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MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

Index

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Information about CIEAEM 69 and presentation of the Volume
Informations sur la CIEAEM 69 et présentation du Volume

The 69th CIEAEM conference was held from 15th to 19th July 2017 at Freie Universität Berlin, Germany. It successfully involved 100 participants from 20 countries all over the world. CIEAEM 69 was dedicated to Professor Christine Keitel, president of CIEAEM from 1997 to 2003, who tragically passed away one year before the conference. The programme of the conference started with a panel that revisited "Mathematics (Education) and Common Sense", the theme of the 47th CIEAEM conference, which was held in Berlin in July 1995 and which was hosted by Christine.

At the conference, researchers, teachers, educators, and students met to discuss, in a collaborative and inspiring environment, the most prominent problems, obstacles and resources in mathematics education; they also presented their latest research findings in the several conference activities: plenary and semi-plenary talks, two round tables, working groups, workshops, and poster presentations (forum of ideas).

As in previous CIEAEM meetings, Working Groups constituted the beating heart of the conference, allowing the participants to fruitfully discuss in critical and constructive ways, in the true CIEAEM spirit, research studies and approaches from different perspectives on the conference theme: *Mathematisation: social process & didactic principle*. There were four Working Groups: *(A) Mathematisation as a didactic principle: mathematizing and modelling of everyday contexts*; *(B) Mathematisation as a didactic principle: representation and generalization within mathematics*; *(C) Interconnecting mathematisation as a social process and as a didactic principle*; and *(D) Mathematisation as a didactic principle: looking at teachers of mathematics*. Each Working Group discussed nine papers, and addressed the conference theme from complementary viewpoints (see the Discussion Paper), under the guidance of the group animators. The conference schedule allowed time also to deepen the plenary talks in the dedicated "Meet the plenary speaker" sessions, and to engage participants in workshops, where actual dialogue between research and practice could be fostered.

This volume contains the final versions of the 53 papers presented during the conference.

We thank all the contributors and the participants to the conference, because they made it such a unique experience, in which we had the good fortune to take part. We are grateful to the International Programme Committee and the Local Organizing Committee that made possible the realization of the conference in every detail with great care. Particularly, we want to thank the Working Group animators, who organized each day the sessions in inclusive as well high-quality ways. A special thanks to all the people who contributed to the realization of the conference, and to Daria Fischer, who helped in editing this volume.

As a result, the CIEAEM 69 Proceedings offer a wide overview on national and international studies on the conference theme *Mathematisation: social process & didactic principle.* We hope that it can constitute an inspiring resource for the research community, for teachers, and for stakeholders in mathematics education. From this perspective, the possibility of free downloading offers to CIEAEM 69 participants, and also to interested people who could not take part in the Conference in Berlin, the possibility of developing a fruitful network of contacts that year after year is becoming richer and wider.

November, 2017

Benedetto Di Paola
Uwe Gellert

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G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Discussion Paper

Math(é)matisation: social process and didactic principle / processus social and principe didactique

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Abstract / Resumée. The first part of the Discussion Paper is in English (Ch. 1 and 2), et la deuxième partie du document de discussion est en française (ch. 3 et 4).

1. Introduction (English)

The intention of CIEAEM 69 is to interrogate the concept of mathematisation which is commonly and undoubtedly accepted as a desirable outcome of formal mathematics education. One of the aims of the 69th CIEAEM conference is to make the mathematisation of social, economic, ecologic, etc. conditions explicit. The second aim of the 69th CIEAEM conference is to reflect on experience with curricular conceptions that pay particular attention to the relation of mathematical and everyday knowledge.

In this call for papers, mathematisation is used in its broadest sense. It may then include people's active use of some kind of mathematics, for example by interpreting notions (including mathematical objects) in the world mathematically, or by expressing one's ideas in a mathematical way. It may also include the way that people encounter mathematics as being used "on" them and their context, for example mathematics as being at the core of how a certain activity is described, or how decisions are made on a mathematically informed basis.

1.1 Making the mathematisation of social, economic, ecologic, etc. conditions explicit

Mathematisation —in its broad range— is a concept that has received CIEAEM's attention for more than half a century. We can trace the occupation of CIEAEM and its members back to 1954, when Servais describes the global changes of society that he expects in the following words:

> Our time marks the beginning of the mathematical era. [...] This fact, whatever the reactions, the opinions and the judgments it may provoke, increases the responsibility of every teacher, who, no matter on which level, teaches mathematics. [...] If it befits to be worthy of a mathematical tradition, it is also important to allow the mathematization [of the world] to come. As much as it is true that he [sic] who devotes his life to teaching, accepts a mission of a world gone-by to build a world being born. The responsibility towards the future is greater than loyalty towards the past. (Servais, 1954, p. 89; quoted in Vanpaemel, De Bock, & Verschaffel, 2011)

