Solutions and positive solutions for superlinear Robin problems

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ABSTRACT

We consider nonlinear, nonhomogeneous Robin problems with a $(p-1)$-superlinear reaction term, which need not satisfy the Ambrosetti-Rabinowitz condition. We look for positive solutions and prove existence and multiplicity theorems. For the particular case of the p-Laplacian, we prove existence results under a different geometry near the origin.

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I. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this work, we study the following nonlinear, nonhomogeneous Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone continuous map, which satisfies certain other regularity and growth conditions, listed in hypotheses $H(a)_1$. These conditions are general enough to incorporate in our framework many nonlinear differential operators of interest, such as the p-Laplacian $(1 < p < \infty)$ and the (p, q)-Laplacian $(1 < q < p < \infty)$. The potential function $\xi \in L^\infty(\Omega)$ is, in general, indefinite [that is, $\xi(\cdot)$ is sign changing]. The reaction term $f(z, x)$ is a Carathéodory function [that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. (almost all) $z \in \Omega$, $x \to f(z, x)$ is continuous]. We assume that $f(z, \cdot)$ is $(p-1)$-superlinear but without satisfying the usual conditions, in such cases the Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we employ a less restrictive condition, which fits in our setting $(p-1)$-superlinear functions with slower growth near $\pm\infty$, which fail to satisfy the AR-condition. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the generalized directional derivative (conormal derivative) of u defined by extension to $W^{1,p}(\Omega)$ of the map $C^1(\overline{\Omega}) \ni u \to \frac{\partial u}{\partial n_a} = (a(\nabla u), n)_{\mathbb{R}^N}$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This directional derivative is dictated by the nonlinear Green's identity (see, for example, Ref. 12, p. 210). The boundary coefficient $\beta(\cdot)$ is a non-negative Hölder continuous on $\partial\Omega$ function. If $\beta \equiv 0$, then we recover the Neumann problem. We are looking for positive solutions. In Sec. III, we prove two such results under different geometries near the origin. First, $u = 0$ is a local minimizer of the energy (Euler) functional and we prove an existence theorem. Second, instead we assume that $f(z, \cdot)$ admits a z-dependent zero on the positive semiaxis and we prove the existence of an ordered pair of positive solutions. We also show the existence of a smallest positive solution. In Sec. IV, we deal with the special case of the p-Laplacian differential operator [that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$] and a reaction term near zero stays above the principal eigenvalue of the p-Laplacian with the Robin boundary condition.

So, now $u = 0$ is no longer a local minimizer of the energy functional leading to a different geometry for the problem and consequently to a different approach. As a particular case of the existence theorem in Sec. IV, we consider a parametric Robin problem with general potential and prove the existence of a positive smooth solution for every value of the parameter $\lambda > 0$, extending this way an earlier such result for semilinear Dirichlet problems by Miyagaki-Souto.[26] Our existence and multiplicity results in this paper extend in different ways the works of Gao-Tang,[11] Iturriaga-Souto-Ubilla,[20] Iturriaga-Lorca-Ubilla,[21] Liu,[23] Li-Yang,[24] Miyagaki-Souto,[26] and Sun.[37] We also mention the related recent work of Papageorgiou-Smyrlis[35] on positive solutions of nonlinear logistic equations.

Our method of proof uses variational tools based on the critical point theory together with suitable truncation and perturbation techniques, comparison principles, and Morse theory (critical groups).

II. MATHEMATICAL BACKGROUND

Let X be a Banach space. By X^*, we denote the topological dual of X, and by $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair (X^*, X). Suppose that $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), provided the following property holds: "Every sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \| u_n \|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$ admits a strongly convergent subsequence." Using this compactness-type condition on the functional φ, one can prove a deformation theorem from which follows the minimax theory of the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem" which we recall here as follows:

Theorem 1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, there exist $u_0, u_1 \in X$ and $\rho > 0$ such that $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho,$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \ge m_\rho$ and c is a critical value of φ [that is, there exists $\widehat{u} \in X$ such that $\varphi'(\widehat{u}) = 0$ and $\varphi(\widehat{u}) = c$].

The study of problem (1) will be based on the following spaces:

- The Sobolev space $W^{1,p}(\Omega)$.
- The Banach space $C^1(\overline{\Omega})$.
- The boundary Lebesgue spaces $L^q(\partial\Omega)$, $1 \le q \le \infty$.

By $\| \cdot \|$, we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p \right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with the positive (order) cone given by $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior which is given by the set $D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}$. On $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \le q \le \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^q(\partial\Omega)$, known as the "trace map," such that $\gamma_0(u) = u\big|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Therefore, the trace map gives meaning to the "boundary values" on $\partial\Omega$ of any Sobolev function. We know that γ_0 is compact in $L^q(\partial\Omega)$ for $q \in \left[1, \frac{(N-1)p}{N-p}\right)$ if $N > p$ and into $L^q(\partial\Omega)$ for all $q \ge 1$ if $N \le p$. Moreover, we have $\ker \gamma_0 = W_0^{1,p}(\Omega)$ and $\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. From now on, for the sake of notational simplicity, we drop the use of the map γ_0. The restriction of any Sobolev function on $\partial\Omega$ is understood in the sense of traces. Let $\vartheta \in C^1(0, \infty)$ with $\vartheta(t) > 0$ for all $t > 0$ and assume that

$$\begin{cases} 0 < \widehat{c} \le \frac{\vartheta'(t)t}{\vartheta(t)} \le c_0 \text{ for all } t > 0, \\ c_1 t^{p-1} \le \vartheta(t) \le c_2(t^{s-1} + t^{p-1}) \text{ for all } t > 0, \text{ some } c_1, c_2 > 0 \quad 1 \le s < p. \end{cases} \tag{2}$$

We introduce the following conditions on the map $a(\cdot)$ [see (1)]:
$H(a): a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \to a_0(t)t$ is strictly increasing, $a_0(t)t \to 0^+$ as $t \to 0^+$ and $\lim\limits_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \le c_3 \frac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \backslash \{0\}$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{\vartheta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \backslash \{0\}$, all $\xi \in \mathbb{R}^N$.

Remark 1. These conditions on $a(\cdot)$ are dictated by the nonlinear regularity theory of Lieberman[22] and the nonlinear maximum principle of Pucci-Serrin[36] (pp. 111 and 120).

We set $G_0(t) = \int_0^t a_0(s)s\, ds$ and define $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently, $G(\cdot)$ is convex and $G(0) = 0$. Also, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \backslash \{0\},$$

$\nabla G(0) = 0$. Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$. Then, the convexity of $G(\cdot)$, and since $G(0) = 0$, implies that

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{3}$$

Lemma 1 summarizes the basic properties of the map $a(\cdot)$ and is an easy consequence of hypotheses $H(a)$.

Lemma 1. If hypotheses $H(a)$ hold, then

(a) *$y \to a(y)$ is continuous, strictly monotone, hence maximal monotone too;*

(b) *$|a(y)| \le c_4(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$, some $c_4 > 0$;*

(c) *$(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

This lemma together with (3) lead to the following growth estimates for the primitive $G(\cdot)$:

Corollary 1. If hypotheses $H(a)$ hold, then $\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_5(1 + |y|^p)$ for all $y \in \mathbb{R}^N$, some $c_5 > 0$.

In addition, we impose the following conditions on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$:

$H(\xi)$: $\xi \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$ with $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

Remark 2. When $\beta \equiv 0$, we have the usual Neumann problem.

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfies

$$|f_0(z, x)| \le a_0(z)(1 + |x|^{p^*-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \le p \end{cases}$ (the critical Sobolev exponent).

We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and also define the C^1-functional $\mu : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mu(u) = \int_\Omega pG(\nabla u)dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$$

for all $u \in W^{1,p}(\Omega)$. Then, we consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\mu(u) - \int_\Omega F(z, u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We set $V = C^1(\overline{\Omega})$ and $X = W^{1,p}(\Omega) = \overline{C^1(\Omega)}^{\|\cdot\|}$. The following result can be found in Ref. 30 (subcritical case) and Ref. 32 (critical case).

Proposition 1. If $u_0 \in W^{1,p}(\Omega)$ is a local V-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0+h)$ for all $h \in V$, $\|h\|_V \leq \rho_0$, then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is a local X-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0+h)$ for all $h \in X$, $\|h\| \leq \rho_1$.

Consider the nonlinear map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), h \rangle = \int_\Omega (a(\nabla u), \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasiński-Papageorgiou,[13] we have the following proposition:

Proposition 2. The map $A(\cdot)$ is bounded (that is, it maps bounded sets to bounded sets), and it is continuous monotone (hence maximal monotone too) and of type $(S)_+$ which means that "$u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n \to +\infty} \langle A(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$."

We will also need some basic facts about the spectrum of the operator $-\Delta_p + \xi(z)I$ with the Robin boundary condition. Recall that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for all $u \in W^{1,p}(\Omega)$ (the p-Laplace differential operator). So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}\,|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In this case, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ and so $\frac{\partial u}{\partial n_p}$ is defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to \frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}(\nabla u, n)_{\mathbb{R}^N} = |\nabla u|^{p-2}\frac{\partial u}{\partial n}.$$

By an "eigenvalue," we mean a $\widehat{\lambda} \in \mathbb{R}$ for which problem (4) admits a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$ known as an "eigenfunction" corresponding to the eigenvalue $\widehat{\lambda}$. From Ref. 32 (see also Ref. 18), we have that $\widehat{u} \in L^\infty(\Omega)$. Then, from the nonlinear regularity theory of Lieberman,[22] we have that $\widehat{u} \in C^1(\overline{\Omega})$. We know that there exists a smallest eigenvalue $\widehat{\lambda}_1(\xi, \beta) \in \mathbb{R}$ given by

$$\widehat{\lambda}_1(\xi, \beta) = \inf\left[\frac{\mu(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right]. \tag{5}$$

This eigenvalue has the following properties:

- $\widehat{\lambda}_1(\xi, \beta)$ is isolated in the spectrum $\widehat{\sigma}(p)$ of (4) (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(\xi, \beta), \widehat{\lambda}_1(\xi, \beta) + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
- $\widehat{\lambda}_1(\xi, \beta)$ is simple (that is, if $\widehat{u}, \widehat{v}$ are eigenfunctions corresponding to the eigenvalue $\widehat{\lambda}_1(\xi, \beta)$, then $\widehat{u} = \eta\widehat{v}$ with $\eta \in \mathbb{R}\backslash\{0\}$).
- The eigenfunctions corresponding to $\widehat{\lambda}_1(\xi, \beta)$ have constant sign.

Let $\widehat{u}_1(\xi, \beta)$ be the L^p-normalized (that is, $\|\widehat{u}_1(\xi, \beta)\|_p = 1$) positive eigenfunction, then we have $\widehat{u}_1(\xi, \beta) \in C_+$. Moreover, from the nonlinear maximum principle (see Ref. 36), we have $\widehat{u}_1(\xi, \beta) \in D_+$. Since $\widehat{\lambda}_1(\xi, \beta)$ is isolated and $\widehat{\sigma}(p) \subseteq \mathbb{R}$ is closed, the second eigenvalue $\widehat{\lambda}_2(\xi, \beta)$ is well-defined by $\widehat{\lambda}_2(\xi, \beta) = \inf[\widehat{\lambda} : \widehat{\lambda} \in \widehat{\sigma}(p), \widehat{\lambda} > \widehat{\lambda}_1(\xi, \beta)]$.

