QUOTIENTS OF HYPERSURFACES IN WEIGHTED PROJECTIVE SPACE

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ABSTRACT. In [1] some quotients of one-parameter families of Calabi-Yau varieties are related to the family of Mirror Quintics by using a construction due to Shioda. In this paper, we generalize this construction to a wider class of varieties. More specifically, let A be an invertible matrix with non-negative integer entries. We introduce varieties X_A and $\overline{M}_A$ in weighted projective space and in $\mathbb{P}^n$, respectively. The variety $\overline{M}_A$ turns out to be a quotient of a Fermat variety by a finite group. As a by-product, X_A is a quotient of a Fermat variety and $\overline{M}_A$ is a quotient of X_A by a finite group. We apply this construction to some families of Calabi-Yau manifolds in order to show their birationality.

1. INTRODUCTION

Hypersurfaces in weighted projective space have been investigated by many authors in connection with Physics, in particular Mirror Symmetry: see, for instance, [3], [4] and [21]. The usual quintic threefold is an example of Calabi-Yau manifold in ordinary projective space. Its mirror can be described in terms of quotients of a one-parameter family of quintics, the Dwork pencil. In [1], we investigated other one-parameter families of Calabi-Yau manifolds and related them to the family of Mirror quintics. Our main tool was a construction due to Shioda [19], which clarified previous work in [8].

In the present paper we generalize this construction to hypersurfaces in weighted projective space. Let A be an invertible matrix of size n with non-negative integer entries. For such a matrix we define a weighted homogeneous polynomial F_A, a sum of n monomials. Its zero locus gives a projective scheme X_A in weighted projective space $W\mathbb{P}^{n-1}(q_1, \ldots, q_n)$. The weights are determined by the relation $A\mathbf{q} = d\mathbf{e}$, where $\mathbf{q} = (q_1, \ldots, q_n)$, $\mathbf{e} = (1, 1, \ldots, 1)$ and d is the smallest positive integer such that dA^{-1} has integer entries. If the total degree of F_A equals the degree of the anticanonical bundle and the singularities of X_A are canonical, then X_A is a Calabi-Yau manifold as explained in Section 2. It is therefore interesting to study the quotients of X_A in relation with the mirror varieties of X_A. For this purpose, we introduce a manifold $\overline{M}_A \subset \mathbb{P}^n$, which we refer to as the Shioda quotient. It is the image of a suitable map (first introduced by Shioda in [19] and applied to a similar context in [1]) of a Fermat variety. Actually, we prove that the Shioda quotient is the quotient of a Fermat variety by a finite group. As a by-product, it is also the quotient of X_A by a finite group. We describe these groups in detail.

It is natural to investigate deformation families of X_A defined by $F_{A,t} = F_A - tx_1 \ldots x_n$, where x_i are variables of degree q_i. They are particularly meaningful

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when the Hodge number $h^{2,1}$ is one, since then $F_{A,t}$ gives a versal deformation. Some examples of deformation families of X_A when X_A is a Calabi-Yau manifold in weighted projective space are given in [20]. Their mirror all have $h^{2,1} = 1$. We investigate the birational classes associated to these families and prove they are birational to the mirror families of the first four hypergeometric Calabi-Yau families studied by Rodriguez Villegas [18] and listed in [15], p. 134.

2. Preliminaries

2.1. **Weighted projective spaces.** We briefly recall the definition of weighted projective space. Let $(q_1, \ldots, q_n)$ be a sequence of positive integers. As customary, set

$$W\mathbb{P}^{n-1}(q_1, \ldots, q_n) := (\mathbb{C}^n \setminus \{0\})/\sim,$$

where the equivalence relation is

$$(x_1, \ldots, x_n) \sim (\lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n)$$

whenever $\lambda \in \mathbb{C} \setminus \{0\}$. Recall that

$$W\mathbb{P}^{n-1}(q_1, \ldots, q_n) \cong W\mathbb{P}^{n-1}(q_1/a_1, \ldots, q_n/a_n),$$

where $a_i = l.c.m.(d_1, \ldots, \widehat{d_i}, \ldots, d_n)$ and $d_i = g.c.d.(q_1, \ldots, \widehat{q_i}, \ldots, q_n)$. Weighted projective space are almost always singular. As proved in [6], the singular locus of $W\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ can be described in the following way. Let p be a prime. Let $I(x) = \{j; x_j \neq 0\}$. Then define

$$Sing_p(W\mathbb{P}^{n-1}) := \{x \in W\mathbb{P}^{n-1}(q_1, \ldots, q_n) : p|q_i \text{ for any } i \in I(x)\}.$$

The singular locus of the weighted projective space is given by the union over all primes of $Sing_p(W\mathbb{P}^{n-1})$.

As explained, for instance, in [5], weighted projective space is a toric variety. Set $Q := \sum_i q_i$. We recall that the canonical sheaf of $W\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ is given by $\mathcal{O}(-Q)$, which is not always a line bundle. As proved in [5], Lemma 3.5.6., the canonical sheaf is a line bundle if and only if $q_i|Q$ for all $i = 1, \ldots, n$. Under this assumption, weighted projective space is a Fano toric variety.