This statement is informed by the prevailing optimism that by basing social and technological development on a mathematical tradition the future would be more prosperous than the past. Indeed, as Davis and Hersh show thoroughly 30 years later, "the social and physical worlds are being mathematized at an increasing rate" (1986, p. xv). The extent of the ongoing mathematisation makes Davis and Hersh warn us that "we'd better watch it, because too much of it may not be good for us" (ibid.). Keitel, Kotzmann and Skovsmose substantiate this warning by describing a circular process:

> On the one side society becomes formalized and mathematized by the influence of the self-produced technological environment and economic structures respectively; on the other, mathematics is "naturally" a magnificent help in dealing with technological and quantified surroundings. Society, therefore, needs more and more techno-mathematical help. In this process, many structures of human activity are recognized as having formal character. Hence, one can use mathematics to control or change these structures. It is a characteristic of modern technology and science that not only the purpose determines the means but also the other way around: the means determine or create the ends. (1993, p. 249)

The mathematisation of social, economic and technological relations in the form of formal structures is a double-edged sword. On the one hand, it has proven effective and efficient in terms of developing more and more complex structures. As Fischer points out, "[t]he more mathematics is used to construct a reality, the better it can be applied to describe and handle exactly that reality" (1993, p. 118). On the other hand, once established as the standard (or only) way of describing, predicting and prescribing social, economic, ecologic, etc. processes, it severely reduces the possibilities of finding non-formal, non-quantifiable, non-mathematical solutions to the problems we face (Straehler-Pohl, 2017).

Moreover, the mathematisation of social, economic and technological relations cannot be fully understood without taking into account a process occurring in parallel (Gellert & Jablonka, 2007) — the demathematisation of social practices, for instance, the fact that taxes are nowadays deducted automatically from salaries and no longer calculated in the historical form of labour or grain to be given to the authorities:

> The greatest achievement of mathematics, one which is immediately geared to their intrinsic progress, can paradoxically be seen in the never-ending, twofold process of (explicit) demathematising of social practices and (implicit) mathematizing of socially produced objects and techniques. (Chevallard, 1989, p. 52)

For Keitel, mathematics-based technology as a form of implicit mathematics "makes mathematics disappear from ordinary social practice" (1989, p. 10). As a consequence, the (explicit) demathematisation of social practices leads to a devaluation of the mathematical knowledge involved in these practices. What kind of mathematical knowledge, then, is helpful so that citizens can do more than simply "obey" the structures which seem so "inseparably connected with our social organization" (Fischer, 1993, p. 114)? A threat to the democratic character of our political fundament is thus posed, which Skovsmose translates into the relation between technological and reflective knowledge:

> Technological knowledge itself is insufficient for predicting and analysing the results and consequences of its own production; reflections building upon different competencies are needed. The competence in constructing a car is not adequate for the evaluation of the social consequences of car production. (1994, p. 99)

From a pedagogic point of view, in which democracy and critical citizenship are taken into consideration as the overarching aim of education, the mathematisation/demathematisation of social relations, of economic and technological development can count as a starting point for curricular reflection and imagination. However, what do we really know about the structures and effects of mathematisation and demathematisation? Taken to an extreme, might it even be necessary to actively work toward preserving the capacity and confidence to reject, at least some of the time, the "solv[ing of] problems of social significance by means of mathematics" (Straehler-Pohl, 2017, p. 49)?

Turning from the discussion of making mathematisation explicit, we now consider the second aim.

1.2 Reflecting on experience with curricular conceptions

The second aim of the 69th CIEAEM conference is related to a practice where, in most countries, school mathematics, particularly elementary school mathematics, is, and has historically been, constructed as a subject in which everyday knowledge and scientific knowledge are somehow brought together. In these practices, it seems to be a commonplace assumption that mathematical knowledge may be useful in all kinds of professional and occupational contexts. See, for instance, an old German mathematics textbook for seventh-graders, on the cover of which mathematics is constructed as prevalent in manual work (Figure 1). Examples like this abound. Keitel refers to a US textbook of 1937, in whose table of contents mathematics is overtly related to the supposed community needs, when arguing that "a trivial though dogmatic social-needs orientation" (1987, p. 398) is often the driving force for curriculum construction.

Non-trivial considerations on the relationship of mathematics and the everyday have served, and continue to serve, as the cornerstone of several curriculum conceptions in mathematics education (Jablonka, 2003; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). In some of these conceptions, mathematisation is taken as a key didactic principle for the teaching and learning of mathematics.