By the Ljusternik-Schnirelmann minimax scheme, we know that $\widehat{\sigma}(p)$ has a whole strictly increasing sequence $\{\widehat{\lambda}_k(\xi, \beta)\}_{k \in \mathbb{N}} \subseteq \mathbb{R}$ such that $\widehat{\lambda}_k(\xi, \beta) \to +\infty$ as $k \to +\infty$. These are known as "LS-eigenvalues" or "variational eigenvalues" of (4). We do not know if this sequence exhausts $\widehat{\sigma}(p)$. The Ljusternik-Schnirelmann scheme provides minimax expressions for these eigenvalues. For $\widehat{\lambda}_2(\xi, \beta)$, we have an alternative minimax characterization which is more suitable for our purposes. So, let

$$\partial B_1^{L^p} = \left\{ u \in L^p(\Omega) : \|u\|_p = 1 \right\}, \quad M = W^{1,p}(\Omega) \cap \partial B_1^{L^p}$$

and

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1(\xi, \beta), \widehat{\gamma}(1) = \widehat{u}_1(\xi, \beta)\}.$$

Proposition 3. $\widehat{\lambda}_2(\xi, \beta) = \inf_{\widehat{\gamma} \in \widehat{\Gamma}} \max_{-1 \leq t \leq 1} \mu(\widehat{\gamma}(t))$.

For more information on these issues, we refer to Refs. 28 and 32. Next, let us recall a few basic definitions and facts from Morse theory (critical groups) which we will need in the sequel. So, let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$, we denote the kth-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $k \in -\mathbb{N}$, then $H_k(Y_1, Y_2) = 0$. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}, \quad K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Suppose that $u_0 \in K_\varphi^c$ is isolated. The critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u_0 such that $K_\varphi \cap U \cap \varphi^c = \{u_0\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the choice of the neighborhood U. Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $\inf \varphi(K_\varphi) > c$. The critical groups of φ at infinity are defined by $C_k(\varphi, \infty) = H_k(X, \varphi^c)$ for all $k \in \mathbb{N}_0$. This definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$. To see this, let $d < c < \inf \varphi(K_\varphi)$. From the noncritical interval theorem (see, for example, Ref. 27, p. 110), we know that φ^d is a strong deformation retract of φ^c. Therefore, $H_k(X, \varphi^c) = H_k(X, \varphi^d)$ for all $k \in \mathbb{N}_0$. The following proposition is useful in producing additional critical points for a functional $\varphi \in C^1(X, \mathbb{R})$ (see Ref. 27):

Proposition 4. If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, for some $k \in \mathbb{N}_0$ $C_k(\varphi, 0) \neq 0$, $C_k(\varphi, \infty) = 0$, then there exists $u \in K_\varphi$, $u \neq 0$.

Finally, let us fix our notation. So, for $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W^{1,p}(\Omega)$, $u = u^+ - u^-$, and $|u| = u^+ + u^-$. By $|\cdot|_N$, we denote the Lebesgue measure on $\mathbb{R}^N$. If $u, v \in W^{1,p}(\Omega)$, then

$$[u, v] = \{h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

For $\Sigma_0 \subseteq \partial \Omega$ a closed set, we introduce the following set of functions:

$$\widehat{D}_+(\Sigma_0) = \left\{ h \in C^1(\overline{\Omega}) : h(z) > 0 \text{ for all } z \in \Omega, \left. \frac{\partial h}{\partial n} \right|_{\Sigma_0} < 0 \right\}.$$

Evidently, this set is open in $C^1(\overline{\Omega})$ and we have $D_+ \subseteq \widehat{D}_+(\Sigma_0)$. Note that $\widehat{D}_+(\Sigma_0)$ is the interior of the positive (order) cone of $C^1_*(\overline{\Omega})$ $= \left\{ u \in C^1(\overline{\Omega}) : u \big|_{\Sigma_0} = 0 \right\}$. So, alternatively the analysis below can be done using the pair $C^1_*(\overline{\Omega})$, $W^{1,p}_*(\Omega) = \overline{C^1_*(\overline{\Omega})}^{\|\cdot\|}$.

III. NONHOMOGENEOUS EQUATION

In this section, we deal with problem (1) and we look for positive solutions. We prove an existence theorem and a multiplicity theorem producing two nontrivial smooth solutions. The two results differ in the geometry of the equation near zero.

For the existence theorem, our hypotheses on the map $a(\cdot)$ are the following:

$H(a)_1$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, and hypotheses $H(a)_1$ (i)–(iii) are the same as the corresponding hypotheses $H(a)$ (i)–(iii) and

(iv) if $G_0(t) = \int_0^t a_0(s)s \, ds$, then $pG_0(t) - a_0(t)t^2 \geq -\overline{c}$ for all $t > 0$ and some $\overline{c} > 0$.

Remark 3. As we already mentioned in Sec. II, hypotheses $H(a)_1(i)$–(iii) come from the nonlinear regularity theory of Lieberman[22] and the nonlinear maximum principle of Pucci-Serrin[36] (pp. 110 and 120). Hypothesis $H(a)_1(iv)$ serves the particular needs of our problem, but it is very mild and it is satisfied in all cases of interest as the examples which follow illustrate.

Example 1. The following maps $a : \mathbb{R}^N \to \mathbb{R}^N$ satisfy hypotheses $H(a)_1$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p-Laplacian defined by $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ for all $u \in W^{1,p}(\Omega)$.

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$. This map corresponds to the (p, q)-Laplacian defined by $\Delta_p u + \Delta_q u$ for all $u \in W^{1,p}(\Omega)$. Such an operator arises in problems of mathematical physics (see, for example, Ref. 5). Recently, there have been some existence and multiplicity results for such equations. We refer to the works of Aizicovici-Papageorgiou-Staicu,[2] Barile-Figueiredo,[3] Candito-Livrea-Papageorgiou,[4] Cingolani-Degiovanni,[6] Gasiński-Papageorgiou,[14,15] Marano-Mosconi-Papageorgiou,[25] Papageorgiou-Rădulescu,[29,31,33,34] Sun,[38] and Sun-Zhang.[39]

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}} y$ with $1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator defined by $\mathrm{div}\, ((1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u)$ for all $u \in W^{1,p}(\Omega)$.

(d) $a(y) = |y|^{p-2}\left[1 + \frac{1}{1+|y|^p}\right]$ with $1 < p < \infty$. This map corresponds to the following differential operator $-\Delta_p u + \mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{1+|\nabla u|^p}\right)$ for all $u \in W^{1,p}(\Omega)$. This operator arises in problems of plasticity.

Our hypotheses on the reaction term $f(z, x)$ are the following:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \le a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \ge 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii)   if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii)   if $e(z,x) = f(z,x)x - pF(z,x)$, then there exists $d \in L^1(\Omega)$ such that $e(z,x) \le e(z,y) + d(z)$ for a.a. $z \in \Omega$, all $0 \le x \le y$;

(iv)   if $\widehat{\xi} = \frac{p-1}{c_1}\xi$, $\widehat{\beta} = \frac{p-1}{c_1}\beta$ [see (2)], then there exist functions $\eta, \tilde{\eta} \in L^\infty(\Omega)$ such that $\eta(z) \le \frac{c_1}{p-1}\widehat{\lambda}_1(\widehat{\xi}, \widehat{\beta})$ for a.a. $z \in \Omega$, $\eta \not\equiv \frac{c_1}{p-1}\widehat{\lambda}_1(\widehat{\xi}, \widehat{\beta})$, $\tilde{\eta}(z) \le \liminf_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \le \limsup_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \le \eta(z)$ uniformly for a.a. $z \in \Omega$.

Remark 4. Since we look for positive solutions and the above hypotheses concern the positive semiaxis, we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$. Hypotheses H_1 (ii) and (iii) imply that $\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$ uniformly for a.a. $z \in \Omega$. So, the reaction term $f(z, \cdot)$ is $(p-1)$-superlinear. However, the superlinearity condition of $f(z, \cdot)$ is not formulated using the so-called Ambrosetti-Rabinowitz condition. We recall that the AR-condition says that there exist $s > p$ and $M > 0$ such that

$$0 < sF(z,x) \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \tag{6}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{6a}$$

This is a unilateral version of the usual AR-condition since we have assumed that $f(z, 0) = 0$ for a.a. $z \in \Omega$, all $x \le 0$. Integrating (6) and using (6a), we obtain the following weaker condition:

$$c_6 x^s \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \text{ some } c_6 > 0. \tag{7}$$

From (4), we see that, for a.a. $z \in \Omega$, $F(z, \cdot)$ eventually has s-polynomial growth. From (6), this implies that for a.a. $z \in \Omega$, $f(z, \cdot)$ eventually has $(s-1)$-polynomial growth. This excludes from consideration $(p-1)$-superlinear nonlinearities with slower growth near $+\infty$ (see the examples below). Here, instead of (6) and (6a), we use hypothesis H_1(iii) which includes such nonlinearities. Hypothesis H_1(iii) is a quasimonotonicity condition on the function $e(z, \cdot)$. This condition is a little more general than the one used by Li-Yang.[24] Hypothesis H_1(iii) is satisfied if there exists $M > 0$ such that for a.a. $z \in \Omega$, the function $x \to \frac{f(z,x)}{x^{p-1}}$ is nondecreasing on $[M, +\infty)$ or if for a.a. $z \in \Omega$, the function $x \to e(z, x)$ is nondecreasing on $[M, +\infty)$.

Example 2. The following functions satisfy hypotheses H_1. For the sake of simplicity, we drop the z-dependence,

$$f_1(x) = \begin{cases} \vartheta x^{p-1} & \text{if } x \in [0, 1], \\ x^{r-1} + (\vartheta - 1)x^{\tau-1} & \text{if } x > 1, \end{cases}$$

with $\vartheta < \widehat{\lambda}_1(\widehat{\xi}, \widehat{\beta})$ and $p, \tau < r$. Also, $f_2(x) = x^{p-1}[\ln(1 + x) + \vartheta]$ with $\vartheta < \widehat{\lambda}_1(\widehat{\xi}, \widehat{\beta})$. Note that f_2 does not satisfy the AR-condition.

Let $\vartheta > \|\xi\|_\infty$ [see hypothesis $H(\xi)$]. We introduce the following Carathéodory function:

$$\widehat{f}(z,x) = f(z,x) + \vartheta(x^+)^{p-1} = \begin{cases} 0 & \text{if } x \le 0, \\ f(z,x) + \vartheta x^{p-1} & \text{if } 0 < x. \end{cases} \tag{8}$$

We set $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)ds$ and consider the C^1-functional $\widehat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\mu(u) + \frac{\vartheta}{p}\|u\|_p^p - \int_\Omega \widehat{F}(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 5. If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_1 (i)–(iii) hold, then $\widehat{\varphi}$ satisfies the C-condition.

