

MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR A CLASS OF COERCIVE $(p, 2)$ -EQUATIONS

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ABSTRACT. We consider a nonlinear Dirichlet equation driven by the sum of a p -Laplacian and of a Laplacian (a $(p, 2)$ -equation). The hypotheses on the reaction $f(z, x)$ are minimal and make the energy (Euler) functional of the problem coercive. We prove two multiplicity theorems producing three and four nontrivial smooth solutions respectively, all with sign information. We apply our multiplicity results to the particular case of a class of parametric $(p, 2)$ -equations.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following nonlinear Dirichlet problem:

$$(1) \quad -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad 2 < p.$$

In this problem Δ_p denotes the p -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

So, in the left hand side of (1) we have the sum of two differential operators of different nature. One is the p -Laplacian (a nonlinear operator) and the other is the Laplacian (a linear operator). The resulting nonlinear differential operator is nonhomogeneous and this is a source of difficulties in the analysis of problem (1). The resulting equation is known in the literature as a $(p, 2)$ -equation and is a particular case of the class of the so-called double phase equations which arise in mathematical models of various physical processes. As a characteristic example of such a model, we mention the work of Zhikov [27] on the homogenization of composites consisting of two materials with distinct hardening exponents which come up in elasticity theory. In problem (1) the reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \rightarrow f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \rightarrow f(z, x)$ is continuous). The conditions on $f(z, \cdot)$ are minimal and make the energy (Euler) functional of the problem coercive. Using variational methods based on the critical point theory combined with suitable truncation and comparison techniques, we prove the existence of three nontrivial smooth solutions all with sign information, namely a positive solution, a negative solution and a nodal (sign changing) solution. If we strengthen the regularity of $f(z, \cdot)$, more precisely, if we assume that for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and use the theory of critical groups (Morse theory), then we produce a second nodal solution, for a total of four nontrivial

Key words and phrases. $(p, 2)$ -differential operator, critical groups, constant sign solutions, nodal solutions, homotopy.

2010 Mathematics Subject Classification: 35J20, 35J60, 58E05.

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smooth solutions, all with sign information. Moreover, when the reaction term has the following particular form

$$f(z, x) = \lambda x - g(z, x)$$

with $\lambda > 0$ being a parameter, then our multiplicity results applied to this particular case, guarantee that if $\lambda > \widehat{\lambda}_2(2)$ (with $\widehat{\lambda}_2(2) > 0$ being the second eigenvalue of $(-\Delta, H_0^1(\Omega))$), then the equation has three or four nontrivial smooth solutions all with sign information, depending on the regularity of $g(z, \cdot)$. This particular right hand side is a typical subdiffusive reaction in a nonlinear nonhomogeneous logistic equation.

We mention that recently $(p, 2)$ -equations attracted considerable interest and there have been various existence and multiplicity results for such equations. We mention the works of Aizicovici-Papageorgiou-Staicu [1], Gasiński-Papageorgiou [6], He-Lei-Zhang-Sun [9], Liang-Han-Li [12], Liang-Song-Su [13], Papageorgiou-Rădulescu [15, 16], Papageorgiou-Vetro-Vetro [19, 20, 21], Papageorgiou-Zhang [23], Sun-Zhang-Su [25], Zhang-Liang [26]. In particular [1, 15] also deal with coercive problems but under more restrictive conditions on the source term $f(z, x)$ which exclude from consideration logistic equations. In [6, 16, 21], the authors deal with equations in which the reaction term exhibits asymmetric behavior as $x \rightarrow \pm\infty$. In [12, 13, 26] the authors prove on existence results for $(p, 2)$ -equations. In [9] the equation is parametric and the approach is based on flow invariant arguments and in [23] the authors deal with equations involving a parametric concave term.

2. MATHEMATICAL BACKGROUND

The main spaces in the analysis of problem (1) are the Sobolev spaces $W_0^{1,p}(\Omega)$, $H_0^1(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$ and by $\|\cdot\|_{1,2}$ the norm of $H_0^1(\Omega)$. On account of the Poincaré inequality we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_{1,2} = \|\nabla u\|_2 \quad \text{for all } u \in H_0^1(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered, with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n} \Big|_{\partial\Omega} < 0 \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Recall that $C_0^1(\overline{\Omega})$ is dense in both $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$.

Let $A_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$ be the nonlinear map defined by

$$\langle A_p(u), h \rangle = \int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

The next proposition summarizes the well-known properties of this map (see Gasiński-Papageorgiou [4], p. 746).

Proposition 1. *The map $A_p(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone) and of type $(S)_+$ (this means that if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n \rightarrow +\infty} \langle A_p(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$).*

Also $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is the linear operator defined by

$$\langle A(u), h \rangle = \int_{\Omega} (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H_0^1(\Omega).$$

We will use the spectra of $(-\Delta_p, W_0^{1,p}(\Omega))$ and of $(-\Delta, H_0^1(\Omega))$. So, let us recall some basic facts about them.

We start by considering the following nonlinear eigenvalue problem

$$(2) \quad -\Delta_p u(z) = \widehat{\lambda} |u(z)|^{p-2} u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an “eigenvalue” of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$, known as an “eigenfunction” corresponding to the eigenvalue $\widehat{\lambda}$. There is a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which has the following properties:

- (a) $\widehat{\lambda}_1(p)$ is isolated, that is, there exists $\varepsilon > 0$ such that the interval $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon)$ contains no eigenvalues;
- (b) $\widehat{\lambda}_1(p)$ is simple, that is, if \widehat{u}, \widehat{v} are two eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u} = \xi \widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$;
- (c)

$$(3) \quad \widehat{\lambda}_1(p) = \inf \left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right].$$

The infimum in (c) is realized on the corresponding one-dimensional eigenspace (see (b)). The above properties imply that the elements of this eigenspace do not change sign. By $\widehat{u}_1(p)$ we denote the positive, L^p -normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p) > 0$. From the nonlinear regularity theory and from the nonlinear maximum principle (see Gasiński-Papageorgiou [4], pp. 737-738), we have that $\widehat{u}_1 \in \text{int } C_+$.

Using the Ljusternik-Schnirelmann minimax scheme, we can generate a whole strictly increasing sequence $\{\widehat{\lambda}_k(p)\}_{k \geq 1}$ of eigenvalues such that $\widehat{\lambda}_k(p) \rightarrow +\infty$ as $k \rightarrow +\infty$. These are known as variational eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$. We do not know if they exhaust the spectrum $\widehat{\sigma}(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. The isolation of $\widehat{\lambda}_1(p)$ (see (a) above) and the closedness of $\widehat{\sigma}(p)$, imply that the second eigenvalue $\widehat{\lambda}_2^*(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ is well-defined by

$$\widehat{\lambda}_2^*(p) = \min \left\{ \widehat{\lambda} \in \widehat{\sigma}(p) : \widehat{\lambda}_1(p) < \widehat{\lambda} \right\}.$$

We know that $\widehat{\lambda}_2^*(p) = \widehat{\lambda}_2(p)$, that is, the second eigenvalue and the second variational eigenvalue coincide. Let $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$ and let $M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}$. Using this Banach manifold, we can have the following well-known minimax characterization of $\widehat{\lambda}_2(p)$ (see, for example, Gasiński-Papageorgiou [7], p. 840).

Proposition 2. $\widehat{\lambda}_2(p) = \inf_{\widehat{\gamma} \in \widehat{\Gamma}} \max_{-1 \leq t \leq 1} \|\nabla \widehat{\gamma}(t)\|_p^p$ where $\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1(p), \widehat{\gamma}(1) = \widehat{u}_1(p)\}$.

