\omega,\theta}^\dagger g := M_{\bar{m},\theta,\omega} g = \int_X \overline{m(x)} \langle g | \theta_x \rangle \omega_x d\mu \quad \forall g \in \mathcal{D},$$

is well-defined. One has:

$$\langle M_{m,\omega,\theta} f | g \rangle = \langle f | M_{m,\omega,\theta}^\dagger g \rangle, \quad \forall f, g \in \mathcal{D}.$$

Then $M_{m,\omega,\theta}$ is a closable operator in \mathcal{H} . It is not difficult to show that the domain of the closure $D(\overline{M}_{m,\omega,\theta})$ is $\{f \in \mathcal{H} : \int_X |m(x) \langle f | \omega_x \rangle|^2 d\mu < \infty\}$. In general, the operators $M_{m,\omega,\theta}$ are unbounded, so their product is not always defined. However, if they belong to the space $\mathcal{L}^\dagger(\mathcal{D})$, they can be multiplied. In the following example, some cases of unbounded multipliers in $\mathcal{L}^\dagger(\mathcal{D})$ are considered.

Example 6 Let us consider the rigged Hilbert space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\times$. We write \widehat{f} and \check{f} for the Fourier transform and inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R})$, respectively.

Define: $\omega_x = \delta_x$ and $\theta_x(y) := e^{-2\pi i xy}$ for $y \in \mathbb{R}$, thus $\delta_x, \theta_x \in \mathcal{S}(\mathbb{R})^\times$. Let $O_M(\mathbb{R})$ be the space of C^∞ -functions which, together with their derivatives, are polynomially bounded (see [32]). If $m \in O_M(\mathbb{R})$, for $f \in \mathcal{S}(\mathbb{R})$ we have:

$$M_{m,\omega,\omega} f = m f, \quad M_{m,\omega,\theta} f = \check{m} * \check{f}, \quad M_{m,\theta,\omega} f = m \widehat{f} \quad \text{and} \quad M_{m,\theta,\theta} f = \check{m} * f.$$

The above considerations lead in particular to the case of a Riesz basis ω and its dual θ of Example 4.2 [46]. To simplify the notation, we denote the multiplier as $M_m := M_{m,\omega,\omega}$. In Example 4.2 [46], it is shown that $m(x)$ is a generalized eigenvalue of M_m , i.e.:

$$\langle (M_m^\dagger)^\times \theta_x | g \rangle = m(x) \langle \theta_x | g \rangle, \quad \forall g \in \mathcal{D}, \quad \mu\text{-a.e. } x \in X,$$

and $\bar{m}(x)$ is a generalized eigenvalue of M_m^\dagger i.e.:

$$\langle (M_m)^\times \omega_x | g \rangle = \bar{m}(x) \langle \omega_x | g \rangle \quad \forall g \in \mathcal{D}, \quad \mu\text{-a.e. } x \in X.$$

A consequence is the following:

Proposition 8 *Let M_{m_1} and M_{m_2} be multipliers of a Riesz basis ω and its dual θ , such that $M_{m_i} \in \mathcal{L}^\dagger(\mathcal{D})$, $i = 1, 2$. Then $M_{m_1} M_{m_2} = M_{m_1 m_2}$.*

Proof For all $f, g \in \mathcal{D}$

$$\begin{aligned} \langle (M_{m_2} M_{m_1}) f | g \rangle &= \langle M_{m_1} f | M_{m_2}^\dagger g \rangle = \int_X m_1(x) \langle f | \omega_x \rangle \langle \theta_x | M_{m_2}^\dagger g \rangle d\mu = \\ &= \int_X m_1(x) m_2(x) \langle f | \omega_x \rangle \langle \theta_x | g \rangle d\mu = \langle M_{m_1 m_2} f | g \rangle \end{aligned}$$

and the proof is completed. \square

Analogously, $M_{m_1}^\dagger$ and $M_{m_2}^\dagger$ can be multiplied, and $(M_{m_1} M_{m_2})^\dagger = M_{m_2}^\dagger M_{m_1}^\dagger$. This shows that the multipliers in $\mathcal{L}^\dagger(\mathcal{D})$ of dual Riesz bases is a \dagger -subalgebra in $\mathcal{L}^\dagger(\mathcal{D})$.

