TRIPLE SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS DRIVEN BY A NON-HOMOGENEOUS OPERATOR

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ABSTRACT. Some multiplicity results for a parametric nonlinear Dirichlet problem involving a nonhomogeneous differential operator of p-Laplacian type are given. Via variational methods, the article furnishes new contributions and completes some previous results obtained for problems considering other types of differential operators and/or nonlinear terms satisfying different asymptotic conditions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the following problem

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x, \nabla u) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a function admitting a potential and satisfying some natural conditions such that the differential operator $\mathrm{div}\boldsymbol{A}(x, \nabla(\cdot))$ includes the usual p-Laplacian $(p > 1)$, λ is a positive parameter, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable Carathéodory function.

There is a wide literature dealing with parameter dependent Dirichlet nonlinear problems. As is natural, such kind of problems have been studied starting with the case when the differential operator reduces to the classical Laplacian (semilinear case), then the interest has been focused on equations driven by the p-Laplacian and finally the more general nonhomogeneous framework has been treated.

The papers [1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 14, 15, 16], and the references therein, can help to have an essential idea about the remarkable developments of the research on this topic. In particular, in [1, 2, 3, 10, 11, 14, 15, 16] existence, non existence and multiplicity of solutions of the considered Dirichlet problem have been studied requiring that the reaction term exhibits a so-called 'concave-convex' behaviour. In [5], a perturbation of the p-Laplacian eigenvalue problem is treated, considering the cases when the perturbing term is either $(p-1)$-sublinear or $(p-1)$-superlinear both near at zero and near at infinity. The paper [9] introduced the notion of uniformly convex functional in order to establish the existence and the multiplicity of solutions to a class of nonlinear elliptic problem involving a differential operator that is more general than the classical p-Laplacian, provided that $p \geq 2$; moreover, the assumptions on the nonlinear term make use of the usual Ambrosetti-Rabinowitz condition and a suitable $(p-1)$-linearity at zero. In [12] the authors considered

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a counterpart of the case treated in [9] and allow the reaction term to be $(p-1)$-sublinear near at infinity and $(p-1)$-superlinear near at zero. In their arguments, the presence of the parameter plays a crucial role in proving the existence of at least three solutions for equations that are of p-Laplacian type, but the condition $p \geq 2$ persists. This restriction was removed in [8] where a more general class of elliptic differential operators has been considered, so that Δ_p can be covered for all $p > 1$.

In the present paper, inspired by [8] and [12], we study problem (P_λ) when $\boldsymbol{A}$ admits a potential $\mathscr{A} : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, such that

($\mathcal{A}$) *$\mathscr{A} = \mathscr{A}(x, \xi)$ is a continuous function on $\bar{\Omega} \times \mathbb{R}^N$, with a continuous derivative with respect to ξ and $\mathbf{A} = \partial_\xi \mathscr{A}$. Moreover*

(i) *$\mathscr{A}(x, 0) = 0$ and $\mathscr{A}(x, \xi) = \mathscr{A}(x, -\xi)$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.*

(ii) *$\mathscr{A}(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.*

(iii) *There exist two constants a_1, a_2, with $0 < a_1 \leq a_2$ such that*

$$\boldsymbol{A}(x, \xi) \cdot \xi \geq a_1 |\xi|^p \quad \textit{and} \quad |\boldsymbol{A}(x, \xi)| \leq a_2 |\xi|^{p-1}$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Our reaction term f is not assumed to satisfy the classical Ambrosetti-Rabinowitz condition. Moreover, as a relevant novelty with respect to the most part of the already known results in the literature, in the more general cases that we treat, explicit asymptotic conditions at zero are avoided in order to establish a concrete interval Λ of parameters for which (P_λ) admits at least three (weak) solutions (see Theorem 3.1). Indeed, as a particular situation, we can study the case when f is $(p-1)$-superlinear at zero, obtaining a more precise conclusion (see Theorem 3.3) with respect to that of [12] where a multiplicity theorem was established, but the interval Λ was estimated, but not explicitly computed.
In [8], the authors studied the problem

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x, \nabla u) = \lambda(a(x)|u|^{p-2}u + f(x, u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \in L^\alpha(\Omega)$ is a positive weight and f is $(p-1)$-sublinear at infinity and $(p-1)$-superlinear at zero. Our approach allows us to give a multiplicity result also in this framework (see Theorem 3.5) and, as a consequence, to furnish a complement to a non existence theorem proved in [8] when $\lambda \in [0, \lambda_\star)$ for a suitable $\lambda_\star$, since a simpler estimate from above of $\lambda_\star$ is here established.

Finally, we wish to point out that our main results are given in Section 3 and their proofs are fully based on variational methods. In particular, a very important tool is a suitable version of a critical point theorem, proved in [4], that is here recalled in Section 2, as well as some other preliminaries.

2. Basic notations and auxiliary results

Throughout the paper Ω, is a bounded domain of $\mathbb{R}^N$, $1 < p < N$, $W_0^{1,p}(\Omega)$ is the usual Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_p$$

and $W^{-1,p'}(\Omega)$ is its dual space. It is well known that, if $1 < p < N$ and $p^* = \frac{Np}{N-p}$ there is a constant $T = T(N, p)$ such that

$$\|u\|_{p^*} \leq T\|u\| \tag{2.1}$$

for every $u \in W_0^{1,p}(\Omega)$. Such a constant has been sharply determined by Talenti in [18] by the formula

$$T = \pi^{-1/2} N^{-1/p} \left(\frac{p-1}{N-p} \right)^{1-1/p} \left\{ \frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)} \right\}^{1/N}, \tag{2.2}$$

where Γ is the gamma function. Clearly, (2.1), in conjunction with the Hölder's inequality, implies that for every $s \in [1, p^*]$

$$\|u\|_s \leq T |\Omega|^{(p^*-s)/(p^* s)} \|u\| \tag{2.3}$$

for all $u \in X$, where $|\Omega|$ is the Lebesgue measure of Ω and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ is compact provided $s \in [1, p^*[$.

Following [8], we will assume that $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a function admitting a smooth potential $\mathscr{A} : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ as given in the Introduction.
In [8], it has been explicitly observed that the preceding assumptions $(\mathcal{A})$(i) and $(\mathcal{A})$(iii) imply that

$$a_1 |\xi|^p \leq p\, \mathscr{A}(x, \xi) \leq a_2 |\xi|^p \tag{2.4}$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^N$. Moreover, it is possible to obtain the following

Lemma 2.1. [8, Lemma 2.5] *Let $\mathscr{A}$ satisfy $(\mathcal{A})$(i)–$(\mathcal{A})$(iii). Then the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Phi(u) = \int_\Omega \mathscr{A}(x, \nabla u(x))\, dx \tag{2.5}$$

is convex, weakly lower semicontinuous and of class C^1 in $W_0^{1,p}(\Omega)$, being

$$\Phi'(u)(v) = \int_\Omega \boldsymbol{A}(x, \nabla u) \cdot \nabla v\, dx$$

for every $u, v \in W_0^{1,p}(\Omega)$.