Some of our hypersurfaces are Calabi-Yau varieties. Following [5], a (possibly singular) Calabi-Yau variety is an m-dimensional normal compact variety X which satisfies the following conditions:
(i) X has at most Gorenstein canonical singularities;
(ii) the dualizing sheaf of X is trivial;
(iii) $H^1(X, \mathcal{O}_X) = \ldots = H^{m-1}(X, \mathcal{O}_X) = 0$.

Let f be a weighted homogeneous polynomial such that the zero locus $\{f = 0\}$ is quasi-smooth (according to Definition 3.1.5 in [7]). Since a quasi-smooth scheme has finite quotient singularities ([7], Thm. 3.1.6), the locus $\{f = 0\}$ is normal and Cohen-Macaulay; furthermore, it is Gorenstein with dualizing sheaf $\mathcal{O}(d)$, where d is the weighted degree of f. Assume that $\{f = 0\}$ has at most canonical singularities. Following the arguments in Proposition 4.1.3 in [5], it follows that $\{f = 0\}$ is a Calabi-Yau variety when $d = Q$. Notice that if there exists a crepant resolution of $\{f = 0\}$, the singularities are canonical by definition.

2.2. **The Shioda maps.** Let A be an invertible matrix with non-negative integer entries:

$$A = (a_{ij}) \quad (\in M_n(\mathbb{Z})), \quad a_{ij} \in \mathbb{Z}_{\geq 0}, \quad det(A) \neq 0.$$

For such a matrix we define a polynomial in n variables, a sum of n monomials:

$$F_A := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}} = x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}} + x_1^{a_{21}} x_2^{a_{22}} \dots x_n^{a_{2n}} + \dots$$

Let d be the smallest positive integer such that $B := dA^{-1}$ is in $M_n(\mathbb{Z})$. Then set

$$\mathbf{q} := B\mathbf{e}, \tag{2.1}$$

where $\mathbf{e} = (1, \dots, 1)$ and $\mathbf{q} = (q_1, \dots, q_n)$. Clearly, this implies that

$$A\mathbf{q} = d\mathbf{e}. \tag{2.2}$$

The zero locus $X_A = Z(F_A)$ is a (not necessarily smooth or irreducible) projective variety X_A, which is contained in $W\mathbb{P}^{n-1}(q_1, \dots, q_n)$. Let m be the greatest common divisor of the q_i's. Define $a_i = q_i/m$. Thus we have

$$X_A \subset W\mathbb{P}^{n-1}(q_1, \dots, q_n) \cong W\mathbb{P}^{n-1}(a_1, \dots, a_n).$$

We have a rational map ϕ_A from $\mathbb{P}^{n-1}$ of degree d to X_A defined by:

$$\phi_A : \mathbb{P}^{n-1} \to W\mathbb{P}^{n-1}(q_1, \dots, q_n),$$
$$(y_1 : \dots : y_n) \to (x_1 : \dots : x_n), \quad x_j = \prod_{k=1}^{n} y_k^{b_{jk}}.$$

Notice that each y_j has degree one, so $deg(x_j) = \sum_k b_{jk} = q_j$. Hence ϕ_A is indeed a rational map from $\mathbb{P}^{n-1}$ to $W\mathbb{P}^{n-1}(q_1, \dots, q_n)$.

Assume further that $d = Q = \sum_j q_j$, so that the variety X_A gives a Calabi-Yau provided the singularities are canonical. We can read this condition on the coefficients of the matrix A^{-1}. In fact, we have:

$$d = Q = \sum_j q_j = {}^t\mathbf{e}\mathbf{q} = d\,{}^t\mathbf{e}A^{-1}\mathbf{e},$$

which gives

$${}^t\mathbf{e}A^{-1}\mathbf{e} = 1; \tag{2.3}$$

in other words, $\sum_{ij} a'_{ij} = 1$, where a'_{ij} are the entries of A^{-1}.

We define a rational map

$$q_A : W\mathbb{P}^{n-1}(q_1, \dots, q_n) \to \overline{Im(q_A)} := \overline{M}_A \subset \mathbb{P}^n$$

in the following way:

$$(x_1 : \dots : x_n) \to (u_0 : u_1 : \dots : u_n) := \left(\prod_{j=1}^{n} x_j : \prod_{j=1}^{n} x_j^{a_{1j}} : \dots : \prod_{j=1}^{n} x_j^{a_{nj}} \right).$$

Since $deg(x_i) = q_j$ and $A\mathbf{q} = d\mathbf{e}$, we have

$$deg(u_0) = \sum_j q_j = d, \qquad deg(u_k) = \sum_j a_{kj} q_j = d,$$

hence q_A is well-defined.

Finally, we describe the composition $q_A \circ \phi_A$, which will be used in the next section. First, as $BA = AB = dI_n$ (where I_n is the identity matrix) we have $\sum_j a_{lj} b_{jk} = d\delta_{lk}$, where δ_{kl} is the Kronecker delta. Second, we set

$$\mathbf{q}' := d\,{}^tA^{-1}\mathbf{e} = d\,{}^t\mathbf{e}A^{-1} = {}^t\mathbf{e}B;$$

so $q'_k = \sum_j b_{jk}$. This said, it is easy to check that the composition $q_A \circ \phi_A : X_d \subset \mathbb{P}^{n-1} \to \overline{M}_A \subset \mathbb{P}^n$ is given by

$$(u_0 : u_1 : \ldots : u_n) = \left(\prod_{k=1}^{n} y_k^{q'_k} : y_1^d : \ldots : y_n^d \right), \tag{2.4}$$

where X_d is the Fermat variety $\{\sum_{i=1}^n y_i^d = 0\}$ and $\overline{M}_A$ is the closure of the image of X_d.

