Our friend and mathematician Karl Strambach

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In memory of Prof. Karl Strambach (1939–2016)

Abstract. This paper is dedicated to Karl Strambach on the occasion of his 80th birthday. Here we want to describe our work with Prof. Karl Strambach.

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1. Introduction

Prof. Karl Strambach was the mathematician with a wide range of interests. He was engaged in research in many different areas of mathematics (group theory, differential geometry, etc.). Karl Strambah was one of the founders of the biannual workshop *"Groups and topological groups"*. Since the 1970's GTG meetings are organized by different universities in Central Europe and take place twice a year.

Karl Strambach was an unpretentious person in a life respecting and appreciating work of usual people. He was a person of the World. He knew many foreign languages, but the surprisingly melodious and beautiful Italian language was the favourite for him. The fact is that for his scientific work professor Strambach was very often in Italy. He even named one of his scientific friends (Adriano Barlotti) his quasi-father.

Peter Plaumann was one of the closest of his friends. They knew each other since student times.

Karl Strambach recalled with great gratitude the support which Peter rendered to him.

Karl Strambach was a very good mathematician and friend. He was kind and easy to communicate with. He always supported young scientists. Prof. K. Strambach cooperated with many people from different countries (Hungary, Italy, Germany, Czech Republic, Russia and others).

with Peter Plaumann
2015, Oaxaca, Mexico

Among his students were Gábor P. Nagy and Ágota Figula. His cooperation with Hungarian mathematicians was especially long and productive and the result of this work has been marked: in November 2007 K. Strambach became the honourable doctor of Debrecen university.

Mathematics was his life. Sometimes he was vulnerable as a child. But if he was fascinated by a problem, he was strong and he thought about it day and night. And he surely found a perfect solution.

2. Some Memories on Karl Strambach

The first time Heinrich Wefelscheid met Karl must have been in 1965 at the conference "Fundamentals of Geometry" in Oberwolfach. At that time, the old hunting lodge was still standing and because of the many participants, most were housed in the nearby guest houses. These meetings were first led by Emanuel Sperner and Friedrich Bachmann, Hans Freudenthal joined later.

The field: Fundamentals of Geometry is a continuation of the axiomatic approach, as Hilbert did in his famous book of the same name in 1899 in continuation of the thoughts of Euclid in an exemplary canonical manner. Expressed more simply: take some (reasonable) mathematical axioms, which taken together must be consistent, and try to determine all mathematical structures that fulfill this axiom system. Of course, this approach differs fundamentally from mathematics based on rich basic assumptions, such as those found in calculus, differential geometry, analytic numbers theory, and so on.

These meetings on the foundations of geometry took place annually, bringing together, as it were, all the older and younger researchers who were interested in it. Among the "old" would be to name: in addition to Friedrich Bachmann, Hans Freudenthal, Emanuel Sperner of course, still Reinhold Bear and Günther Pickert.

Helmut Karzel, Heinz Lüneburg, Jakob Joussen, Johannes André, Helmut Salzmann (PhD thesis supervisor of Karl), Günter Ewald and Walter Benz were the next generation, all of whom were already lecturers or shortly before their habilitation. Then the "young" people like Manfred Meurer, Karl Strambach, Peter Plaumann, Eberhard Schröder, Kay Sörensen, Dieter Betten, Hermann Hähl, Rainer Löwen, Steffen Timm, Hans-Joachim Arnold, Wilhelm Junkers, Martin Götzky, Armin Herzer, Werner Heise, Hans-Joachim Kroll, Heinz Wähling, William Kerby, Werner Leißner, Irene Pieper-Seier, Werner Seier, and Theo Grundhöfer came. We to be excused, if we have not mentioned someone who should be mentioned.

Of course, this circle fluctuated. One of the main interests in the foundations of geometry is, on the one hand, to find algebraic structures for geometric structures, with which one can describe these geometries, if possible in a clear manner, and vice versa. A variant of this approach is to determine the structure itself, which has G as an automorphism group, backwards from the automorphism group G of a mathematical structure.

After Sperner's death in 1980, the geometry conferences in Oberwolfach became rarer. But in 1974, the Karzel workshops at the Technical University of Munich gained in importance. The geographical proximity of Munich to

Italy naturally led to an ever closer contact with the Italian colleagues (Adriano Barlotti and Giuseppe Tallini, as well as Tullio Ceccherini-Silberstein and Mario Marchi).

An especially close bond developed between Karl and the Barlotti family. Karl loved the Italian way of life and the Italian language, which he spoke fluently. There were many combinatoric conferences, held every two years by different universities and generously supported by the provincial governments.

Since Völklein, who was student of Karl Strambach, has received a professorship in Essen, Karl came frequently, often for several weeks, to represent Völklein in lectures or to work as a guest professor. Karl usually lived in Heinrich Wefelscheid's guest house of the Dierks von Zweck Foundation. Some of Karl's essays of the last ten years were conceived here.

with Heinrich Wefelscheid
2013, Berlin, Germany

Karl was a restless spirit, almost always on the move. In fact, he traveled constantly, to Palermo to Giovanni Falcone, to Debrecen to Ágota Figula, to

Olomouc to Josef Mikeš, to Kaliningrad to Olga Belova. One could have the impression that he needed traveling to actively practice mathematics.

with Péter T. Nagy
2000, Oberwolfach, Germany

3. Loops

Péter T. Nagy began joint research with Karl Strambach in the end of 80th. At that time he worked in a research project with K. H. Hofmann on the development of the theory of topological and analytical loops and P. T. Nagy was interested in the differential-geometric theory of 3-webs.

Since differentiable 3-webs can be locally coordinatized using differentiable local loops, Karl proposed developing interesting classes for smooth loops using various differential geometric theories.

A loop is a non-associative generalization of the notion of a group. An invertible multiplication together with the left and right division and a unit element on a set is called a loop. The study of non-associative structures started in the first decades of the last century motivated by the foundation of geometry, particularly by the investigation of coordinate systems of non-Desarguesian planes.