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\widehat{\varphi}(u_n)| \le M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{9}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{10}$$

From (10), we have

$$\left| \langle A(u_n), h \rangle + \int_\Omega (\xi(z) + \vartheta)|u_n|^{p-2} u_n h \, dz \right.$$
$$+ \int_{\partial\Omega} \beta(z)|u_n|^{p-2} u_n h \, d\sigma - \int_\Omega \widehat{f}(z, u_n) h \, dz \Big| \tag{11}$$
$$\le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}, \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+.$$

In (11), we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then,

$$\frac{c_0}{p-1} \|\nabla u_n^-\|_p^p + \int_\Omega (\xi(z) + \vartheta)(u_n^-)^p \, dz \le \varepsilon_n \text{ for all } n \in \mathbb{N}$$

$$[\text{see Lemma 1, hypothesis } H(\beta) \text{ and } (8)]$$

$$\Rightarrow \quad u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to +\infty \text{ (recall that } \vartheta > \|\xi\|_\infty). \tag{12}$$

From (9) and (12), we have

$$\int_\Omega pG(\nabla u_n^+) dz + \int_\Omega (\xi(z) + \vartheta)(u_n^+)^p dz + \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma$$
$$- \int_\Omega p\widehat{F}(z, u_n^+) dz \le M_2, \tag{13}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

In (11), we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then,

$$-\int_\Omega (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N} dz - \int_\Omega (\xi(z) + \vartheta)(u_n^+)^p dz$$
$$- \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma + \int_\Omega \widehat{f}(z, u_n^+) u_n^+ dz \le \varepsilon_n, \tag{14}$$

for all $n \in \mathbb{N}$. Adding (13) and (14) and using hypothesis $H(a)_1$(iv), we obtain

$$\int_\Omega e(z, u_n^+) dz \le M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}. \tag{15}$$

Claim: $\{u_n^+\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that the claim is not true. By passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to +\infty \quad \text{as } n \to +\infty. \tag{16}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then, $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega), \ y \ge 0. \tag{17}$$

First, assume that $y \ne 0$ and let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$ {recall that $y \ge 0$ [see (17)]}. Then, $|\Omega_+|_N > 0$ and we have

$$u_n^+(z) \to +\infty \quad \text{for a.a. } z \in \Omega_+. \tag{18}$$

From (18) and hypothesis H_1(ii), it follows that

$$\frac{F(z, u_n^+(z))}{u_n^+(z)^p} \to +\infty \quad \text{for a.a. } z \in \Omega_+. \tag{19}$$

Using (19), we have

$$\lim_{n \to +\infty} \frac{F(z, u_n^+(z))}{\|u_n^+\|^p} = \lim_{n \to +\infty} \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p = +\infty$$

for a.a. $z \in \Omega_+$. Then, invoking Fatou's lemma [hypotheses H_1 (i) and (ii) permit its use], we obtain

$$\lim_{n \to +\infty} \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz = +\infty. \tag{20}$$

Hypothesis H_1(ii) implies that we can find $M_4 > 0$ such that

$$F(z, x) \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_4. \tag{21}$$

Then,

$$\int_{\Omega \setminus \Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz = \int_{(\Omega \setminus \Omega_+) \cap \{u_n^+ \geq M_4\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz \tag{22}$$
$$+ \int_{(\Omega \setminus \Omega_+) \cap \{u_n^+ < M_4\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz \geq -c_7$$

for all $n \in \mathbb{N}$, some $c_7 > 0$ [see (21) and use hypothesis H_1(i)]. Hence, we have

$$\int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz = \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz + \int_{\Omega \setminus \Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz$$
$$\geq \int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz - c_7 \quad \text{for all } n \in \mathbb{N} \left[\text{see (22)} \right]$$
$$\Rightarrow \lim_{n \to +\infty} \int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz = +\infty \quad [\text{see (20)}]. \tag{23}$$

From (9) and (12), we have

$$\int_{\Omega} \widehat{F}(z, u_n^+) dz \leq M_5 + \int_{\Omega} G(\nabla u_n^+) dz$$
$$+ \frac{1}{p} \int_{\Omega} (\xi(z) + \vartheta)(u_n^+)^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma,$$
$$\text{for some } M_5 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \int_{\Omega} \frac{\widehat{F}(z, u_n^+)}{\|u_n^+\|^p} dz \leq \frac{M_5}{\|u_n^+\|^p} + c_5 \left(\frac{1}{\|u_n^+\|^p} + \|\nabla y\|_p^p \right)$$
$$+ \frac{1}{p} \int_{\Omega} (\xi(z) + \vartheta) y_n^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) y_n^p d\sigma,$$
$$\text{for all } n \in \mathbb{N} \text{ (see Corollary 1)}$$
$$\Rightarrow \int_{\Omega} \frac{\widehat{F}(z, u_n^+)}{\|u_n^+\|^p} dz \leq M_6 \quad \text{for some } M_6 > 0, \text{ all } n \in \mathbb{N}. \tag{24}$$

Comparing (23) and (24), we have a contradiction. Next, we assume that $y = 0$. For $k > 0$, we set $v_n = k^{\frac{1}{p}} y_n$ for all $n \in \mathbb{N}$. Evidently, we have

$$v_n \to 0 \text{ in } L^r(\Omega) \quad [\text{see (17) and recall that } y = 0]. \tag{25}$$

Hypothesis H_1(i) and Krasnoselskii's theorem (see, for example, Ref. 12, p. 407) imply that

$$\int_{\Omega} F(z, v_n) dz \to 0 \quad \text{as } n \to +\infty \quad [\text{see (25)}]. \tag{26}$$

Because of (16), we see that we can find $n_0 \in \mathbb{N}$ such that

$$0 < k^{\frac{1}{p}} \frac{1}{\|u_n^+\|^p} \leq 1 \quad \text{for all } n \geq n_0. \tag{27}$$

Consider the C^1-functional $\widehat{\varphi}_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_0(u) = \frac{c_1}{p(p-1)} \|\nabla u\|_p^p + \frac{1}{p}\int_\Omega (\xi(z) + \vartheta)|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \widehat{F}(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$. Let $t_n \in [0,1]$ be such that

$$\widehat{\varphi}_0(t_n u_n^+) = \max[\widehat{\varphi}_0(t u_n^+) : 0 \leq t \leq 1] \quad \text{for all } n \in \mathbb{N}. \tag{28}$$

On account of (27) and (28), we have

$$\widehat{\varphi}_0(t_n u_n^+) \geq \widehat{\varphi}_0(v_n)$$

$$= \frac{c_1}{p(p-1)} \|\nabla v_n\|_p^p + \frac{1}{p}\int_\Omega \xi(z)v_n^p dz$$

$$+ \frac{1}{p}\int_{\partial\Omega} \beta(z)v_n^p d\sigma - \int_\Omega F(z,v_n)dz \quad [\text{see } (8)]$$

$$\geq \frac{1}{p}\left[\frac{c_1}{(p-1)}\|\nabla v_n\|_p^p + \int_\Omega (\xi(z)+\vartheta)v_n^p dz \right]$$

$$- \frac{\vartheta}{p}\|v_n\|_p^p - \int_\Omega F(z,v_n)dz$$

$$(\text{see hypothesis } H(\beta), \text{ recall that } \vartheta > \|\xi\|_\infty)$$

$$\geq c_8 \|v_n\|^p - \frac{\vartheta}{p}\|v_n\|_p^p - \int_\Omega F(z,v_n)dz$$

$$\text{for all } n \geq n_0, \text{ some } c_8 > 0,$$

$$\geq c_8 k - \frac{\vartheta}{p}\|v_n\|_p^p - \int_\Omega F(z,v_n)dz \quad \text{for all } n \geq n_0$$

$$(\text{recall that } \|y_n\| = 1 \text{ for all } n \in \mathbb{N})$$

$$\geq \frac{c_8}{2}k \quad \text{for all } n \geq n_1 \geq n_0 \quad [\text{see } (25) \text{ and } (26)].$$

However, $k > 0$ is arbitrary. So, we infer that

$$\widehat{\varphi}_0(t_n u_n^+) \to +\infty \quad \text{as } n \to +\infty. \tag{29}$$

Note that

$$\widehat{\varphi}_0(0) = 0 \quad \text{and} \quad \widehat{\varphi}_0(u_n^+) \leq M_1 \text{ for all } n \in \mathbb{N} \tag{30}$$

[see (9) and note that $\widehat{\varphi}_0(u_n^+) \leq \widehat{\varphi}_0(u_n) \leq \widehat{\varphi}(u_n)$ for all $n \in \mathbb{N}$].

Then, (29) and (30) imply that

$$t_n \in (0,1) \quad \text{for all } n \geq n_2$$

$$\Rightarrow \quad \frac{d}{dt}\widehat{\varphi}_0(t u_n^+)\Big|_{t=t_n} = 0 \quad \text{for all } n \geq n_2 \quad [\text{see } (28)]$$

$$\Rightarrow \quad \langle \widehat{\varphi}_0'(t_n u_n^+), t_n u_n^+ \rangle = 0 \quad (\text{by the chain rule})$$

$$\Rightarrow \quad \frac{c_0}{p-1}\|\nabla(t_n u_n^+)\|_p^p + \int_\Omega \xi(z)(t_n u_n^+)^p dz$$

$$+ \int_{\partial\Omega} \beta(z)(t_n u_n^+)^p d\sigma = \int_\Omega \widehat{f}(z, t_n u_n^+)(t_n u_n^+)dz,$$

$$\Rightarrow \quad p\widehat{\varphi}_0(t_n u_n^+) = \int_\Omega e(z, t_n u_n^+)dz \leq \int_\Omega e(z, u_n^+)dz + \|d\|_1$$

$$\text{for all } n \geq n_2 [\text{see } H_1 \text{ (iii)}]$$

$$\Rightarrow \quad p\widehat{\varphi}_0(t_n u_n^+) \leq M_7 \text{ for some } M_7 \geq 0, \text{ all } n \geq n_2 [\text{see } (15)]. \tag{31}$$

Comparing (29) and (31), we have a contradiction. This proves the claim. The claim and (12) imply that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{32}$$

In (11), we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$, and use (32). Then,

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u \rangle = 0$$
$$\Rightarrow \quad u_n \to u \text{ in } W^{1,p}(\Omega)$$
$$\Rightarrow \quad \widehat{\varphi} \text{ satisfies the } C-\text{condition.}$$

$\square$

To proceed further, we will need Lemma 2. This lemma will help us to establish the geometry near the origin.

Lemma 2. *If $\eta \in L^\infty(\Omega)$ is as in hypothesis $H_1(iv)$, then we can find $c_9 > 0$ such that*

$$\frac{c_1}{p-1}\|\nabla u\|_p^p + \int_\Omega (\xi(z) - \eta(z))|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$$
$$\geq c_9\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proof. We have

$$\frac{c_1}{p-1}\left[\|\nabla u\|_p^p + \int_\Omega (\widehat{\xi}(z) - \widehat{\eta}(z))|u|^p dz + \int_{\partial\Omega} \widehat{\beta}(z)|u|^p d\sigma\right]$$

with $\widehat{\xi} = \frac{p-1}{c_1}\xi, \widehat{\beta} = \frac{p-1}{c_1}\beta, \widehat{\eta} = \frac{p-1}{c_1}\eta$ [see also hypothesis $H_1(iv)$].

Let $\widehat{\psi} : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\widehat{\psi}(u) = \|\nabla u\|_p^p + \int_\Omega (\widehat{\xi}(z) - \widehat{\eta}(z))|u|^p dz + \int_{\partial\Omega} \widehat{\beta}(z)|u|^p d\sigma$$

for all $u \in W^{1,p}(\Omega)$. We have

$$\widehat{\psi}(u) \geq \int_\Omega [\widehat{\lambda_1}(\widehat{\xi},\widehat{\beta}) - \eta(z)]|u|^p dz \geq 0$$

[see (5) and hypothesis H_1 (iv)].