Let us consider the projection

$$\begin{array}{rcc} \pi : \overline{M}_A \subset \mathbb{P}^n & \to & V \cong \mathbb{P}^n \\ (u_0 : u_1 : \ldots : u_n) & \to & (u_1 : \ldots : u_n), \end{array} \tag{2.5}$$

where V is the closure of $\pi(\overline{M}_A)$. It is easy to check that V is isomorphic to the $\mathbb{P}^{n-1}$ given by $\sum_{i=1}^n u_i = 0$ as the Fermat equation has to be satisfied.

From now on, we will assume that q'_k is strictly positive for any k. By direct inspection, we obtain the following equations for the image of the Fermat variety X_d under $q_A \circ \phi_A$:

$$u_1 + \ldots + u_n = 0, \quad u_0^d = u_1^{q'_1} \ldots u_n^{q'_n}. \tag{2.6}$$

Set $m' := g.c.d(d, q'_1, \ldots, q'_n)$. Hence $d = m'a'$ and $q'_k = a'_k m'$, so the composition $q_A \circ \phi_A$ is given by

$$(u_0 : \ldots : u_n) = \left(\prod_{k=1}^{n} y_k^{q'_k} : y_1^d : \ldots : y_n^d \right) = \left(\prod_{k=1}^{n} (y_k^{m'})^{a'_k} : (y_1^{m'})^{a'} : \ldots : (y_n^{m'})^{a'} \right). \tag{2.7}$$

By composing (2.7) with the map $t_k = y_k^{m'}$ for $k = 1, \ldots, n$, we get

$$(u_0 : \ldots : u_n) = \left(\prod_{k=1}^{n} t_k^{a'_k} : t_1^{a'} : \ldots : t_n^{a'} \right),$$

so the equations defining $\overline{M}_A$ are the following:

$$u_1 + \ldots + u_n = 0, \quad u_0^{a'} = u_1^{a'_1} \ldots u_n^{a'_n}. \tag{2.8}$$

In the next section, we will show that under our assumptions on the q_j's, the equation (2.8) define a very singular model in $(n-1)$-dimensional projective space for a manifold with $h^{n-2,0} = 1$ of degree $a' (> n$ in general).

3. The Shioda quotient

In this section we assume that $A \in M_n(\mathbb{Z}_{\geq 0})$ is an invertible matrix such that (2.3) holds. We will introduce "natural" automorphism groups and study the quotients by these groups.

3.1. **The automorphism groups.** Let $\zeta = \zeta_d$ be a generator of the cyclic group of d-th roots of unity, where d is the smallest positive integer such that dA^{-1} has integer entries. For $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}/d\mathbb{Z})^n$ we define an automorphism $g_{\mathbf{k}}$ of $\mathbb{P}^{n-1}$ by

$$g_{\mathbf{k}}(y_1 : \ldots : y_n) := \left(\zeta^{k_1} y_1 : \ldots : \zeta^{k_n} y_n\right).$$

Note that $\mathbf{a}, \mathbf{b} \in \mu_d^n$ define the same automorphism iff $\mathbf{a} - \mathbf{b} = (k, \ldots, k)$ for some $k \in \mu_d$. Define Γ_d to be the quotient group

$$\Gamma_d := \mu_d^n / \langle g_{(1,1,\ldots,1)} \rangle \ \left(\subset Aut(\mathbb{P}^{n-1})\right)$$

Notice that $\Gamma_d \cong \mu_d^{n-1}$, hence $\#\Gamma_d = d^{n-1}$. The group Γ_d is a subgroup of the automorphism group of the Fermat variety X_d.

The map $q_A \circ \phi_A$ (2.4) is invariant under the subgroup of Γ_d given by

$$\Gamma(\mathbf{q}') := \left\{ g_{\mathbf{k}} : \mathbf{k} = (k_1, \ldots, k_n); \sum_j k_j q'_j \equiv 0 \bmod d \right\} \Big/ \langle g_{(1,\ldots,1)} \rangle.$$

In other words, we have

$$(q_A \circ \phi_A)(g_{\mathbf{k}}(y_1 : \ldots : y_n)) = (q_A \circ \phi_A)(y_1 : \ldots : y_n)$$

for all $g_{\mathbf{k}}$ in $\Gamma(\mathbf{q}')$ and all $(y_1, \ldots, y_n) \in X_d$. Notice that $g_{(1,\ldots,1)}$ is an element of $\Gamma(\mathbf{q}')$. In fact, by (2.3):

$$\sum_j q'_j = {}^t\mathbf{e}\mathbf{q}' = d\,{}^t\mathbf{e}\,{}^tA^{-1}\mathbf{e} = d.$$

The coordinate functions of the Shioda map ϕ_A are products of the y_i. If $\phi_A(y_1 : \ldots : y_n) = (x_1 : \ldots : x_n)$, then

$$\phi_A(g_{\mathbf{k}}(y_1 : \ldots : y_n)) = \left(\zeta^{k'_1} x_1 : \ldots : \zeta^{k'_n} x_n\right).$$

As $x_j = \prod y_k^{b_{jk}}$, the column vector $\mathbf{k}'$ is obtained from the column vector $\mathbf{k}$ as $\mathbf{k}' = B\mathbf{k}$. Thus we get a homomorphism

$$\Gamma(\mathbf{q}') \to Aut(X_A), \quad g_{\mathbf{k}} \to g_{B\mathbf{k}},$$

which is well defined since $B\mathbf{e} \equiv 0 \bmod d$.

