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Keywords (separated by '-')	Analytical dynamics - Transfer function matrix - Power spectral density matrix - Singular matrix	
Footnote Information

Stochastic response determination of structural systems modeled via dependent coordinates: a frequency domain treatment based on generalized modal analysis

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Abstract Generalized independent coordinates are typically utilized within an analytical dynamics framework to model the motion of structural and mechanical engineering systems. Nevertheless, for complex systems, such as multi-body structures, an explicit formulation of the equations of motion by utilizing generalized, independent, coordinates can be a daunting task. In this regard, employing a set of redundant coordinates can facilitate the formulation of the governing dynamics equations. In this setting, however, standard response analysis techniques cannot be applied in a straightforward manner. For instance, defining and determining a transfer function within a frequency domain response analysis framework is challenging due to the presence of singular

matrices, and thus, the machinery of generalized matrix inverses needs to be employed. An efficient frequency domain response analysis methodology for structural dynamical systems modeled via dependent coordinates is developed herein. This is done by resorting to the Moore–Penrose generalized matrix inverse in conjunction with a recently proposed extended modal analysis treatment. It is shown that not only the formulation is efficient in drastically reducing the computational cost when compared to a straightforward numerical evaluation of the involved generalized inverses, but also facilitates the derivation and implementation of the celebrated random vibration input–output frequency domain relationship between the excitation and the response power spectrum matrices. The validity of the methodology is demonstrated by considering a multi-degree-of-freedom shear type structure and a multi-body structural system as numerical examples.

Keywords Analytical dynamics · Transfer function matrix · Power spectral density matrix · Singular matrix

1 Introduction

In the field of mechanical vibration theory [1], it is not uncommon for continuous systems to be modeled as

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discrete systems under the assumption of lumped masses at specific points in the spatial domain. In this regard, the governing dynamics of such multi-degree-of-freedom (MDOF) systems is modeled by a set of coupled differential equations to be solved for determining the system response. One well-established approach in standard structural dynamics is modal analysis, which decouples the original system of differential equations. In fact, the response of the original n -DOF system can be expressed as a superposition of the responses of n appropriately defined single-DOF oscillators. Note that besides the significant decrease in computational cost related to the solution of the original set of coupled equations, modal analysis has the additional advantage of yielding quantities of clear physical interpretation, such as mode shapes and modal frequencies.

Further, generalized independent coordinates are typically utilized within the standard structural dynamics analysis framework. Nevertheless, for complex systems, such as multi-body structures, an explicit formulation of the equations of motion, by utilizing the minimum number of independent coordinates, can be a non-trivial task. In this regard, employing a redundant set of coordinates and introducing appropriate constraints can facilitate the modeling of the governing equations in a differential-algebraic form [2]. In this setting, however, standard response analysis techniques such as modal analysis, cannot be applied in a straightforward manner due to the occurrence of singular matrices; and thus, generalizations of standard vibration theory concepts and tools are required (e.g. [3–5]).

In this paper, an efficient frequency domain response analysis methodology is developed for structural dynamic systems modeled via dependent coordinates. To this aim, employing the Moore–Penrose generalized matrix inverse in conjunction with a recently proposed extended modal analysis treatment [6] yields an input–output relationship in the frequency domain involving generalized modal matrices. It is shown that not only the formulation is efficient in drastically reducing the computational cost when compared to a straightforward numerical evaluation of the involved generalized inverses, but also facilitates the derivation and implementation of one of the most important relationships in linear random vibration theory [7]: the input–output frequency domain relationship between the excitation and the

response power spectrum matrices. The validity of the methodology is demonstrated by considering a multi-degree-of-freedom shear type structure and a multi-body system as numerical examples.

2 Multi-degree-of-freedom structural systems modeled via non-generalized (dependent) coordinates

Consider a lumped mass linear system whose dynamics are governed by the set of second order coupled differential equations

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{f}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \\ \dot{\mathbf{x}}(0) &= \dot{\mathbf{x}}_0 \end{aligned} \quad (1)$$

where the dot denotes differentiation with respect to time, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the viscous damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{f}(t)$ is the force vector, $\mathbf{x}^T(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ is the transpose of the displacement vector and $\mathbf{x}_0$, $\dot{\mathbf{x}}_0$ the initial conditions in terms of displacement and velocity, respectively.