Later, W. Blaschke formulated a scientific project to investigate topological questions of differential geometry, in particular topological local behavior of foliations.

In the half century 1940–1990 the theory of loops became an independent algebraic theory. The investigation of loops within the frame of topological algebra, topological geometry and differential geometry gained importance by the work of K. H. Hofmann, M. A. Akivis and L. V. Sabinin.

The basic idea of K. Strambach was to explore loops using the tools of group theory and Lie theory, describing the multiplication of loops by the set of left multiplication maps in the group generated by these maps, that is, considering the set of left multiplication maps as sections in a group, topologically generated by the left multiplications. In the first years of the cooperation of P. T. Nagy with K. Strambach, they systematically investigated one-dimensional and two-dimensional loops as sections in the possible transformation groups. P. T. Nagy and K. Strambach constructed many interesting examples which motivated them for the further research.

After the initial steps of their work, they devoted their study to the class of left conjugacy closed loops, which can be given by invariant sections in the group generated by their left multiplications, cf. [22]. These loops generalize the previously introduced conjugacy closed loops, assuming only from one side the invariance property of the corresponding section, just as the Bol condition for loops is an asymmetric version of Moufang condition. P. T. Nagy and K. Strambach studied the relationship of these loops to the common classes of loops and discussed many classical loop constructions. For example, it was proved that the proper topological left conjugacy closed loops on a connected manifold, satisfying the the left Bol condition, are necessarily groups. The isotopy classes of left conjugacy closed loops were investigated and it was showed that the corresponding configurational condition in the 3-net has the same importance in the geometry of 3-nets as the Reidemeister or the Bol condition.

The main tool for further loop theoretical studies was the construction of geodesic loops, defined by a linear connection on a manifold. Namely, the parallel translation of a geodesic segment defines a natural local loop structure on a manifold equipped with a linear connection. P. T. Nagy and K. Strambach investigated the interesting algebraic properties of geodesic loops of linear connections having a large group of affine automorphisms. In this way, the relationship between affine symmetric spaces, smooth Bol and Moufang loops, left distributive quasi-groups and differentiable 3-nets were explored, cf. [23]. P. T. Nagy and K. Strambach applied their method for the proof of analyticity of smooth Moufang loops and left distributive quasi-groups with involutive left multiplications, and also for the study of the Lie nature of the transformation groups naturally associated with some classes of smooth binary systems and

3-nets. In addition, P. T. Nagy and K. Strambach investigated the power series extension of geodesic loops satisfying some weak associativity condition and obtained the description of geodesic loops having Euclidean lines, either as their geodesic lines, or as geodesic lines of their core.

After more than 10 years of intensive cooperation, a monograph [24] on the theory of topological, differentiable, or analytical loops was published. P. T. Nagy and K. Strambach conducted a parallel study of loops where the multiplication, left and right division had topological, differentiable or analytic properties. Using the analogous constructions, algebraic loops were also investigated. The most important assumption was that the group G topologically generated by the left multiplications of a loop L is a Lie group. If H is the subgroup of G stabilizing the unit element of L and $\sigma\colon G/H \to G$ is the sharply transitive section determined by the left multiplications of L then the loop can be identified with the factor manifold G/H in such a way that the multiplication $(x_1H)\cdot(x_2H) = \sigma(x_1H)x_2H$ on G/H corresponds to the multiplication of L. P. T. Nagy and K. Strambach systematically investigated the generalization of the opposite construction: for an arbitrary triple $(G; H; \sigma)$ of a Lie group G, closed subgroup H and a sharply transitive section $\sigma\colon G/H \to G$ a loop multiplication is defined on G/H by the rule $(x_1H)\cdot(x_2H) = \sigma(x_1H)x_2H$ having the differentiability properties determined by the properties of σ.

The main results of the book are related to the study of new constructions of interesting classes of loops that satisfy various conditions of weak associativity, as well as the classification of related Lie algebraic, geometric, and group theoretical structures. It was known that the theory of differentiable Moufang loops can be classified by their tangential structures. Similarly, analytic Bol local loops can be classified by Bol algebras, but a local Bol loop may not necessarily be embedded into a global one. Therefore, the study of global differentiable loops cannot be reduced to that of local loops. But the procedure of examining suitable sections in Lie groups provide a very effective method for classifying differentiable global loops.

The first part of the book is devoted to the investigation of differentiable Bol loops and related algebraic and differential geometric structures: Bruck loops, symmetric spaces and quasi-groups, classification of compact Bol loops and loops with a compact group generated by left multiplications, etc. The second part contains a systematic study of low-dimensional differentiable loops and related structures.

After completing the book, P. T. Nagy and K. Strambach continued to work together in the covering theory of topological loops [25] and in the theory of loop extensions [26].

Their last joint work [27] was devoted to the memory of L. V. Sabinin, who formulated a generalization of systems of geodesic loops defined by a linear connection on a manifold. They formulated an abstract version of Sabinin's theory of transitive families of diffeomorphisms. P. T. Nagy and K. Strambach proved an isomorphism theorem for holonomy groups associated with a transitive family of transformations, and studied how the automorphism group of a transitive family affects the differential geometric properties and the algebraic structure of loops defined by this family.

Ágota Figula met Karl Strambach at the first time at University of Debrecen in 1998 when Karl visited Péter T. Nagy to collaborate on loops. She was a student of P. T. Nagy and had begun writing her diploma thesis on geodesic loops. Karl Strambach invited her to do scientific work at the Institute of Mathematics in Erlangen. There she took part in the weekly organized "Kandidaten Seminar". In the seminar K. Strambach, P. Plaumann, H. Kurzweil gave interesting problem questions and their students presented their scientific results. This relationship with K. Strambach was very fruitful and decisive for her whole life.