The kernel (image resp.) of this homomorphism will be denoted by Γ_A (H_A resp.). Notice that Γ_A is the subgroup of $\Gamma(\mathbf{q}')$, which is generated by the images of the $g_{\mathbf{k}}$ such that $B\mathbf{k} \equiv 0 \bmod d$.

3.2. **The birational model.** We recall that two rational maps between algebraic varieties $f_i : X \to Y_i$ for $i = 1, 2$ are said to be birationally equivalent if there is a Zariski open subset U of X and there are Zariski open subsets $U_i \subset Y_i$ with an isomorphism $\phi : U_1 \to U_2$ such that $\phi \circ f_1 = f_2$ on U.

Theorem 3.1. *Let A be an invertible $n \times n$ matrix with integer entries such that X_A is irreducible and* (2.3) *holds. Then the composition $q_A \circ \phi_A$ is birational to the quotient map $X_d \to X_d/\Gamma(\mathbf{q}')$; hence $X_d/\Gamma(\mathbf{q}')$ is birational to $\overline{M}_A$.*

Proof. The composition $q_A \circ \phi_A$ is given by (2.4). Also, recall the map π defined in (2.5). The composition of $q_A \circ \phi_A$ and π yields a map from the Fermat variety X_d to $V \cong \mathbb{P}^{n-1}$, which corresponds to an abelian extension with group Γ_d of function fields - recall that X_d is the Fermat variety and $u_i = y_i^d$. This means that $X_d \to V$ is the quotient for the group Γ_d, namely $X_d/\Gamma_d = V$. Therefore, by abelian Galois theory, each subfield is obtained as an invariant field under a finite subgroup of Γ_d. Thus, the map $X_d \to \overline{M}_A$ corresponds to a quotient by a finite subgroup of Γ_d. Now, we show that this subgroup is isomorphic to $\Gamma(\mathbf{q}')$. The map π corresponds to an abelian extension of function fields with group $\mathbb{Z}/a'\mathbb{Z}$, where $d = m'a'$ and $m' = g.c.d.(d, q'_1, \ldots, q'_n)$. The group Γ_d acts on $\overline{M}_A$ only through the variable u_0, and $g_{\mathbf{k}} : u_0 \to \zeta_{m'}^{\mathbf{q}'\cdot\mathbf{k}} u_0$, where $\zeta_{m'} = \zeta^{a'}$ is a primitive m'-th root of unity. The kernel of this action is exactly $\Gamma(\mathbf{q}')$, hence the map $X_d \to \overline{M}_A$ corresponds to an extension of function fields with group $\Gamma(\mathbf{q}')$. Hence, the claim follows. □

By the Theorem above, the order of the group $\Gamma(\mathbf{q}')$ is $d^{n-2}m'$.

Corollary 3.2. *The maps $\phi_A : X_d \to X_A$ and $q_A : X_A \to \overline{M}_A$ are birational to quotient maps. In particular, X_A ($\overline{M}_A$ resp.) is birational to X_d/Γ_A (X_A/H_A resp.).*

Proof. By Theorem 3.1, the composition of ϕ_A and q_A is a quotient map, namely:

$$X_d \to X_A \to \overline{M}_A \approx X_d/\Gamma(\mathbf{q}').$$

The proof follows easily from arguments similar to those in Theorem 2.6 in [1]. □

Now, we prove that the equations (2.6) and (2.8) give a very singular model for a manifold with $h^{n-2,0} = 1$. Assume, $q'_i > 0$. We recall that the vector space of holomorphic $(n-2)$-forms on a smooth hypersurface $X = Z(F)$ of degree $d \geq n$ in $\mathbb{P}^{n-1}$ has a basis of the form

$$\omega_{\mathbf{b},F} = Res_X \left(y_1^{b_1} \ldots y_n^{b_n} \frac{\sum_i^n (-1)^i y_i dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_n}{F} \right),$$

where $\mathbf{b} = (b_1, \ldots, b_n)$ and $b_i \in \mathbb{Z}_{\geq 0}$, $\sum_i b_i = d - n$.