An internationally influential example of a curriculum conception drawing explicitly on mathematisation(s) is Realistic Mathematics Education (e.g., de Lange, 1996; Treffers, 1987). RME distinguishes between a horizontal and a vertical mathematisation. A horizontal mathematisation denotes the students' activity of expressing mathematically a realistic everyday situation from which mathematical meaning can be developed. This can be interpreted as a sideways shift between discourses. However, the everyday situations are valued mostly for their didactic potential as a starting point for the mathematisation to occur. Their purpose is illustrative and motivational, and authenticity is not the main criterion for the design of the everyday situations. Once a mathematical formulation of the everyday situation has been arrived at, the next step is a vertical mathematisation, in which the organised structure of mathematical knowledge is the focus. The students get 'deeper' into the mathematics, or arrive at 'higher' levels of abstraction.

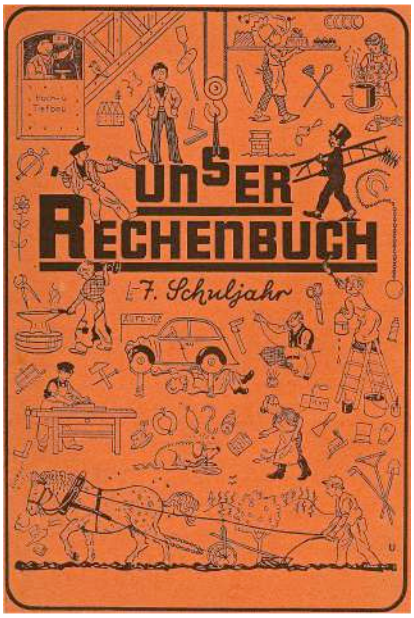

Figure 1. Front cover of *unser Rechenbuch*, Baßler et al. (1949).

Mathematical Modelling (e.g., Blum, Galbraith, Henn, & Niss, 2007; Stillman, Blum, & Salett Biembengut, 2015) is another orientation for curriculum construction that attracts worldwide attention. Within Mathematical Modelling, the authenticity of everyday situations is of relevance. From these everyday situations, a 'real world model' is generated and, further the 'real world model' is translated into a 'mathematical model', which can be used for calculation or other mathematical procedures. This translation is called mathematisation. In this curricular perspective, mathematics education is constructed as a didactically simplified version of applied mathematics.

In relation to this second aim concerning curriculum, two things should not go unnoticed. First, from a psychological perspective on cognitive development mathematisation is strongly related to abstraction, or reflective abstraction, and decontextualisation. The issue has been substantially developed by Vergnaud, who describes the process of dissecting mathematical concepts from sets of problems via concepts such as operational invariants, theorems-in-action, and schemes. Students' symbolic representations and processes of instrumentation represent a major focus in this field (e.g., Vergnaud, 1999). It is of interest that Piaget's work, as a central reference for Vergnaud's theoretical developments, has been a long-time influence on discussions in CIEAEM. See for instance Servais (1968), in which a shift from mathematisation-of-the-world to mathematisation-of-a-situation is visible.

> The true involvement of students in mathematical work can only be assured by an adequate motivation at their level: pleasure of playing or of competition, interest for application, satisfaction of the appetite for discovery, the affirmation of themselves, a taste for mathematics itself. In order to learn mathematics in an active manner, it is best to present to the students a situation to be mathematized. So, today's didactic is based, as far as possible, on mathematical initiations to situations easy to approach at the basic level and sufficiently interesting and problematic to create and sustain investigations by the students. They learn by experience to schematicize, to untangle

> the structures, to define, to demonstrate, to apply themselves instead of listening to and memorizing ready-made results. (p. 798)

Second, much of the conceptual work that draws on mathematisation as a didactic principle refers explicitly to the writings of Freudenthal. In *Mathematics as an Educational Task*, his point of departure is an analysis of what mathematisation, or mathematizing, might mean on different mathematical levels:

> Today many would agree that the student should also learn mathematizing unmathematical (or insufficiently mathematical) matters, that is, to learn to organize it into a structure that is accessible to mathematical refinements. Grasping spatial gestalts as figures is mathematizing space. Arranging the properties of a parallelogram such that a particular one pops up to base the others on it in order to arrive at a definition of parallelogram, that is mathematizing the conceptual field of the parallelogram. Arranging the geometrical theorems to get all of them from a few, that is mathematizing (or axiomatizing) geometry. Organizing this system by linguistic means is again mathematizing of a subject, now called formalizing. (Freudenthal, 1973, p. 133)

In this quote, the RME-concepts of horizontal mathematisation (as mathematizing the unmathematical) and vertical mathematisation (as axiomatizing and formalizing) are already elaborately preformed.

2. Subthemes and Questions

The theme of the conference *Mathematisation: social process & didactic principle* aims to attract contributions based on experience and analysis of a diverse nature and broad variety. Four subthemes, which represent possible thematic foci and will thus be used as a basis for the composition of the working groups, help to orientate and to categorize the contributions.