It is worth noting that Eq. (1) corresponds to the minimum number of independent coordinates (generalized coordinates) [8, 9]. However, for complex systems, such as multi-body structures, an explicit formulation of the equations of motion by utilizing generalized coordinates can be a non-trivial task. Thus, employing a set of redundant coordinates can facilitate the formulation of the equations of motion [2, 10–12]. In this regard, consider an l -DOF ($l > n$) system of the form

$$\begin{aligned} \mathbf{M}_u\ddot{\mathbf{u}}(t) + \mathbf{C}_u\dot{\mathbf{u}}(t) + \mathbf{K}_u\mathbf{u}(t) &= \mathbf{f}_u(t) \\ \mathbf{u}(0) &= \mathbf{u}_0 \\ \dot{\mathbf{u}}(0) &= \dot{\mathbf{u}}_0 \end{aligned} \quad (2)$$

where $\mathbf{u}$ is the l -vector of the coordinates, $\mathbf{f}_u(t)$ the l -vector of external forces, and $\mathbf{M}_u$, $\mathbf{C}_u$, $\mathbf{K}_u$ are the mass, damping and stiffness matrices ($l \times l$), respectively. Further, the above system is subjected to m -constraints of the form

$$\mathbf{A}(\mathbf{u}, \dot{\mathbf{u}}, t)\ddot{\mathbf{u}} = \mathbf{b}(\mathbf{u}, \dot{\mathbf{u}}, t) \quad (3)$$

where $\mathbf{A}$ is an $(m \times l)$ matrix and $\mathbf{b}$ is an m -vector. For simplicity and without loss of generality, consider

134 $\mathbf{b}(\mathbf{u}, \dot{\mathbf{u}}, t) = -\mathbf{E}\dot{\mathbf{u}} - \mathbf{L}\mathbf{u} + \mathbf{F}$, where $\mathbf{E}$, $\mathbf{L}$ are $(m \times l)$
 135 matrices and $\mathbf{F}$ is an m -vector. Then, Eq. (3) is
 136 rewritten as

$$\mathbf{A}(\mathbf{u}, \dot{\mathbf{u}}, t)\ddot{\mathbf{u}} = -\mathbf{E}\dot{\mathbf{u}} - \mathbf{L}\mathbf{u} + \mathbf{F}. \quad (4)$$

138 Next, combining Eqs. (2, 4), the augmented system
 139 can be written in the form

$$\begin{aligned} \bar{\mathbf{M}}_u \ddot{\mathbf{u}}(t) + \bar{\mathbf{C}}_u \dot{\mathbf{u}}(t) + \bar{\mathbf{K}}_u \mathbf{u}(t) &= \bar{\mathbf{f}}_u(t) \\ \mathbf{u}(0) &= \mathbf{u}_0 \\ \dot{\mathbf{u}}(0) &= \dot{\mathbf{u}}_0 \end{aligned} \quad (5)$$

141 with

$$\begin{aligned} \bar{\mathbf{M}}_u &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{M}_u \\ \mathbf{A} \end{bmatrix}_{((l+m) \times l)}; \\ \bar{\mathbf{C}}_u &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{C}_u \\ \mathbf{E} \end{bmatrix}_{((l+m) \times l)}; \\ \bar{\mathbf{K}}_u &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{K}_u \\ \mathbf{L} \end{bmatrix}_{((l+m) \times l)}; \\ \bar{\mathbf{f}}_u &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{f}_u \\ \mathbf{F} \end{bmatrix}_{((l+m) \times 1)}; \end{aligned} \quad (6a - d)$$

143 where $\mathbf{A}^+(l \times m)$ is the so-called Moore–Penrose
 144 inverse of $\mathbf{A}$.