Moreover, $\Phi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the $(\mathscr{S}_+)$ condition, i.e., for every sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty} \int_\Omega \boldsymbol{A}(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx \leq 0,$$

then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

Given a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, a positive function $a \in L^\alpha(\Omega)$, $\alpha > N/p$, and $1 < q \leq p$, we say that f is of type $(\mathscr{G}_{f,a,q})$ if it satisfies the following growth condition

$(\mathscr{G}_{f,a,q})$ *There exist two positive constants M_1 and M_2 such that*

$$|f(x,t)| \leq a(x)(M_1 + M_2 |t|^{q-1}) \quad \textit{for a.a. } x \in \Omega \textit{ and all } t \in \mathbb{R}.$$

According to [8, Lemma 3.2] one can state the following

Lemma 2.2. *Assume that f is of type $(\mathscr{G}_{f,a,q})$ and put $F(x,t) = \int_0^t f(x,s)\, ds$. Then, the functionals $\Psi_1,\ \Psi_2 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Psi_1(u) = \frac{1}{p} \int_\Omega a(x) |u(x)|^p\, dx, \quad \Psi_{2,f}(u) = \int_\Omega F(x, u(x))\, dx \tag{2.6}$$

are of class C^1 being

$$\Psi_1'(u)(v) = \int_\Omega a(x)|u(x)|^{p-2}u(x)v(x)\ dx, \quad \Psi_{2,f}'(u)(v) = \int_\Omega f(x,u(x))v(x)\ dx.$$

Moreover the operators $\Psi_1',\ \Psi_{2,f}' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ are compact and $\Psi_1,\ \Psi_{2,f}$ are sequentially weakly continuous in $W_0^{1,p}(\Omega)$.

Remark 2.1. In Lemma 3.2 of [8] the compactness of $\Psi_{2,f}'$ is proved when $1 < q < p$, but the same arguments can be adopted in order to still assure the same property also if the case $q = p$ occurs.

We explicitly recall that, if f is a function of type $(\mathscr{G}_{f,a,q})$, a weak solution of problem (P_λ) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \mathbf{A}(x, \nabla u(x)) \cdot \nabla v(x)\ dx - \lambda \int_\Omega f(x,u(x))v(x)\ dx = 0$$

for every $v \in W_0^{1,p}(\Omega)$. Hence, in view of Lemma 2.1 and Lemma 2.2, if for $\lambda > 0$ we consider the functional $\mathscr{I}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $\mathscr{I}_\lambda(u) = \Phi(u) - \lambda\Psi_2(u)$, it is obvious that

$$\textit{The critical points of } \mathscr{I}_\lambda \textit{ are weak solutions of problem } (P_\lambda). \tag{2.7}$$

Let us conclude this section stating the main tool that we will use in studying problem (P_λ). It is a critical point result and represents a suitable version of Theorem 7.1 of [4], since it can be derived immediately from [4, Remark 7.1].

Theorem 2.1. *Let X be a real Banach space and let $\Phi,\ \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals with Φ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there exist $r > 0$ and $\bar{u} \in X$, with $\Phi(\bar{u}) > r$, such that*

$$\frac{\sup_{\Phi(u) \le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}. \tag{2.8}$$

Moreover, for each $\lambda \in \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$ the functional $\mathscr{I}_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.

Then, for each $\lambda \in \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$ the functional $\mathscr{I}_\lambda$ admits at least three critical points.

3. Main results

In the present section, we are going to show some multiplicity results for problem (P_λ) when different conditions about the behavior at infinity of f are considered coupled with some suitable local assumptions.

Let us first introduce few further notations. Let $\mathscr{R} : \Omega \to [0, +\infty[$ be the function defined by $\mathscr{R}(x) = d(x, \partial\Omega)$ for each $x \in \Omega$. Thus, for every fixed $x_0 \in \Omega$ one has $B(x_0, \mathscr{R}(x_0)) = \{x \in \Omega : \ |x - x_0| < \mathscr{R}(x_0)\} \subseteq \Omega$ and, for $a \in L^\alpha(\Omega)$ (with $\alpha > N/p$), we put

$$\kappa = \kappa(x_0) = \frac{T}{|\Omega|^{1/p^*}} \left[\left(\frac{\mathscr{R}(x_0)}{2} \right)^{(N-p)} (2^N - 1)|B(0,1)| \right]^{1/p} \tag{3.1}$$

and

$$\mathscr{H} = \mathscr{H}(x_0) = \frac{a_2\|a\|_\alpha (2T)^p(2^N-1)|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1[\mathscr{R}(x_0)]^p}, \tag{3.2}$$

where T is the Talenti's constant introduced in (2.2), a_1 and a_2 are the constants considered in $(\mathcal{A})$(iii), $|B(0,1)|$ is the Lebesgue measure of the N-dimensional unit ball and α' is the conjugate exponent of α.

Here is our first main result.

Theorem 3.1. *Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfils $(\mathscr{G}_{f,a,q})$ and that there exist $x_0 \in \Omega$ and $c,\ d > 0$ with*

$$c < \kappa d, \tag{3.3}$$

such that

(H$_1$) $F(x,t) \geq 0$ *for a.a.* $x \in B(x_0, \mathscr{R}(x_0))$ *and for every* $t \in [0,d]$

(H$_2$) $\mathscr{H}\left(M_1 c^{1-p} + \dfrac{M_2}{q} c^{q-p}\right) < \dfrac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p}.$

Put

$$\lambda_* = \frac{a_2 2^p(2^N-1)}{p[\mathscr{R}(x_0)]^p}\frac{d^p}{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)} \tag{3.4}$$

and

$$\lambda^* = \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}} \frac{1}{M_1 c^{1-p} + \frac{M_2}{q} c^{q-p}}. \tag{3.5}$$

Then, for each $\lambda \in]\lambda_*, \lambda^*[$ problem (P_λ) admits at least three weak solutions.

Proof. For the sake of completeness, first of all let us explicitly observe that $\lambda_* < \lambda^*$. Indeed, from (H$_2$) and the definition of $\mathscr{H}$ one has

$$\frac{a_2\|a\|_\alpha (2T)^p(2^N-1)|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1[\mathscr{R}(x_0)]^p}\left(M_1 c^{1-p} + \frac{M_2}{q} c^{q-p}\right) < \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p}$$

hence

$$\frac{\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1}\left(M_1 c^{1-p} + \frac{M_2}{q} c^{q-p}\right) < \frac{[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)}\frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p},$$

that is

$$\frac{1}{p\lambda^*} < \frac{1}{p\lambda_*},$$

namely $\lambda_* < \lambda^*$.

We apply Theorem 2.1 with $X = W_0^{1,p}(\Omega)$, $\Phi : X \to \mathbb{R}$ as defined in (2.5) and $\Psi = \Psi_{2,f}$ as introduced in (2.6).

Lemma 2.1 and Lemma 2.2 establish that Φ and Ψ are of class C^1, while condition (2.4) assures the following control for Φ

$$\frac{a_1}{p}\|u\|^p \leq \Phi(u) \leq \frac{a_2}{p}\|u\|^p \tag{3.6}$$

for every $u \in X$ and Φ is bounded from below. Clearly $\Phi(0) = \Psi(0) = 0$.