The associative law forces that the product $\lambda_a\lambda_b$ of any two left translations of a group G is the left translation λ_{ab}. Also the identity $(ab)^{-1}ab = 1$ holds for all elements a, b of a group G. For loops these behaviours change radically. This observation led to a broader research of loops L in which either for any two left translations λ_a, λ_b the product $\lambda_a\lambda_b\lambda_a$ is again a left translation $\lambda_{a(ba)}$ of L or the mapping $x \mapsto [(ab)^{-1}(a(bx))]$ is an automorphism of L. The loops with the first property have been called (left) Bol loops, with the second property left A-loops. Following the common research of P. T. Nagy and K. Strambach, K. Strambach and Á. Figula initiated the investigation and classification of the three-dimensional connected differentiable proper Bol loops and left A-loops having a non-solvable Lie group G as the group topologically generated by their left translations. The tangential objects of Bol loops are in one-to-one correspondence to global simply connected symmetric spaces, whereas those of left A-loops belong to affine reductive spaces, which are essential objects in differential geometry. If the group G is an at most nine-dimensional semi-simple Lie group, then the corresponding Bol loops L are either simple, or isotopic to the direct products of Bruck loops of hyperbolic type or to Scheerer extensions of Lie groups by Bruck loops of hyperbolic type. In [16] they determined all simple differentiable Bol loops L having the direct product $G_1 \times G_2$ of two groups with simple Lie algebras as the group topologically generated by their left translations such that the stabilizer of the identity element of L is the direct product $H_1 \times H_2$ with $H_i < G_i$. To obtain this classification they used intersection-free factorizations of simple Lie groups. The three-dimensional differentiable Bol and left A-loops L have the connected component of the motion group of the three-dimensional hyperbolic or pseudo-Euclidean geometry as the group topologically generated by the left translations and the set of the left translations of L induces on the plane at infinity the set of left translations of a loop isotopic to the hyperbolic plane loop. In [13] they seek a simple geometric procedure for an extension of a loop realized as the image Σ^* of a sharply transitive section in a subgroup G^* of the projective linear group $PGL(n-1, \mathbb{K})$ to a loop realized as the image of a sharply transitive section in a group $\Delta = T' \rtimes C$ of affinities of the n-dimensional space $\mathcal{A}_n = \mathbb{K}^n$ over a commutative field $\mathbb{K}$ such that T' is a large subgroup of the affine translations and $\alpha(C) = G^*$ holds for the canonical homomorphism $\alpha : GL(n, \mathbb{K}) \to PGL(n, \mathbb{K})$. The given construction is applied to sharply transitive sections in unitary and orthogonal groups $SU_{p_2}(n, \mathbb{F})$ of positive index p_2 over ordered pythagorean n-real fields $\mathbb{F}$ and in unitary or orthogonal Lie groups of any positive index. In [14] we construct a wide class of proper loops L which are semidirect products of groups of translations of an affine space $\mathcal{A}$ of dimension $2n$ over a commutative field $\mathbb{K}$ by suitable subgroups Γ_0' of $GL(2n, \mathbb{K})$. In many cases the elements of L are affine n-dimensional transversal subspaces of $\mathcal{A}$. This representation of L depends on the existence of a regular orbit in the hyperplane at infinity of $\mathcal{A}$ for the group Γ_0'. If the field $\mathbb{K}$ is a topological field then we obtain topological loops; for real or complex numbers the constructed loops are analytic. For smooth proper loops the group topologically generated by the left translations is a Lie group, but the groups topologically generated by the right translations of these loops are smooth groups having a normal Abelian subgroup of infinite dimension. Hence the groups topologically generated by all translations of these analytic loops are smooth transformation groups of infinite dimension. To any differentiable loop L one can associate an Akivis algebra which is realized in the tangent space of L at the identity $e \in L$ and plays a similar role as the Lie algebra in the case of a Lie group. The Akivis algebras of the constructed smooth loops are semidirect products of Lie algebras which shows that there are non-connected proper smooth loops having Lie algebras as their Akivis algebras.

The topological, respectively differentiable loops which are realized on compact manifolds are investigated by P. T. Nagy and K. Strambach in Sections 14 and 16 in [24]. In particular, in Section 16 the connected, simply connected differentiable compact proper Bol loops are classified. All these loops are either extended core loops of a pair (K, K_1) of non-trivial simply connected Lie groups K and K_1 or a Scheerer extension of a simply connected compact Lie group K by a Moufang loop or a Scheerer extension of an extended core loop by a Moufang loop. The group topologically generated by their left translations are direct product of at least three factors (cf. Theorem 16.7, in [24, p. 198]).

Topological loops are compact if the group topologically generated by their left translations is compact. The converse is not true already for differentiable loops defined on the circle (cf. Section 18 in [24]). These loops have a finite covering of the group $PSL_2(\mathbb{R})$ as the group topologically generated by their left translations and the stabilizer of their identities has no non-trivial compact subgroups. The differentiable one-dimensional loops can be classified by pairs of real functions which satisfy a differential inequality containing these functions and their first derivatives. In [15] K. Strambach and Á. Figula determined the functions satisfying this inequality explicitly in terms of Fourier series. Moreover, they showed that any topological loop L homeomorphic to a sphere or to a real projective space and having a compact-free Lie subgroup as the stabilizer of the identity of L in the Lie group topologically generated by all left translations is homeomorphic to the circle. Applying the investigation of H. Scheerer, who has clarified for which compact connected Lie groups G and for which closed subgroups H the natural projection $G \to G/H$ has a continuous section σ, in [18] K. Strambach and Á. Figula proved that there does not exist any connected topological proper loop homeomorphic to a quasi-simple Lie group and having a compact Lie group as the group topologically generated by its left translations. Similarly, any connected topological loop L homeomorphic to the 7-sphere and having a compact Lie group as the group topologically generated by its left translations is either the Moufang loop $\mathcal{O}$ of octonions of norm 1 or the factor loop $\mathcal{O}/Z$, where Z is the centre of $\mathcal{O}$. In contrast to this they gave a particular simple general construction for proper topological loops such that the compact group generated by their left translations is direct product of at least 3 factors.