145 3 Input–output frequency domain relationship 146 between the excitation and response power 147 spectra

148 Regarding a frequency domain system response
 149 analysis, defining a transfer function between response
 150 displacements and excitation force vectors is not a
 151 straightforward task due to the presence of singular
 152 matrices. In this regard, a Moore–Penrose inverse
 153 operation can be employed (see also [4] for more
 154 details), yielding

$$\bar{\mathbf{H}}_u(\omega) = [-\omega^2 \bar{\mathbf{M}}_u + i\omega \bar{\mathbf{C}}_u + \bar{\mathbf{K}}_u]^+. \quad (7)$$

156 Note, however, that the numerical evaluation of the
 157 Moore–Penrose inverse matrix can be highly compu-
 158 tationally demanding, especially for large values of
 159 l . To address this issue, the modal analysis procedure
 160 introduced in [6] is utilized in the ensuing analysis.

161 For completeness, the main elements of the modal
 162 analysis developed in [6] are outlined as follows. The

first step is to evaluate the eigenvalues $\bar{\omega}_j^2$ and
 eigenvectors $\bar{\phi}_j$ ($j = 1, 2, \dots, l$) of the matrix $\bar{\mathbf{D}}_u$ of
 order $(l \times l)$ defined as:

$$\bar{\mathbf{D}}_u = [\bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u]. \quad (8)$$

Next, taking into account the modal matrix $\bar{\Phi}$
 $(l \times l)$ containing the eigenvectors $\bar{\phi}_j$ as columns, the
 modal transformation $\mathbf{u}(t) = \bar{\Phi} \mathbf{p}(t)$ decouples the
 system of Eq. (5) in the form

$$\begin{aligned} \bar{\mathbf{I}} \ddot{\mathbf{p}}(t) + \bar{\Lambda} \dot{\mathbf{p}}(t) + \bar{\Omega} \mathbf{p}(t) &= \bar{\mathbf{f}}(t) \\ \mathbf{p}(0) &= \bar{\Phi}^{-1} \mathbf{u}_0 \\ \dot{\mathbf{p}}(0) &= \bar{\Phi}^{-1} \dot{\mathbf{u}}_0 \end{aligned} \quad (9)$$

where $\bar{\mathbf{f}}(t)$ is an l -vector, $\mathbf{I}(l \times l)$ is the identity matrix,
 and $\bar{\Omega}(l \times l)$ and $\bar{\Lambda}(l \times l)$ are diagonal matrices given
 by

$$\begin{aligned} \mathbf{I} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{M}}_u \bar{\Phi} = \text{diag}[1] \\ \bar{\Omega} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{K}}_u \bar{\Phi} = \text{diag}[\bar{\omega}_j^2] \\ \bar{\Lambda} &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{C}}_u \bar{\Phi} = \text{diag}[2\bar{\zeta}_j \bar{\omega}_j] \\ \bar{\mathbf{f}}(t) &= \bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{f}}_u(t) \end{aligned} \quad (10a - d)$$

and $\bar{\omega}_j$ and $\bar{\zeta}_j$ are the undamped natural frequencies
 and damping ratios of the augmented system, respec-
 tively. Next, taking the Fourier transform on both sides
 of Eq. (9) yields

$$\mathbf{P}(\omega) = \mathbf{H}(\omega) [\bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{F}}_u(\omega) + (i\omega \mathbf{I} + \bar{\Lambda}) \mathbf{p}_0 + \dot{\mathbf{p}}_0] \quad (11)$$

where $\bar{\mathbf{H}}(\omega)$ is the transfer function matrix, relating
 the displacement and force vectors corresponding to
 the system of Eq. (9), and given in the form

$$\begin{aligned} \bar{\mathbf{H}}(\omega) &= [\bar{\Omega} - \omega^2 \mathbf{I} + i\omega \bar{\Lambda}]^{-1} \\ &= \text{diag}[\bar{\omega}_j^2 - \omega^2 + i2\bar{\zeta}_j \bar{\omega}_j]^{-1}. \end{aligned} \quad (12)$$

Once $\mathbf{P}(\omega)$ has been evaluated, the structural
 response of the system with singular matrices can be
 determined in the frequency domain as

$$\begin{aligned} \mathbf{U}(\omega) &= \bar{\Phi} \mathbf{P}(\omega) = \bar{\Phi} \bar{\mathbf{H}}(\omega) \\ &\times [\bar{\Phi}^{-1} \bar{\mathbf{M}}_u^+ \bar{\mathbf{F}}_u(\omega) + (i\omega \mathbf{I} + \bar{\Lambda}) (\bar{\Phi}^{-1} \mathbf{u}_0) + \bar{\Phi}^{-1} \dot{\mathbf{u}}_0]. \end{aligned} \quad (13)$$

Clearly, the inverse Fourier transform of $\mathbf{U}(\omega)$ yields the structural response of the system of Eq. (5) in the time domain. It is worth noting that Eq. (13) does take into account the effect of the initial conditions, corresponding to the terms $(\bar{\Phi}^{-1}\mathbf{u}_0)$ and $(\bar{\Phi}^{-1}\dot{\mathbf{u}}_0)$, which is required for computing the system response in the time domain as the inverse Fourier transform of $\mathbf{U}(\omega)$.