For the linear eigenvalue problem ($p = 2$), the situation is much better since we have full knowledge of the spectrum $\widehat{\sigma}(2)$ of $(-\Delta, H_0^1(\Omega))$. We know that $\widehat{\sigma}(2) = \{\widehat{\lambda}_k(2)\}_{k \geq 1}$, that is, the variational eigenvalues exhaust the spectrum. Each eigenvalue $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$,

has an eigenspace $E(\widehat{\lambda}_k(2))$ which is a finite dimensional subspace of $H_0^1(\Omega)$. We have the following orthogonal direct sum decomposition

$$\overline{H}_0^1 = \overline{\bigoplus_{k \geq 1} E(\widehat{\lambda}_k(2))}.$$

For every $m \geq 1$ we set

$$\overline{H}_m = \bigoplus_{k=1}^m E(\widehat{\lambda}_k(2)) \text{ and } \widehat{H}_{m+1} = \overline{\bigoplus_{k \geq m+1} E(\widehat{\lambda}_k(2))}.$$

Evidently, $\widehat{H}_{m+1} = \overline{H}_m^\perp$ and so we have

$$H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_{m+1}.$$

All eigenvalues have variational characterizations. More precisely we have

$$(4) \quad \widehat{\lambda}_1(2) = \inf \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_0^1(\Omega), u \neq 0 \right] \quad (\text{see (3)}),$$

and for $m \geq 2$

$$(5) \quad \begin{aligned} \widehat{\lambda}_m(2) &= \sup \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_m, u \neq 0 \right] \\ &= \inf \left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_m, u \neq 0 \right]. \end{aligned}$$

In (4) the infimum is realized on $E(\widehat{\lambda}_1(2))$, while in (5) both the infimum and the supremum are realized on $E(\widehat{\lambda}_m(2))$. The eigenspaces $E(\widehat{\lambda}_k(2))$ have the ‘‘Unique Continuation Property’’, that is, if $u \in E(\widehat{\lambda}_k(2))$ and $u(\cdot)$ vanishes on a set of positive measure, then $u \equiv 0$. Moreover, standard regularity theory implies that $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ for all $k \in \mathbb{N}$.

These properties lead to the following useful inequalities (see Gasiński-Papageorgiou [7], Problem 5.117, p. 870).

Proposition 3. (a) *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \widehat{\lambda}_m(2)$ for a.a. $z \in \Omega$, $\theta \not\equiv \widehat{\lambda}_m(2)$, then there exists $c_1 > 0$ such that*

$$\|\nabla u\|_2^2 - \int_{\Omega} \theta(z) u^2 dz \geq c_1 \|u\|_{1,2}^2 \quad \text{for all } u \in \widehat{H}_m.$$

(b) *If $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \widehat{\lambda}_m(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_m(2)$, then there exists $c_2 > 0$ such that*

$$\|\nabla u\|_2^2 - \int_{\Omega} \eta(z) u^2 dz \leq -c_2 \|u\|_{1,2}^2 \quad \text{for all } u \in \overline{H}_m.$$

As we already mentioned in the Introduction, in order to produce additional nodal solutions, we will use the theory of critical groups. So, let us briefly recall some basic definitions and facts from that theory. For details we refer to the book of Papageorgiou-Rădulescu-Repovš [18].

So, let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$

$$\begin{aligned} K_\varphi &= \{u \in X : \varphi'(u) = 0\} \quad (\text{the critical set of } \varphi), \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k^{th} -relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Given an isolated $u \in K_\varphi^c$, the “critical groups” of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being an isolating neighborhood, that is, $\varphi^c \cap U \cap K_\varphi = \{u\}$. The excision property of singular homology implies that the above definition is independent of the choice of the isolating neighborhood U .

We say that $\varphi \in C^1(X, \mathbb{R})$ satisfies the “ C -condition”, if the following property holds:

“Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow +\infty$, admits a strongly convergent subsequence”.

Note that if $\varphi \in C^1(X, \mathbb{R})$, it is coercive and $\varphi' = A + K$ with $A(\cdot)$ of $(S)_+$ -type and $K(\cdot)$ completely continuous, then φ satisfies the C -condition (see [18], p. 369). Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C -condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed if $\eta < c < \inf \varphi(K_\varphi)$, then from Corollary 5.3.13, p. 322, of Papageorgiou-Rădulescu-Repovš [18], we know that φ^η is a strong deformation retract of φ^c . Hence Corollary 6.1.24, p. 384, of Papageorgiou-Rădulescu-Repovš [18] implies that

$$H_k(X, \varphi^c) = H_k(X, \varphi^\eta) \quad \text{for all } k \in \mathbb{N}_0.$$

Now, suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C -condition and K_φ is finite. We define

$$\begin{aligned} M(t, u) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Then the Morse relation says that

$$(6) \quad \sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t) \quad \text{for all } t \in \mathbb{R},$$

with $Q(t) = \sum_{k \geq 0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally, let us fix our notation. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$ we set $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, if $u, v \in W_0^{1,p}(\Omega)$, with $u \leq v$, then we set

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

By $\text{int}_{C_0^1(\overline{\Omega})}[u, v]$, we denote the interior in the $C_0^1(\overline{\Omega})$ -topology of $[u, v] \cap C_0^1(\overline{\Omega})$. Finally given $k, m \in \mathbb{N}_0$, by $\delta_{k,m}$ we denote the Kronecker symbol defined by $\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m, \end{cases}$ and $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p, \end{cases}$ (the Sobolev critical exponent).

3. THREE NONTRIVIAL SOLUTIONS

In this section assuming only continuity on $f(z, \cdot)$, we prove the existence of three nontrivial smooth solutions all with sign information.

We start with the following conditions on the reaction $f(z, x)$.

$H(f)_1$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

- (i) $|f(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;
- (ii) $\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = -\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \geq \widehat{\lambda}_1(2) \quad \text{for a.a. } z \in \Omega, \theta \not\equiv \widehat{\lambda}_1(2),$$

$$\liminf_{x \rightarrow 0} \frac{f(z, x)}{x} \geq \theta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Under these general conditions on the reaction $f(z, x)$, we can establish the existence of constant sign solutions.

Proposition 4. *If hypothesis $H(f)_1$ hold, then problem (1) has two constant sign smooth solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$.*

Proof. Hypothesis $H(f)_1$ (ii) implies that we can find $M_0 > 0$ such that

$$(7) \quad f(z, x)x \leq -1 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_0.$$

Let $\beta \geq M_0$ and consider the Carathéodory function $\widehat{f}_+(z, x)$ defined by

$$(8) \quad \widehat{f}_+(z, x) = \begin{cases} f(z, x^+) & \text{if } x \leq \beta, \\ f(z, \beta) & \text{if } \beta < x. \end{cases}$$

We set $\widehat{F}_+(z, x) = \int_0^x \widehat{f}_+(z, s)ds$ and consider the C^1 -functional $\widehat{\varphi}_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega \widehat{F}_+(z, u)du \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (8) it is clear that $\widehat{\varphi}_+(\cdot)$ is coercive. Also, using the Sobolev embedding we see that $\widehat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$(9) \quad \widehat{\varphi}_+(u_0) = \inf[\widehat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)].$$

On account of hypothesis $H(f)_1$ (iii), given $\varepsilon > 0$, we can find $\delta \in (0, \beta)$ such that

$$F(z, x) \geq \frac{1}{2}[\theta(z) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta,$$

where $F(z, x) = \int_0^x f(z, s)ds$. Since $\widehat{u}_1(2) \in \text{int } C_+$, we can find $t \in (0, 1)$ small such that $0 \leq t\widehat{u}_1(2)(z) \leq \delta$ for all $z \in \overline{\Omega}$. Then we have

$$\begin{aligned}
 \widehat{\varphi}_+(t\widehat{u}_1(2)) &= \frac{t^p}{p} \|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2} \widehat{\lambda}_1(2) \|\widehat{u}_1(2)\|_2^2 - \int_{\Omega} \widehat{F}_+(z, t\widehat{u}_1(2)) dz \\
 (10) \quad &\leq \frac{t^p}{p} \|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2} \left[\int_{\Omega} [\widehat{\lambda}_1(2) - \theta(z)] \widehat{u}_1(2)^2 dz + \varepsilon \right] \\
 &\quad \text{(recall that } \|\widehat{u}_1(2)\|_2 = 1).
 \end{aligned}$$

The hypothesis on $\theta(\cdot)$ (see $H(f)_1$ (ii)) implies that

$$\gamma_0 = \int_{\Omega} [\theta(z) - \widehat{\lambda}_1(2)] \widehat{u}_1(2) dz > 0.$$

Choosing $\varepsilon \in (0, \gamma_0)$, from (10) we obtain

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) \leq c_3 t^p - c_4 t^2 \quad \text{for some } c_3, c_4 > 0.$$

Since $2 < p$, choosing $t \in (0, 1)$ even smaller if necessary, we have

$$\begin{aligned}
 &\widehat{\varphi}_+(t\widehat{u}_1(2)) < 0, \\
 \Rightarrow &\widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0) \quad (\text{see (9)}), \\
 \Rightarrow &u_0 \neq 0.
 \end{aligned}$$