Consider the function $u_d \in X$ defined by

$$u_d(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \bar{B}(x_0, \mathscr{R}(x_0)) \\ \frac{2d}{\mathscr{R}(x_0)}\left(\mathscr{R}(x_0) - |x - x_0|\right) & \text{if } x \in B(x_0, \mathscr{R}(x_0)) \setminus \bar{B}(x_0, \mathscr{R}(x_0)/2) \\ d & \text{if } x \in B(x_0, \mathscr{R}(x_0)/2), \end{cases} \tag{3.7}$$

and put

$$r=\frac{a_1|\Omega|^{p/p^*}}{pT^p}c^p.$$

A direct computation based on (3.6) and (3.3) shows that

$$\begin{aligned}\Phi(u_d) &\geq \frac{a_1}{p}\frac{2^p}{[\mathscr{R}(x_0)]^p}|B(x_0,\mathscr{R}(x_0))\setminus \bar{B}(x_0,\mathscr{R}(x_0)/2)|d^p\\ &= \frac{a_1}{p}\frac{2^p}{[\mathscr{R}(x_0)]^p}|B(0,1)|([\mathscr{R}(x_0)]^N-[\mathscr{R}(x_0)/2]^N)d^p\\ &= \frac{a_1}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d^p\\ &> \frac{a_1|\Omega|^{p/p^*}}{pT^p}c^p=r.\end{aligned}$$

Moreover

$$\Phi(u_d)\leq \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d^p, \tag{3.8}$$

as well as, in view of assumption (H_1), one has

$$\begin{aligned}\Psi(u_d) &= \int_\Omega F(x,u_d(x))\,dx\\ &\geq \int_{B(x_0,\mathscr{R}(x_0)/2)} F(x,d)dx\\ &\geq |B(x_0,\mathscr{R}(x_0)/2)|\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)\\ &= \left(\frac{\mathscr{R}(x_0)}{2}\right)^N|B(0,1)|\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d).\end{aligned} \tag{3.9}$$

From (3.8) and (3.9) one infers

$$\frac{\Psi(u_d)}{\Phi(u_d)}\geq \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)}\frac{\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p}=\frac{1}{\lambda_*}. \tag{3.10}$$

On the other hand, since $\alpha>N/p$ implies that $1<\alpha'<\alpha' q\leq \alpha' p<p^*$, condition $(\mathscr{G}_{f,a,q})$, the Hölder's inequality and (2.3) lead to

$$\begin{aligned}\Psi(u) &\leq M_1\int_\Omega a(x)|u(x)|\,dx+\frac{M_2}{q}\int_\Omega a(x)|u(x)|^q\,dx\\ &\leq M_1\|a\|_\alpha\|u\|_{\alpha'}+\frac{M_2}{q}\|a\|_\alpha\|u\|^q_{\alpha' q}\\ &\leq M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\|u\|+\frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\|u\|^q\end{aligned} \tag{3.11}$$

for every $u\in X$. Hence, observed that in view of (3.6)

$$\{u\in X:\ \Phi(u)\leq r\}\subseteq\left\{u\in X:\ \|u\|\leq\left(\frac{pr}{a_1}\right)^{1/p}\right\}, \tag{3.12}$$

condition (3.11) assures that

$$\sup_{\Phi(u)\leq r}\Psi(u)\leq M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\left(\frac{pr}{a_1}\right)^{1/p}+\frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\left(\frac{pr}{a_1}\right)^{q/p}.$$

Thus,

$$\begin{aligned}
\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} &\le M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\left(\frac{p}{a_1}\right)^{1/p} r^{1/p-1}+\\
&\qquad +\frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\left(\frac{p}{a_1}\right)^{q/p} r^{q/p-1}\\
&= \|a\|_\alpha\frac{p}{a_1}T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\left[M_1\left(\frac{T^p pr}{a_1|\Omega|^{p/p^*}}\right)^{(1-p)/p}+\right.\\
&\qquad\left.+\frac{M_2}{q}\left(\frac{T^p pr}{a_1|\Omega|^{p/p^*}}\right)^{(q-p)/p}\right]\\
&= \|a\|_\alpha\frac{p}{a_1}T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\left[M_1c^{1-p}+\frac{M_2}{q}c^{q-p}\right]\\
&= \frac{1}{\lambda^*}. \qquad (3.13)
\end{aligned}$$

Putting together (3.13) and (3.10) one achieves

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r}\le\frac{1}{\lambda^*}<\frac{1}{\lambda_*}\le\frac{\Psi(u_d)}{\Phi(u_d)},$$

namely assumption (2.8) of Theorem 2.1 is verified with $\bar{u}=u_d$ and, in particular, one has

$$]\lambda_*,\lambda^*[\subseteq\left]\frac{\Phi(u_d)}{\Psi(u_d)},\frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[. \qquad (3.14)$$

Finally, let us proof that for every $\lambda\in[0,\lambda^*[$ the functional $\mathscr{I}_\lambda=\Phi-\lambda\Psi$ is bounded from below and satisfies the (PS)-condition. Indeed, when $1<q<p$, conditions (3.6) and (3.11) assures that for every $\lambda\ge 0$ the functional $\mathscr{I}_\lambda$ is coercive. Otherwise, if $q=p$, one has

$$\begin{aligned}
\mathscr{I}_\lambda(u) &\ge \frac{a_1}{p}\|u\|^p-\lambda M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\|u\|+\\
&\qquad -\lambda\frac{M_2}{p}\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\|u\|^p\\
&= \frac{1}{p}\left(a_1-\lambda M_2\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\right)\|u\|^p+ \qquad (3.15)\\
&\qquad -\lambda M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\|u\|
\end{aligned}$$

that implies the coercivity also in this case, being

$$\lambda<\lambda^*=\frac{a_1}{\|a\|_\alpha pT^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}\frac{1}{M_1c^{1-p}+\frac{M_2}{p}}<\frac{a_1}{M_2\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}.$$

Since $\mathscr{I}_\lambda$ is sequentially weakly lower semicontinuous (see Lemma 2.1 and Lemma 2.2), it is bounded from below. Fix $\{u_n\}$ in X such that $\{\mathscr{I}_\lambda(u_n)\}$ is bounded and $\mathscr{I}_\lambda'(u_n)\to 0$ in X^*. Thus, $\{u_n\}$ is bounded and the reflexivity of X as well as the compactness of Ψ' (see Remark 2.1) assures that $u_n\rightharpoonup u$ weakly in X and $\Psi'(u_n)\to S^*$ in X^*, where a subsequence is considered if necessary. Hence, for every $n \in \mathbb{N}$

$$\begin{aligned}\Phi'(u_n)(u_n-u) &= \mathscr{I}'_\lambda(u_n)(u_n-u)+\lambda\Psi'(u_n)(u_n-u)\\ &\le \|\mathscr{I}'_\lambda(u_n)\|_{X^*}\|u_n-u\|+\lambda\Psi'(u_n)(u_n-u)\\ &\le \|\mathscr{I}'_\lambda(u_n)\|_{X^*}\|u_n-u\|+\lambda\|\Psi'(u_n)-S^*\|_{X^*}\|u_n-u\|+\lambda S^*(u_n-u).\end{aligned}$$

Passing to the limsup in the preceding inequality, one has

$$\limsup_{n\to\infty}\Phi'(u_n)(u_n-u)\le 0,$$

and the conclusion is obtained because Φ' satisfies the $(\mathscr{S}_+)$ condition (see Lemma 2.1).