Evidently, it is enough to prove the lemma for $\widehat{\psi}$. Arguing indirectly, and since $\widehat{\psi}$ is p-homogeneous, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| = 1 \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \widehat{\psi}(u_n) \downarrow 0 \text{ as } n \to +\infty. \tag{33}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{34}$$

The functional $\widehat{\psi}(\cdot)$ is sequentially weakly lower semicontinuous. So, from (33) and (34), it follows that

$$\widehat{\psi}(u) \leq 0,$$
$$\|\nabla u\|_p^p + \int_\Omega \widehat{\xi}(z)|u|^p dz + \int_{\partial\Omega} \widehat{\beta}(z)|u|^p d\sigma \leq \int_\Omega \widehat{\eta}(z)|u|^p dz$$
$$\Rightarrow \quad \mu(u) \leq \int_\Omega \widehat{\eta}(z)|u|^p dz \leq \widehat{\lambda_1}(\widehat{\xi},\widehat{\beta})\|u\|_p^p \tag{35}$$
$$\Rightarrow \quad u = \lambda\widehat{u_1}(\widehat{\xi},\widehat{\beta}) \quad \text{for some } \lambda \in \mathbb{R} \quad [\text{see (5)}].$$

If $\lambda = 0$, then $u = 0$ and so $u_n \to 0$ in $W^{1,p}(\Omega)$, a contradiction to the fact that $\|u_n\| = 1$ for all $n \in \mathbb{N}$. If $\lambda \neq 0$, then to fix things we assume that $\lambda > 0$ (the reasoning is similar if $\lambda < 0$). We have $u = \lambda\widehat{u_1}(\widehat{\xi},\widehat{\beta}) \in D_+$. So, from (35) and the hypothesis on η [see hypothesis $H_1(iv)$], we obtain $\mu(u) \leq \widehat{\lambda_1}(\widehat{\xi},\widehat{\beta})\|u\|_p^p$, which contradicts (5). $\square$

Using this lemma, we can determine the geometry near zero for $\widehat{\varphi}$. This is the first step in establishing the mountain pass geometry for $\widehat{\varphi}$.

Proposition 6. If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_1 hold, then $u = 0$ is a local minimizer of $\widehat{\varphi}$.

Proof. Hypotheses H_1(i), (iv) imply that given $\varepsilon > 0$, we can find $c_{10} = c_{10}(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{p}(\eta(z) + \varepsilon)x^p + c_{10}x^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{36}$$

Then, for $u \in W^{1,p}(\Omega)$, we have

$$\widehat{\varphi}(u)$$
$$= \int_\Omega G(\nabla u)dz + \frac{1}{p}\int_\Omega (\xi(z) + \mu)|u|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma$$
$$\quad - \int_\Omega \widehat{F}(z, u)dz$$
$$\geq \frac{1}{p}\left[\frac{c_1}{p-1}\|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega}\beta(z)|u|^p d\sigma\right]$$
$$\quad - \int_\Omega F(z, u^+)dz \quad [\text{see Corollary 1 and } (8)]$$
$$\geq \frac{1}{p}\left[\frac{c_1}{p-1}\|\nabla u\|_p^p + \int_\Omega (\xi(z) - \eta(z))|u|^p dz + \int_{\partial\Omega}\beta(z)|u|^p d\sigma\right]$$
$$\quad - \frac{\varepsilon}{p}\|u\|^p - c_{11}\|u\|^r \quad \text{for some } c_{11} > 0 \, [\text{see } (36)]$$
$$\geq \frac{1}{p}[c_9 - \varepsilon]\|u\|^p - c_{11}\|u\|^r \quad (\text{see Lemma 2}). \tag{37}$$

Choosing $\varepsilon \in (0, c_9)$, from (37), we infer that

$$\widehat{\varphi}(u) \geq c_{12}\|u\|^p - c_{11}\|u\|^r \quad \text{for all } u \in W^{1,p}(\Omega), \text{ some } c_{12} > 0.$$

Since $p < r$, we see that we can find $\rho \in (0, 1)$ small such that

$$\widehat{\varphi}(u) > 0 = \widehat{\varphi}(0) \quad \text{for all } u \in W^{1,p}(\Omega), 0 < \|u\| \leq \rho$$
$$\Rightarrow \quad u = 0 \text{ is a (strict) local minimizer of } \widehat{\varphi}.$$

$\square$

As a consequence of hypothesis H_1 (ii), we have the following proposition:

Proposition 7. If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_1 hold and $u \in D_+$, then $\widehat{\varphi}(tu) \to -\infty$ as $t \to +\infty$.

Proposition 8. If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_1 hold, then $K_{\widehat{\varphi}} \subseteq C_+$.

Proof. Let $u_0 \in K_{\widehat{\varphi}}$. Then,

$$\widehat{\varphi}'(u_0) = 0$$
$$\Rightarrow \quad \langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \vartheta)|u_0|^{p-2}u_0 h dz$$
$$\quad + \int_{\partial\Omega}\beta(z)|u_0|^{p-2}u_0 h d\sigma = \int_\Omega \widehat{f}(z, u_0)h dz \tag{38}$$

for all $h \in W^{1,p}(\Omega)$. In (38), we choose $h = -u_0^- \in W^{1,p}(\Omega)$, and using Lemma 1, we have

$$\frac{c_1}{p-1}\|\nabla u_0^-\|_p^p + \int_\Omega (\xi(z) + \vartheta)(u_0^-)^p dz \leq 0$$
$$[\text{see } (8) \text{ and hypothesis } H(\beta)]$$
$$\Rightarrow \quad c_{13}\|u_0^-\|^p \leq 0 \quad \text{for some } c_{13} > 0 \quad (\text{recall } \vartheta > \|\xi\|_\infty)$$
$$\Rightarrow \quad u_0 \geq 0.$$

So, we have the following equation (see Ref. 30):

$$\langle A(u_0), h \rangle + \int_\Omega \xi(z) u_0^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) u_0^{p-1} h \, d\sigma$$

$$= \int_\Omega f(z, u_0) h \, dz \text{ for all } h \in W^{1,p}(\Omega)$$

$$\Rightarrow \begin{cases} -\operatorname{div} a(\nabla u_0(z)) + \xi(z) u_0(z)^{p-1} = f(z, u_0(z)) \\ \qquad\qquad\qquad\qquad \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_a} + \beta(z) u_0^{p-1} u = 0 \text{ on } \partial\Omega \end{cases} \tag{39}$$

(see Ref. 30). Now, we can use Proposition 2.10 of Ref. 32 (see also Ref. 18) on (39) and have that

$$u_0 \in L^\infty(\Omega). \tag{40}$$

Then, (40) and the nonlinear regularity theory of Lieberman[22] imply that

$$u_0 \in C_+$$
$$\Rightarrow K_{\widehat{\varphi}} \subseteq C_+.$$

$\square$

In what follows, we assume that $K_{\widehat{\varphi}}$ is finite. Otherwise, we already have infinitely many positive smooth solutions for problem (1) (see Proposition 8). Now, we are ready to produce positive solutions.

Theorem 2. *If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_1 hold, then problem (1) admits at least one positive solution $u_0 \in D_+$.*

Proof. Proposition 6 and since $K_{\widehat{\varphi}}$ is finite imply that we can find $\rho \in (0, 1)$ small such that

$$\widehat{\varphi}(0) = 0 < \inf[\widehat{\varphi}(u) : \|u\| = \rho] = \widehat{m}_\rho \tag{41}$$

(see Ref. 1, Proof of Proposition 29). Combining (41) with Propositions 5 and 7, we see that we can apply Theorem 1 (the mountain pass theorem) and find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in K_{\widehat{\varphi}} \quad \text{and} \quad \widehat{m}_\rho \le \widehat{\varphi}(u_0). \tag{42}$$

From (41), (42), and Proposition 8, we infer that $u_0 \in C_+ \backslash \{0\}$. Let $\rho = \|u_0\|_\infty$. Hypotheses $H_1(i), (iv)$ imply that we can find $\widehat{\xi}_\rho > 0$ such that $f(z, x) + \widehat{\xi}_\rho x^{p-1} \ge 0$ for a.a. $z \in \Omega$, all $x \in [0, \rho]$. Using this in (39), we obtain the following equation (see Ref. 36, p. 120):

$$\operatorname{div} a(\nabla u_0(z)) \le [\|\xi\|_\infty + \widehat{\xi}_\rho] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$
$$\Rightarrow u_0 \in D_+.$$

$\square$

By changing the geometry of the problem near zero, we can have a multiplicity theorem for the positive solutions of (1). So, we strengthen a little the conditions on $a(\cdot)$ and modify the hypotheses on the reaction term $f(z, \cdot)$. The new hypotheses on the map $y \to a(y)$ are the following:
$H(a)_2$: $a(y) = a_0(|y|) y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, and hypotheses $H(a)_2(i)-(iii)$ are the same as the corresponding hypotheses $H(a)(i)-(iii)$ and
(iv) if $G_0(t) = \int_0^t a_0(s) s \, ds$, then $p G_0(t) - a_0(t) t^2 \ge -\bar{c}$ for all $t \ge 0$, some $\bar{c} > 0$, and there exists $q \in (1, p]$ such that $\limsup_{t \to 0^+} \frac{G_0(t)}{t^q} < +\infty$ and $t \to G_0(t^{1/q})$ is convex.

Remark 5. *All the examples given after hypotheses $H(a)_1$ also satisfy the new conditions $H(a)_2$.*

The new hypotheses on the reaction term $f(z, x)$ are the following:
H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H_2 (i)–(iii) are the same as the corresponding hypotheses H_1 (i)–(iii) and
(iv) with $q \in (1, p]$ as in hypothesis $H(a)_2(iv)$, we have $\lim_{x \to 0^+} \frac{f(z, x)}{x^{q-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;
(v) there exists $w_+ \in C^1(\overline{\Omega})$ such that

$$w_+(z) \geq c_+ > 0 \quad \text{for all } z \in \overline{\Omega},$$

$$f(z, w_+(z)) - \xi(z)w_+(z)^{p-1} \leq -c_{14} < 0 \quad \text{for a.a. } z \in \Omega,$$

$$\operatorname{div} a(\nabla w_+) \in L^{r'}(\Omega) \quad \left(\frac{1}{r} + \frac{1}{r'} = 1\right), \quad 0 \leq -\operatorname{div} a(\nabla w_+(z))$$

$$\text{for a.a. } z \in \Omega,$$

(vi) there exists $\widehat{\xi}_+ > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x) + \widehat{\xi}_+ x^{p-1}$ is nondecreasing on $[0, \|w_+\|_\infty]$.

Remark 6. Hypothesis H_2(iv) *implies the presence of a concave term near the origin. If* $\xi(z) \geq 0$ *for a.a.* $z \in \Omega$ *and* $f(z, c_+) \leq 0$ *for a.a.* $z \in \Omega$, *then* $w_+ \equiv c_+$ *satisfies hypothesis* H_2(v).

Theorem 3. *If hypotheses* $H(a)_2$, $H(\xi)$, $H(\beta)$, *and* H_2 *hold, then problem* (1) *has at least two positive solutions* $u_0, \widehat{u} \in D_+$ *and* $u_0 \leq \widehat{u}$, $u_0 \neq \widehat{u}$.