From (9) we have

$$\begin{aligned}
 &\widehat{\varphi}'_+(u_0) = 0, \\
 (11) \quad \Rightarrow &\langle A_p(u_0), h \rangle + \langle A(u_0), h \rangle = \int_{\Omega} \widehat{f}_+(z, u_0) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).
 \end{aligned}$$

In (11) we choose $h = (u_0 - \beta)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
 &\langle A_p(u_0), (u_0 - \beta)^+ \rangle + \langle A(u_0), (u_0 - \beta)^+ \rangle \\
 &= \int_{\Omega} f(z, \beta) (u_0 - \beta)^+ dz \quad (\text{see (8)}) \\
 &\leq 0 \quad (\text{see (7)}) \\
 &= \langle A_p(\beta), (u_0 - \beta)^+ \rangle + \langle A(\beta), (u_0 - \beta)^+ \rangle, \\
 \Rightarrow &u_0 \leq \beta.
 \end{aligned}$$

On the other hand, if in (11) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$, we obtain $0 \leq u_0$. Therefore we have

$$(12) \quad u_0 \in [0, \beta], \quad u_0 \neq 0.$$

From (8), (11) and (12) it follows that

$$(13) \quad -\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega, \quad u_0|_{\partial\Omega} = 0.$$

From (13) and Theorem 7.1, p. 286, of Ladyzhenskaya-Ural'tseva [11] we have that $u_0 \in L^\infty(\Omega)$. So, we can apply Theorem 1 of Lieberman [14] and have that $u_0 \in C_+ \setminus \{0\}$. Hypotheses $H(f)_1$ (i), (iii) imply that if $\rho = \|u_0\|_\infty$, then we can find $\widehat{\xi}_\rho > 0$ such that

$$(14) \quad f(z, x) + \widehat{\xi}_\rho x^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

From (13) and (14) it follows that

$$\Delta_p u_0(z) + \Delta u_0(z) \leq \widehat{\xi}_\rho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega.$$

Then the nonlinear maximum principle of Pucci-Serrin [24] (pp. 111, 120), implies that $u_0 \in \text{int } C_+$.

For the negative solution, we consider the Carathéodory function $\widehat{f}_-(z, x)$ defined by

$$(15) \quad \widehat{f}_-(z, x) = \begin{cases} f(z, -\beta) & \text{if } x \leq -\beta, \\ f(z, -x^-) & \text{if } -\beta < x. \end{cases}$$

We set $\widehat{F}_-(z, x) = \int_0^x \widehat{f}_-(z, s) ds$ and consider the C^1 -functional $\widehat{\varphi}_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega \widehat{F}_-(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as we did with $\widehat{\varphi}_+(\cdot)$, using this time (15), we produce a negative solution $v_0 \in -\text{int } C_+$ for problem (1). \square

In fact we can produce extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution. These solutions will be used to produce a nodal solution (see Section 4).

To generate the extremal constant sign solutions, we need to do some preliminary work. Note that on account of hypotheses $H(f)_1$ (i), (iii), given $\varepsilon > 0$ we can find $c_5 > 0$ such that

$$(16) \quad f(z, x)x \geq [\theta(z) - \varepsilon]x^2 - c_5|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

This unilateral growth estimate on the reaction term, leads to the consideration of the following auxiliary Dirichlet problem

$$(17) \quad -\Delta_p u(z) - \Delta u(z) = [\theta(z) - \varepsilon]u(z) - c_5|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Proposition 5. *For all $\varepsilon \in (0, \gamma_0)$ (see the proof of Proposition 4), problem (17) has a unique positive solution $\widetilde{u} \in \text{int } C_+$ and since the equation is odd $\widetilde{v} = -\widetilde{u} \in \text{int } C_+$ is the unique negative solution of problem (17).*

Proof. First we prove the existence of a positive solution. So, we consider the C^1 -functional $\widehat{\psi}_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 + \frac{c_5}{r} \|u^+\|_r^r - \frac{1}{2} \int_\Omega [\theta(z) - \varepsilon](u^+)^2 dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $r > p > 2$, we see that $\widehat{\psi}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$(18) \quad \widehat{\psi}_+(\widetilde{u}) = \inf[\widehat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)].$$

As in the proof of Proposition 4, we show that for $t \in (0, 1)$ small and $\varepsilon \in (0, \gamma_0)$ we have

$$\begin{aligned} & \widehat{\psi}_+(t\widetilde{u}_1(2)) < 0, \\ \Rightarrow & \widehat{\psi}_+(\widetilde{u}) < 0 = \widehat{\psi}_+(0), \\ \Rightarrow & \widetilde{u} \neq 0. \end{aligned}$$

From (18), we have

$$\begin{aligned} \widehat{\psi}'_+(\tilde{u}) &= 0, \\ (19) \quad \Rightarrow \quad \langle A_p(\tilde{u}), h \rangle + \langle A(\tilde{u}), h \rangle &= \int_{\Omega} [\theta(z) - \varepsilon](\tilde{u}^+) h dz - c_5 \int_{\Omega} (\tilde{u}^+)^{r-1} h dz \end{aligned}$$

for all $h \in W_0^{1,p}(\Omega)$. Choosing $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ in (19), we obtain $\tilde{u} \geq 0$, $\tilde{u} \neq 0$. Therefore from (19) we infer that \tilde{u} is a positive solution of (17). As before the nonlinear regularity theory (see [14]) and the nonlinear maximum principle (see [24]), imply that $\tilde{u} \in \text{int } C_+$.

We will show that this positive solution is unique. For this purpose we consider the integral functional $j : L^1(\Omega) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p} \|\nabla u^{1/2}\|_p^p + \frac{1}{2} \|\nabla u^{1/2}\|_2^2 & \text{if } u \geq 0, u^{1/2} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\text{dom } j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$), let $u_1, u_2 \in \text{dom } j$ and set $u = [(1-t)u_1 + tu_2]^{1/2}$ for $t \in [0, 1]$. In what follows for notational economy we use

$$G_0(t) = \frac{1}{p} t^p + \frac{1}{2} t^2 \quad \text{for all } t \geq 0 \text{ and } G(y) = G_0(|y|) \text{ for all } y \in \mathbb{R}^N.$$

Note that $G(\cdot) \in C^1(\mathbb{R}^N, \mathbb{R})$ (recall that $p > 2$) and $\nabla G(y) = |y|^{p-2} y + y$ (that is, for all $u \in W_0^{1,p}(\Omega)$, $\text{div } \nabla G(u) = \Delta_p u + \Delta u$). From Lemma 4 of Benguria-Brezis-Lieb [2] we have

$$\begin{aligned} |\nabla u| &\leq \left[(1-t) \left| \nabla u_1^{1/2} \right|^2 + t \left| \nabla u_2^{1/2} \right|^2 \right]^{1/2} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad G_0(|\nabla u|) &\leq G_0 \left(\left[(1-t) \left| \nabla u_1^{1/2} \right|^2 + t \left| \nabla u_2^{1/2} \right|^2 \right]^{1/2} \right) \quad (\text{since } G_0(\cdot) \text{ is increasing}). \end{aligned}$$

The function $t \rightarrow G_0(t^{1/2})$ is convex (recall $p > 2$). Therefore we have

$$\begin{aligned} &G_0 \left(\left[(1-t) \left| \nabla u_1^{1/2} \right|^2 + t \left| \nabla u_2^{1/2} \right|^2 \right]^{1/2} \right) \\ &\leq (1-t) G_0 \left(\left| \nabla u_1^{1/2} \right| \right) + t G_0 \left(\left| \nabla u_2^{1/2} \right| \right), \\ \Rightarrow \quad G(\nabla u) &\leq (1-t) G \left(\left| \nabla u_1^{1/2} \right| \right) + t G \left(\left| \nabla u_2^{1/2} \right| \right), \\ \Rightarrow \quad j(\cdot) &\text{ is convex.} \end{aligned}$$

Also by Fatou's lemma $j(\cdot)$ is lower semicontinuous.