Considering Eq. (13), the transfer function matrix related to the system of Eq. (5) is given in the form

$$\bar{\mathbf{H}}_u(\omega) = \bar{\Phi}\bar{\mathbf{H}}(\omega)\bar{\Phi}^{-1}\bar{\mathbf{M}}_u^+. \quad (14)$$

Note that the expression of the transfer function given in Eq. (14) is equivalent to the one in Eq. (7) expressed by directly considering the system of Eq. (5). Nevertheless, it can be readily seen that evaluating Eq. (14) is significantly more computationally efficient as compared to a straightforward numerical determination of Eq. (7), particularly for large values of l . Further, in addition to the enhanced efficiency, the modal analysis treatment accounts for the effect of the initial conditions as well.

Also, generalization of the input-output frequency domain relationship to account for a stochastic excitation vector described by a *power spectral density* (PSD) matrix is a straightforward task. In this regard, the celebrated input-output frequency domain relationship of the linear random vibration theory takes the form

$$\mathbf{S}_u(\omega) = \bar{\Phi} \left[\bar{\mathbf{H}}(\omega) (\bar{\Phi}^{-1}\bar{\mathbf{M}}_u^+) \bar{\mathbf{S}}_{fuF} (\bar{\Phi}^{-1}\bar{\mathbf{M}}_u^+)^T \bar{\mathbf{H}}^*(\omega) \right] \bar{\Phi}^T. \quad (15)$$

Equation (15) determines the response PSD matrix $\mathbf{S}_u(\omega)$ ($l \times l$) of the augmented system given the excitation PSD matrix $\bar{\mathbf{S}}_{fuF}$ ($(l+m) \times (l+m)$) expressed via the block matrix

$$\bar{\mathbf{S}}_{fuF} = \begin{bmatrix} [(\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{S}_{fu} (\mathbf{I} - \mathbf{A}^+ \mathbf{A})^T]_{l \times l} & [(\mathbf{I} - \mathbf{A}^+ \mathbf{A}) \mathbf{S}_{fuF}]_{l \times m} \\ [\mathbf{S}_{Ffu} (\mathbf{I} - \mathbf{A}^+ \mathbf{A})^T]_{m \times l} & [\mathbf{S}_F]_{m \times m} \end{bmatrix}_{(l+m) \times (l+m)}. \quad (16)$$

In Eq. (16) $\mathbf{S}_{fu}$, $\mathbf{S}_F$ denote PSD matrices, whereas $\mathbf{S}_{fuF}$, $\mathbf{S}_{Ffu}$ denote cross spectral density matrices [in Eq. (15) the upper asterisk denotes the complex conjugate transform].

4 Numerical examples

4.1 Shear-type structural model

To assess the validity of the proposed methodology, consider a shear-type frame with n stories (Fig. 1), so that the solution obtained via the developed methodology can be compared with the standard solution available in classical structural dynamics based on a generalized coordinates model.

For this model, the motion of a given story-DOF depends only on the motion of the stories above and below. The underlying assumption relates to the fact that the floors are much stiffer than the walls, and thus, the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{K}$ and the damping matrix $\mathbf{C}$ in Eq. (1), can be defined as

$$\mathbf{M} = \text{diag}[m_i] \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & k_4 + k_{i+1} & -k_{i+1} & 0 \\ 0 & 0 & 0 & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & 0 & 0 & 0 & -k_n & k_n \end{bmatrix} \quad (17)$$

$$\mathbf{C} = \begin{bmatrix} C_1 + C_2 & -C_2 & 0 & 0 & 0 & 0 \\ -C_2 & C_2 + C_3 & -C_3 & 0 & 0 & 0 \\ 0 & -C_3 & C_3 + C_4 & -C_4 & 0 & 0 \\ 0 & 0 & -C_4 & C_4 + C_{i+1} & -C_{i+1} & 0 \\ 0 & 0 & 0 & -C_{n-1} & C_{n-1} + C_n & -C_n \\ 0 & 0 & 0 & 0 & -C_n & C_n \end{bmatrix}$$

Fig. 1 Shear-type frame

where k_i, C_i and m_i are the stiffness, damping and the mass of the story i .