At this point, all the assumptions of Theorem 2.1 are fulfilled, hence, taking in mind (3.14), for every $\lambda \in]\lambda_*,\lambda^*[$ the functional $\mathscr{I}_\lambda$ admits at least three critical points, namely, because of claim (2.7), problem (P_λ) admits at least three weak solutions and the proof is complete. □

Let $x_0 \in \Omega$ be such that $\mathscr{R}(x_0) = \max_{x\in\Omega}\mathscr{R}(x)$ and put

$$\Theta = \Theta(x_0) = \frac{a_2(2T)^p(2^N-1)|\Omega|^{(p^*-p)/(p^*)}}{a_1[\mathscr{R}(x_0)]^p}.$$

If $f:\mathbb{R}\to\mathbb{R}$ is a continuous and nonnegative function and there exist $M_1,\ M_2 > 0$ and $1 < q \le p$ such that

$$|f(t)| \le M_1 + M_2|t|^{q-1} \tag{3.16}$$

for every $t \in \mathbb{R}$, the previous result takes the following form.

Corollary 3.1. *Put $F(t) = \int_0^t f(s)\,ds$ for every $t \in \mathbb{R}$ and assume that there exist $c,\ d > 0$ satisfying (3.3) such that*

$$\Theta\left(M_1c^{1-p}+\frac{M_2}{q}c^{q-p}\right) < \frac{F(d)}{d^p}. \tag{3.17}$$

Then, for every $\lambda \in \Lambda = \left]\frac{a_2 2^p(2^N-1)}{p[\mathscr{R}(x_0)]^p}\frac{d^p}{F(d)},\ \frac{a_1}{pT^p|\Omega|^{(p^*-\alpha'p)/(\alpha'p^*)}}\frac{1}{M_1c^{1-p}+\frac{M_2}{q}c^{q-p}},\right[$ *the problem*

$$\begin{cases} -\mathrm{div}\mathbf{A}(x,\nabla u) = \lambda f(u) & \textit{in } \Omega,\\ u = 0 & \textit{on } \partial\Omega,\end{cases}$$

admits at least three weak solutions.

Proof. It is enough to apply Theorem 3.1 with $a \equiv 1$, $\alpha = \infty$, $\alpha' = 1$ and observing that the sign condition on f assures in particular that $F(t) \ge 0$ for every $t \in \mathbb{R}$. □

Remark 3.1. Because of the choice of $\mathscr{R}(x_0)$ in the preceding corollary, one can observe that the interval Λ of parameters is the largest that the technique involving the function u_d permits to obtain. To the best of our knowledge in paper [6], for the first time, such a kind of u_d has been introduced for studying a Dirichlet problem with the p-Laplacian.

A careful analysis of the crucial assumption (H_2) permits to use Theorem 3.1 in order to obtain multiple solutions of problem (P_λ) when Ω is any fixed ball with radius large enough. To be precise, one can state the following

Theorem 3.2. *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a nonnegative Carathéodory function, with $f \not\equiv 0$, let $a \in L^\alpha(\mathbb{R}^N)$, $\alpha > N/p$, be a positive function and assume that there exist two positive constants M_1, M_2 and $q \in]1, p]$ such that*

$$|f(x,t)| \leq a(x)(M_1 + M_2|t|^{q-1})$$

for a. a. $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$. Moreover, assume that there exists $\bar{d} > 0$ such that

$$\mathrm{essinf}_{x\in\mathbf{R}} F(x, \bar{d}) > 0.$$

Then there exists $\bar{R} = \bar{R}(\bar{d}) > 0$ such that, for every $R > \bar{R}$ there exist $\tau_* = \tau_*(R)$, $\tau^* = \tau^*(R) > 0$ with $\tau_* < \tau^*$ such that for every $x_0 \in \mathbb{R}^N$ and every $\lambda \in]\tau_*, \tau^*[$ the problem

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x, \nabla u) = \lambda f(x,u) & \text{in } B(x_0, R), \\ u = 0 & \text{on } \partial B(x_0, R), \end{cases} \tag{3.18}$$

admits at least three weak solutions.

Proof. First, for $R > 0$ put

$$\tilde{\mathscr{H}}_R = \frac{a_2\|a\|_{L^\alpha(\mathbb{R}^N)}(2T)^p(2^N-1)|B(0,1)|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1}\frac{R^{[N(p^*-\alpha' p)]/(\alpha' p^*)}}{R^p}$$

and observe that

$$N\frac{p^* - \alpha' p}{\alpha' p^*} < p.$$

Indeed,

$$\begin{aligned} N\frac{p^*-\alpha' p}{\alpha' p^*} &= N\frac{p^* - \frac{\alpha}{\alpha-1}p}{\frac{\alpha}{\alpha-1}p^*} \\ &= N\frac{\alpha p^* - p^* - \alpha p}{\alpha p^*} \\ &= N\left(1 - \frac{1}{\alpha} - \frac{p}{p^*}\right) \\ &= N - N\left(\frac{1}{\alpha} + \frac{p}{p^*}\right) \\ &= N - p - N\left(\frac{1}{\alpha} + \frac{p}{p^*}\right) + p \\ &= N\left[\frac{N-p}{N} - \left(\frac{1}{\alpha} + \frac{p}{p^*}\right)\right] + p \\ &= N\left[\frac{p}{p^*} - \left(\frac{1}{\alpha} + \frac{p}{p^*}\right)\right] + p \\ &= p - \frac{N}{\alpha} \\ &< p. \end{aligned}$$

Hence, it is clear that

$$\lim_{R\to+\infty} \tilde{\mathscr{H}}_R = 0. \tag{3.19}$$

Let $d = d(\bar{d})$ be a positive number such that

$$d > \max\left\{\bar{d}, \frac{1}{T|B(0,1)|^{(p^*-p)/(p^*p)}}\left(\frac{2^{N-p}}{2^N-1}\right)^{1/p}\right\}. \tag{3.20}$$

Obviously one has that

$$\frac{\text{essinf}_{\mathbb{R}} F(x,d)}{d^p} \geq \frac{\text{essinf}_{\mathbb{R}} F(x,\bar{d})}{d^p} > 0,$$

and, in view of (3.19), there exists $\bar{R} = \bar{R}(\bar{d})$ large enough such that

$$\tilde{\mathscr{H}}_R\left(M_1 + \frac{M_2}{q}\right) < \frac{\text{essinf}_{\mathbb{R}} F(x,d)}{d^p} \leq \frac{\text{essinf}_{x\in B(x_0,R/2)} F(x,d)}{d^p} \tag{3.21}$$

for every $R > \bar{R}$ and for every $x_0 \in \mathbb{R}^N$.