Suppose that $\tilde{y} \in W_0^{1,p}(\Omega)$ is another positive solution of (17). Again we have $\tilde{y} \in \text{int } C_+$. Therefore $\tilde{u}^2, \tilde{y}^2 \in \text{dom } j$. Also, if $h \in C_0^1(\overline{\Omega})$, then for $|t| < 1$ small we have $\tilde{u}^2 + t(h^2 - \tilde{u}^2) \in \text{dom } j$ and $\tilde{y}^2 + t(h^2 - \tilde{y}^2) \in \text{dom } j$, and we can easily check that the Gâteaux derivatives of $j(\cdot)$ at \tilde{u}^2 and at \tilde{y}^2 in the direction $\tilde{u}^2 - \tilde{y}^2$ exist. In particular note that on account of Proposition 4.1.22, p. 274, of [18], we can find $\tilde{c} > 0$ such that

$$\tilde{y} \leq \tilde{c} \tilde{u}.$$

Therefore we have

$$0 \leq \frac{\tilde{y}}{\tilde{u}} \leq \tilde{c}.$$

Recalling that $\tilde{u}, \tilde{y} \in \text{int } C_+$, it follows that

$$\begin{aligned} \int_{\Omega} \frac{1}{\tilde{u}} |\nabla \tilde{u}|^{p-2} (\nabla \tilde{u}, \nabla(\tilde{u}^2 - \tilde{y}^2))_{\mathbb{R}^N} dz &< +\infty, \\ \int_{\Omega} \frac{\tilde{u}^2 - \tilde{y}^2}{\tilde{u}^2} |\nabla \tilde{u}|^p dz &< +\infty. \end{aligned}$$

Moreover, using the chain rule and the nonlinear Green's identity (see Gasiński-Papageorgiou [4], p. 210), we have

$$\begin{aligned} j'(\tilde{u}^2)(\tilde{u}^2 - \tilde{y}^2) &= \frac{1}{2} \int_{\Omega} \frac{-\Delta_p \tilde{u} - \Delta \tilde{u}}{\tilde{u}} (\tilde{u}^2 - \tilde{y}^2) dz, \\ j'(\tilde{y}^2)(\tilde{u}^2 - \tilde{y}^2) &= \frac{1}{2} \int_{\Omega} \frac{-\Delta_p \tilde{y} - \Delta \tilde{y}}{\tilde{y}} (\tilde{u}^2 - \tilde{y}^2) dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Recalling that \tilde{u}, \tilde{y} are solutions of (17), we obtain

$$\begin{aligned} 0 &\leq \int_{\Omega} c_5 [\tilde{y}^{r-2} - \tilde{u}^{r-2}] (\tilde{u}^2 - \tilde{y}^2) dz \\ \Rightarrow \quad \tilde{u} &= \tilde{y}. \end{aligned}$$

This proves the uniqueness of the positive solution $\tilde{u} \in \text{int } C_+$ of problem (17). Since the equation is odd, we infer that $\tilde{v} = -\tilde{u} \in -\text{int } C_+$ is the unique negative solution of (17). \square

In what follows by S_+ (resp. S_-) we denote the set of positive (resp. negative) solutions of problem (1). From Proposition 4 and its proof, we have $\emptyset \neq S_+ \subseteq \text{int } C_+$ and $\emptyset \neq S_- \subseteq -\text{int } C_+$.

Proposition 6. *If hypotheses $H(f)_1$ hold, then $\tilde{u} \leq u$ for all $u \in S_+$ and $v \leq \tilde{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+ \subseteq \text{int } C_+$ and consider the following Carathéodory function

$$(20) \quad e_+(z, x) = \begin{cases} [\theta(z) - \varepsilon]x^+ - c_5(x^+)^{r-1} & \text{if } x \leq u(z), \\ [\theta(z) - \varepsilon]u(z) - c_5u(z)^{r-1} & \text{if } u(z) < x, \end{cases}$$

with $\varepsilon \in (0, \gamma_0)$ (see the proof of Proposition 4). We set $E_+(z, x) = \int_0^x e_+(z, s) ds$ and consider the C^1 -functional $\psi_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_{\Omega} E_+(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (20) it is clear that $\psi_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semi-continuous. So, we can find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$(21) \quad \psi_+(\tilde{u}_0) = \inf[\psi_+(u) : u \in W_0^{1,p}(\Omega)].$$

As before (see the proof of Proposition 4), we have

$$\begin{aligned} \psi_+(t\hat{u}_1(2)) &< 0 \quad \text{for } t > 0 \text{ small,} \\ \Rightarrow \quad \psi_+(\tilde{u}_0) &< 0 = \psi_+(0) \quad (\text{see (21)}), \end{aligned}$$

$$\Rightarrow \tilde{u}_0 \neq 0.$$

From (21) we have

$$\psi'_+(\tilde{u}_0) = 0,$$

$$(22) \quad \Rightarrow \quad \langle A_p(\tilde{u}_0), h \rangle + \langle A(\tilde{u}_0), h \rangle = \int_{\Omega} e_+(z, \tilde{u}_0) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

In (22) we choose $h = -\tilde{u}_0^- \in W_0^{1,p}(\Omega)$ and obtain that $\tilde{u}_0 \geq 0$. Next in (22) we choose $h = (\tilde{u}_0 - u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} & \langle A_p(\tilde{u}_0), (\tilde{u}_0 - u)^+ \rangle + \langle A(\tilde{u}_0), (\tilde{u}_0 - u)^+ \rangle \\ &= \int_{\Omega} ([\theta(z) - \varepsilon]u - c_5 u^{r-1}) (\tilde{u}_0 - u)^+ dz \quad (\text{see (20)}) \\ &\leq \int_{\Omega} f(z, u) (\tilde{u}_0 - u)^+ dz \quad (\text{see (16)}) \\ &= \langle A_p(u), (\tilde{u}_0 - u)^+ \rangle + \langle A(u), (\tilde{u}_0 - u)^+ \rangle \quad (\text{since } u \in S_+), \\ &\Rightarrow \tilde{u}_0 \leq u. \end{aligned}$$

So, finally we have

$$(23) \quad \tilde{u}_0 \in [0, u], \quad \tilde{u}_0 \neq 0.$$

From (20), (22), (23) it follows that \tilde{u}_0 is a positive solution of (17), hence $\tilde{u}_0 = \tilde{u} \in \text{int } C_+$ (see Proposition 5). Therefore $\tilde{u} \leq u$ for all $u \in S_+$. Similarly we show that $v \leq \tilde{v}$ for all $v \in S_-$. \square

Now we are ready to produce extremal constant sign solutions for problem (1).

Proposition 7. *If hypotheses $H(f)_1$ hold, then problem (1) has a smallest positive solution $u_* \in \text{int } C_+$ and a biggest negative solution $v_* \in -\text{int } C_+$.*

Proof. As in Filippakis-Papageorgiou [3] (see also Papageorgiou-Rădulescu-Repovš [17]), we have that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1, u \leq u_2$). Then invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [10], we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n \geq 1} u_n.$$

We have

$$(24) \quad \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle = \int_{\Omega} f(z, u_n) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N},$$

$$(25) \quad \tilde{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 6).}$$

If in (24) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (25) and hypothesis $H(f)_1$ (i), then we have that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$(26) \quad u_n \xrightarrow{w} u_* \text{ in } W_0^{1,p}(\Omega).$$

If in (24) we choose $h = u_n - u_* \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \rightarrow +\infty$, then using (26) we obtain

$$\lim_{n \rightarrow +\infty} [\langle A_p(u_n), u_n - u_* \rangle + \langle A(u_n), u_n - u_* \rangle] = 0,$$

$$\begin{aligned} &\Rightarrow \limsup_{n \rightarrow +\infty} [\langle A_p(u_n), u_n - u_* \rangle + \langle A(u_*), u_n - u_* \rangle] \leq 0 \text{ (since } A(\cdot) \text{ is monotone),} \\ &\Rightarrow \limsup_{n \rightarrow +\infty} \langle A_p(u_n), u_n - u_* \rangle \leq 0 \text{ (see (26)),} \end{aligned}$$

$$(27) \Rightarrow u_n \rightarrow u_* \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1).}$$

So, if in (24) we pass to the limit as $n \rightarrow +\infty$ and use (27), then

$$(28) \quad \langle A_p(u_*), h \rangle + \langle A(u_*), h \rangle = \int_{\Omega} f(z, u_*) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

In addition from (25) we have

$$(29) \quad \tilde{u} \leq u_*.$$

Then from (28), (29), we conclude that $u_* \in S_+ \subseteq \text{int } C_+$, $u_* = \inf S_+$. Similarly, we produce the biggest negative solution $v_* \in -\text{int } C_+$ of (1). Note that the set S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$, see [3]). \square

Now that we have the extremal constant sign solutions, we can produce a nodal (sign changing) solution for problem (1). To do this, we need to strengthen a little the condition on $f(z, \cdot)$ near zero. So, the new hypotheses on $f(z, x)$ are the following:

$H(f)_2$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(f)_2$ (i), (ii) are the same as the corresponding hypotheses $H(f)_1$ (i), (ii) and

$$(iii) \quad \widehat{\lambda}_1(2) < \eta_0 \leq \liminf_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \limsup_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \widehat{\eta}_0 \text{ uniformly for a.a. } z \in \Omega.$$

In what follows $u_* \in \text{int } C_+$ and $v_* \in -\text{int } C_+$ are the two extremal constant sign solutions of problem (1) produced in Proposition 7. The idea is to produce a nontrivial solution y_0 of (1) such that $y_0 \in [v_*, u_*]$, $y_0 \notin \{v_*, u_*\}$. On account of the extremality of u_* and v_* , such a nontrivial solution will necessarily be nodal.