Proof. As before, let $\vartheta > \|\xi\|_\infty$ [see hypothesis $H(\xi)$] and consider the Carathéodory function $\widehat{k} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{k}(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z,x) + \vartheta x^{p-1} & \text{if } 0 \leq x \leq w_+(z), \\ f(z, w_+(z)) + \vartheta w_+(z)^{p-1} & \text{if } w_+(z) < x. \end{cases} \tag{43}$$

Let $\widehat{K}(z,x) = \int_0^x \widehat{k}(z,s)\,ds$ and consider the C^1-functional $\widehat{\psi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}(u) = \frac{1}{p}\mu(u) + \frac{\vartheta}{p}\|u\|_p^p - \int_\Omega \widehat{K}(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (43) and since $\vartheta > \|\xi\|_\infty$, we see that $\widehat{\psi}$ is coercive. Also, via the Sobolev embedding theorem and the compactness of the trace map, we check that $\widehat{\psi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\psi(u_0) = \inf[\psi(u) : u \in W^{1,p}(\Omega)]. \tag{44}$$

Hypothesis $H(a)_2$(iv) implies that we can find $c_{15} \geq 1$ and $\delta \in (0, c_+)$ such that

$$G(y) \leq c_{15}|y|^q \quad \text{for all } |y| \leq \delta. \tag{45}$$

Moreover, hypothesis H_2(iv) implies that given any $\tau > 0$, we can find $\delta_1 \in (0, \delta]$ such that

$$F(z,x) \geq \tau x^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta_1]. \tag{46}$$

Let $\widehat{u}_1(q)(\xi^+, \beta)$ be the principal eigenvalue for the eigenvalue problem (4), when the differential operator is $u \to -\Delta_q u + \xi^+(z)u$. We know that $\widehat{u}_1(q)(\xi^+, \beta) \in D_+$ and so we can find $t \in (0,1)$ small such that

$$t\widehat{u}_1(q)(\xi^+, \beta)(z) \in (0, \delta_1] \text{ and } t|\nabla \widehat{u}_1(q)(\xi^+, \beta)(z)| \leq \delta_1 \tag{47}$$

for all $z \in \overline{\Omega}$.

Then, we have

$$\psi(t\widehat{u}_1(q)(\xi^+, \beta))$$

$$\leq c_{15}t^q \left[\|\nabla \widehat{u}_1(q)(\xi^+, \beta)\|_q^q + \int_\Omega \xi^+(z)\widehat{u}_1(q)(\xi^+, \beta)^q dz \right.$$

$$\left. + \int_{\partial\Omega} \beta(z)\widehat{u}_1(q)(\xi^+, \beta)^q d\sigma \right] - \int_\Omega F(z, t\widehat{u}_1(q)(\xi^+, \beta))dz$$

$$[\text{see } (45), (47) \text{ and recall } c_{14} \geq 1]$$

$$= t^q[c_{15}\widehat{\lambda}_1(q)(\xi^+, \beta) - \tau].$$

Choosing $\tau > c_{15}\widehat{\lambda}_1(q)(\xi^+, \beta)$, we see that

$$\psi(f\widehat{u}_1(q)(\xi^+, \beta)) < 0$$
$$\Rightarrow \quad \psi(u_0) < 0 = \psi(0) \quad [\text{see } (44)]$$
$$\Rightarrow \quad u_0 \neq 0.$$

From (44), we have

$$\psi'(u_0) = 0$$
$$\Rightarrow \quad \langle A(u_0), h \rangle + \int_\Omega (\xi(z) + \vartheta)|u_0|^{p-2} u_0 h dz \tag{48}$$
$$+ \int_{\partial\Omega} \beta(z)|u_0|^{p-2} u_0 h d\sigma = \int_\Omega \widehat{k}(z, u_0) h dz$$

for all $h \in W^{1,p}(\Omega)$. In (48), we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then,

$$\frac{c_1}{p-1} \|\nabla u_0^-\|_p^p + \int_\Omega (\xi(z) + \vartheta)(u_0^-)^p dz \leq 0$$
$$[\text{see Lemma 1, hypothesis } H(\beta) \text{ and } (43)]$$
$$\Rightarrow \quad c_{16} \|u_0^-\|^p \leq 0 \text{ for some } c_{16} > 0 \text{ (recall that } \vartheta > \|\xi\|_\infty)$$
$$\Rightarrow \quad u_0 \geq 0, u_0 \neq 0.$$

Next, in (48), we choose $h = (u_0 - w_+)^+ \in W^{1,p}(\Omega)$. Then,

$$\langle A(u_0), (u_0 - w_+)^+ \rangle + \int_\Omega (\xi(z) + \vartheta) u_0^{p-1}(u_0 - w_+)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - w_+)^+ d\sigma$$
$$= \int_\Omega \widehat{k}(z, u_0)(u_0 - w_+)^+ dz$$
$$= \int_\Omega [f(z, w_+) + \vartheta w_+^{p-1}](u_0 - w_+)^+ dz \quad [\text{see } (43)]$$
$$\leq \langle A(w_+), (u_0 - w_+)^+ \rangle + \int_\Omega (\xi(z) + \vartheta) w_+^{p-1}(u_0 - w_+)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z) w_+^{p-1}(u_0 - w_+)^+ d\sigma$$
$$[\text{see hypotheses } H_2(v) \text{ and } H(\beta)]$$
$$\Rightarrow \quad u_0 \leq w_+ \quad (\text{since } \vartheta > \|\xi\|_\infty).$$

So, we have proved that

$$u_0 \in [0, w_+], \quad u_0 \neq 0. \tag{49}$$

From (49) and (43), Eq. (48) becomes (see Ref. 30)

$$\langle A(u_0), h \rangle + \int_\Omega \xi(z) u_0^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma$$
$$= \int_\Omega f(z, u_0) h dz$$
$$\text{for all } h \in W^{1,p}(\Omega)$$
$$\Rightarrow \quad \begin{cases} -\text{div } a(\nabla u_0(z)) + \xi(z) u_0(z)^{p-1} = f(z, u_0(z)) \\ \qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_a} + \beta(z) u_0^{p-1} = 0 \text{ on } \partial\Omega \end{cases} \tag{50}$$

As before from (50) and the nonlinear regularity theory (see Refs. 18, 22, and 32), we have $u_0 \in C_+\backslash\{0\}$. On account of hypothesis $H_2(vi)$, there is $\tilde{\xi}_+ > \max\{\widehat{\tilde{\xi}}_+, \|\xi\|_\infty\}$ such that $f(z, x) + \tilde{\xi}_+ x^{p-1} \geq 0$ for a.a. $z \in \Omega$, all $x \in [0, \|w_+\|_\infty]$. From (50), we have the following equation (see Ref. 30, pp. 111 and 120):

$$\operatorname{div} a(\nabla u_0(z)) \le \left[\|\xi\|_\infty + \tilde{\xi}_+ \right] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega$$

$$\Rightarrow \quad u_0 \in D_+.$$

In addition, note that

$$- \operatorname{div} a(\nabla u_0(z)) + (\xi(z) + \tilde{\xi}_+) u_0(z)^{p-1}$$

$$= f(z, u_0(z)) + \tilde{\xi}_+ u_0(z)^{p-1}$$

$$\le f(z, w_+(z)) + \tilde{\xi}_+ w_+(z)^{p-1}$$

$$\left[\text{see hypothesis } H_2 \text{ (vi) and recall that } u_0 \le w_+ \right]$$

$$\le -c_{14} + (\xi(z) + \tilde{\xi}_+) w_+(z)^{p-1}$$

$$\le - \operatorname{div} a(\nabla w_+(z)) + (\xi(z) + \tilde{\xi}_+) w_+(z)^{p-1} \text{ for a.a. } z \in \Omega \tag{51}$$

$$\left[\text{see hypothesis } H_2 \text{ (iv)} \right].$$

From (51) and Proposition 5 of Ref. 10, we have

$$(w_+ - u_0)(z) > 0 \text{ for all } z \in \Omega,$$

$$\left. \frac{\partial (w_+ - u_0)}{\partial n} \right|_{\Sigma_0} < 0 \text{ with } \Sigma_0 = \{ z \in \partial\Omega : u_0(z) = w_+(z) \}$$

$$\Rightarrow w_+ - u_0 \in \widehat{D}_+(\Sigma_0). \tag{52}$$

Using u_0, we introduce the following Carathéodory function:

$$\widehat{\gamma}_0(z, x) = \begin{cases} f(z, u_0(z)) + \vartheta u_0(z)^{p-1} & \text{if } x \le u_0(z), \\ f(z, x) + \vartheta x^{p-1} & \text{if } u_0(z) < x. \end{cases} \tag{53}$$

We set $\widehat{\Gamma}_0(z, x) = \int_0^x \widehat{\gamma}_0(z, s) ds$ and consider the C^1-functional $\widehat{\tau}_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\tau}_0(u) = \frac{1}{p} \mu(u) + \frac{\vartheta}{p} \|u\|_p^p - \int_\Omega \widehat{\Gamma}_0(z, u) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using (53), we can show that

$$K_{\widehat{\tau}_0} \subseteq [u_0) \cap D_+ = \{ u \in C^1(\overline{\Omega}) : u_0(z) \le u(z) \text{ for all } z \in \overline{\Omega} \}. \tag{54}$$

We may assume that

$$K_{\widehat{\tau}_0} \cap [u_0, w_+] = \{u_0\}. \tag{55}$$

Indeed, if we can find $\tilde{u}_0 \in K_{\widehat{\tau}_0} \cap [u_0, w_+]$, $\tilde{u}_0 \ne u_0$, then from (54) we have

$$u_0 \le \tilde{u}_0, \ \tilde{u}_0 \in D_+$$

$$\Rightarrow \quad \tilde{u}_0 \text{ is the desired second positive solution of (1) } [\text{see (53)}].$$

We consider the Carathéodory function $\tilde{\gamma}_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{\gamma}_0(z, x) = \begin{cases} \widehat{\gamma}_0(z, x) & \text{if } x \le w_+(z), \\ \widehat{\gamma}_0(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases} \tag{56}$$

We set $\tilde{\Gamma}_0(z, x) = \int_0^x \tilde{\gamma}_0(z, s) ds$ and consider the C^1-functional $\tilde{\tau}_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\tau}_0(u) = \frac{1}{p} \mu(u) + \frac{\vartheta}{p} \|u\|_p^p - \int_\Omega \tilde{\Gamma}_0(z, u) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

As before, since $\vartheta > \| \xi \|_\infty$ and using (53) and (56), we have that $\widehat{\tau}_0(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, via the Weierstrass-Tonelli theorem, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\tilde{\tau}_0(\tilde{u}_0) = \inf[\tilde{\tau}_0(u) : u \in W^{1,p}(\Omega)]. \tag{57}$$

Using (53) and (56), we see that

$$K_{\widehat{\tau}_0} \subseteq [u_0, w_+] \cap D_+$$
$$\Rightarrow \quad \tilde{u}_0 \in [u_0, w_+] \cap D_+ \quad [\text{see } (57)]. \tag{58}$$

Note that

$$\widehat{\tau}_0'\big|_{[0,w_+]} = \tilde{\tau}_0'\big|_{[0,w_+]} \quad [\text{see } (53) \text{ and } (56)]. \tag{59}$$

From (55), (58), and (59), it follows that $\tilde{u} = u_0$. Then, (52) and Proposition 1 imply that

$$u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \widehat{\tau}_0. \tag{60}$$

We assume that $K_{\widehat{\tau}_0}$ is finite [otherwise, on account of (54), we have an infinity of positive smooth solutions of (1) bigger than u_0 and so we are done]. Then, we can find $\rho \in (0, 1)$ small such that

$$\widehat{\tau}_0(u_0) < \inf[\widehat{\tau}_0(u) : \|u - u_0\| = \rho] = \widehat{m}_\rho \tag{61}$$

(see Ref. 1, Proof of Proposition 29). Hypothesis $H_2(\text{ii})$ implies that for all $u \in D_+$, we have

$$\widehat{\tau}_0(tu) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{62}$$

Note that

$$\widehat{\tau}_0\big|_{[u_0]} = \widehat{\varphi}\big|_{[u_0]} + \widehat{\xi}_* \text{ for some } \widehat{\xi}_* \in \mathbb{R} \ [\text{see } (53) \text{ and } (8)]$$
$$\Rightarrow \quad \widehat{\tau}_0 \quad \text{satisfies the } C - \text{condition (see Proposition 5)}. \tag{63}$$

Then (61), (62), and (63) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\widehat{\tau}_0} \subseteq [u_0] \cap D_+ \quad [\text{see } (54)], \quad \widehat{m}_\rho \leq \widehat{\tau}_0(\widehat{u}). \tag{64}$$

From (61) and (64), it follows that $\widehat{u} \neq u_0$, $u_0 \leq \widehat{u}$, and $\widehat{u}$ is a positive smooth solution of (1). $\square$

In the current setting, we can prove the existence of a smallest positive solution for problem (1). Let S_+ be the set of positive solutions for problem (1). We have seen that under hypotheses $H(a)$, $H(\beta)$, and H_2, we have $\emptyset \neq S_+ \subseteq D_+$. Moreover, from Ref. 33 (see also Ref. 9), we have that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$).