Hence, fixed $R > \bar{R}$ and $x_0 \in \mathbb{R}^N$, we can apply Theorem 3.1 with $\mathscr{R}(x_0) = R$, $\Omega = B(x_0, R)$, $c = 1$ and d as considered above. Indeed, $a \in L^\alpha(\Omega)$, the restriction of f on $\Omega \times \mathbb{R}$ satisfies condition ($\mathscr{G}_{f,a,q}$) and assumption (H_1) is trivially satisfied because of the nonnegativity of f. Recalling (3.1), one has

$$\begin{aligned}
\kappa &= \frac{T}{|\Omega|^{1/p^*}}\left[\left(\frac{\mathscr{R}(x_0)}{2}\right)^{(N-p)}(2^N-1)|B(0,1)|\right]^{1/p} \\
&= \frac{T}{R^{N/p^*}|B(0,1)|^{1/p^*}} R^{(N-p)/p}\left[\frac{2^N-1}{2^{N-p}}|B(0,1)|\right]^{1/p} \\
&= T\left[\frac{2^N-1}{2^{N-p}}\right]^{1/p}|B(0,1)|^{(p^*-p)/(p^*p)}.
\end{aligned}$$

Condition (3.20) assures that $d > 1/\kappa$, namely $c = 1 < \kappa d$ and (3.3) holds. Moreover, from (3.2) and the positivity of a, one has

$$\begin{aligned}
\mathscr{H} &= \frac{a_2\|a\|_{L^\alpha(\Omega)}(2T)^p(2^N-1)|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1[\mathscr{R}(x_0)]^p} \\
&= \frac{a_2\|a\|_{L^\alpha(B(x_0,R))}(2T)^p(2^N-1)|B(0,1)|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1}\frac{R^{[N(p^*-\alpha' p)]/(\alpha' p^*)}}{R^p} \\
&< \frac{a_2\|a\|_{L^\alpha(\mathbb{R}^N)}(2T)^p(2^N-1)|B(0,1)|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1}\frac{R^{[N(p^*-\alpha' p)]/(\alpha' p^*)}}{R^p} \\
&= \tilde{\mathscr{H}}_R,
\end{aligned}$$

namely, taking in mind (3.21)

$$\begin{aligned}
\mathscr{H}\left(M_1 + \frac{M_2}{q}\right) &< \tilde{\mathscr{H}}_R\left(M_1 + \frac{M_2}{q}\right) \\
&< \frac{\text{essinf}_{\mathbb{R}} F(x,d)}{d^p} \leq \frac{\text{essinf}_{x\in B(x_0,R/2)} F(x,d)}{d^p}
\end{aligned}$$

and (H_2) holds. All the assumptions of Theorem 3.1 are satisfied and if we put

$$\tau_* = \frac{a_2 2^p(2^N-1)}{pR^p}\frac{d^p}{\text{essinf}_{x\in B(x_0,R/2)} F(x,d)}$$

and

$$\tau^* = \frac{a_1}{\|a\|_{L^\alpha(B(x_0,R))} pT^p(R^N|B(0,1)|)^{(p^*-\alpha' p)/(\alpha' p^*)}} \frac{1}{M_1 + \frac{M_2}{q}},$$

it is clear that $\tau_* = \lambda_*$ and $\tau^* = \lambda^*$, where λ_* and λ^* are the constant defined in (3.4) and (3.5) respectively, so that the conclusion follows at once. $\square$

Remark 3.2. Looking at the proof of the preceding theorem, the definition of $\tilde{\mathscr{H}}_R$ as well as conditions (3.20) and (3.21) give an estimate of the size of the constant $\bar{R}$ and, as a consequence, of the largeness of the balls that can be considered as well as of the interval of parameters λ.

We can observe that the assumptions of Theorem 3.1 do not require any particular behavior of $f(x,\cdot)$ at zero. If one adds a kind of $(p-1)$-superlinearity at zero it is possible to obtain an unbounded interval of λ for which problem (P_λ) admits multiple solutions. More precisely, the following result holds

Theorem 3.3. *Assume that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies condition* $(\mathscr{G}_{f,a,q})$ *with* $1 < q < p$ *and that there exist* $x_0 \in \Omega$, $d > 0$ *such that* (H_1) *holds in addition to*

$(\mathrm{H}_2)'$ $\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d) > 0.$

Moreover, suppose that

(H_3) $\lim_{t\to 0} \frac{f(x,t)}{a(x)|t|^{p-1}} = 0$ *uniformly a.e. in* Ω.

Then, for every $\lambda > \lambda_*$ *(with* λ_* *defined in (3.4)) problem* (P_λ) *admits at least two non trivial weak solutions.*

Proof. First observe that, in view of $(\mathrm{H}_2)'$, $\lambda_* > 0$. Fix $\bar\lambda \in]\lambda_*, +\infty[$. Because of claim (2.7) it will be enough to prove that $\mathscr{I}_{\bar\lambda} = \Phi - \bar\lambda\Psi$ admits at least three critical points, where Φ and $\Psi = \Psi_{2,f}$ as defined in (2.5) and (2.6) respectively.

Let us begin pointing out that, adapting the arguments of the proof of [12, Lemma 3.3], assumption (H_3) and $(\mathscr{G}_{f,a,q})$ assure that

$$\lim_{r\to 0^+} \frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} = 0. \tag{3.22}$$

Indeed, fixed $\varepsilon > 0$ from (H_3) there exists $\delta = \delta(\varepsilon) \in]0,1[$ such that

$$|f(x,t)| \le \varepsilon a(x)|t|^{p-1} \tag{3.23}$$

a.e. in Ω and for every $t \in]-\delta,\delta[$.
Thus, if $\gamma \in]p, p^*/\alpha'[$ and $M_3 = \max\left\{\varepsilon, \frac{M_1+M_2}{\delta^{\gamma-1}}\right\}$, we can verify that

$$|f(x,t)| \le \varepsilon a(x)|t|^{p-1} + M_3 a(x)|t|^{\gamma-1} \tag{3.24}$$

a.e. in Ω and for every $t \in \mathbb{R}$. Indeed, if $|t| < \delta$ condition (3.24) follows immediately from (3.23). If $\delta \le |t| < 1$ from $(\mathscr{G}_{f,a,q})$ one has

$$|f(x,t)| \le a(x)(M_1+M_2) = a(x)\frac{M_1+M_2}{|t|^{\gamma-1}}|t|^{\gamma-1} \le a(x)\frac{M_1+M_2}{\delta^{\gamma-1}}|t|^{\gamma-1},$$

a.e. in Ω and (3.24) holds. While if $|t| \ge 1$, again from $(\mathscr{G}_{f,a,q})$ one has

$$|f(x,t)| \le a(x)\left(\frac{M_1}{|t|^{q-1}} + M_2\right)|t|^{q-1} \le a(x)(M_1+M_2)|t|^{q-1} \le a(x)(M_1+M_2)|t|^{\gamma-1}$$

a.e. in Ω, so that also in this case (3.24) is satisfied.