Furthermore, if both M_m^{-1} and $M_{\frac{1}{m}}$ are defined in $\mathcal{L}^\dagger(\mathcal{D})$, by Proposition 8 one has that $M_m^{-1} = M_{\frac{1}{m}}$. The considered case of multipliers in $\mathcal{L}^\dagger(\mathcal{D})$ allows to handle easily the symbolic calculus, but operators in $\mathcal{L}^\dagger(\mathcal{D})$ are, in general, unbounded. What can be said about the case of bounded operators? Obviously, if $m \in L^\infty(X, \mu)$ and ω and θ are Riesz bases, M_m is bounded. Vice versa, the following proposition holds:

Proposition 9 *Let ω be a distribution Riesz basis with dual θ . If the multiplier $M_{m,\omega,\theta}$ is bounded then $m \in L^\infty(X, \mu)$.*

Proof The proposition is true for the diagonal operator $A_{m,\zeta}$, i.e. a multiplier of the Gel'fand basis: $\omega = \theta = \zeta$ (see Example 4.1 of [46]). The same holds for $M_{m,\omega,\theta}$: in fact $M_{m,\omega,\theta}$ and $A_{m,\zeta}$ are similar via W of b) of Proposition 4, i.e. $\overline{M}_{m,\omega,\theta} = W\overline{A}_{m,\zeta}W^{-1}$ (see Example 4.2 of [46]), where $\overline{M}_{m,\omega,\theta}$ and $\overline{A}_{m,\zeta}$ are their closure. \square

For conditions of invertibility of a multiplier, we can state the following in a more general form.

Proposition 10 *Let $\omega, \theta : X \rightarrow \mathcal{D}^\times$ be weakly measurable maps and $m(x) \neq 0$ μ -a.e. in X . Then*

1. *If ω is μ -independent and θ is total, then $M_{m,\omega,\theta}$ is injective.*
2. *If ω is total and θ is μ -independent, then $M_{m,\omega,\theta}$ has dense range in \mathcal{H} .*

Proof Assume that $M_{m,\omega,\theta}f = \int_X m(x)\langle f|\omega_x\rangle\theta_x d\mu = 0$. Since θ is μ -independent, we have $m(x)\langle f|\omega_x\rangle = 0$ a.e., that is $\langle f|\omega_x\rangle = 0$ a.e.; but ω is total, then $f = 0$. For the range, let $g \in \mathcal{D}$ such that $\langle M_{m,\omega,\theta}f|g\rangle = 0$ for all $f \in \mathcal{D}$, that is $\langle M_{m,\omega,\theta}f|g\rangle = \int_X m(x)\langle f|\omega_x\rangle\langle \theta_x|g\rangle d\mu = 0$. Since ω is μ -independent, we have $m(x)\langle \theta_x|g\rangle = 0$ a.e. and $g = 0$, because θ is total. \square

In particular, if ω, θ are Riesz bases and $m(x)$ is nonzero a.e., then $M_{m,\omega,\theta}$ is invertible with densely defined inverse. To have a bounded inverse we can make use of an additional assumption.

Proposition 11 *Let $M_{m,\omega,\theta}$ be a Riesz distribution multiplier. If there exists $C > 0$ such that $0 < C \leq |m(x)|$ for all $x \in X$, the inverse of $M_{m,\omega,\theta}$ is bounded.*

Proof By Proposition 10, the inverse exists. Let A_ω, B_ω and A_θ, B_θ be the lower and upper bounds of ω and θ , respectively. The operator $M_{m,\omega,\theta}$ has closure extension $\widehat{M}_{m,\omega,\theta} := T_\theta D_m T_\omega^*$. By Proposition 4, the operators $T_\omega^*, T_\omega, T_\theta, T_\theta^*$ are bounded, invertible with bounded inverses, so we have:

$$A_\theta A_\omega C \|f\| \leq A_\theta C \|T_\omega^* f\|_2 \leq \|T_\theta D_m T_\omega^* f\|, \quad \forall f \in D(M_{m,\omega,\theta}),$$

and the proof is completed. \square

6 Conclusions

Some questions about symbolic calculus in a more general set-up (not only in the case of dual Riesz bases) are open. For instance, it is known that in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ a partial multiplication is defined (see [2, 47]), thus a symbolic calculus may be developed for multipliers in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$. The idea behind Definition 8 are connected to localization frames which were introduced in [29] and further studied (sometimes with variations) in [6, 7, 11, 22, 23, 28]. More precisely, some result in Section 4 can be extended considering a certain decay of $x \mapsto \langle f | \omega_x \rangle$ instead of assuming that $\langle f | \omega_x \rangle$ is null outside a bounded set.

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