In [26] P. T. Nagy and K. Strambach thoroughly investigate a variation of extensions which yields loops as extensions of groups by loops such that these extensions are the most natural generalization of Schreier's extension theory for groups. These extensions of groups A by loops S are given by two functional equations describing the action of S on A. In [17] K. Strambach and Á. Figula solved these equations for extensions of a group A by a weighted Steiner loop S. Hence they obtained concrete description for all loops with interesting weak associativity properties if the Steiner loop S induces only the trivial automorphism on A. They showed that the (restricted) Fischer groups and their geometry play an important role for loop extension with right alternative property.

Locally compact connected topological non-Desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century. These planes are coordinatized by locally compact quasifields Q such that the kernel of Q is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The classification of topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$. In this way D. Betten has determined all 4-dimensional planes having an at least 7-dimensional collineation group. H. Hähl has classified the 8-dimensional topological translation planes admitting an at least 16-dimensional automorphism group and coordinatizing by quasifields having the field $\mathbb{C}$ as their kernel. N. Knarr has determined the 8-dimensional planes coordinatizing by semifields having the field $\mathbb{C}$ as their kernel. The papers [10,11] are devoted to the determination of the algebraic structure of the multiplicative loops for these quasifields. A complete description of those multiplicative loops which have either a normal subloop of dimension one or which contain the compact subgroup $SO_2(\mathbb{R})$ or $Spin_3(\mathbb{R})$ is given. P. T. Nagy and K. Strambach have proved that the group G topologically generated by the left translations of the 2-dimensional proper multiplicative loops Q^* is the connected component of $GL_2(\mathbb{R})$ but the group topologically generated by the right translations of Q^* has infinite dimension (cf. [24, Section 29, p. 345]).

with Ágota Figula
2015, Chongqing, China

G. Falcone, K. Strambach and Á. Figula found that the group G topologically generated by the left translations of the multiplicative loops Q^* of 4-dimensional quasifields having the field $\mathbb{C}$ as their kernel is one of the following groups: $Spin_3(\mathbb{R}) \times \mathbb{R}$, $Spin_3(\mathbb{R}) \times \mathbb{C}$, $SL_2(\mathbb{C}) \times \mathbb{R}$, $GL_2(\mathbb{C})$. The classification of Hähl and Knarr shows that all these Lie groups are realized as the group generated by the left translations of a multiplicative loop Q^*. If G is the group $Spin_3(\mathbb{R}) \times \mathbb{R}$, then Q^* is associative and Q is either a proper Kalscheuer's near field or the skewfield of quaternions. P. Plaumann and K. Strambach have showed that any locally compact 2-dimensional semifield is the field of complex numbers (cf. [28, XI. 12.2 Proposition, p. 348]). G. Falcone, K. Strambach and Á. Figula proved that a 2-dimensional locally compact quasifield is the field of complex numbers if and only if the multiplicative loop Q^* contains a 1-dimensional compact normal subloop. Using the Betten's, Hähl's and Knarr's classifications they determined the multiplicative loops of the quasifields which coordinatize these translation planes. The multiplicative loops Q^* of the proper 4-dimensional semifields having $\mathbb{C}$ as their kernels are direct products of $\mathbb{R}$ and a compact loop K homeomorphic to the 3-sphere and the group generated by all translations of these loops Q^* are Lie groups, in contrast to the 2-dimensional quasifields. Also, the group generated by the left translations of K has a remarkable structure: it is the group of complex (2×2) matrices the determinants of which have absolute value 1.

4. Karl's Work in Palermo

The collaboration that Karl Strambach carried on for more than 30 years in Palermo began in 1984, when he met Claudio Bartolone at the NATO Advanced Study Institute conference *Rings and Geometry* in Istanbul. In 1979 C. Bartolone had published a paper *"A remark on the projectivities of the projective line over a commutative ring"* on the projective mappings of the line over a commutative ring which Karl found interesting. One year later they started a project, that consisted in the classification of imprimitive groups of transformations, acting sharply m-transitively on the system of imprimitivity, and sharply n-transitively on each block. Clearly, the project had its origin in the characterization of sharply 2-transitive groups as groups Γ of affine mappings $x \longmapsto xa + b$ of a (left) *near-domain*, which was still a very stimulating area at that period.

A near-domain is a set $(D, +, *)$ endowed with two binary operations, such that $(D, +)$ is a loop, $(D^*, *)$ is a group, and the one-side distributive law $a * (b + c) = a * b + a * c$ is fulfilled. For more than 40 years, many mathematicians looked for a near-domain which was not a *near-field*, that is, a near domain where $(D, +)$ is a group. In fact, finite near-domains are near-fields, algebraic near-domains are near-fields, topological near-domains are near-fields. The answer came only in 2014 when E. Rips, Y. Segev, and K. Tent constructed a sharply 2-transitive infinite group without a non-trivial Abelian normal subgroup, thus proving that there exist near-domains which are not a near-fields.

The collaboration that begun in Istanbul ended up in a deep friendship and in the training of a series of scholars, among whom was G. Falcone. As the manuscript was growing, the young collaborators began to feel the necessity of proving their work with a publication, thus in 2004 (sic!) they came out with the first one [1], and, after five years, with a second one [2]. In the first one, they showed that the case where $m > 2$ has only few cases. In 2012, during a *Research in pairs* period at Oberwolfach, K. Strambach and G. Falcone proved that, with $m = 2$, such finite transformation groups exist only for $n = 2$. Thus, the investigation can be confined to the case of what we called a $(2, 2)$-group, which can be illustrated as follows: let Ω be a set operated on imprimitively by a group G, in such a way that the stabilizer G_Δ of a block Δ acts sharply 2-transitive on Δ, that G/N is sharply 2-transitive on the system of imprimitivity $\bar{\Omega}$, that is, the set of blocks, where N is the inertia group, that is, the normal subgroup that leaves each block invariant, and, finally, that G is sharply transitive on the set $\Lambda := \{(P_1, P_2) \in \Omega^2 : \Delta_{P_1} \neq \Delta_{P_2}\}$. Algebraic $(2, 2)$-transformation groups had been characterized in 2009 [2]. In this paper, they exhibit an interesting class of examples where the group G has a non-Abelian, regular, subgroup T of *translations*: for an algebraically closed field k of characteristic $p > 0$, this group is defined on the set $(k_+)^3 \rtimes k_*$ by the multiplication

$$\begin{aligned}&(x_1, x_2, x_3, 1) * (y_1, y_2, y_3, 1)\\&= \left(x_1 + y_1 + y_2 x_3^p + \frac{1}{2} x_3^{2p} y_3^{p^2}, x_2 + y_2 + x_3^p y_3^{p^2}, x_3 + y_3\right)\end{aligned}$$

and $(x_1, x_2, x_3, 1)^{(0,0,0,b)} = (b^{p^2+2p} x_1, b^{p^2+p} x_2, b x_3)$. This group operates on the points (α, β) of the affine plane as follows:

$$(\alpha, \beta) \longmapsto \left(x_1 + a^{p^2+2p}\alpha + \frac{1}{2} x_3^{2p} (a y_3)^{p^2}, x_3 + ay\right).$$