Proposition 3.3. *There exists a unique holomorphic $(n-2)$ form on any resolution of $\overline{M}_A$.*

Proof. The action of an element $g_{\mathbf{k}}$ in $\Gamma(\mathbf{q}')$ on $\omega_{\mathbf{b},F}$ is given by $\zeta^{\sum_i (b_i+1)k_i}$, where $b_i \in \mathbb{Z}_{\geq 0}$. A form is invariant with respect to $\Gamma(\mathbf{q}')$ if and only if $b_i + 1 = q'_i$ for all i. The unique invariant form $\omega_{\mathbf{b},F}$ with vector $\mathbf{b} = \mathbf{q}' - \mathbf{e}$ descends to a form on the quotient $X_d/\Gamma(\mathbf{q}') \approx \overline{M}_A$. □

3.3. **Some examples.** *Example A.* Let us consider $F_A := x_1^5 + x_2^{10} + x_3^{10} + x_4^{10} + x_5^2 = 0$ in weighted projective space $W\mathbb{P}^4(2,1,1,1,5)$. It is easy to check that the corresponding hypersurface is smooth and does not intersect the singularities of $W\mathbb{P}^4(2,1,1,1,5)$, which are two isolated points.

The matrix A is given by $diag(5, 10, 10, 10, 2)$. The matrix B is $diag(2, 1, 1, 1, 5)$ and $d = 10$. The condition ${}^t\mathbf{e}A^{-1}\mathbf{e} = 1$ is satisfied. Moreover,

$$\mathbf{q} = (2, 1, 1, 1, 5), \quad \mathbf{q}' = (2, 1, 1, 1, 5)$$

so $q'_i > 0$ for any $i = 1, \ldots, 5$. The equations cutting out $\overline{M}_A$ in $\mathbb{P}^5$ are given by

$$u_1 + u_2 + u_3 + u_4 + u_5 = 0,$$

$$u_0^{10} = u_1^2 u_2 u_3 u_4 u_5^5.$$

The integer $d = 10$. Generators for the groups $\Gamma(\mathbf{q}')$, Γ_A and H_A are as follows. Consider the elements $\mathbf{v_1}, \ldots, \mathbf{v_4} \in (\mathbb{Z}/10\mathbb{Z})^5$ given by:

$$\mathbf{v}_1 = (0,0,0,5,1), \qquad \mathbf{v}_2 = (0,0,1,4,1),$$

$$\mathbf{v}_3 = (1,0,0,3,1), \qquad \mathbf{v}_4 = (0,1,0,4,1),$$

then ${}^t\mathbf{q}' \cdot \mathbf{v}_i \equiv 0 \bmod 10$ for $i = 1, \ldots, 4$. Moreover, we have $\mathbf{e} = 8\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$; hence

$$\Gamma(\mathbf{q}') = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \cong \mu_{10}^3.$$

The group Γ_A is isomorphic to μ_{10} and is generated by $(5,0,0,0,6)$. Finally, set

$$\mathbf{w}_1 = (0,0,1,4,5), \quad \mathbf{w}_2 = (2,0,0,3,5), \quad \mathbf{w}_3 = (0,1,0,4,5).$$

As $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = \mathbf{q}$, we have

$$H_A = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle \cong \mu_{10}^2 \subset Aut(X_A).$$

The isomorphism $H_A \cong \mu_{10}^2$ was first suggested in [11]. Finally, as mentioned in [16], notice that $h^{2,1}(X_A) = 1$.

Example B. Let us consider the equation $F_A := x_1^{15}x_5 + x_2^5 + x_3^5 + x_3x_4^5 + x_2x_4^2 = 0$ in weighted projective space $W\mathbb{P}^4(1,5,5,4,10)$. In this case, we do not have a Fano toric veariety since 4 does not divide $25 = 1+5+5+4+10$. The zero locus does not intersect the singularities of $W\mathbb{P}^4(1,5,5,4,10)$ and F_A is a smooth variety. The matrices A and B are given by

$$A := \begin{pmatrix} 15 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 10 & 1 & 0 & 0 & -5 \\ 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & -6 & 30 & 0 \\ 0 & -15 & 0 & 0 & 75 \end{pmatrix}.$$

The integer d equals 150. Moreover, we have

$$\mathbf{q} = 6(1,5,5,4,10), \qquad \mathbf{q}' = (10,16,24,30,70)$$

and $a' = g.c.d.(d, q_1', \ldots, q_5') = 2$. Notice that $q_i' > 0$ for any $i = 1, \ldots, 5$.

A birational model of $\overline{M}_A$ is cut out by the equations

$$u_0^{75} = u_1^5 u_2^8 u_3^{12} u_4^{15} u_5^{35}, \quad u_1 + \ldots + u_5 = 0. \tag{3.1}$$

Consider the vectors $\mathbf{r_i} \in \mathbb{Z}/d\mathbb{Z}$ given by:

$$\mathbf{r}_1 = (0,0,75,0,0), \qquad \mathbf{r}_2 = (0,1,1,0,8),$$

$$\mathbf{r}_3 = (1,0,0,0,2), \quad \mathbf{r}_4 = (0,0,0,1,6), \quad \mathbf{r}_5 = (0,0,5,0,9);$$

then $9\mathbf{e} = \mathbf{r}_1 + 9\mathbf{r}_2 + 9\mathbf{r}_3 + 9\mathbf{r}_4 - 15\mathbf{r}_5$. The vectors $\mathbf{r}_i$ generate $\Gamma(\mathbf{q}')$; in fact, the following holds:

$$\Gamma(\mathbf{q}') \cong \langle \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5 \rangle / < \mathbf{r}_1 + 9\mathbf{r}_2 + 9\mathbf{r}_3 + 9\mathbf{r}_4 - 15\mathbf{r}_5 > \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/150\mathbb{Z})^3$$

By using Magma, it is possible to check that

$$\Gamma_A \cong (\mathbb{Z}/150\mathbb{Z})^3,$$

$$H_A \cong \mathbb{Z}/2\mathbb{Z}.$$

The group H_A is generated by $g_{(0,75,75,0,75)}$, which maps $(x_1 : \ldots : x_5)$ to $(x_1 : -x_2 : -x_3 : x_4 : -x_5)$. Recall that $X_A \to \overline{M}_A \approx X_A/H_A$ is a double cover.