Further, the force vector and the initial conditions in Eq. (1) are given in the form

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}; \quad \mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}; \quad \dot{\mathbf{x}}_0 = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{bmatrix} \quad (18)$$

respectively. If a non-generalized, redundant coordinates modeling is adopted, the aforementioned shear-type model is decomposed as in Fig. 2, and is studied next as an augmented system having $l = 2n - 1$ DOFs, related to the previous model via the relationships

$$\begin{aligned} x_1(t) &= u_1(t) = u_2(t) \\ x_2(t) &= u_2(t) + u_3(t) = u_4(t) \\ &\vdots \\ x_{n-1}(t) &= u_{2n-4}(t) + u_{2n-3}(t) = u_{2n-2}(t) \\ x_n(t) &= u_{2n-2}(t) + u_{2n-1}(t). \end{aligned} \quad (19)$$

Further, the system matrices in Eq. (2), of dimension $(2n - 1 \times 2n - 1)$, can be written in the generalized form

$$\mathbf{M}_u = \begin{bmatrix} m_1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & m_2 & m_2 & & 0 & 0 & 0 & 0 \\ 0 & m_2 & m_2 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & m_{n-1} & m_{n-1} & 0 \\ 0 & 0 & 0 & & 0 & m_{n-1} & m_{n-1} & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & m_n \\ 0 & 0 & 0 & & 0 & 0 & m_n & m_n \end{bmatrix}$$

$$\mathbf{K}_u = \begin{bmatrix} k_1 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & k_{n-1} & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & k_n \end{bmatrix}$$

Fig. 2 Modeling of the system shown in Fig. 1 using redundant coordinates

$$\mathbf{C}_u = \begin{bmatrix} C_1 & 0 & 0 & 0 & & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \ddots & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & C_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & 0 & 0 & C_n & 0 \end{bmatrix} \quad (20a - c)$$

262 and

$$\mathbf{f}_u(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_2(t) \\ \vdots \\ f_{n-1}(t) \\ f_{n-1}(t) \\ f_n(t) \\ f_n(t) \end{bmatrix}; \quad \mathbf{u}_0 = \begin{bmatrix} x_1(0) \\ x_1(0) \\ x_2(0) - x_1(0) \\ x_2(0) \\ x_3(0) - x_2(0) \\ \vdots \\ x_{n-1}(0) \\ x_n(0) - x_{n-1}(0) \end{bmatrix};$$

$$\dot{\mathbf{u}}_0 = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_1(0) \\ \dot{x}_2(0) - \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) - \dot{x}_2(0) \\ \vdots \\ \dot{x}_{n-1}(0) \\ \dot{x}_n(0) - \dot{x}_{n-1}(0) \end{bmatrix} \quad (21)$$

264 where Eq. (21) is consistent with Eq. (19).

265 As far as the $m = n - 1$ constraints in Eq. (3) are
266 concerned, the following system can be considered;
267 that is,

$$\begin{aligned} \ddot{u}_2(t) - \ddot{u}_1(t) &= 0 \\ \ddot{u}_4(t) - \ddot{u}_3(t) - \ddot{u}_2(t) &= 0 \\ \ddot{u}_6(t) - \ddot{u}_5(t) - \ddot{u}_4(t) &= 0 \\ &\vdots \\ \ddot{u}_{2n-2}(t) - \ddot{u}_{2n-3}(t) - \ddot{u}_{2n-4}(t) &= 0 \end{aligned} \quad (22)$$