Other interesting classes of examples come from the construction of affine mappings $x \mapsto xa + b$ on the ring R of dual numbers $R = \{a + \epsilon b : a, b \in L, \epsilon^2 = 0\}$, where L is a not necessarily commutative field, as the group of such affine mappings is an imprimitive transformation group on the set R, which is a $(2,2)$-group. Thus, with L the skew-field of Hamilton quaternions, we find the locally compact $(2,2)$-transformation group G, acting on a eight-dimensional space Ω, of affine mappings of the line over the (non-commutative) real algebra of dual quaternions $\epsilon_{\mathbb{H}} = \{a + b\epsilon : a, b \in \mathbb{H}, \epsilon^2 = 0\}$ over the skew-field of quaternions $\mathbb{H}$ (cf. Clifford, Study). For $a, b \in \epsilon_{\mathbb{H}}$, a invertible, the group of affine mappings $\{x \mapsto xa + b\}$ of $\epsilon_{\mathbb{H}}$ is a 16-dimensional locally compact topological $(2,2)$-transformation group with eight-dimensional inertia subgroup consisting of the mappings $\{x \mapsto x(1 + b\epsilon) + d\epsilon\}$. The only other locally compact $(2,2)$-transformation groups have been classified in [8]:

1. there is one family of four-dimensional groups, parametrized by a nonzero real number s, which give deformations of the group of affine mappings over the dual numbers over $\mathbb{R}$, which is obtained for $s = 1$;
2. there is one family of eight-dimensional groups, parametrized by a real number s and an integer number $n \geq 0$, and for $s = n = 1$ one obtains the group of affine mappings of the affine line over the algebra of dual numbers over the complex field.

The general situation was clarified in a paper [9], where the authors introduced *(partial) paradual* rings and proved that $\mathfrak{R} = (R, U_R)$ is a partial paradual near-ring if and only if the group $\Gamma_{\mathfrak{R}}$ of its affine mappings $x \mapsto xa + b$ is a $(2,2)$-transformation group: let $R(+, \cdot)$ be endowed with a binary operation $+$ and a *partial* binary operation $\cdot : R \times S \longrightarrow R$, where S is a subset of R. The authors say that $\mathfrak{R} = (R, S)$ is a *right partial near-ring* with identity $1 \in S$, if

(a) $(R, +)$ is a not necessarily commutative group with neutral element 0;
(b) S is closed with respect to the restriction of $\cdot$ at $S \times S$;
(c) $r(s_1 s_2) = (r s_1) s_2$ for all $r \in R$ and $s_1, s_2 \in S$;
(d) $(r_1 + r_2)s = r_1 s + r_2 s$ for all $r_1, r_2 \in R$ and $s \in S$;
(e) $1s = s$ and $r1 = r$ for all $s \in S$ and $r \in R$.

A. Di Bartolo, G. Falcone and K. Strambach denoted by U_R the group of invertible elements of the monoid S. Notice that, if $S = R$, then $\mathfrak{R} = (R, R)$ is a right near-ring.

If $\mathfrak{R} = (R, U_R)$ is a partial near-ring with identity 1 and the set $I = R\backslash U_R$ is an ideal of $\mathfrak{R}$, then $\mathfrak{R}$ is local, $R\backslash U_R$ is the maximal ideal of $\mathfrak{R}$ and $\mathfrak{R}/I$ is a near-field.

Let $\mathfrak{R} = (R, U_R)$ be a right partial near-ring with identity. The group

$$\Gamma_{\mathfrak{R}} = \{\gamma_{u,r} : x \mapsto xu + r : u \in U_R,\ r \in R\}$$

of affine mappings of $\mathfrak{R}$ is the semidirect product $\Gamma_{\mathfrak{R}} = M \ltimes T = \{\gamma_{u,r} = \mu_u \tau_r : \mu_u \in M, \tau_r \in T\}$ of its translations subgroup $T = \{\tau_r = \gamma_{1,r} : x \mapsto x+r, r \in R\}$ by its multiplications subgroup $M = \{\mu_u = \gamma_{u,0} : x \mapsto xu, u \in U_R\}$.

The group $\Gamma_{\mathfrak{R}}$ acts imprimitively on the points of R, taking as blocks the cosets of I in the additive group of R. The set of mappings $N = \{\gamma_{1+x,y} : x, y \in I\}$ is a normal subgroup of $\Gamma_{\mathfrak{R}}$, which is the inertia subgroup of the system of imprimitivity of $\Gamma_{\mathfrak{R}}$ defined by R/I.

The set of mappings $H = \{\gamma_{1+x,y} : x \in I, y \in R\}$ is a normal subgroup of $\Gamma_{\mathfrak{R}}$, such that $\Gamma_{\mathfrak{R}}/H$ fixes a point X_0 and acts sharply transitively on the blocks not containing X_0.

If the multiplicative group of R/I acts sharply transitively on $I\backslash\{0\}$, then $\Gamma_{\mathfrak{R}}$ is a $(2,2)$-transformation group. Accordingly, we define a (partial) near-ring with identity 1, such that $I = R\backslash U_R$ is an ideal and the multiplicative group of R/I acts sharply transitively on $I\backslash\{0\}$, a *(partial) paradual* near-ring.