Theorem 4. *If hypotheses $H(a)_2$, $H(\xi)$, $H(\beta)$, and H_2 hold, then problem (1) admits a smallest positive solution $u_* \in D_+$.*

Proof. From Lemma 3.10, p. 178 of Ref. 17, we know that we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

Evidently, we may assume that $u_n \in [0, w_+]$ for all $n \in \mathbb{N}$. So, it follows that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{65}$$

We have

$$\langle A(u_n), h \rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \tag{66}$$
$$= \int_\Omega f(z, u_n) h dz$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (66), we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$, and use (65). Then,

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_* \rangle = 0$$

$$\Rightarrow u_n \to u_* \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 2).} \tag{67}$$

So, if in (66) we pass to the limit as $n \to +\infty$ and we use (67), then

$$\langle A(u_*), h \rangle + \int_\Omega \xi(z) u_*^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma$$

$$= \int_\Omega f(z, u_*) h dz \text{ for all } h \in W^{1,p}(\Omega)$$

$$\Rightarrow u_* \in S_+ \cup \{0\}.$$

We need to show that $u_* \neq 0$. Hypotheses H_2 (i) and (iv) imply that given any $\lambda > 0$, we can find $c_{17} = c_{17}(\lambda) > 0$ such that

$$f(z, x) \geq \lambda x^{q-1} - c_{17} x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{68}$$

This unilateral growth estimate on the reaction term $f(z, \cdot)$ leads to the following auxiliary nonlinear Robin problem. Here, as before $\vartheta > \|\xi\|_\infty$,

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) + \xi^+(z) u(z)^{p-1} = \lambda u(z)^{q-1} - c_{17} u(z)^{r-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z) u^{p-1} = 0 \text{ on } \partial\Omega, \ u \geq 0. \end{cases} \tag{69}$$

$\square$

Claim 1. For $\lambda > 0$ big, problem (69) admits a unique positive solution $\overline{u} \in D_+$.

First, we show the existence of a positive solution for problem (69). To this end, let $J : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$J(u) = \frac{1}{p} \mu(u) + \frac{\vartheta}{p} \|u^-\|_p^p + \frac{c_{17}}{r} \|u^+\|_r^r - \frac{\lambda}{q} \|u^+\|_q^q.$$

Recall that $\vartheta > \|\xi\|_\infty$. Therefore, since $q \leq p < r$, it follows that $J(\cdot)$ is coercive. In addition, the Sobolev embedding theorem and the compactness of the trace map imply that $J(\cdot)$ is sequentially weakly lower semicontinuous. So, using the Weierstrass-Tonelli theorem, we can find $\overline{u} \in W^{1,p}(\Omega)$ such that

$$J(\overline{u}) = \inf[J(u) : u \in W^{1,p}(\Omega)]. \tag{70}$$

Hypothesis $H(a)_2(iv)$ and Corollary 1 imply that there exists $c_{18} > 0$ such that

$$G(y) \leq c_{18}(|y|^q + |y|^p) \quad \text{for all } y \in \mathbb{R}^N. \tag{71}$$

Then, for $u \in W^{1,p}(\Omega)$ with $u \geq 0$, $0 < \|u\| \leq 1$, we have

$$J(u) \leq c_{18}(\|\nabla u\|_q^q + \|\nabla u\|_p^p) + \frac{1}{p} \int_\Omega \xi^+(z) u^p dz$$

$$+ \frac{1}{p} \int_{\partial\Omega} \beta(z) u^p d\sigma + c_{19} \|u\|^r - \frac{\lambda}{q} \|u\|^q \text{ for some } c_{19} > 0$$

$$\leq \left[c_{20} - \frac{\lambda}{q} \right] \|u\|^q \quad (\text{recall } q \leq p < r \text{ and } \|u\| = 1).$$

However, $\lambda > 0$ is arbitrary. So, choosing $\lambda > q c_{20} > 0$, we have

$$J(u) < 0 = J(0)$$

$$\Rightarrow J(\overline{u}) < 0 = J(0) \quad [\text{see (70)}]$$

$$\Rightarrow \overline{u} \neq 0.$$

From (70), we have

$$J'(\overline{u}) = 0$$

$$\Rightarrow \langle A(\overline{u}), h \rangle + \int_\Omega \xi^+(z) |\overline{u}|^{p-2} \overline{u} h dz + \int_{\partial\Omega} \beta(z) |\overline{u}|^{p-2} \overline{u} h d\sigma$$

$$= \lambda \int_\Omega (\overline{u}^+)^{p-1} h dz - c_{17} \int_\Omega (\overline{u}^+)^{r-1} h dz \tag{72}$$

$$\text{for all } h \in W^{1,p}(\Omega).$$

In (72), we choose $h = -\overline{u}^- \in W^{1,p}(\Omega)$. Then,

$$\frac{c_1}{p-1}\|\nabla\overline{u}^-\|_p^p + \int_\Omega \xi^+(z)(\overline{u}^-)^p dz + \int_{\partial\Omega}\beta(z)(\overline{u}^-)^p d\sigma \leq 0$$

(see Lemma 1)

$$\Rightarrow c_{21}\|\overline{u}^-\|^p \leq 0 \text{ for some } c_{21} > 0 \text{ (recall that } \vartheta > \|\xi\|_\infty)$$

$$\Rightarrow \overline{u} \geq 0, \ \overline{u} \neq 0.$$

Therefore, (72) becomes (see Ref. 30)

$$\langle A(\overline{u}), h\rangle + \int_\Omega \xi^+(z)\overline{u}^{p-1}hdz + \int_{\partial\Omega}\beta(z)\overline{u}^{p-1}hd\sigma$$

$$= \lambda\int_\Omega \overline{u}^{q-1}hdz - c_{17}\int_\Omega \overline{u}^{r-1}hdz \text{ for all } h \in W^{1,p}(\Omega)$$

$$\Rightarrow \begin{cases} -\text{div } a(\nabla\overline{u}(z)) + \xi^+(z)\overline{u}(z)^{p-1} = \lambda\overline{u}(z)^{q-1} - c_{17}\overline{u}(z)^{r-1} \\ \qquad\qquad\qquad \text{ for a.a. } z \in \Omega, \\ \dfrac{\partial\overline{u}}{\partial n_a} + \beta(z)\overline{u}^{p-1} = 0 \text{ on } \partial\Omega, \end{cases} \tag{73}$$

$$\Rightarrow \overline{u} \in C_+\backslash\{0\} \quad \text{(by the nonlinear regularity theory)}.$$

From (73), we have the following equation (see Ref. 36, p. 111 and 120):

$$\text{div } a(\nabla\overline{u}(z)) \leq \left[\|\xi\|_\infty + c_{17}\|\overline{u}\|_\infty^{r-p}\right]\overline{u}(z)^{p-1} \text{ for a.a. } z \in \Omega$$

$$\Rightarrow \quad \overline{u} \in D_+.$$

This proves the existence of a positive solution in D_+ for the auxiliary problem (69), when $\lambda > 0$ is big. Next, we show the uniqueness of this positive solution. To this end, we introduce the integral $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, defined by

$$j(u) = \begin{cases} \int_\Omega G(\nabla u^{1/q})dz + \frac{1}{p}\int_\Omega \xi^+(z)u^{p/q}dz \\ \quad + \frac{1}{p}\int_{\partial\Omega}\beta(z)u^{p/q}d\sigma \quad \text{if } u^{1/q} \in W^{1,p}(\Omega), \ u \geq 0, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \text{dom}j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ [the effective domain of $j(\cdot)$]. We set

$$u = [tu_1 + (1-t)u_2]^{1/q} \quad \text{for } t \in [0,1].$$

Using Lemma 1 of Ref. 7, we have

$$|\nabla u(z)| \leq \left[t|\nabla u_1(z)^{1/q}|^q + (1-t)|\nabla u_2(z)^{1/q}|^q\right]^{1/q}$$

$$\text{for a.a. } z \in \Omega$$

$$\Rightarrow \quad G_0(|\nabla u(z)|)$$

$$\leq G_0\left(\left[t|\nabla u_1(z)^{1/q}|^q + (1-t)|\nabla u_2(z)^{1/q}|^q\right]^{1/q}\right)$$

$$\text{for a.a. } z \in \Omega \ [\text{recall that } G_0(\cdot) \text{ is increasing}]$$

$$\leq tG_0(|\nabla u_1(z)^{1/q}|) + (1-t)G_0(|\nabla u_2(z)^{1/q}|)$$

$$\text{for a.a. } z \in \Omega$$

$$[\text{see hypothesis } H(a)_2(\text{iv})]$$

$$\Rightarrow \quad G(\nabla u(z)) \leq tG(\nabla u_1(z)^{1/q}) + (1-t)G(\nabla u_2(z)^{1/q})$$

$$\text{for a.a. } z \in \Omega,$$

$$\Rightarrow \quad j(\cdot) \text{ is convex}$$

$$[\text{recall that } q \leq p \text{ and see hypothesis } H(\beta)].$$

Also by Fatou's lemma, $j(\cdot)$ is lower semicontinuous. Suppose that $\bar{v} \in W^{1,p}(\Omega)$ is another positive solution of the auxiliary problem (69). Again we show that $\bar{v} \in D_+$. Then, for $h = \bar{u}^q - \bar{v}^q$ and $t \in (0, 1]$, small, we have $\bar{u}^q - th$, $\bar{v}^q + th \in \text{dom} \, j$. Evidently, $j(\cdot)$ is Gateaux differentiable at $\bar{u}^q$ and at $\bar{v}^q$ in the direction h. So, from the chain rule and the nonlinear Green's identity (see Ref. 12, p. 210), we obtain

$$j'(\bar{u}^q)(h) = \frac{1}{q} \int_\Omega \frac{-\text{div}\, a(\nabla \bar{u}) + \xi^+(z)\bar{u}^{p-1}}{\bar{u}^{q-1}} h dz,$$

$$j'(\bar{v}^q)(h) = \frac{1}{q} \int_\Omega \frac{-\text{div}\, a(\nabla \bar{v}) + \xi^+(z)\bar{v}^{p-1}}{\bar{v}^{q-1}} h dz,$$

for this $h \in C^1(\overline{\Omega})$. From the convexity of $j(\cdot)$, we have the monotonicity of $j'(\cdot)$. Hence,

$$0 \le \int_\Omega \left(\frac{-\text{div}\, a(\nabla \bar{u})}{\bar{u}^{q-1}} - \frac{-\text{div}\, a(\nabla \bar{v})}{\bar{v}^{q-1}} \right)(\bar{u}^q - \bar{v}^q) dz$$

$$= -\int_\Omega \xi^+(z)(\bar{u}^{p-q} - \bar{v}^{p-q})(\bar{u}^q - \bar{v}^q) dz$$

$$- \int_\Omega c_{17}(\bar{u}^{r-q} - \bar{v}^{r-q})(\bar{u}^q - \bar{v}^q) dz$$

$$\Rightarrow \quad \bar{u} = \bar{v} \quad (\text{since } q \le p < r).$$

This proves Claim 1.