269 yielding the $(n - 1 \times 2n - 1)$ matrix $\mathbf{A}$ in Eq. (4) as

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \quad (23)$$

At this point, it is straightforward to determine the
transfer function matrix $\bar{\mathbf{H}}_u(\omega)$ through Eq. (14),
while the following relationships can be used as a
check to ensure the validity of the results, i.e.,

$$\begin{aligned} H_{1,1} &= \bar{H}_{u_{1,1}} = \bar{H}_{u_{2,2}} \\ &\vdots \\ H_{n-1,n-1} &= \bar{H}_{u_{2n-2,2n-2}} \\ H_{n,n} &= \bar{H}_{u_{2n-2,2n-2}} + \bar{H}_{u_{2n-1,2n-1}} + 2\bar{H}_{u_{2n-2,2n-1}} \end{aligned} \quad (24)$$

On the left-hand side of Eq. (24), the components
of the transfer function matrix related to the system of
Eq. (1) can be determined via the relationship

$$\mathbf{H}(\omega) = [-\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K}]^{-1}. \quad (25)$$

On the right-hand side of Eq. (24) the response
function matrix is determined via Eq. (14), while the
response PSD matrix can be given by Eq. (15); thus,
reducing the required computational time drastically.

With reference to the above developed analysis,
consider a 3-DOF shear-type frame with
 $\mathbf{x}^T(t) = [x_1(t) \ x_2(t) \ x_3(t)]$, $m_i = 1$, $k_i = 2$ and
 $C_i = 0.1$ ($i = 1, 2, 3$). Further, the top story is sub-
jected to a zero-mean Gaussian white noise process
 $w(t)$ with constant PSD $S_0 = 10$, thus $f_1(t) = f_2(t) = 0$
and $f_3(t) = w(t)$. Following the aforementioned anal-
ysis, this shear-type frame can be decomposed in a 5-
DOF system, with $\mathbf{u}^T(t) = [u_1(t) \ u_2(t) \ u_3(t) \ u_4(t) \ u_5(t)]$.
Therefore, the relationships connecting $\mathbf{x}(t)$ to $\mathbf{u}(t)$ are given in Eq. (19),
while the additional constraints equations are expressed in Eq. (22).

Applying the procedure described in Sect. 3, the
transfer function matrix $\bar{\mathbf{H}}_u(\omega)$ of the augmented
system can be determined by Eq. (14), while Eq. (15)
leads to the response PSD matrix $\mathbf{S}_u(\omega)$, once the
modal matrix $\bar{\Phi}$ has been computed. Numerical results
obtained by this procedure are directly compared to
corresponding results obtained via a standard gener-
alized coordinates modelling scheme related to the
system of Eq. (1). Specifically, Fig. 3 shows the
comparison in terms of transfer functions and PSD
functions of the three main DOFs.

As it can be readily seen, the two solutions overlap
with each other. Further, note that once $\bar{\mathbf{H}}_u(\omega)$ is
obtained, an inverse Fourier transform of $\mathbf{U}(\omega)$ given
in Eq. (13) yields the time domain response of the
considered system. In this regard, to show the

Fig. 3 Comparison of the transfer functions and PSD functions of the 3-DOFs shear-type frame: **a, c, e** Transfer functions $H_{1,1}(\omega)$, $H_{2,2}(\omega)$ and $H_{3,3}(\omega)$ respectively; **b, d, f** PSD response functions $S_{1,1}(\omega)$, $S_{2,2}(\omega)$ and $S_{3,3}(\omega)$ respectively. Black line-

classical solution; red dashed line-non-generalized coordinates modeling solution

accuracy of the procedure, in Fig. 4 the time domain responses of the 3-DOFs are plotted for the case $f_3(t) = \delta(t)$, with $\delta(\cdot)$ being the Dirac's delta function, and with initial conditions given by $\mathbf{x}_0^T(t) =$

$[0.5 \ 0.1 \ 0]$ and $\dot{\mathbf{x}}_0^T(t) = [1 \ 0.5 \ 0]$. As anticipated, also in this case the two solutions overlap with each other.