On the other hand, if a group G is such that $G = M \ltimes T$, where T acts sharply transitively on the points and $M = G_0$ is the stabilizer of the point $0 \in T$, if $J < T$ is normal in G, and M acts sharply transitively on $T\backslash J$ by conjugation, then, fixing an element ι in $T\backslash J$ and denoting by k_s the element of M such that $s = \iota^{k_s}$, the addition of $\mathfrak{R} = (T, T\backslash J)$ given by the multiplication in T and the *partial* multiplication of $\mathfrak{R}$ given, for any $t \in T$ and any $s \in T\backslash J$ by $t * s = t^{k_s}$, turn $\mathfrak{R}$ into a partial near-ring, such that G is isomorphic to the group of affine mappings of the affine line over $\mathfrak{R}$ and the set $T\backslash J$ consists of the invertible elements U_T of $\mathfrak{R}$. If G is a $(2,2)$-transformation group, then R is paradual.

Since G. Falcone and K. Strambach proved in Oberwolfach (2012) that, if G is a *finite* $(2,2)$-transformation group, then $G = T \rtimes G_X$, where T is a normal subgroup of G acting regularly, and is either an elementary Abelian group of order p^{2r} or the direct product of r copies of a cyclic group of order p^2, the investigation was reduced to the near-fields obtained as the quotient R/I, G. Falcone will always remember the big efforts made in Milano 2014 with the seven Zassenhaus exceptional near-fields, but it was two years later in Palermo that he and K. Strambach realized that, more generally, for any paradual near-ring R with maximal ideal I and the corresponding near-field $L = R/I$, there exists a not necessarily commutative field F, an isomorphism

$\phi : L^* \longrightarrow F^*$, and an anti-homomorphism $\gamma : L^* \longrightarrow \mathrm{Aut}(F_+)$, fulfilling

$$\phi(x_1 + x_2)y^{\gamma(x_1+x_2)} = \phi(x_1)y^{\gamma(x_1)} + \phi(x_2)y^{\gamma(x_2)},$$

such that $R \approx L \times F$ with the addition

$$(x_1, y_1) + (x_2, y_2) = \Big(x_1 + x_2, y_1 + y_2 + \Phi(x_1, x_2)\Big),$$

and with the multiplication

$$(x_1, y_1)(x_2, y_2) = \Big(x_1x_2, y_1\phi(x_2) + \phi(x_1)(y_2\phi(x_2^{-1}))^{\gamma(x_1)}\phi(x_2) + \Psi(x_1, x_2)\Big)$$

for a multiplicative factor system $\Psi : L \times L \longrightarrow F$ fulfilling $\Psi(1, t) = \Psi(t, 1) = \Psi(0, t) = \Psi(t, 0) = 0$, and an additive factor system $\Phi : L_+ \times L_+ \longrightarrow F_+$, fulfilling

$$\Phi(x_1, x_2)\phi(x_3) + \Psi(x_1 + x_2, x_3) = \Phi(x_1x_3, x_2x_3) + \Psi(x_1, x_3) + \Psi(x_2, x_3).$$

In the case of a *finite* paradual near-ring $R = L \times F$, with L near-field, $F \approx \mathrm{GF}(p^n)$, from the isomorphism $\phi : L^* \longrightarrow F^*$ we infer that $L = F$ and that the epimorphism $\phi : L^* \longrightarrow F^*$ is $\phi(x) = x^t$, with $\gcd(t, p^n - 1) = 1$. Moreover, by the theorem of Zassenhaus, the multiplicative group of R splits over the normal subgroup $(1+I) \ni (1, y)$, and the multiplicative factor system Ψ is a coboundary, that is, $\Psi(x_1, x_2) = \sigma(x_1x_2)\sigma(x_1)^{-1}\sigma(x_2)^{-1}$, for a suitable function $\sigma : L^* \longrightarrow R^*$.

Thus, we can say that, if R is a finite paradual near-ring, I is its maximal ideal, and $L = R/I \approx \mathrm{GF}(p^n)$, then $R := R_{l,\Phi} \equiv L \times L$ with the addition

$$(x_1, y_1) + (x_2, y_2) = \Big(x_1 + x_2, y_1 + y_2 + \Psi(x_1, x_2)\Big),$$

where $\Psi(x_1, x_2)$ is either zero, or $\Phi_1(x_1, x_2) = \frac{x_1^p + x_2^p - (x_1+x_2)^p}{p}$, or $-\Phi_1(x_1, x_2)^{p^{n-1}}$, where Φ_1 is the factor system of the Witt extension

$$1 \longrightarrow \mathfrak{W}_1 \longrightarrow \mathfrak{W}_2 \longrightarrow \mathfrak{W}_1 \longrightarrow 1,$$

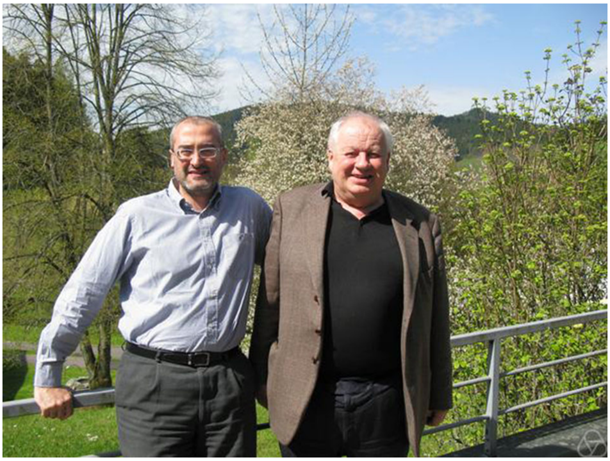
and with the multiplication $(x_1, y_1)(x_2, y_2) = \Big(x_1x_2, y_1x_2 + x_1^{p^l}y_2\Big)$ where $0 \leq l < n$. The near-ring R is a ring if and only if $l = 0$.