Proposition 8. *If hypotheses $H(f)_2$ hold, then problem (1) admits a nodal solution $y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega})$.*

Proof. Using the extremal constant sign solutions $u_* \in \text{int } C_+$ and $v_* \in -\text{int } C_+$ (see Proposition 7), we introduce the following truncation of the reaction $f(z, \cdot)$

$$(30) \quad k(z, x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z), \\ f(z, x) & \text{if } v_*(z) \leq x \leq u_*(z), \\ f(z, u_*(z)) & \text{if } u_*(z) < x. \end{cases}$$

We also consider the positive and negative truncations of $k(z, \cdot)$, namely the functions

$$(31) \quad k_{\pm}(z, x) = k(z, \pm x^{\pm}).$$

All three functions are Carathéodory functions. We set

$$K(z, x) = \int_0^x k(z, s) ds \text{ and } K_{\pm}(z, x) = \int_0^x k_{\pm}(z, s) ds$$

and consider the C^1 -functionals $\varphi, \varphi_{\pm} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_{\Omega} K(z, u) dz,$$

$$\varphi_{\pm}(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_{\Omega} K_{\pm}(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (30) and (31), we can easily show that

$$K_{\varphi} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad K_{\varphi_+} \subseteq [0, u_*] \cap C_+, \quad K_{\varphi_-} \subseteq [v_*, 0] \cap (-C_+).$$

The extremality of u_* and v_* implies that

$$(32) \quad K_{\varphi} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad K_{\varphi_+} = \{0, u_*\}, \quad K_{\varphi_-} = \{0, v_*\}.$$

Note that φ_+ is coercive and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \varphi_+(\widehat{u}_*) &= \inf [\varphi_+(u) : u \in W_0^{1,p}(\Omega)] < 0 = \varphi_+(0) \quad (\text{see hypothesis } H(f)_2 \text{ (iii)}), \\ \Rightarrow \widehat{u}_* &\neq 0. \end{aligned}$$

Since $\widehat{u}_* \in K_{\varphi_+}$, from (32) it follows that $\widehat{u}_* = u_* \in \text{int } C_+$. Note that $\varphi|_{C_+} = \varphi_+|_{C_+}$. Hence we have

$$\begin{aligned} &u_* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi, \\ (33) \quad &\Rightarrow u_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi \end{aligned}$$

(see Gasiński-Papageorgiou [5], Proposition 2.6). Similarly using this time the functional φ_- , we show that

$$(34) \quad v_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi.$$

We may assume that $\varphi(v_*) \leq \varphi(u_*)$ (the reasoning is similar if the opposite inequality holds using this time (29) instead of (28)) and also that K_{φ} is finite (otherwise on account of (32) we already have an infinity of smooth nodal solutions). Using Theorem 5.7.6, p. 367, of Papageorgiou-Rădulescu-Repovš [18], we can find $\rho \in (0, 1)$ small such that

$$(35) \quad \varphi(v_*) \leq \varphi(u_*) < \inf [\varphi(u) : \|u - u_*\| = \rho] = m_{\rho}, \quad \|v_* - u_*\| > \rho.$$

The functional $\varphi(\cdot)$ is coercive (see (30)). Therefore

$$(36) \quad \varphi(\cdot) \text{ satisfies the } C\text{-condition.}$$

Then (35) and (36) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$(37) \quad y_0 \in K_{\varphi} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}) \text{ and } m_{\rho} \leq \varphi(y_0).$$

From (37) we see that if we can show that $y_0 \neq 0$, then $y_0 \in C_0^1(\overline{\Omega})$ will be the desired smooth nodal solution of (1). Therefore in what follows, we show that $y_0 \neq 0$. From the mountain pass theorem, we have

$$(38) \quad \varphi(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = v_*, \gamma(1) = u_*\}$. From (38) we see that if we can produce a path $\gamma_* \in \Gamma$ such that $\varphi|_{\gamma_*} < 0$, then $\varphi(y_0) < 0 = \varphi(0)$ (see (38)) and so we have $y_0 \neq 0$. Hence our aim is to construct such a path $\gamma_* \in \Gamma$. We define $M = H_0^1(\Omega) \cap \partial B_1^{L^2}$ and let $M_c = M \cap C_0^1(\overline{\Omega})$. We consider the following two sets of continuous paths

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1(2), \widehat{\gamma}(1) = \widehat{u}_1(2)\},$$

$$\widehat{\Gamma}_c = \{\widehat{\gamma} \in C([-1, 1], M_c) : \widehat{\gamma}(-1) = -\widehat{u}_1(2), \widehat{\gamma}(1) = \widehat{u}_1(2)\}.$$

From Papageorgiou-Winkert [22] (see the Claim in the proof of Theorem 4.5), we have that $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$. So, using Proposition 2, we see that given $\delta > 0$, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ such that

$$(39) \quad \max_{-1 \leq t \leq 1} \|\nabla \widehat{\gamma}_0(t)\|_2^2 \leq \widehat{\lambda}_2(2) + \delta.$$

On account of hypothesis $H(f)_2$ (iii), we see that if $\delta > 0$ is small, we can find $\widetilde{\eta}_0 > \widehat{\lambda}_2(2) + \delta$ and $\delta_0 \in (0, \delta)$ such that

$$(40) \quad F(z, x) \geq \frac{1}{2} \widetilde{\eta}_0 x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.$$

Since $\widehat{\gamma}_0 \in \Gamma_c$ and $u_* \in \text{int } C_+$ and $v_* \in -\text{int } C_+$, we can find $\lambda \in (0, 1)$ small such that

$$(41) \quad \lambda \widehat{\gamma}_0(t) \in [v_*, u_*] \text{ and } \lambda |\widehat{\gamma}_0(t)| \leq \delta_0 \quad \text{for all } t \in [-1, 1].$$

Then we have

$$\begin{aligned} \varphi(\lambda \widehat{\gamma}_0(t)) &\leq \frac{\lambda^p}{p} \|\nabla \widehat{\gamma}_0(t)\|_p^p + \frac{\lambda^2}{2} [\widehat{\lambda}_2(2) + \delta - \widetilde{\eta}_0] \quad (\text{see (39), (40), (41)}) \\ &\leq \lambda^p c_6 - c_7 \lambda^2 \quad \text{for some } c_6, c_7 > 0, \text{ all } t \in [-1, 1]. \end{aligned}$$

Since $p > 2$, choosing $\lambda \in (0, 1)$ even smaller if necessary, we have

$$\varphi(\lambda \widehat{\gamma}_0(t)) < 0 \quad \text{for all } t \in [-1, 1].$$

If we set $\gamma_0 = \lambda \widehat{\gamma}_0$, then γ_0 is a continuous path in $M_c \subseteq W_0^{1,p}(\Omega)$ which connects $-\lambda \widehat{u}_1(2)$ and $\lambda \widehat{u}_1(2)$ and we have

$$(42) \quad \varphi|_{\gamma_0} < 0.$$

Next we will produce a continuous path in $W_0^{1,p}(\Omega)$ which connects $\lambda \widehat{u}_1(2)$ and u_* and along which the functional φ is negative. To this end let

$$a = \varphi_+(u_*) = \inf \varphi_+ < 0 = \varphi_+(0).$$

From (32) we see that

$$K_{\varphi_+}^0 = \{0\} \text{ and } \varphi_+^a = \{u_*\}.$$

Invoking the second deformation theorem (see Papageorgiou-Rădulescu-Repovš [18], Theorem 5.3.12, p. 317), we can find a deformation $h : [0, 1] \times (\varphi_+^0 \setminus K_{\varphi_+}^0) \rightarrow \varphi_+^0$ such that

$$(43) \quad h(0, u) = u \quad \text{for all } u \in \varphi_+^0 \setminus K_{\varphi_+}^0 = \varphi_+^0 \setminus \{0\},$$

$$(44) \quad h(1, \varphi_+^0 \setminus \{0\}) = u_* = \varphi_+^a,$$

$$(45) \quad \varphi_+(h(t, u)) \leq \varphi_+(h(s, u)) \quad \text{for all } 0 \leq s \leq t, \text{ all } u \in \varphi_+^0 \setminus \{0\}.$$