Example C. Consider the weighted homogeneous polynomial

$$F_A := x_1^2 + x_2^3 + x_3^{18} + x_4^{18} + x_5^{18}$$

. It gives a quasi-smooth locus in weighted projective space $W\mathbb{P}^4(9,6,1,1,1)$. The matrices A and B are given by $diag(2,3,18,18,18)$ and $diag(9,6,1,1,1)$, respectively.

The integer d is equal to 18 and we have

$$\mathbf{q} = \mathbf{q}' = (9,6,1,1,1).$$

Notice that $q_i' > 0$ for any i. Moreover, using Magma we found that

$$\Gamma(\mathbf{q}) \cong (\mathbb{Z}/18\mathbb{Z})^3\,;$$
$$H_A \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z};$$
$$\Gamma_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}.$$

The Calabi-Yau X_A has a singularity at $[-1,1,0,0,0]$, which is a singularity of $W\mathbb{P}^4(9,6,1,1,1)$. As explained in [13], this singularity can be blown-up so as to get a Calabi-Yau in the (toric) blow-up of $W\mathbb{P}^4(9,6,1,1,1)$.

Example D. When the group H_A is trivial, it is possible to write down an explicit birational inverse between from $\overline{M}_A$ to X_A. We show it in one specific example. Let us consider the polynomial

$$F_A := x_1^5 + x_2^9x_3 + x_3^9x_4 + x_4^{10} + x_5^2$$

where A is the matrix

$$A := \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \\ 0 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The variety $X_A := Z(F_A)$ is contained in $W\mathbb{P}^4(2,1,1,1,5)$. The matrix B is given by

$$B := \begin{pmatrix} 162 & 0 & 0 & 0 & 0 \\ 0 & 90 & -10 & 1 & 0 \\ 0 & 0 & 90 & -9 & 0 \\ 0 & 0 & 0 & 81 & 0 \\ 0 & 0 & 0 & 0 & 405 \end{pmatrix},$$

where $AB = BA = 810I$. The map $q_A : W\mathbb{P}^4(2,1,1,1,5) \to \mathbb{P}^5$

$$(x_1 : \ldots : x_5) \to (u_0 : \ldots : u_5) := (x_1x_2\ldots x_5 : x_1^5 : x_2^9x_3 : x_3^9x_4 : x_4^{10}x_2 : x_5^2)$$

maps X_A to the variety

$$\overline{M}_A := Z(u_1 + \ldots + u_5, -u^{810} + u_1^{162}u_2^{90}u_3^{80}u_4^{73}u_5^{405}) \subset \mathbb{P}^5.$$

An explicit birational inverse for q_A is given by the following map:

$$\begin{cases} M^{162}x_1 = u_1^{65}u_2^{54}u_3^{12}u_4^{15}u_5^{162}, \\ M^{81}x_2 = u_1^{-2}u_2^{8}u_3^{-11}u_4^{-8}u_5^{-5}, \\ M^{81}x_3 = u_1^{18}u_2^{19}u_3^{-1}u_4u_5^{45}, \\ M^{81}x_4 = u_2^{9}u_3^{-10}u_4^{-7}, \\ M^{405}x_5 = u_1^{81}u_2^{90}u_3^{-10}u_4u_5^{203}, \end{cases}$$

where $M = x_1^2 x_2^3 x_3 x_4 x_5^2$.

4. A one-dimensional family

Let us consider the one-parameter family $\mathcal{X} \to \mathbb{P}^1_t$ of degree d hypersurfaces in $\mathbb{P}^{n-1}$ with $\mathcal{X}_t = X_{d,t} = Z(F_{d,t})$, where

$$F_{d,t} = \sum_i^n y_i^d - t\, y_1^{q_1'} \dots y_n^{q_n'}. \tag{4.1}$$

Clearly, this is a one-dimensional deformation of the Fermat variety X_d. If we apply the map $\phi_A : X_{d,t} \subset \mathbb{P}^{n-1} \to X_{A,t} \subset W\mathbb{P}(q_1, \dots, q_n)$, the image of $X_{d,t}$ is given by $X_{A,t} = Z(F_{A,t})$, where

$$F_{A,t} = F_A - t\, x_1 x_2 \dots x_n.$$

Under the composition $q_A \circ \phi_A$ the image of (4.1) is given by the equations

$$\sum_i u_i - t u_0 = 0, \qquad u_0^d = u_1^{q_1'} \dots u_n^{q_n'}. \tag{4.2}$$

If $t \neq 0$, we solve for u_0 and get the equation

$$\left(\sum_i^n u_i \right)^d = t^d u_1^{q_1'} \dots u_n^{q_n'}. \tag{4.3}$$

The group $\Gamma(\mathbf{q}')$ acts on each $X_{A,t}$ since $\sum_i q_i'$ equals d. Denote by $\overline{\mathcal{M}} \to \mathbb{P}^1_t$ the family given by (4.3). By the universal property of the quotient there exists a map Ψ between $\mathcal{X}/\Gamma(\mathbf{q}')$ and $\overline{\mathcal{M}}$, which commutes with the projection map on $\mathbb{P}^1_t$.