Since $\alpha'\gamma < p^*$, from (3.24) and the Hölder's inequality one has

$$\begin{aligned}\Psi(u) &\leq \frac{\varepsilon}{p}\int_\Omega a(x)|u(x)|^p\,dx + \frac{M_3}{\gamma}\int_\Omega a(x)|u(x)|^\gamma\,dx \\ &\leq \frac{\varepsilon}{p}\|a\|_\alpha\|u\|^p_{\alpha' p} + \frac{M_3}{\gamma}\|a\|_\alpha\|u\|^\gamma_{\alpha'\gamma} \\ &\leq \frac{\varepsilon}{p}\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\|u\|^p + \frac{M_3}{\gamma}\|a\|_\alpha T^\gamma|\Omega|^{(p^*-\alpha'\gamma)/(p^*\alpha')}\|u\|^\gamma\end{aligned} \tag{3.25}$$

for every $u \in X$. Putting together (3.12) and (3.25) one has that for every $r > 0$

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} &\leq \frac{\varepsilon}{a_1}\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')} + \\ &\quad + \frac{M_3}{\gamma}\|a\|_\alpha T^\gamma|\Omega|^{(p^*-\alpha'\gamma)/(p^*\alpha')}\left(\frac{p}{a_1}\right)^{\gamma/p} r^{(\gamma-p)/p}.\end{aligned}$$

Taking in mind that $\gamma > p$, the preceding condition assures that

$$\limsup_{r\to 0^+}\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{\varepsilon}{a_1}\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}$$

and, in view of the arbitrary of $\varepsilon > 0$ it is clear that (3.22) holds.

Hence, if $u_d \in X$ is the function defined in (3.7), in correspondence of the fixed $\bar\lambda \in]\lambda^*, +\infty[$ there exists $\bar r = \bar r(\bar\lambda) \in]0, \Phi(u_d)[$ small enough such that

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq \bar r}\Psi(u)}{\bar r} &< \frac{1}{\bar\lambda} < \frac{1}{\lambda_*} \\ &= \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)}\frac{\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p} \\ &\leq \frac{\Psi(u_d)}{\Phi(u_d)},\end{aligned} \tag{3.26}$$

where, thanks to assumption (H_1), we have also used condition (3.10). Thus, condition (2.8) is satisfied with $r = \bar r$ and $\bar u = u_d$. Moreover, in the proof of Theorem 3.1 it has been already observed that when $1 < q < p$ the functional $\mathscr{I}_\lambda$ is coercive for every $\lambda \geq 0$, hence it is bounded from below and satisfies the (PS)-condition . Finally, we can apply Theorem 3.1 and conclude that for every $\lambda \in \left]\frac{\Phi(u_d)}{\Psi(u_d)}, \frac{\bar r}{\sup_{\Phi(u)\leq \bar r}\Psi(u)}\right[$ the functional $\mathscr{I}_\lambda$ admits at least three critical points. The proof is completed once observed that (3.26) implies that $\bar\lambda \in \left]\frac{\Phi(u_d)}{\Psi(u_d)}, \frac{\bar r}{\sup_{\Phi(u)\leq \bar r}\Psi(u)}\right[$. □

A particular version of Theorem 3.3 is now pointed out when the non linearity f does not depend on x and, for simplicity, it is assumed to satisfy a sign condition.

Corollary 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t) \geq 0$ for every $t \in [0, +\infty[$, with $f \not\equiv 0$, and there exist $M_1,\ M_2 > 0$ and $q \in]1, p[$ such that*

$$f(t) \leq M_1 + M_2|t|^{q-1}$$

for every $t \in \mathbb{R}$. Assume that

$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}} = 0. \tag{3.27}$$

Put $F(t)=\int_0^t f(s)\,ds$ *for every* $t\in\mathbb{R}$ *and let* $d>0$ *be such that*

$$\frac{F(d)}{d^p}=\max_{t\in]0,+\infty[}\frac{F(t)}{t^p}.$$

Moreover, let $x_0\in\Omega$ *such that* $\mathscr{R}(x_0)=\max_{x\in\Omega}\mathscr{R}(x)$.
Then, for every $\lambda>\beta_*=\dfrac{a_2 2^p(2^N-1)}{p[\mathscr{R}(x_0)]^p}\dfrac{d^p}{F(d)}$ *the problem*

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x,\nabla u)=\lambda f(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

admits at least two positive solutions.

Proof. Put

$$f_+(x,t)=\begin{cases} f(t) & \text{if } (x,t)\in\Omega\times[0,+\infty[\\ 0 & \text{if } (x,t)\in\Omega\times]-\infty,0[\end{cases}$$

and $F_+(x,t)=\int_0^t f(x,s)\,ds$ for every $(x,t)\in\Omega\times\mathbb{R}$. It is very simple to verify that f_+ and F_+ satisfy all the assumptions of Theorem 3.3 with $a\equiv 1$. Hence, for every $\lambda>\beta_*$ the following problem

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x,\nabla u)=\lambda f_+(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases} \tag{3.29}$$

admits at least two nontrivial weak solutions. We can conclude the proof once we point out that every nontrivial weak solution of (3.29) is positive in Ω, hence, in view of the definition of f_+, it also solves (3.28).

First, observe that if u is a nontrivial solution of (3.29), then it must be nonnegative. Indeed, since

$$\int_\Omega \boldsymbol{A}(x,\nabla u(x))\cdot\nabla v(x)\,dx=\lambda\int_\Omega f_+(x,u(x))v(x)\,dx$$

for every $v\in W_0^{1,p}(\Omega)$, taking $v=-u^-$ and exploiting $(\mathcal{A})$(i), $(\mathcal{A})$(iii) one has

$$\begin{aligned} a_1\|u^-\|^p &\le \int_\Omega \boldsymbol{A}(x,-\nabla u^-(x))\cdot(-\nabla u^-(x))\,dx\\ &= \int_\Omega \boldsymbol{A}(x,\nabla u(x))\cdot(-\nabla u^-(x))\\ &= -\lambda\int_\Omega f_+(x,u(x))u^-(x)\,dx\\ &= -\lambda f(0)\int_\Omega u^-(x)\,dx=0, \end{aligned}$$

namely $u\ge 0$ a.e. in Ω. Taking in mind [13, Theorem 7.1, pag. 286] and [13, Theorem 1.1, pag. 251] one has that u is continuous and applying the strong maximum principle as in [17, Theorem 11.1] the conclusion follows. □

Remark 3.3. The preceding Corollary 3.2 suggests a comparison with Theorem 2.1 of the very nice paper [12], where the existence of multiple solutions of problem (3.28) is studied, being the differential operator of p-Laplacian type with $p\ge 2$. In particular, in [12] the authors require that the continuous function f is both super $(p-1)$-linear at zero and sub $(p-1)$-linear at infinity, while F is positive at some $s_0>0$ in order to state the existence of a *bounded* interval, that is localized, but not explicitly determined, of positive parameters for which the problem under examination has at least three distinct weak solutions that, in addition, satisfy an a priori estimate. Here, under the more restrictive sign condition on f the multiplicity result is still assured, but the interval of parameters is *computed* and *unbounded*, while the solutions are positive.

From a carefully read of the proofs of Theorem 3.3 one can observe that a further multiplicity result can be stated when $f(x,\cdot)$ is super $(p-1)$-linear at zero and $(p-1)$-linear, as it is shown in the following

Theorem 3.4. *Assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies condition $(\mathscr{G}_{f,a,p})$ and that there exist $x_0\in\Omega$, $d>0$ such that* (H_1) *holds in addition to*

$$(\mathrm{H}_2)'' \quad \frac{M_2}{p}\mathscr{H} < \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p},$$

where $\mathscr{H}$ is the constant defined in (3.2). Moreover, suppose that f satisfies (H_3)*. Put*

$$\lambda^{**}=\frac{a_1}{M_2\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha'p)/(\alpha'p^*)}}.$$

Then, for every $\lambda\in]\lambda_*,\lambda^{**}[$ problem (P_λ) admits at least two non trivial weak solutions.