Fig. 4 Comparison of the time domain responses of the 3-DOFs shear-type frame: **a** Response of the first floor; **b** response of the second floor; **c** response of the third floor. Black line- classical solution; red dashed line- non-generalized coordinates modeling solution

(a)

(b)

(c)

Fig. 5 Analyzed system: **a** Initial configuration; **b** free body diagram

4.2 Application to multi-body systems

Besides the previously considered case of building structures, the herein proposed technique can be beneficial also for a broad range of multi-body systems. This term refers to the dynamic behavior of interconnected rigid or flexible bodies with translational or rotational DOFs. This kind of systems is generally studied using the minimum number of DOFs, which leads to a non-singular mass matrix allowing for the application of the classical dynamics relations for the evaluation of eigenvalues and eigenvectors.

Nevertheless, in some cases it may appear natural to describe the system with more than the strictly sufficient number of DOFs. For these cases, however, a singular mass matrix can appear, and thus, the

response analysis can be performed by employing the herein developed technique.

In the following, an indicative example is considered in detail.

Consider the system, shown in Fig. 5a, composed of a rigid massless bar linked, through a spring of rigidity k_1 , to a rigid beam of mass m connected to the ground with a spring of rigidity k_2 and a damper c_2 .

Note that this is a SDOF system which could be fully described considering only the vertical displacement $x_1(t)$ of the point A_2 . In this case, however, the derivation of the corresponding equation of motion may be challenging. Therefore, it may be convenient to select an additional DOF, represented by the vertical displacement $x_2(t)$ of the point B_1 , so that the corresponding free body diagram can be easily derived, as sketched in Fig. 5b. In this manner,

Fig. 6 Comparison of the transfer function and PSD response: **a** Transfer functions $H_{1,1}(\omega)$; **b** PSD response function $S_{1,1}(\omega)$. Black line-exact solution; red dashed line- proposed solution

summation of moments with respect to the point A_1 and B_2 yields the equilibrium equations as

$$\begin{cases} \left(\frac{I_0}{3a} + \frac{3}{4}ma \right) \ddot{x}_1(t) + \frac{a}{3}c_2\dot{x}_1(t) + k_1\left(\frac{2}{3}x_1(t) - x_2(t)\right)2a + 3ak_2x_1(t) = 0 \\ k_1\left(\frac{2}{3}x_1(t) - x_2(t)\right)3a = 2aF(t) \end{cases} \quad (26a, b)$$

Equations (26) can be rewritten in the form of Eq. (2), with

$$\begin{aligned} \mathbf{M}_u &= \begin{bmatrix} \frac{I_0}{3a} + \frac{3}{4}ma & 0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{K}_u &= \begin{bmatrix} \frac{4}{3}ak_1 + 3ak_2 & -2ak_1 \\ -2ak_1 & 3ak_1 \end{bmatrix} \\ \mathbf{C}_u &= \begin{bmatrix} \frac{1}{3}ac_2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (27a - c)$$

where $\mathbf{M}_u$ appears to be singular. Further, considering the last row of $\mathbf{K}_u$, corresponding to the null row in $\mathbf{M}_u$,

$$\mathbf{A} = [-2ak_1 \quad 3ak_1] \quad (28)$$

Therefore, the structural response of the system can be obtained through the proposed procedure by employing Eqs. (14–16).

In this regard, Fig. 6 shows the transfer function and PSD of $x_1(t)$ considering: $m = 1$, $I_0 = 100$, $a = 1$, $k_1 = 100$, $k_2 = 2k_1$, $c_2 = 10$, zero initial conditions and $F(t)$ being a zero mean white noise process with constant PSD $S_0 = 10$.

Note that, for this simple case, an exact solution can be obtained by rewriting Eq. (26) in terms of $x_1(t)$ only, deriving $x_2(t)$ from Eq. (26b) and substituting in Eq. (26a); thus, leading to a SDOF system.

As it can be observed in Fig. 6, a perfect agreement is obtained between the exact solution of the corresponding SDOF system (black line) and the proposed procedure (red dashed line).

5 Concluding remarks

An efficient frequency domain response analysis methodology for structural dynamical systems modeled via non-generalized coordinates has been developed herein by resorting to the Moore–Penrose generalized matrix inverse in conjunction with a recently proposed generalized modal analysis treatment. In this regard, the corresponding transfer function relating the system excitation and response vectors has been derived and cast into a compact closed form expression, which can be evaluated in a computationally efficient manner. Further, the celebrated random vibration input–output frequency domain relationship between the excitation and the response power spectrum matrices has been recovered. The validity of the methodology has been demonstrated by considering a classical multi-degree-of-freedom shear kind building structure, as well as a multi-body system. The latter has been considered as an indicative case where an explicit formulation of the equations of motion by utilizing generalized, independent, coordinates can be a daunting task.