Note that

$$\begin{aligned} \varphi_+(\lambda \widehat{u}_1(2)) &= \varphi(\lambda \widehat{u}_1(2)) = \varphi(\gamma_0(1)) < 0 \quad (\text{see (42)}), \\ \Rightarrow \lambda \widehat{u}_1(2) &\in \varphi_+^0 \setminus \{0\}. \end{aligned}$$

So, we can define

$$\gamma_+(t) = h(t, \lambda \widehat{u}_1(2))^+ \quad \text{for all } t \in [0, 1].$$

This is a continuous path in $W_0^{1,p}(\Omega)$ and $\gamma_+(t) \geq 0$ for all $t \in [0, 1]$. Moreover, we have

$$\gamma_+(0) = \lambda \widehat{u}_1(2) \text{ (see (43)), } \gamma_+(1) = u_* \text{ (see (44)).}$$

So, the continuous path γ_+ connects $\lambda \widehat{u}_1(2)$ and u_* . Finally along this path we have

$$\begin{aligned} \varphi(\gamma_+(t)) = \varphi_+(\gamma_+(t)) &\leq \varphi_+(\lambda \widehat{u}_1(2)) = \varphi(\lambda \widehat{u}_1(2)) < 0 \\ &\text{for all } t \in [0, 1] \text{ (see (42), (45)),} \end{aligned}$$

$$(46) \quad \Rightarrow \quad \varphi|_{\gamma_+} < 0.$$

Similarly we produce a continuous path γ_- in $W_0^{1,p}(\Omega)$ which connects $-\lambda \widehat{u}_1(2)$ and v_* and along which we have

$$(47) \quad \varphi|_{\gamma_-} < 0.$$

We concatenate γ_- , γ_0 , γ_+ and we have a continuous path γ_* in $W_0^{1,p}(\Omega)$ which connects v_* and u_* and along which we have

$$\varphi|_{\gamma_*} < 0 \quad (\text{see (42), (46), (47)}).$$

Therefore we conclude that

$$\begin{aligned} y_0 &\neq 0 \quad (\text{see (38)}), \\ \Rightarrow y_0 &\in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1), } y_0 \in [v_*, u_*]. \end{aligned}$$

□

We can improve the conclusion of this proposition, if we strengthen further the conditions on $f(z, \cdot)$ near zero.

So, the new conditions on the reaction $f(z, x)$ are the following:

$H(f)_3$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(f)_3$ (i), (ii), (iii) are the same as the corresponding hypotheses $H(f)_2$ (i), (ii), (iii) and

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \rightarrow f(z, x) + \widehat{\xi}_\rho |x|^{p-2} x$$

is nondecreasing on $[-\rho, \rho]$

Proposition 9. *If hypotheses $H(f)_3$ hold, then problem (1) has a nodal solution $y_0 \in \text{int } C_0^1(\overline{\Omega})[v_*, u_*]$.*

Proof. From Proposition 8, we already have a nodal solution $y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega})$.

Let $\rho = \max\{\|u_*\|_\infty, \|v_*\|_\infty\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)_3$ (iv). Take $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$. Then we have

$$\begin{aligned} & -\Delta_p y_0 - \Delta y_0 + \widetilde{\xi}_\rho |y_0|^{p-2} y_0 \\ &= f(z, y_0) + \widetilde{\xi}_\rho |y_0|^{p-2} y_0 \\ &\leq f(z, u_*) + \widetilde{\xi}_\rho u_*^{p-1} \quad (\text{see hypothesis } H(f)_3 \text{ (iv)}) \\ (48) \quad &= -\Delta_p u_* - \Delta u_* + \widetilde{\xi}_\rho u_*^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

Let $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be defined by

$$a(y) = |y|^{p-2} y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Hence $\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u$ for all $u \in W_0^{1,p}(\Omega)$. Also, $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and

$$\begin{aligned} \nabla a(y) &= |y|^{p-2} \left[I + (p-2) \frac{y \otimes y}{|y|^2} \right] + I \quad \text{for all } y \in \mathbb{R}^N, \\ \Rightarrow (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} &\geq |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N. \end{aligned}$$

Since $u_*, y \in C_0^1(\overline{\Omega})$ are solutions of problem (1) and $y \leq u_*$, we can apply the tangency principle of Pucci-Serrin [24] (Theorem 2.5.2, p. 35) and have

$$(49) \quad y(z) < u_*(z) \quad \text{for all } z \in \Omega.$$

Note that

$$\begin{aligned} f(z, y_0) + \widehat{\xi}_\rho |y_0|^{p-2} y_0 + (\widetilde{\xi}_\rho - \widehat{\xi}_\rho) |y_0|^{p-2} y_0 \\ \leq f(z, u_*) + \widehat{\xi}_\rho u_*^{p-1} + (\widetilde{\xi}_\rho - \widehat{\xi}_\rho) u_*^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

If we set

$$h = (\widetilde{\xi}_\rho - \widehat{\xi}_\rho) [u_*^{p-1} - |y_0|^{p-2} y_0],$$

then we have that $0 \prec h$ in the sense that for every compact $K \subseteq \Omega$, we can find $c_K > 0$ such that

$$c_K \leq h(z) \quad \text{for all } z \in K \text{ (see (49)).}$$

Then from (48) and Proposition 3.2 of Gasiński-Papageorgiou [8], we infer that $u_* - y_0 \in \operatorname{int} C_+$. Similarly, we show that $y_0 - v_* \in \operatorname{int} C_+$. We conclude that $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$. \square

Now we can state our first multiplicity theorem for problem (1).

Theorem 1. (a) *If hypotheses $H(f)_2$ hold, then problem (1) has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, $y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega})$ nodal.*

(b) *If hypotheses $H(f)_3$ hold, then problem (1) has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0]$ nodal.*

4. FOUR NONTRIVIAL SOLUTIONS

In this section by improving the regularity of $f(z, \cdot)$ and by using tools from the theory of critical groups (Morse theory), we produce a second smooth nodal solution, for a total of four nontrivial smooth solutions, all with sign information.

The new hypotheses on the reaction $f(z, x)$ are the following:

$H(f)_4$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

- (i) $|f'_x(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
- (ii) $\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = -\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$ uniformly for a.a. $z \in \Omega$ and there exist $m \in \mathbb{N}$, $m \geq 2$, $\delta > 0$ and $\theta \in L^\infty(\Omega)$ such that

$$\text{if } m \geq 3, \text{ then } \widehat{\lambda}_m(2) \leq \theta(z) \text{ for a.a. } z \in \Omega, \theta \not\equiv \widehat{\lambda}_m(2),$$

$$\text{if } m = 2, \text{ then } \widehat{\lambda}_2(2) < \operatorname{ess\,inf}_\Omega \theta,$$

$$\theta(z)x^2 \leq f(z, x)x \leq \widehat{\lambda}_{m+1}(2)x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta$$

and when $x \neq 0$ the last inequality is strict on a set of positive measure;
 (iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \rightarrow f(z, x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$

Remark 1. Evidently hypothesis $H(f)_4$ (iii) covers also the case where

$$\lim_{x \rightarrow 0} \frac{f(z, x)}{x} = \widehat{\lambda}_{m+1}(2) \quad \text{uniformly for a.a. } z \in \Omega.$$

So, we can have resonance at zero with respect to any eigenvalue $\widehat{\lambda}_{m+1}(2)$ with $m \in \mathbb{N}$, $m \geq 2$.