Proof. Assumption $(\mathrm{H}_2)''$ means

$$\frac{M_2a_2\|a\|_\alpha(2T)^p(2^N-1)|\Omega|^{(p^*-\alpha'p)/(\alpha'p^*)}}{pa_1[\mathscr{R}(x_0)]^p} < \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p},$$

so that

$$\begin{aligned}\frac{1}{\lambda^*} &= \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)}\frac{\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p}\\ &> \frac{M_2\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha'p)/(\alpha'p^*)}}{a_1}=\frac{1}{\lambda^{**}},\end{aligned}$$

namely $0<\lambda^*<\lambda^{**}$. Fix $\bar\lambda\in]\lambda_*,\lambda^{**}[$ and, arguing as in Theorem 3.3, let us verify that $\mathscr{I}_{\bar\lambda}=\Phi-\bar\lambda\Psi$ admits at least three critical points.

Reasoning in analogy with the proof of Theorem 3.3 one can find $\bar r=\bar r(\lambda^{**})\in]0,\Phi(u_d)[$ small enough such that

$$\begin{aligned}\frac{\sup_{\Phi(u)\le\bar r}\Psi(u)}{\bar r} &< \frac{1}{\lambda^{**}}<\frac{1}{\bar\lambda}<\frac{1}{\lambda_*}\\ &= \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)}\frac{\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)}{d^p}\\ &\le \frac{\Psi(u_d)}{\Phi(u_d)}.\end{aligned}$$

Moreover, from (3.15) it follows that for every $\lambda\in[0,\lambda^{**}[$ the functional $\mathscr{I}_\lambda$ is bounded from below and satisfies the (PS)-condition, because it is coercive. Hence, we can apply Theorem 2.1 and conclude that for every $\lambda\in\left]\frac{\Phi(u_d)}{\Psi(u_d)},\frac{\bar r}{\sup_{\Phi(u)\le\bar r}\Psi(u)}\right[$ the functional $\mathscr{I}_\lambda$ admits at least three critical points. The proof is complete observing that, in particular, $\bar\lambda\in\left]\frac{\Phi(u_d)}{\Psi(u_d)},\frac{\bar r}{\sup_{\Phi(u)\le\bar r}\Psi(u)}\right[$. □

Inspired by [8], we wish to conclude this note considering a case when the nonlinear term of problem (P_λ) has a suitable structure that implies its $(p-1)$-linearity both at zero and at infinity.

Theorem 3.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying condition $(\mathscr{G}_{f,a,q})$ with $1 < q < p$ and assume that there exist $x_0 \in \Omega$ and $d > 0$ such that* (H_1) *and* (H_3) *hold in addition to*

$$(\mathrm{H}_2)''' \quad \frac{1}{p}\left(\mathscr{H} - \operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} a(x)\right) < \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p},$$

where $\mathscr{H}$ is the constant defined in (3.2). *Put*

$$\mu_* = \left[\frac{[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \operatorname*{ess\,inf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} a(x) + \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} \mathcal{L}(x,d)}{d^p}\right]^{-1}$$

and

$$\mu^* = \frac{a_1}{\|a\|_\alpha T^p |\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}.$$

Then, for every $\lambda \in]\mu_*, \mu^*[$ problem

$$\begin{cases} -\operatorname{div}\boldsymbol{A}(x,\nabla u) = \lambda(a(x)|u|^{p-2}u + f(x,u)) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two non trivial weak solutions.

Proof. First of all, let us point out that $0 < \mu_* < \mu^*$. Indeed, from assumption $(\mathrm{H}_2)'''$ and the definition of $\mathscr{H}$ one has

$$p\frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p} > \frac{a_2\|a\|_\alpha (2T)^p (2^N-1)|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1[\mathscr{R}(x_0)]^p} + -\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} a(x),$$

so that

$$\frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p} > \frac{\|a\|_\alpha T^p |\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1} + -\frac{[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \operatorname*{ess\,inf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} a(x).$$

Hence,

$$\begin{aligned}\frac{1}{\mu_*} &= \frac{[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \operatorname*{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} a(x) + \frac{p[\mathscr{R}(x_0)]^p}{a_2 2^p(2^N-1)} \frac{\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)} F(x,d)}{d^p}\\ &> \frac{\|a\|_\alpha T^p |\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1} = \frac{1}{\mu^*} > 0\end{aligned}$$

that is $0 < \mu_* < \mu^*$. Fix now $\bar\lambda \in]\mu_*, \mu^*[$ and let us prove that $\mathscr{I}_{\bar\lambda} = \Phi - \bar\lambda\Psi$, where Φ is as in (2.5) and $\Psi = \Psi_1 + \Psi_{2,f}$ according to (2.6), admits at least three critical points. Lemma 2.2 assures that Ψ' is a compact operator and that Ψ is of class C^1 as well as sequentially weakly continuous.

Assumption (H_3) implies condition (3.24). Hence, following the reasoning adopted in (3.25) one has

$$\begin{aligned}\Psi(u) &\le \frac{1+\varepsilon}{p}\int_\Omega a(x)|u(x)|^p\,dx + \frac{M_3}{\gamma}\int_\Omega a(x)|u(x)|^\gamma\,dx \\ &\le \frac{1+\varepsilon}{p}\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}\|u\|^p + \frac{M_3}{\gamma}\|a\|_\alpha T^\gamma|\Omega|^{(p^*-\alpha'\gamma)/(\alpha' p^*)}\|u\|^\gamma.\end{aligned} \tag{3.30}$$

Hence, for every $r > 0$, in view of (3.6), one has

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le (1+\varepsilon)\frac{\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{a_1} + \\ +\frac{M_3}{\gamma}\|a\|_\alpha T^\gamma|\Omega|^{(p^*-\alpha'\gamma)/(\alpha' p^*)}\left(\frac{p}{a_1}\right)^{\gamma/p} r^{(\gamma-p)/p}.$$

Thus, passing to the limsup as $r \to 0^+$ and then exploiting the arbitrary of $\varepsilon > 0$,

$$\limsup_{r\to 0^+}\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le \frac{\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}}{pa_1} = \frac{1}{\mu^*} < \frac{1}{\bar{\lambda}}.$$

If $u_d \in X$ is the function defined in (3.7), one can find $\bar{r} = \bar{r}(\bar{\lambda}) \in]0, \Phi(u_d)[$ such that

$$\frac{\sup_{\Phi(u)\le \bar{r}}\Psi(u)}{\bar{r}} < \frac{1}{\bar{\lambda}}. \tag{3.31}$$

Moreover, a computation based on assumption (H_1) shows that

$$\begin{aligned}\Psi(u_d) &= \frac{1}{p}\int_\Omega a(x)|u_d(x)|^p\,dx + \int_\Omega F(x,u_d(x))\,dx \\ &\ge |B(x_0,\mathscr{R}(x_0)/2)|\left[\frac{d^p}{p}\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)}a(x) + \operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)\right] \\ &= \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\left[\frac{d^p}{p}\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)}a(x) + \operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)\right].\end{aligned}$$

Putting together (3.8) and the preceding inequality, one has

$$\begin{aligned}\frac{\Psi(u_d)}{\Phi(u_d)} &\ge \frac{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N|B(0,1)|\left[\frac{d^p}{p}\operatorname{essinf}_{x\in B(x_0,\mathscr{R}(x_0)/2)}a(x) + \operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d)\right]}{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d^p} \\ &= \frac{1}{\mu_*} > \frac{1}{\bar{\lambda}}.\end{aligned} \tag{3.32}$$

Hence, from (3.31) and (3.32) it follows that assumption (2.8) holds with $\bar{u} = u_d$ and $r = \bar{r}$.