This was a short survey of what was just one half of the contribution given by Karl Strambach to the academic training of G. Falcone, who cannot help but mention in what follows one of the many results contained in the monograph [7]. While climbing the stairs of the Erlangen Mathematics Institute in Bismarckstraße 1 1/2, Karl asked G. Falcone whether connected algebraic groups where every connected subgroup was normal always have one-dimensional commutator subgroups. This led G. Falcone to a broad project, funded by DFG STR97/9-1, focused on the concept of *chains*, that is, connected algebraic groups with a unique connected algebraic subgroup for any dimension. The smallest commutative chain is the Witt group $\mathfrak{W}_2$, and the smallest non-commutative chain is the group $\mathfrak{J}_2(\mathbf{F}^r)$ of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & x^{p^r} \\ 0 & 0 & 1 \end{pmatrix},$$

which J. Dieudonné found of some interest. The authors of the paper [7] generalized this group to the n-dimensional unipotent algebraic group $\mathfrak{J}_n$, of maximal nilpotent class n, having the following linear representation

$$M_{n-1}(x_0,\ldots,x_{n-1}) = \begin{pmatrix} 1 & x_0 & x_1 & \cdot & \cdot & \cdot & x_{n-1} \\ & 1 & x_0^p & x_1^p & \cdot & \cdot & x_{n-2}^p \\ & & 1 & x_0^{p^2} & x_1^{p^2} & \cdot & x_{n-3}^{p^2} \\ & & & & & & \vdots \\ & & & & & & 1 \end{pmatrix}.$$



with Giovanni Falcone
2012, Oberwolfach, Germany

These examples answered in the negative the question that K. Strambach posed to G. Falcone about twenty years ago. Finally, on October 7th, 2016, Karl sent to G. Falcone his contribution to an introductory note to an edited volume [12] and asked him to check the details, because he was not very well.

5. Elementary Geometry and Geodesics

In differential geometry affine connections and metric tensors play a central role. An affine connection on a connected differentiable manifold M generates geodesics, and M endowed with a system S of geodesics may be considered as an incidence geometry (M, S) with the geodesics as lines. Following Klein's point of view the study of transformation groups of M related to metric tensors, affine connections or of transformation groups leaving S invariant has been a research area of great importance for more than hundred years.

The group Γ of diffeomorphisms of M leaving the system S of geodesics invariant, i.e., the group of geodesic transformations, is a Lie group of dimension $\leq n^2 + 3n$ if M has dimension n. The group Θ of affine transformations leaving an affine connection ∇ invariant as well as the group Ω of isometries leaving the metric tensor g of an Riemannian manifold invariant are closed Lie subgroups of Γ.

If M is Riemannian and ∇ is the unique affine connection compatible with g then one has $\Omega \leq \Theta \leq \Gamma$. In order to find the relations between the groups Ω, Θ and Γ one needs geometric properties of the manifold M. For instance, if M is a Riemannian manifold then the group Ω of isometries is compact. If M itself is compact then $\Omega = \Theta$ is a compact Lie group.

For a 2-dimensional differentiable manifold M E. Beltrami had already derived an Abelian differential equation having as coefficients expressions in the Christoffel symbols of an affine connection ∇ such that its solutions are local geodesics for ∇. If, in particular, the coefficients of this differential equation are constant, then its solutions can be obtained explicitly.

For a suitable system S of curves the Beltrami differential equation allows one to decide whether there exists an affine connection ∇ on M such that S consists of global geodesics with respect to ∇. Thoroughly investigated systems of differentiable curves occur as lines of 2-dimensional topological affine planes.

This motivated G. Gerlich to ask for which such planes M there exists an affine connection ∇ generating the lines of M and to study when there exists a Riemannian metric such that ∇ is its associated connection. He has shown that the validity of Desargues' theorem is essential for the existence of ∇. The only important non-Desarguesian exceptions are the Moulton planes.

The affine connections with this property are classified for the Grünwald planes as well as for the Moulton planes. Moreover, the groups of isometries and of affine transformations are determined. It turns out that in the Moulton planes M for any dimension $0 \leq n \leq 4$ there exists in the collineation group Γ a subgroup Θ of dimension n such that Θ is the group of affine mappings with respect to an affine connection having Γ as the group of geodesic transformations. There is even an affine connection for M such that the group Θ of affine mappings coincides with the group Γ of geodesic transformations.

The Euclidean plane, Grünwald's models of the real affine plane and the affine Moulton planes are the only known examples of differentiable $\mathbb{R}^2$-planes in which the group of geodesic transformations can coincide with the group of affine transformations.

J. Mikeš and K. Strambach (see [19]) have shown that the real affine plane (with straight lines or as a Grünwald model of the real affine plane) is the only generalized shift $\mathbb{R}^2$-plane (this class forms a generalization of translation planes) such that their lines are geodesics with respect to an affine connection

∇, the components of which can be calculated. Among the generalized Moulton planes only the Moulton planes themself admit affine connections ∇ such that their lines are geodesics with respect to ∇. Moreover, all such connections ∇ can be classified.

E. Beltrami has shown that a differentiable curve is a local geodesic with respect to an affine connection ∇ precisely if it is a solution of an Abelian differential equation having as coefficients expressions in Christoffel symbols associated with ∇, the use of differential geometry for study of $\mathbb{R}^2$-planes having differentiable curves as lines started only 2000 by G. Gerlich.

G. Gerlich asked for which $\mathbb{R}^2$-planes A with differentiable lines there exists an affine connection ∇ generating the lines of A and for affine planes A with an at least three-dimensional collineation group he proved that ∇ exists if and only if A is either Desarguesian or a Moulton plane. Moreover it is shown that the differentiable lines of a generalized shift $\mathbb{R}^2$-plane A are geodesics with respect to an affine connection ∇ precisely if A is either the Euclidean plane or a Grünwald model of the real affine plane.

The extension of the investigation from $\mathbb{R}^2$-planes to geometries on $\mathbb{R}^n$ having as lines a system S of curves such that any two different points are incident with precisely one curve of S turns out to be surprisingly difficult as one can see in Gerlich's papers, where, for example, D. Betten created a theory of 3-dimensional topological incidence geometries.