Proposition 10. *If hypotheses $H(f)_4$ hold, then problem (1) has a second nodal solution $\widehat{y} \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$.*

Proof. From Theorem 1(b) we already have three nontrivial solutions

$$u_* \in \text{int } C_+, v_* \in -\text{int } C_+, y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*] \text{ nodal.}$$

We use the functional $\varphi \in C^1(W_0^{1,p}(\Omega))$ from the proof of Proposition 8. Recall that $u_* \in \text{int } C_+$ and $v_* \in -\text{int } C_+$ are local minimizers of φ (see (33), (34)). Therefore

$$(50) \quad C_k(\varphi, u_*) = C_k(\varphi, v_*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Let $\rho = \max\{\|v_*\|_\infty, \|u_*\|_\infty\}$ and choose a cut-off function $\xi \in C_c^2(\mathbb{R})$ such that

$$0 \leq \xi \leq 1 \text{ and } \xi|_{[-\rho, \rho]} \equiv 1.$$

We set $\widehat{f}_c(z, x) = f(z, \xi(x))$. Then $\widehat{f}_c(\cdot, \cdot)$ is measurable and for a.a. $z \in \Omega$, $\widehat{f}_c(z, 0) = 0$, $\widehat{f}_c(z, \cdot) \in C^1(\mathbb{R})$. We set $\widehat{F}_c(z, x) = \int_0^x \widehat{f}_c(z, s)ds$ and consider the functional $\widehat{\varphi}_c : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_c(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega \widehat{F}_c(z, u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have $\widehat{\varphi}_c \in C^2(W_0^{1,p}(\Omega))$, it is coercive (hence satisfies the C -condition) and

$$\widehat{\varphi}_c|_{[v_*, u_*]} = \varphi|_{[v_*, u_*]}, \quad \widehat{\varphi}_c'|_{[v_*, u_*]} = \varphi'|_{[v_*, u_*]}.$$

Recall that we assume that K_φ is finite. Otherwise on account of (32) we already have an infinity of smooth nodal solutions and so we are done. We consider the homotopy

$$h_t(u) = h(t, u) = (1-t)\varphi(u) + t\widehat{\varphi}_c(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(51) \quad t_n \rightarrow t, u_n \rightarrow y_0 \text{ in } W_0^{1,p}(\Omega) \text{ and } (h_{t_n})'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

We have

$$\langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle = \int_\Omega [(1-t_n)f(z, u_n) + t_n\widehat{f}_c(z, u_n)]h dz \quad \text{for all } W_0^{1,p}(\Omega),$$

$$\Rightarrow -\Delta_p u_n(z) - \Delta u_n(z) = (1-t_n)f(z, u_n(z)) + t_n\widehat{f}_c(z, u_n(z)) \quad \text{for a.a. } z \in \Omega,$$

$$u_n|_{\partial\Omega} = 0 \quad \text{for all } n \in \mathbb{N}.$$

As before (see the proof of Proposition 4), from Ladyzhenskaya-Ural'tseva [11] (p. 286), we have

$$\|u_n\|_\infty \leq c_8 \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N}.$$

Invoking Theorem 1 of Lieberman [14], we can find $\alpha \in (0, 1)$ and $c_9 > 0$ such that

$$(52) \quad u_n \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_9 \quad \text{for all } n \in \mathbb{N}.$$

Since $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly, from (51) and (52) it follows that

$$(53) \quad u_n \rightarrow y_0 \text{ in } C_0^1(\overline{\Omega}).$$

We know that $y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$. So, we will have

$$u_n \in [v_*, u_*] \quad \text{for all } n \geq n_0 \text{ (see (53)).}$$

This contradicts the fact that K_φ is finite (see (32)). Therefore (51) cannot happen and then the homotopy invariance of critical groups (see Papageorgiou-Rădulescu-Repovš [18], Theorem 6.3.6, p. 413) implies that

$$(54) \quad \begin{aligned} C_k(h_0, y_0) &= C_k(h_1, y_0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow C_k(\varphi, y_0) &= C_k(\widehat{\varphi}_c, y_0) \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

We know that y_0 is a critical point of φ of mountain pass type (see the proof of Proposition 8). Invoking Theorem 6.5.8, p. 431, of Papageorgiou-Rădulescu-Repovš [18], we have

$$(55) \quad \begin{aligned} C_1(\varphi, y_0) &\neq 0, \\ \Rightarrow C_1(\widehat{\varphi}_c, y_0) &\neq 0 \quad \text{(see (54)).} \end{aligned}$$

Recall that $\widehat{\varphi}_c \in C^2(W_0^{1,p}(\Omega))$. Then from (55) and Papageorgiou-Rădulescu [15] (see the proof of Proposition 3.5), we have

$$(56) \quad \begin{aligned} C_k(\widehat{\varphi}_c, y_0) &= \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow C_k(\varphi, y_0) &= \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (54)).} \end{aligned}$$

Consider the coercive C^2 -functional $\widehat{\sigma} : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\sigma}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z, u)dz \quad \text{for all } u \in H_0^1(\Omega).$$

Let $\sigma = \widehat{\sigma}|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely, from Theorem 6.6.26, p. 444, of Papageorgiou-Rădulescu-Repovš [18], we have

$$(57) \quad C_k(\sigma, 0) = C_k(\widehat{\sigma}, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

Claim: $C_k(\sigma, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_m = \dim \overline{H}_m$, $\overline{H}_m = \bigoplus_{k=1}^m E(\widehat{\lambda}_k(2))$ (see $H(f)_4$ (iii)). Let $\beta \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ and consider the C^2 -functional $\widehat{\sigma}_0 : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\sigma}_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\beta}{2}\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

We consider the homotopy $\widehat{h}(t, u)$ defined by

$$\widehat{h}(t, u) = (1-t)\widehat{\sigma}(u) + t\widehat{\sigma}_0(u) \quad \text{for all } (t, u) \in [0, 1] \times H_0^1(\Omega).$$

Let $0 < t \leq 1$ and let $u \in C_0^1(\bar{\Omega})$ with $\|u\|_{C_0^1(\bar{\Omega})} \leq \delta$ where $\delta > 0$ is as in hypothesis $H(f)_3$ (iii). In what follows by $\langle \cdot, \cdot \rangle_H$ we denote the duality brackets for the pair $(H_0^1(\Omega), H^{-1}(\Omega) = H_0^1(\Omega)^*)$. We have

$$(58) \quad \langle \hat{h}'_u(t, u), h \rangle_H = (1-t) \langle \hat{\sigma}'(u), h \rangle_H + t \langle \hat{\sigma}'_0(u), h \rangle_H.$$

We consider the following orthogonal direct sum decomposition:

$$H_0^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1} \quad \text{with } \bar{H}_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k(2)), \quad \hat{H}_{m+1} = \bar{H}_m^\perp.$$

So, if $u \in H_0^1(\Omega)$, then it admits a unique sum decomposition

$$u = \bar{u} + \hat{u} \quad \text{with } \bar{u} \in \bar{H}_m, \quad \hat{u} \in \hat{H}_{m+1}.$$

In (58) we choose $h = \hat{u} - \bar{u} \in H_0^1(\Omega)$. Exploiting the orthogonality of the component spaces, we have

$$(59) \quad \langle \hat{\sigma}'(u), \hat{u} - \bar{u} \rangle_H = \|\nabla \hat{u}\|_2^2 - \|\nabla \bar{u}\|_2^2 - \int_{\Omega} f(z, u)(\hat{u} - \bar{u}) dz.$$

From hypothesis $H(f)_3$ (iii) we have

$$\theta(z) \leq \frac{f(z, x)}{x} \leq \hat{\lambda}_{m+1}(2) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 < |x| \leq \delta.$$

Setting $y = \hat{u} - \bar{u}$, we have

$$\begin{aligned} f(z, u)(\hat{u} - \bar{u}) &= f(z, u)y \\ &= \frac{f(z, u)}{u} uy \\ &\leq \begin{cases} \hat{\lambda}_{m+1}(2)[\hat{u}^2 - \bar{u}^2] & \text{if } uy \geq 0, \\ \theta(z)[\hat{u}^2 - \bar{u}^2] & \text{if } uy < 0 \end{cases} \quad (\text{see hypothesis } H(f)_4 \text{ (iii)}) \\ &\leq \hat{\lambda}_{m+1}(2)\hat{u}^2 - \theta(z)\bar{u}^2 \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

We use this in (59) and obtain

$$\langle \hat{\sigma}'(u), \hat{u} - \bar{u} \rangle_H \geq \|\nabla \hat{u}\|_2^2 - \hat{\lambda}_{m+1}(2)\|\hat{u}\|_2^2 - \left[\|\nabla \bar{u}\|_2^2 - \int_{\Omega} \theta(z)\bar{u}^2 dz \right] \geq 0 \quad (\text{see (5)}).$$

Also, we have

$$\begin{aligned} \langle \hat{\sigma}'_0(u), \hat{u} - \bar{u} \rangle_H &= \|\nabla \hat{u}\|_2^2 - \beta \|\hat{u}\|_2^2 - [\|\nabla \bar{u}\|_2^2 - \beta \|\bar{u}\|_2^2] \\ &\geq c_{10} \|u\|^2 \quad \text{for some } c_{10} > 0 \quad (\text{see Proposition 3}). \end{aligned}$$

Therefore for $0 < t \leq 1$, we have

$$\langle \hat{h}'_u(t, u), \hat{u} - \bar{u} \rangle_H \geq tc_{10} \|u\|^2 > 0 \quad \text{if } u \neq 0.$$

So, $u = 0$ is isolated in $K_{\hat{h}(t, \cdot)} \subseteq C_0^1(\bar{\Omega})$ (by standard regularity theory) for all $t \in (0, 1]$.