Arguing as in (3.15) one has that for every $\lambda \in [0, \mu^*[$ the functional $\mathscr{I}_\lambda$ is coercive, so that it is bounded from below. Indeed, from (3.6) and ($\mathscr{G}_{f,a,q}$)

$$\mathscr{I}_\lambda(u) \ge \frac{1}{p}\left(a_1 - \lambda\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)}\right)\|u\|^p - \lambda M_1\|a\|_\alpha T|\Omega|^{(p^*-\alpha')/(p^*\alpha')}\|u\| + \\ -\lambda\frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\|u\|^q$$

and the coercivity is verified, being $a_1 - \lambda\|a\|_\alpha T^p|\Omega|^{(p^*-\alpha' p)/(\alpha' p^*)} > 0$ and $1 < q < p$. The same arguments exploited in the proof of Theorem 3.1, based on the coercivity of $\mathscr{I}_\lambda$, the compactness of Ψ' and the ($\mathscr{S}_+$) condition, assure that $\mathscr{I}_\lambda$ satisfies the (PS)-condition for every $\lambda \in [0, \mu^*[$. Hence, all the assumptions of Theorem 2.1 hold and one has that for all $\lambda \in \left]\frac{\Phi(u_d)}{\Psi(u_d)}, \frac{\bar{r}}{\sup_{\Phi(u)\le\bar{r}}\Psi(u)}\right[$ the functional $\mathscr{I}_\lambda$ admits at least three critical points. Since from (3.31) and (3.32) it follows that $\bar{\lambda} \in \left]\frac{\Phi(u_d)}{\Psi(u_d)}, \frac{\bar{r}}{\sup_{\Phi(u)\le\bar{r}}\Psi(u)}\right[$, the proof can be considered complete. □

Remark 3.4. In [8], the problem

$$\begin{cases} -\mathrm{div}\boldsymbol{A}(x,\nabla u) = \lambda(a(x)|u|^{p-2}u + f(x,u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied when f satisfies $(\mathscr{G}_{f,a,q})$ with $1 < q < p$, in addition to

$$\limsup_{t\to 0} \frac{|f(x,t)|}{a(x)|t|^{\gamma-1}} < \infty, \tag{3.33}$$

for some $\gamma \in (p, p^*/\alpha')$, uniformly a.e. in Ω and

$$\int_\Omega F(x, u_1(x))\, dx > \frac{1}{p}\left(\frac{a_2}{a_1} - 1\right), \tag{3.34}$$

where a_1, a_2 are the constants given in $(\mathcal{A})$(iii), while u_1 is the eigenfunction (with $\int_\Omega a(x)|u_1|^p\, dx = 1$) related to the first eigenvalue of the problem

$$-\Delta_p u = \lambda a(x) u^{p-2} u$$

in $W_0^{1,p}(\Omega)$, namely

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega a(x)|u|^p\, dx}.$$

In particular, among the others, in [8], two technical positive constants $\lambda_\star$, $\lambda^\star$, with $\lambda_\star \le \lambda^\star < a_1\lambda_1$, and $\lambda_\star$ depending on λ_1, are considered in such a way that the Dirichlet problem has only the zero solution if $\lambda \in [0, \lambda_\star)$, while there are at least two nontrivial solutions if $\lambda \in (\lambda^\star, a_1\lambda_1)$.

Hence, we can observe that condition (3.33) implies our assumption (H_3) so that Theorem 3.5 is compatible with the result in [8] and it could represent a concrete tool for estimating the constant $\lambda_\star$ in all the cases when it is not simple to compute λ_1.

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References

[1] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Functional Anal. **122** (1994), 519–543.

[2] D. Arcoya and D. Ruiz, *The Ambrosetti-Prodi problem for the p-Laplacian operator*. Commun. Partial Differ. Equ. **31** (2006), 849-865.

[3] L. Boccardo, M. Escobedo and I. Peral, *A Dirichlet problem involving critical exponent*, Nonlinear Anal. TMA **24** (1995) 1639-1648.

[4] G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012) 2992–3007.

[5] G. Bonanno, P. Candito, R. Livrea and N. S. Papageorgiou, *Existence, nonexistence and uniqueness of positive solutions for nonlinear eigenvalue problems*, Commun. Pure Appl. Anal. **16** (2017), no. 4, 1169–1188.

[6] G. Bonanno and R. Livrea, *Multiplicity theorems for the Dirichlet problem involving the p-Laplacian*, Nonlinear Anal. **54** (2003) 1–7.

[7] P. Candito, R. Livrea and N. S. Papageorgiou, *Nonlinear elliptic equations with asymmetric asymptotic behavior at* $\pm\infty$, Nonlinear Anal. Real World Appl. **32** (2016) 159–177.

[8] F. Colasuonno, P. Pucci and C. Varga, *Multiple solutions for an eigenvalue problem involving p-Laplacian type operators*, Nonlinear Anal. **75** (2012) 4496–4512.

[9] P. L. De Nápoli, M. C. Mariani, *Mountain pass solutions to equations of p-Laplacian type*, Nonlinear Anal. **54** (2003) 1205–1219.

[10] J. Garcia Azorero, J. Manfredi and I. A. Peral, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp, Math. **2** (2000), 385–404.

[11] Z. Guo and Z. Zhang, $W^{1,p}$ *versus* C^1 *local minimizers and multiplcity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), 32–50.

[12] A. Kristály, H. Lisei and C. Varga, *Multiple solutions for p-Laplacian type equations*, Nonlinear Anal. **68** (2008) 1375–1381.

[13] O. Ladyzhenskaya and N. Uraltseva, *Linear and Quasilinear Equations*, Accademic Press, New York, London, 1968.

[14] S. Li, S. Wu and H.-S. Zhou, *Solutions to semilinear elliptic problems with combined nonlinearities*, J. Differ. Equ. **185** (1), 200–224 (2002).

[15] S. Marano and N. S. Papageorgiou, *Positive solutions to a Dirichlet problem with the p-Laplacian and concave-convex nonlinearity depending on a parameter*, Commun. Pure Appl. Anal. **12** (2013) 815–829.

[16] N. S. Papageorgiou and P.Winkert, *Positive solutions for nonlinear nonhomogeneous Dirichlet problems with concave-convex nonlinearities*, Positivity **20** (2016), no. 4, 945–979.

[17] P. Pucci and J. Serrin, *The strong maximum principle revisited*, J. Differential Equations **196** (2004) 1–66.

[18] G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976) 353–372.

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