Appendix

Moore–Penrose theory elements

In this Appendix, some elements of the generalized matrix inverse theory pertaining to the Moore–Penrose (M–P) matrix inverse are provided for completeness [13, 14].

Definition: If $\mathbf{A} \in \mathbb{C}^{m \times l}$ then $\mathbf{A}^+$ is the unique matrix in $\mathbb{C}^{l \times m}$ so that

$$\begin{aligned} \mathbf{A}\mathbf{A}^+\mathbf{A} &= \mathbf{A}, \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+, (\mathbf{A}\mathbf{A}^+)^T \\ &= \mathbf{A}\mathbf{A}^+, (\mathbf{A}^+\mathbf{A})^T = \mathbf{A}^+\mathbf{A}. \end{aligned} \quad (29)$$

The matrix $\mathbf{A}^+$ is known as the M–P inverse of $\mathbf{A}$ and satisfies all Eq. (29). In general, the M–P inverse of a square matrix exists for any arbitrary $\mathbf{A} \in \mathbb{C}^{m \times m}$, and if $\mathbf{A}$ is non-singular, $\mathbf{A}^+$ coincides with $\mathbf{A}^{-1}$. Further, the M–P inverse of any $m \times l$ matrix $\mathbf{A}$ can be determined, for instance, via a number of recursive formulae (e.g., [12, 13]), and provides a tool for solving equations of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (30)$$

where $\mathbf{A}$ is a rectangular ($m \times l$) matrix, $\mathbf{x}$ is an l vector and $\mathbf{b}$ is an m vector. For a singular square matrix $\mathbf{A}$, utilizing the M–P inverse, Eq. (2) yields

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{y} \quad (31)$$

where $\mathbf{y}$ is an arbitrary n vector and $\mathbf{I}$ is the identity matrix.

References

1. Clough RW, Penzien J (1993) Dynamics of structures. McGraw Hill Inc, New York

2. Udwadia FE, Phohomsiri P (2006) Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics. Proc R Soc A 462(2071):2097–2117
3. Antoniou EN, Pantelous AA, Kougoumtzoglou IA, Pirrotta A (2017) Response determination of linear dynamical systems with singular matrices: a polynomial matrix theory approach. Appl Math Model 42:423–440
4. Kougoumtzoglou I, Frangkoulis V, Pantelous A, Pirrotta A (2017) Random vibration of linear and nonlinear structural systems with singular matrices: a frequency domain approach. J Sound Vib 404:84–101
5. Moysis L, Pantelous AA, Antoniou E, Karampetakis NP (2017) Closed form solution for the equations of motion for constrained linear mechanical systems and generalizations: an algebraic approach. J Frankl Inst 354(3):1421–1445
6. Pantelous A, Pirrotta A (2017) Modal analysis of multi-degree-of-freedom systems with singular matrices: analytical dynamics approach. ASCE J Eng Mech 143(6):06017005–06017007
7. Roberts JB, Spanos PD (2003) Random vibration and statistical linearization. Dover Publications, New York
8. Lanczos C (1986) The variational principles of mechanics, 4th edn. Dover, New York
9. Paz M, Leigh W (2004) Structural dynamics: Theory and Computation. Springer, Berlin
10. Schutte AD, Udwadia FE (2011) New approach to the modeling of complex multi-body dynamical systems. J Appl Mech 78(2):021018
11. Frangkoulis VC, Kougoumtzoglou IA, Pantelous AA (2016) Linear random vibration of structural systems with singular matrices. ASCE J Eng Mech 142(2):04015081
12. Frangkoulis VC, Kougoumtzoglou IA, Pantelous AA (2016) Statistical linearization of nonlinear structural systems with singular matrices. ASCE J Eng Mech 142(9):04016063
13. Campbell SL Meyer, Jr CD (1979) Generalized inverses of linear transformations. Dover Publications, New York
14. Ben-Israel A, Greville TNE (2003) Generalized inverses: theory and applications, 2nd edn. Springer, New York

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