Claim 2. $\bar{u} \le u$ for all $u \in S_+$.
Let $u \in S_+$ and consider the Carathédory function $k_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \lambda x^{q-1} - c_{17} x^{r-1} + \vartheta x^{p-1} & \text{if } 0 \le x \le u(z), \\ \lambda u(z)^{q-1} - c_{17} u(z)^{r-1} + \vartheta u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{74}$$

We set $K_+(z, x) = \int_0^x k_+(z, s) ds$ and consider the C^1-functional $\psi_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \int_\Omega G(\nabla u) dz + \frac{1}{p} \int_\Omega (\xi^+(z) + \vartheta)|u|^p dz$$

$$+ \frac{1}{p} \int_{\partial \Omega} \beta(z)|u|^p d\sigma - \int_\Omega K_+(z, u) dz$$

for all $u \in W^{1,p}(\Omega)$. Since $\vartheta > \|\xi\|_\infty$, from (74), we infer that ψ_+ is coercive. In addition, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi_+(\tilde{u}) = \inf[\psi_+(u) : u \in W^{1,p}(\Omega)]. \tag{75}$$

As in the prove of Claim 1, we show that

$$\psi_+(\tilde{u}) \le 0 = \psi_+(0)$$

$$\Rightarrow \quad \tilde{u} \ne 0.$$

From (75), we have

$$\psi'_+(\tilde{u}) = 0$$

$$\Rightarrow \quad \langle A(\tilde{u}), h \rangle + \int_\Omega (\xi^+(z) + \vartheta)|\tilde{u}|^{p-2} \tilde{u} h dz$$

$$+ \int_{\partial \Omega} \beta(z)|\tilde{u}|^{p-2} \tilde{u} h d\sigma = \int_\Omega k_+(z, \tilde{u}) h dz \tag{76}$$

for all $h \in W^{1,p}(\Omega)$. In (76), first we choose $h = -\tilde{u}^- \in W^{1,p}(\Omega)$. Then, since $\vartheta > \|\xi\|_\infty$, we obtain

$$c_{22} \|\tilde{u}^-\| \le 0 \quad \text{for some } c_{22} > 0$$

$$\Rightarrow \quad \tilde{u} \ge 0, \, \tilde{u} \ne 0.$$

Also in (76), we choose $h = (\tilde{u} - u)^+ \in W^{1,p}(\Omega)$. Then,

$$\langle A(\tilde{u}), (\tilde{u} - u)^+ \rangle + \int_\Omega (\xi^+(z) + \vartheta) \tilde{u}^{p-1} (\tilde{u} - u)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z) \tilde{u}^{p-1} (\tilde{u} - u)^+ d\sigma$$
$$= \int_\Omega k_+(z, \tilde{u})(\tilde{u} - u)^+ dz$$
$$= \int_\Omega [\lambda u^{q-1} - c_{17} u^{r-1} + \vartheta u^{p-1}](\tilde{u} - u)^+ dz$$
$$\leq \int_\Omega [f(z, u) + \vartheta u^{p-1}](\tilde{u} - u)^+ dz \quad [\text{see } (68)]$$
$$\leq \langle A(u), (\tilde{u} - u)^+ \rangle + \int_\Omega (\xi^+(z) + \vartheta) u^{p-1} (\tilde{u} - u)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z) u^{p-1} (\tilde{u} - u)^+ d\sigma \quad (\text{since } u \in S_+)$$
$$\Rightarrow \quad \langle A(\tilde{u}) - A(u), (\tilde{u} - u)^+ \rangle$$
$$+ \int_\Omega (\xi^+(z) + \vartheta)(\tilde{u}^{p-1} - u^{p-1})(\tilde{u} - u)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z)(\tilde{u}^{p-1} - u^{p-1})(\tilde{u} - u)^+ d\sigma \leq 0$$
$$\Rightarrow \quad \tilde{u} \leq u \, [\text{since } \vartheta \geq \|\xi\|_\infty, \text{ see also hypothesis } H(\beta)].$$

So, we have proved that

$$\tilde{u} \in [0, u], \, \tilde{u} \neq 0. \tag{77}$$

Then, on account of (74), Eq. (76) becomes

$$\langle A(\tilde{u}), h \rangle + \int_\Omega \xi^+(z) \tilde{u}^{p-1} h dz + \int_{\partial\Omega} \beta(z) \tilde{u}^{p-1} h d\sigma$$
$$= \int_\Omega (\lambda \tilde{u}^{q-1} - c_{17} \tilde{u}^{r-1}) h dz \text{ for all } h \in W^{1,p}(\Omega)$$
$$\Rightarrow \quad \tilde{u} \text{ is a positive solution of the auxiliary problem } (69).$$

From Claim 1, we have

$$\tilde{u} = \overline{u} \in D_+$$
$$\Rightarrow \quad \overline{u} \leq u \quad \text{for all } u \in S_+ \, [\text{see } (77)].$$

This proves Claim 2. Now, on account of Claim 2, we have

$$\overline{u} \leq u_n \quad \text{for all } n \in \mathbb{N}$$
$$\Rightarrow \quad \overline{u} \leq u_* \quad [\text{see } (67)]$$
$$\Rightarrow \quad u_* \neq 0 \quad \text{and so} \quad u_* \in S_+, \, u_* = \inf S_+.$$

$\square$

IV. p-LAPLACIAN EQUATION

In the existence theorem of Sec. III (see Theorem 2), we assumed that the quotient $\frac{f(z,x)}{x^{p-1}}$ stayed below a multiple of the principal eigenvalue $\widehat{\lambda}_1(\widehat{\xi}, \widehat{\beta})$. In the particular case of the p-Laplacian [that is, $a(y) = |y|^{p-2} y$ for all $y \in \mathbb{R}^N$], hypothesis $H_1(iv)$ becomes

$$\tilde{\eta}(z) \leq \liminf_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} \leq \eta(z)$$
$$\text{uniformly for a.a. } z \in \Omega,$$
$$\eta(z) \leq \widehat{\lambda}_1(\xi, \beta) \quad \text{for a.a. } z \in \Omega, \, \eta \neq \widehat{\lambda}_1(\xi, \beta).$$

So, at zero, we have nonuniform, nonresonance with respect to the principal eigenvalue $\widehat{\lambda}_1(\xi,\beta)$. This condition made $u = 0$ a local minimizer of $\widehat{\varphi}$ (see Proposition 6) and made possible the use of Theorem 1 (the mountain pass theorem). It is natural to ask what happens if the quotient $\frac{f(z,x)}{x^{p-1}}$ stays above $\widehat{\lambda}_1(\xi,\beta)$ as $x \to 0^+$. We will show that in this case for the p-Laplacian equation, we can still produce a nontrivial smooth solution but without any information on its sign. In this case, the mountain pass theorem cannot be used. Instead we use tools from Morse theory (critical groups). So, the problem under consideration is the following nonlinear Robin equation:

$$(1)_P \quad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Therefore, now we have

$$a(y) = |y|^{p-2}y \quad \text{for all } y \in \mathbb{R}^N \ (1 < p < \infty),$$

$$G(y) = \frac{1}{p}|y|^p \quad \text{for all } y \in \mathbb{R}^N,$$

$$\frac{\partial u}{\partial n_p} = |u|^{p-2}\frac{\partial u}{\partial n} \quad \text{for all } u \in W^{1,p}(\Omega).$$

We introduce the following subspace of $W^{1,p}(\Omega)$:

$$V = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega \widehat{u}_1(\xi,\beta)u\,dz = 0 \right\}. \tag{78}$$

We have the following direct sum decomposition:

$$W^{1,p}(\Omega) = \mathbb{R}\widehat{u}_1(\xi,\beta) \oplus V. \tag{79}$$

In the sequel, for economy in the notation, we write $\widehat{u}_1 = \widehat{u}_1(\xi,\beta) \in D_+$. We set

$$\widehat{\lambda}_V = \inf\left[\frac{\mu_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), \ u \neq 0 \right]. \tag{80}$$

In this case, the C^1-functional $\mu_p : W^{1,p}(\Omega) \to \mathbb{R}$ is given by

$$\mu_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

As in Ref. 28 (see Proposition 3.8), we can show that

$$\widehat{\lambda}_1 < \widehat{\lambda}_V \leq \widehat{\lambda}_2. \tag{81}$$

Moreover, if $p = 2$, then $\widehat{\lambda}_V = \widehat{\lambda}_2$. The hypotheses on the reaction term are the following:

H_3: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p < r < p*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x \to \pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) if $e(z,x) = f(z,x)x - pF(z,x)$, then there exists $d \in L^1(\Omega)$ such that $e(z,x) \leq e(z,y) + d(z)$ for a.a. $z \in \Omega$, all $0 \leq x \leq y$ or $y \leq x \leq 0$;

(iv) there exist $\delta_0 \in (0,1)$ and $\eta \in (\widehat{\lambda}_1, \widehat{\lambda}_V)$ such that $\widehat{\lambda}_1|x|^p \leq f(z,x)x \leq \eta|x|^p$ for a.a. $z \in \Omega$, all $|x| \leq \delta_0$.

Remark 7. Now hypothesis $H_3(iv)$ at zero permits resonance with respect to the principal eigenvalue $\widehat{\lambda}_1$.

Let $\varphi_p : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem $(1)_P$ defined by

$$\varphi_p(u) = \frac{1}{p}\mu_p(u) - \int_\Omega F(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega),$$

with $\mu_p : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma$$

for all $u \in W^{1,p}(\Omega)$. Note that the hypotheses H_3 (i)–(iii) are the same as the corresponding hypotheses H_1 (i)–(iii). Therefore, Proposition 5 remains valid and we have the following proposition:

Proposition 9. If hypotheses $H(\xi)$, $H(\beta)$, and H_3 hold, then the functional φ satisfies the C-condition.

Next, we compute the critical groups of the energy functional at infinity. In fact, we will do this for the general case of the differential operator $\mathrm{div}\, a(\nabla u)$ since the result is of independent interest and can be used in other occasions. In this case, the energy functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ is given by

$$\varphi(u) = \frac{1}{p}\mu(u) - \int_\Omega F(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega),$$

with $\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \int_\Omega pG(\nabla u)dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

As we already mentioned, if $G(y) = \frac{1}{p}|y|^p$ for all $y \in \mathbb{R}^N$, then $\varphi = \varphi_p$ and this energy functional corresponds to the p-Laplace differential operator.

Proposition 10. If hypotheses $H(a)_1$, $H(\xi)$, $H(\beta)$, and H_3 (i)–(iii) hold, then $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$.