Proposition 4.1. *The map Ψ yields a birational morphism from $\mathcal{X}/\Gamma(\mathbf{q}')$ to $\overline{\mathcal{M}}_t$.*

Proof. It suffices to compare the degree of the quotient map $\mathcal{X} \to \mathcal{X}/\Gamma(\mathbf{q}')$, which is $\#\Gamma(\mathbf{q}')$, with that of the map $\mathcal{X} \to \overline{\mathcal{M}}$. Let

$$(l_0 : l_1 \dots : l_n) = (\prod_k^n y_k^{q_k'} : y_1^d : \dots y_n^d)$$

be a generic point in $\overline{Im(q_A \circ \phi_A)}$. Thus, we have $y_j = \zeta^{k_j} \sqrt[d]{l_j}$, where ζ is a primitive d-th root of unity. Hence, we get

$$\zeta^{\sum_j q_j' k_j} \sqrt[d]{l_1^{q_1'} \dots l_n^{q_n'}} = l_0.$$

On the other hand, by the equation of $F_{A,t}$, we must have

$$\sum_j q_j' k_j \equiv 0 \bmod d. \tag{4.4}$$

Recall that $\sum_j q_j' \equiv 0 \bmod d$, so we can take the quotient of the set (4.4) of solutions by $(1, 1, \dots, 1)$. The degree of $\mathcal{X} \to \overline{\mathcal{M}}$ is thus equal to $\#\Gamma(\mathbf{q}')$.

$\square$

4.1. **Some Birational Families.** Let us examine some Calabi-Yau varieties X_A which have a one-dimensional versal deformation family, so it may be described by the family in the section above. In Schimmrigk's list (see [20]), we found twelve entries with $h^{2,1} = 1$. Since the deformation space is one dimensional, we take into account one-dimensional families corresponding to these entries of the list. The generic members of the families, which are denoted by the same letter, have the same Euler characteristic. We have

$$\mathcal{A}_1(t) = x_1^8x_3 + x_2x_3^7 + x_2^7x_4 + x_4^7 + x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(75, 84, 86, 98, 343),$$
$$\mathcal{A}_2(t) = x_1^8x_2 + x_2^7x_3 + x_3^7 + x_4^4 + x_4x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(43, 48, 56, 98, 147),$$
$$\mathcal{A}_3(t) = x_2^8 + x_1^7x_3 + x_3^7 + x_1x_4^4 + x_4x_5^2 - tx_1x_2x_3x_4x_5, \subset \mathbb{A}^1 \times W\mathbb{P}^4(48, 49, 56, 86, 153)$$
$$\mathcal{A}_4(t) = x_1x_2^7 + x_1^7x_3 + x_3^7 + x_2x_4^4 + x_4x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(42, 43, 49, 75, 134),$$

$$\mathcal{B}_1(t) = x_1^{10}x_2 + x_2^9x_3 + x_3^9 + x_4^5 + x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(73, 80, 90, 162, 405),$$
$$\mathcal{B}_2(t) = x_1^9x_2 + x_2^9 + x_1x_3^8 + x_3x_4^5 + x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(64, 72, 73, 115, 324),$$
$$\mathcal{B}_3(t) = x_1^9x_2 + x_2^9 + x_1x_3^5 + x_4^5 + x_3x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(40, 45, 73, 81, 166),$$
$$\mathcal{B}_4(t) = x_1^9 + x_2^8 + x_2x_3^5 + x_1x_4^5 + x_4x_5^2 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(40, 45, 63, 64, 148),$$

$$\mathcal{C}_1(t) = x_1^6x_3 + x_2x_3^5x_2^5 + x_4 + x_4^5 + x_5^3 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(52, 60, 63, 75, 125),$$
$$\mathcal{C}_2(t) = x_2^6 + x_1^5x_3 + x_3^5 + x_1x_4^4 + x_4x_5^3 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(48, 50, 60, 63, 79),$$

$$\mathcal{D}_1(t) = x_1^5x_3 + x_3^4x_4 + x_2x_4^4 + x_2^4x_5 + x_5^4 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(41, 48, 51, 52, 64),$$
$$\mathcal{D}_2(t) = x_3^5 + x_1^5x_4 + x_2x_4^4 + x_2^4x_5 + x_5^4 - tx_1x_2x_3x_4x_5 \subset \mathbb{A}^1 \times W\mathbb{P}^4(51, 60, 64, 65, 80).$$

Let $V_5(t), V_6(t), V_8(t), V_{10}(t)$ be the four hypergeometric families on page 134 in [15]. For each of these families $V_j(t)$, $j = 5, 6, 8, 10$, there is a group acting on the family such that the mirror $W_j(t)$ of $V_j(t)$ can be described as a resolution of the quotient $V_j(t)/G$. The singular members have one orbit of ordinary nodes under the action of G and the resolution of the quotient is a rigid Calabi-Yau threefold, i.e., $h^{2,1} = 0$.