Next let $t = 0$. Again we show that $u = 0$ is isolated in $K_{\hat{h}(0, \cdot)} = K_{\hat{\sigma}} \subseteq C_0^1(\bar{\Omega})$. Arguing indirectly, suppose that we could find $\{u_n\}_{n \geq 1} \subseteq H_0^1(\Omega)$ such that

$$(60) \quad \{u_n\}_{n \geq 1} \subseteq K_{\hat{\sigma}} \text{ and } u_n \rightarrow 0 \quad \text{in } H_0^1(\Omega).$$

Then the regularity theory and the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ ($0 < \alpha < 1$) into $C_0^1(\overline{\Omega})$ imply that

$$\begin{aligned} & u_n \rightarrow 0 \text{ in } C_0^1(\overline{\Omega}) \quad (\text{see (60)}), \\ \Rightarrow & |u_n(z)| \leq \delta \quad \text{for all } z \in \overline{\Omega}, \text{ all } n \geq n_0, \\ \Rightarrow & \theta(z)u_n(z)^2 \leq f(z, u_n(z))u_n(z) \leq \widehat{\lambda}_{m+1}(2)u_n(z)^2 \\ & \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0 \text{ (see hypothesis } H(f)_4 \text{ (iii))}. \end{aligned}$$

Hence we have

$$(61) \quad f(z, u_n)(\widehat{u}_n - \overline{u}_n) \leq \widehat{\lambda}_{m+1}(2)\widehat{u}_n^2 - \theta(z)\overline{u}_n^2 \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0.$$

Since $u_n \in K_{\widehat{\sigma}}$ (see (60)), we have

$$\langle A(u_n), h \rangle = \int_{\Omega} f(z, u_n)h dz \quad \text{for all } h \in H_0^1(\Omega), \text{ all } n \in \mathbb{N}.$$

Choosing $h = \widehat{u} - \overline{u} \in H_0^1(\Omega)$ and using (61), we obtain

$$\begin{aligned} 0 & \leq \|\nabla \widehat{u}_n\|_2^2 - \widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2 = \|\nabla \overline{u}_n\|_2^2 - \int_{\Omega} \theta(z)\overline{u}_n^2 dz \\ & \leq -c_{11}\|\overline{u}_n\|^2 \quad \text{for all } n \geq n_0 \text{ (see Proposition 3)}, \\ \Rightarrow & \overline{u}_n = 0 \text{ and } u_n = \widehat{u}_n \in E(\widehat{\lambda}_{m+1}(2)) \quad \text{for all } n \geq n_0. \end{aligned}$$

The Unique Continuation Property of $E(\widehat{\lambda}_{m+1}(2))$ implies that

$$\widehat{u}_n(z) \neq 0 \quad \text{for a.a. } z \in \Omega, n \geq n_0.$$

Using this fact and hypothesis $H(f)_4$ (iii), we have

$$\widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2 = \int_{\Omega} f(z, \widehat{u}_n)\widehat{u}_n dz < \widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2 \quad \text{for all } n \geq n_0,$$

a contradiction. So, we conclude that $u = 0$ is isolated in $K_{\widehat{\sigma}} = K_{\widehat{h}(0, \cdot)}$.

The homotopy invariance property of critical groups implies

$$(62) \quad C_k(\widehat{\sigma}, 0) = C_k(\widehat{\sigma}_0, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

Recall that $\beta \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$. So, Proposition 6.6.19, p. 440, of Papageorgiou-Rădulescu-Repovš [18], implies that

$$\begin{aligned} & C_k(\widehat{\sigma}_0, 0) = \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = \dim \overline{H}_m, \\ \Rightarrow & C_k(\widehat{\sigma}, 0) = \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (62))}, \\ \Rightarrow & C_k(\sigma, 0) = \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (57))}. \end{aligned}$$

This proves the Claim. Note that

$$\begin{aligned} & |\varphi(u) - \sigma(u)| \leq \frac{1}{p} \|\nabla u\|_p^p, \\ & |\langle \varphi'(u) - \sigma'(u), h \rangle| = \left| \int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla h)_{\mathbb{R}^N} dz \right| \\ & \leq \int_{\Omega} |\nabla u|^{p-1} |\nabla h| dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow & \|\varphi'(u) - \sigma'(u)\|_* \leq \|\nabla u\|_p^{p-1}. \end{aligned}$$

Using the C^1 -continuity property of critical groups (see Papageorgiou-Rădulescu-Repovš [18], Theorem 6.3.4, p. 17), we have

$$(63) \quad \begin{aligned} C_k(\varphi, 0) &= C_k(\sigma, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow C_k(\varphi, 0) &= \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

Since $\varphi(\cdot)$ is coercive, we have

$$(64) \quad C_k(\varphi, \infty) = \delta_{k, 0} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Suppose that $K_\varphi = \{u_*, v_*, y_0, 0\}$. Then (50), (56), (63), (64) and the Morse relation with $t = -1$ (see (6)), imply that

$$\begin{aligned} 2(-1)^0 + (-1)^1 + (-1)^{d_m} &= (-1)^0, \\ \Rightarrow (-1)^{d_m} &= 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\hat{y} \in K_\varphi$, $\hat{y} \notin \{u_*, v_*, y_0, 0\}$. As we did for y_0 (see Proposition 9), we show that

$$\hat{y} \in \text{int}_{C_0^1(\bar{\Omega})}[v_*, u_*] \text{ and } \hat{y} \in C_0^1(\bar{\Omega}) \text{ is nodal.}$$

□

Now we can state the second multiplicity theorem for problem (1).

Theorem 2. *If hypotheses $H(f)_4$ hold, then problem (1) has at least four nontrivial solutions $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$, $y_0, \hat{y} \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0]$ nodal.*

5. A SPECIAL CASE

In this section we present an application of our multiplicity theorems (Theorems 1 and 2), to the following parametric $(p, 2)$ -equation:

$$(65) \quad -\Delta_p u(z) - \Delta u(z) = \lambda u(z) - g(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \lambda > 0.$$

We impose the following conditions on the perturbation $g(z, x)$.

$H(g)_1$: $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for a.a. $z \in \Omega$ and

- (i) $|g(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
- (ii) $\lim_{x \rightarrow \pm\infty} \frac{g(z, x)}{|x|^{p-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $\lim_{x \rightarrow 0} \frac{g(z, x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;
- (iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \rightarrow \hat{\xi}_\rho |x|^{p-2}x - g(z, x)$$

is nondecreasing on $[-\rho, \rho]$.

Then from Theorem 1(b), we infer the following multiplicity theorem for problem (65) (a subdiffusive nonlinear logistic equation).

Proposition 11. *If hypotheses $H(g)_1$ hold, then for all $\lambda > \hat{\lambda}_2(2)$ problem (65) has at least three nontrivial solutions $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$, $y_0 \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0]$ nodal.*

We can improve this multiplicity result by strengthening the regularity of $g(z, \cdot)$. So, the new conditions on $g(z, x)$ are the following:

$H(g)_2$: $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $g(z, 0) = 0$, $g(z, \cdot) \in C^1(\mathbb{R})$ and

- (i) $|g'_x(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
- (ii) $\lim_{x \rightarrow \pm\infty} \frac{g(z, x)}{|x|^{p-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $g'_x(z, 0) = \lim_{x \rightarrow 0} \frac{g(z, x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;
- (iv) same as hypothesis $\tilde{H}(g)_1$ (iv).

Using Theorem 2, we can formulate the following multiplicity result for problem (65).

Proposition 12. *If hypotheses $H(g)_2$ hold, then for all $\lambda > \hat{\lambda}_2(2)$ problem (65) has at least four nontrivial solutions $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$, $y_0, \hat{y} \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0]$.*

Conflict of Interest: The authors declare that they have no conflict of interest.

Acknowledgment: The authors wish to thank the two knowledgeable referees for their corrections and remarks.

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