Proof. Let $\partial B_1 = \{u \in W^{1,p}(\Omega) : \|u\| = 1\}$. Hypotheses H_3(i), (ii) imply that given $k > 0$ we can find $c_{23} = c_{23}(k) > 0$ such that

$$F(z,x) \geq \frac{k}{p}|x|^p - c_{23} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{82}$$

Then, for $u \in \partial B_1$ and $t > 0$, we have

$$\varphi(tu) = \frac{1}{p}\mu(tu) - \int_\Omega F(z,tu)dz$$

$$\leq c_5(1 + t^p\|\nabla u\|_p^p) + \frac{t^p}{p}c_{24}\|u\|_p^p - \frac{t^p}{p}k\|u\|_p^p + c_{23}|\Omega|_N$$

$$\text{for some } c_{24} > 0$$

$$[\text{see Corollary 1, hypotheses } H(\xi), H(\beta) \text{ and } (82)]$$

$$\leq t^p[c_{25} - k\|u\|_p^p] \quad \text{for some } c_{25} > 0 \quad (\text{recall } u \in \partial B_1).$$

Recall that $k > 0$ is arbitrary. So, choosing $k > \frac{c_{25}}{\|u\|_p^p}$, we infer that

$$\varphi(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{83}$$

For $u \in \partial B_1$ and $t > 0$, we have

$$\frac{d}{dt}\varphi(tu) = \langle \varphi'(tu), u \rangle \quad (\text{by the chain rule})$$

$$= \frac{1}{t}\langle \varphi'(tu), tu \rangle$$

$$= \frac{1}{t}\left[\int_\Omega (a(t\nabla u), t\nabla u)_{\mathbb{R}^N}dz + \int_\Omega \xi(z)|tu|^p dz \right.$$

$$\left. + \int_{\partial\Omega} \beta(z)|tu|^p d\sigma - \int_\Omega f(z,tu)(tu)dz\right]$$

$$\leq \frac{1}{t}\left[\int_\Omega p\,G(t\nabla u)dz + \int_\Omega \xi(z)|tu|^p dz + \int_{\partial\Omega}\beta(z)|tu|^p d\sigma - \int_\Omega pF(z,tu)dz + c_{26}\right]$$

$$\text{for some } c_{26} > 0 \, [\text{see hypotheses } H(a)_1 \text{ (iv)}, H_3 \text{ (iii)}]$$

$$= \frac{1}{t}[p\varphi(tu) + c_{26}]. \tag{84}$$

From (83) and (84), it follows that $\frac{d}{dt}\varphi(tu) < 0$ for all $t > 0$ big. The implicit function theorem implies that we can find $\widehat{e} \in C(\partial B_1)$ such that

$$\widehat{e} > 0 \quad \text{and} \quad \varphi(\widehat{e}(u)u) = \rho_0 < -\frac{c_{25}}{p}. \tag{85}$$

We extend $\widehat{e}(\cdot)$ on $W^{1,p}(\Omega)\backslash\{0\}$ by $\widehat{e}_0(u) = \frac{1}{\|u\|}\widehat{e}(\frac{u}{\|u\|})$ for all $u \in W^{1,p}(\Omega)\backslash\{0\}$. We have $\widehat{e}_0 \in C(W^{1,p}(\Omega)\backslash\{0\})$ and $\varphi(\widehat{e}_0(u)u) = \rho_0$. Also, we have

$$\varphi(u) = \rho_0 \Rightarrow \widehat{e}_0(u) = 1. \tag{86}$$

So, if we define

$$\tilde{e}_0(u) = \begin{cases} 1 & \text{if } \varphi(u) \leq \rho_0, \\ \widehat{e}_0(u) & \text{if } \rho_0 < \varphi(u), \end{cases} \tag{87}$$

then we have $\tilde{e}_0 \in C(W^{1,p}(\Omega)\backslash\{0\})$ [see (86)]. We consider the deformation $h : [0,1] \times (W^{1,p}(\Omega)\backslash\{0\}) \to W^{1,p}(\Omega)\backslash\{0\}$ defined by $h(t,u) = (1-t)u + t\tilde{e}_0(u)u$ for all $t \in [0,1]$, all $u \in W^{1,p}(\Omega)$. We have

- $h(0,u) = u$ for all $u \in W^{1,p}(\Omega)\backslash\{0\}$,
- $h(1,u) = \tilde{e}_0(u)u \in \varphi^{\rho_0}$ [see (87)],
- $h(t,\cdot)\big|_{\varphi^{\rho_0}} = id\big|_{\varphi^{\rho_0}}$ [see (86) and (87)].

From these facts, we infer that

$$\varphi^{\rho_0} \quad \text{is a strong deformation retract of } W^{1,p}(\Omega)\backslash\{0\}. \tag{88}$$

Consider the radial retraction $r_1 : W^{1,p}(\Omega)\backslash\{0\} \to \partial B_1$ defined by

$$r_1(u) = \frac{u}{\|u\|} \quad \text{for all } u \in W^{1,p}(\Omega)\backslash\{0\}.$$

We introduce the deformation $\widehat{h} : [0,1] \times (W^{1,p}(\Omega)\backslash\{0\}) \to W^{1,p}(\Omega)\backslash\{0\}$ defined by $\widehat{h}(t,u) = (1-t)u + tr_1(u)$ for all $t \in [0,1]$, all $u \in W^{1,p}(\Omega)\backslash\{0\}$. With this deformation, we see that

$$W^{1,p}(\Omega)\backslash\{0\} \text{ is deformable into } \partial B_1. \tag{89}$$

In addition, using radial retraction $r_1(\cdot)$, we see that

$$\partial B_1 \text{ is a retract of } W^{1,p}(\Omega)\backslash\{0\}. \tag{90}$$

From (89), (90), and Theorem 6.5, p. 325 of Ref. 8, we infer that

$$\partial B_1 \text{ is a deformation retract of } W^{1,p}(\Omega)\backslash\{0\}. \tag{91}$$

From (88) and (91), it follows that (see Ref. 27, p. 143)

$$\varphi^{\rho_0} \text{ and } \partial B_1 \text{ are homotopy equivalent}$$
$$\Rightarrow \quad H_k(W^{1,p}(\Omega), \varphi^{\rho_0}) = H_k(W^{1,p}(\Omega), \partial B_1) \quad \text{for all } k \in \mathbb{N}_0. \tag{92}$$

The Sobolov space $W^{1,p}(\Omega)$ is infinite dimensional. Hence (see Ref. 16, Problems 4.154 and 4.159, and Ref. 27, p. 147),

$$\partial B_1 \text{ is contractible}$$
$$\text{(see Ref. 16, Problems 4.154 and 4.159)}$$
$$\Rightarrow \quad H_k(W^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \in \mathbb{N}_0$$
$$\text{(see Ref. 27, p. 147)}$$
$$\Rightarrow \quad H_k(W^{1,p}(\Omega), \varphi^{\rho_0}) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad \text{[see (92)]}. \tag{93}$$

As usual, we assume that K_φ is finite (or otherwise, we already have an infinity of nontrivial solutions which are in $C^1(\overline{\Omega})$ by the nonlinear regularity theory). So, choosing $\rho_0 < -\frac{c_{25}}{p}$ even more negative, we have

$$C_k(\varphi, \infty) = H_k(W^{1,p}(\Omega), \varphi^{\rho_0}) \quad \text{for all } k \in \mathbb{N}_0$$
$$\Rightarrow \quad C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

$\square$

Now for the functional φ_p, we compute the critical groups at zero.

Proposition 11. *If hypotheses $H(\xi)$, $H(\beta)$, and H_3 hold and $\widehat{\lambda}_V > 0$, then $C_1(\varphi_p, 0) \neq 0$.*

Proof. We consider the direct sum decomposition,

$$W^{1,p}(\Omega) = \mathbb{R}\widehat{u}_1 \oplus V \quad [\text{see } (79)]. \tag{94}$$

Recall that $\widehat{u}_1 \in D_+$. So, for $|t| \leq 1$ small, we have

$$|t\widehat{u}_1(z)| \leq \delta_0, \quad \text{and} \quad |t\nabla \widehat{u}_1(z)| \leq \delta_0 \quad \text{for all } z \in \overline{\Omega}. \tag{95}$$

Then, using (95) and hypothesis H_3(iv), we have

$$\varphi_p(t\widehat{u}_1) \leq \frac{t^p}{p} [\mu_p(\widehat{u}_1) - \widehat{\lambda}_1] = 0 \quad (\text{recall } \|\widehat{u}_1\|_p = 1). $$

So, we can find $\rho_1 > 0$ such that

$$\varphi_p\big|_{\mathbb{R}\widehat{u}_1 \cap \partial \overline{B}_{\rho_1}} \leq 0 \tag{96}$$

with $\overline{B}_{\rho_1} = \{u \in W^{1,p}(\Omega) : \|u\| \leq \rho_1\}$. From hypotheses H_3(i), (iv), we have

$$F(z, x) \leq \frac{\eta}{p} |x|^p + c_{26} |x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{26} > 0. \tag{97}$$

Then, for $u \in V$, we have

$$\varphi_p(u) \geq \frac{1}{p} \big[\mu_p(u) - \eta \|u\|_p^p\big] - c_{27} \|u\|^r$$

$$\text{for some } c_{27} > 0 \ [\text{see } (97)]$$

$$\geq \frac{1}{p}\bigg[1 - \frac{\eta}{\widehat{\lambda}_V}\bigg] \mu_p(u) - c_{27} \|u\|^r \ [\text{see } (80)]$$

$$\geq c_{28} \|u\|^p - c_{27} \|u\|^r \text{ for some } c_{28} > 0$$

$$(\text{since } \eta < \widehat{\lambda}_V \text{ and } \widehat{\lambda}_V > 0).$$

Because $r > p$, choosing $\rho_2 \in (0, 1)$ small, we have

$$\varphi_p(u) > 0 \quad \text{for all } 0 < \|u\| \leq \rho_2, u \in V. \tag{98}$$

From (96) and (98), it follows that φ_p has a local linking at the origin with respect to the decomposition (94). So, from Ref. 27, p. 171, we have $C_1(\varphi_p, 0) \neq 0$. $\square$

Now we are ready for the existence theorem for problem $(1)_P$.

Theorem 5. *If hypotheses $H(\xi)$, $H(\beta)$, and H_3 hold and $\widehat{\lambda}_V > 0$, then problem $(1)_P$ admits a nontrivial solution $\widehat{u} \in C^1(\overline{\Omega})$.*

Proof. From Proposition 11, we have that

$$C_1(\varphi_p, 0) \neq 0. \tag{99}$$

Also, from Proposition 10, we know that

$$C_k(\varphi_p, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{100}$$

From (99) and (100), we infer that there exists $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_p}, \ \widehat{u} \neq 0 \quad (\text{see Proposition 4})$$
$$\Rightarrow \quad \widehat{u} \text{ is a nontrivial solution of } (1)_P.$$

The nonlinear regularity theory implies that $\widehat{u} \in C^1(\overline{\Omega})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We conclude with an existence result concerning the following nonlinear parametric Robin problem:

$$(1)_P^{\lambda} \quad \begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda f_0(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \ u \geq 0. \end{cases}$$

We impose the following conditions on the data of $(1)_P^{\lambda}$:

H_0: $\xi \in L^{\infty}(\Omega)$, $\beta \in C^{0,\alpha}(\partial(\Omega))$ with $\alpha \in (0, 1)$, $\xi \geq 0$, $\beta \geq 0$ and at least one of them is nontrivial.

H_4: $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f_0(z, 0) = 0$ for a.a. $z \in \Omega$, and hypotheses H_4(i)–(iii) are the same as the corresponding hypotheses H_1(i)–(iii) and

(iv) $\quad \lim\limits_{x \to 0^+} \frac{f_0(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 8. *Hypothesis H_0 implies that $\widehat{\lambda}_1 > 0$. Then, hypothesis $H_4(iv)$ is a special case of hypothesis $H_1(iv)$.*
Invoking Theorem 4, we have the following result concerning problem $(1)_P^{\lambda}$:

Theorem 6. *If hypotheses H_0 and H_4 hold, then for every $\lambda > 0$ problem $(1)_P^{\lambda}$ admits a positive solution $\widehat{u}_{\lambda} \in D_+$.*

Remark 9. *For $p = 2$ (semilinear problem) and Dirichlet boundary condition, with stronger hypotheses on the data, such an existence result is the main theorem in the work of Miyagaki-Souto.[26] Their method of proof is more involved. Semilinear Robin problems were studied in the recent work of Hu-Papageorgiou.[19]*

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