If one tries to extend the characterization of differentiable shift spaces having as lines geodesics with respect to an affine connection starting with a Grünwald plane, then one also meets with great difficulties. Namely, J. Mikeš and K. Strambach showed that for at least 3-dimensional differentiable shift spaces S, generalizing in a natural way the 2-dimensional shift spaces corresponding to Grünwald planes, there exists no affine connection ∇ such that the lines of S are geodesics of ∇. This is surprising since there exist n-dimensional shift spaces if the derivatives of their generating functions are homeomorphisms of $\mathbb{R}$.

In contrast to a shift space the set of all images of the system of curves arising by shifting the argument from a Grünwald curve $\mathcal{C}$ under the translation group of $\mathbb{R}^n$ is a system of geodesics with respect to a natural affine connection if and only if $\mathcal{C}$ is a curve corresponding to parabolas in a suitable coordinate system.

J. Mikeš and K. Strambach [20] have proved that an n-dimensional differentiable shift space S, for which in case $n = 2$ there exists an affine connection if S is a Grünwald plane, does not admit for $n \geq 3$ affine connection. In contrast to this the set of all images of the system of curves arising by shifting the argument from a Grünwald curve $\mathcal{C}$ under the translation group of $\mathbb{R}^n$ is a system of geodesics with respect to a metrizable affine connection if and only if $\mathcal{C}$ is a curve corresponding to parabolas in a suitable coordinate system.

Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$ which is described by a set of differentiable strictly monotone and surjective functions. The aim of this line of research is to study under which conditions all images of $\mathcal{C}$ under a group Ω of affine transformations containing all translations of $\mathbb{R}^n$ are geodesics with respect to an affine connection ∇. If Ω coincides with the group T of affine translations, then in this case we already obtain concrete informations on the components of the affine connection ∇ and about the form of the functions describing the curves $\mathcal{C}$. If the groups Ω are semidirect products of the translation group T of $\mathbb{R}^n$ with groups consisting of many dilations, then we determine explicitly the components of ∇ and the form of the curves $\mathcal{C}$, the images of which under Ω consists of geodesics with respect to ∇.

The authors paid special attention to curves $\mathcal{C}$ in the 3-dimensional affine space and to the translation group $\Omega = T$ in $\mathbb{R}^3$. In this case the form of the curves $\mathcal{C}$ was determined, the components of ∇ were calculated as well as the components of the curvature and Weyl tensor. Moreover, they were able to decides when ∇ yields a flat or metrizable space and to calculate the corresponding metric tensor. In the proofs the solutions of Ricatti and Abelian differential equation play an important role. For this reason one can explicitly describe the solutions of Abelian differential equations which are derivatives of strongly monotone and surjective function on $\mathbb{R}$.

In [21] J. Mikeš and K. Strambach have determined the form of curves $\mathcal{C}$ in $\mathbb{R}^n$ corresponding to strictly monotone functions as well as the components of affine connections ∇ for which any image of $\mathcal{C}$ under a compact-free group of affinities containing the translation group is a geodesic with respect to ∇.

O. Belova, J. Mikeš with K. Strambach (see [3]) have investigated Hjelmslev geometries $\mathcal{H}$ having a representation in a complex affine space $\mathbb{C}^n$, the lines of which are given by entire functions. Since, it is possible to define natural complex affine connections ∇, the notion of a geodesic is also available. If the lines of $\mathcal{H}$ are geodesics with respect to ∇ then a detailed classification of them as well as of the corresponding geometries is obtained. Generalizations of complex Grünwald planes play a main role in the classification. Since in the considered geometries the set of lines is invariant under the translation group of $\mathbb{C}^n$, O. Belova, J. Mikeš and K. Strambach classified all complex curves $\mathcal{C}$ in $\mathbb{C}^n$ given by entire functions as well as the connections ∇ such that all images of $\mathcal{C}$ under the translation group of $\mathbb{C}^n$ consist of geodesics with respect to ∇.

Actually O. Belova, J. Mikeš and K. Strambach studied how in $\mathbb{R}^n$, for a system S of differentiable curves which is invariant under the semidirect product of the translation group with an $(n-1)$-dimensional real algebraic split torus, the conditions for the forms of curves of S to be geodesics or almost geodesics differ. In contrast to systems S consisting of geodesic curves, systems S consisting of almost geodesics constitute a wide class. Hence the paper arising on this subject contains tricky calculations.

Within the frame of this research O. Belova, J. Mikeš and K. Strambach wanted to determine differentiable affine planes and n-dimensional differentiable shift spaces the lines of which are almost geodesics with respect to an affine connection with constant coefficients. In the case of affine planes

O. Belova, J. Mikeš and K. Strambach have paid the special attention to non-Desarguesian planes and in the case of differentiable shift spaces the authors have concentrated to such differentiable shift spaces with lines which are not parabolas. Also, for the analogous systems as treated in [21] J. Mikeš and K. Strambach have determined the form of curves $\mathcal{C}$ for which any image of $\mathcal{C}$ under a compact-free group of affinities containing the translation group is an almost geodesic with respect to an affine connection ∇. Furthermore, the authors tried to transport the notion of almost geodesics to complex manifolds and investigate with this tool Hjelmslev geometries having a representation in a complex affine space $\mathbb{C}^n$, the lines of which are given by entire functions. The goal of authors is a classification of Hjelmslev geometries $\mathcal{H}$, the lines of which are almost geodesics with respect to a complex affine connection ∇. The investigations and calculations are not easy, but some goals have been achieved in [4–6].

with Josef Mikeš,
2015, Olomouc, Czech Republic

with Olga Belova and Josef Mikeš,
2013, Kaliningrad, Russia

6. Afterword

In October 2016 Karl Strambach passed away in Erlangen suddenly and unexpectedly.

Karl Strambach was kind and sociable, relaxed and lively, and completely unpretentious. He was deeply rooted in the cultural traditions of Europe.

We are grateful to the destiny that has given to us the luck to communicate and to work with such person.

We really miss him.

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