Theorem 4.2. *The following birational equivalences hold:*

$$\mathcal{A}_1(t) \approx \mathcal{A}_2(t) \approx \mathcal{A}_3(t) \approx \mathcal{A}_4(t) \approx W_8(t),$$
$$\mathcal{B}_1(t) \approx \mathcal{B}_2(t) \approx \mathcal{B}_3(t) \approx \mathcal{B}_4(t) \approx W_{10}(t),$$
$$\mathcal{C}_1(t) \approx \mathcal{C}_2(t) \approx W_6(t),$$
$$\mathcal{D}_1(t) \approx \mathcal{D}_2(t) \approx W_5(t).$$

Proof. It is easy to check that the general member of the families above is a singular Calabi-Yau in four weighted projective space. The singular locus is a rational curve. For some values of t there are extra singularities that are ordinary nodes, namely:

$\mathcal{A}_i(t)$	$t^8 = 2^{16}$	$\mathcal{B}_i(t)$	$t^{10} = 800000$
$\mathcal{C}_i(t)$	$t^6 = 3^6 2^4$	$\mathcal{D}_i(t)$	$t^5 = 5^5$

Let us focus on the two families $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$. The proof for the other cases can be dealt with analogously. Define the two families

$$X_{d_1,t} = \left\{ \sum_{i=1}^{5} y_i^{320} - \prod_i y_i^{64} = 0 \right\},$$

$$X_{d_2,t} = \left\{ \sum_{i=1}^{5} y_i^{1280} - \prod_i y_i^{256} = 0 \right\}.$$

Let A_1 and A_2 be the following matrices:

$$A_1 := \begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Using Magma we found that the groups H_{A_1} and H_{A_2} are trivial - this happens for all 12 families.

We thus have

$$q_{A_1} \circ \phi_{A_1} : X_{d_1,t} \to \overline{M}_{A_1,t}$$
$$(y_1 : y_2 : \ldots : y_5) \to \left(\prod_i y_i^{64} : y_1^{320} : \ldots : y_5^{320} \right)$$

and, similarly,

$$q_{A_2} \circ \phi_{A_2} : X_{d_2,t} \to \overline{M}_{A_2,t}$$
$$(y_1 : y_2 : \ldots : y_5) \to \left(\prod_i y_i^{256} : y_1^{1280} : \ldots : y_5^{1280} \right)$$

It is easy to check that $\overline{M}_{A_1,t} \cong \overline{M}_{A_2,t} \cong \left\{ \sum_i u_i - tu_0,\, u_0^5 = u_1 \ldots u_5 \right\} \approx W_5(t)$.

Since H_{A_1} and H_{A_2} are trivial, $\mathcal{D}_1(t)$ and $\mathcal{D}_1(t)$ are birational since they are both birational to $W_5(t)$.

□

4.2. **Picard-Fuchs equations.** When X_A is a Calabi-Yau hypersurface, the Hodge number $h^{2,1}(X_A)$ gives the number of independent parameters of deformations of X_A. There exists a system of partial differential equations, the so called GKZ-hypergeometric system (see [9]), which yield Picard-Fuchs equations for the variation of periods along families with central fiber X_A. When $h^{2,1}(X_A) = 1$, the Picard-Fuchs equation can also be found via a generalization of the Griffiths-Dwork method for hypersurfaces in weighted projective space: see, for instance, [16].

5. $\overline{M}_A$ and the mirror family of Calabi-Yau hypersurfaces in weighted projective space

Assume X_A is a Calabi-Yau manifold (as defined in Section 2) in weighted projective space $W\mathbb{P}^{n-1}(q_1, \ldots, q_n)$, where $q_i | Q$ and $Q = \sum_j q_j$. Batyrev's mirror construction (see, for instance, [12]) depends only on the polytope Δ associated to the toric variety $W\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ and not on the matrix A. As explained, for instance in [12], the Calabi-Yau varieties in the mirror family $\mathcal{W} \to \mathbb{P}^1_x$ of a general section of the anticanonical bundle $\mathcal{O}(Q)$, with $Q = \sum_j q_j$, can be represented as compactifications of complete intersections of the affine hypersurfaces in $(\mathbb{C}^*)^n$ given by

$$t_1 + \ldots + t_n = 1, \qquad t_1^{q_1} \ldots t_n^{q_n} = x. \tag{5.1}$$

Let W_x be the fiber over the point $x \in \mathbb{P}^1$. By comparing (4.2) and (5.1), the following holds.

Proposition 5.1. *The compactification of $\mathcal{W}_1$ is given by the equations* (4.2) *that define the Shioda quotient $\overline{M}_{{}^tA,1}$ for any matrix A.*

Proof. Let A be a matrix as in Section 1. If we start from the family $F_{{}^tA,t}$, the equations (4.2) become:

$$\sum_i u_i - tu_0 = 0, \qquad u_0^d = u_1^{q_1} \ldots u_n^{q_n}. \tag{5.2}$$

Since $\sum_j q_j = d$, the claim follows. □

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