- Subtheme 1 is concerned with the issue of mathematisation as a didactic principle. It collects research on, and experience with, the teaching and learning of mathematics by mathematisations and in the classroom (or kindergarten, university, …) and also considers curriculum development in this field.
- Subtheme 2, in contrast to Subtheme 1, is not directly related to the learning of mathematics. It engages with the ways in which society is mathematised, and with the recent mathematisations by which the current local and global social, environmental, etc. situation is modelled.
- Subtheme 3 tries to bring the topics of the subthemes 1 and 2 into fertile interaction. The value of such an attempt has been described in the CIEAEM Manifesto 2000:

  > Mathematics education has to provide understanding of the processes of "mathematisation" in society. […] How can mathematics teaching and learning be presented not only as an introduction to some powerful ideas of our culture, but also as a critique of ideas and their application? Do we teach about how mathematics is used in our society? Do we sufficiently understand in what ways, society is becoming increasingly "mathematised"? (CIEAEM 2000, pp. 8–9)

- Subtheme 4 is dedicated to analysis of, and self-reflection on, the effects of mathematisation on pedagogy. At stake are the ways in which the recent political emphasis on standards, assessment and evidence, influence, impact or impair the daily practices of mathematics teachers and researchers in mathematics education.

In the final part of the discussion document of CIEAEM 69, we further develop the four Subthemes. The descriptions as well as the exemplary questions that are posed are intended to stimulate contributions and discussions. They provide a tentative structure to the general topic, while explicitly encouraging the exploration of issues that are located in their intersection or in the space between them.

2.1 Subtheme 1: Mathematisation as a didactic principle

The focus of the Subtheme 1 is on teaching experience with, and research studies on, conceptions of mathematics education that interrelate mathematics and the everyday world. The contributions can be aligned to well-established conceptions such as RME or Mathematical Modelling, can question them or can explore new ways of connecting mathematics and the world. We encourage the contributors to Subtheme 1 to analyse the challenges and the potential of mathematisation as a didactic principle, as we invite critical reflections on historical developments and educational policy. A further issue is the implication of mathematisation as a didactic principle for students' learning and identity formation.

Some questions to start with:

- What qualifies a real-world context as a point of departure and/or point of arrival of a didactic arrangement that builds on mathematisation?
- How relevant is the authenticity of everyday contexts for the learning of mathematics?
- What are specific cognitive, social or discursive processes that occur in learning environments that have mathematisation as a pivot?
- Do all students benefit equally from these conceptions of mathematics education?
- Which material arrangements support students' learning of mathematics by mathematisation (e.g. artefacts, physical experiences, learning spaces, etc.)?
- Which epistemologies of mathematics are built into particular didactical principles of mathematisation?

2.2 Subtheme 2: Mathematisation of society

Subtheme 2 studies the models, in which mathematics is partly or largely adopted, by which social, economic, ecological, etc. processes may be described, predicted and prescribed. These models often inform social and environmental policy on issues such as refugee migration, water, energy, climate change (Hauge & Barwell, 2015), health (Hall & Barwell, 2015); or they may be used for legitimizing political decisions. Subtheme 2 is concerned with the recent developments at the interface of mathematics, technology and globalisation: big data, security, internet of things, mathematisation of urban spaces, etc.; keeping in mind that mathematisation is not a naturally occurring phenomenon that we cannot avoid. It is done on purpose and it might be illuminative to ask whose intentions become realised (Davis, 1989).

Some questions to start with:

- What do we know of and about the mathematical models in use? In what ways are they made public?
- Which experiences and practices are facilitated by mathematisation and would not have been possible without it? Are there experiences and practices that are made unlikely, or even impossible by such mathematisations?
- By comparing competing technologies that use different mathematical models/ algorithms for the same ends, what are or could be the unforeseen side effects?
- How is the mathematisation of society made an object of reflection in the media and popular culture (e.g. in advertisements, newspapers, novels, movies, documentaries)?
- How do mathematical models influence the fundamental conditions of life for particular social groups (e.g. by regulation of social welfare, supplies for refugees, or even transnational restrictions or sanctions for importing food or health supplies) (see, e.g., Alshwaikh & Straehler-Pohl, 2017)?
- Considering the effects of mathematisation on mathematics education research: How does the increasing mathematisation affect the ways research is carried out? What counts as research? What are the "policy implications of developing mathematics education research" (Hoyles & Ferrini-Mundy, 2013)?

2.3 Subtheme 3: Interconnecting mathematisation as a social process and as a didactic principle

It has been argued that we urgently need an "ethic of mathematics for life" (Renert, 2011, p. 25) and that "the political and sociological dimensions of the relationship between mathematics, technology and society are fundamental" (Gellert, 2011, p. 19). For such an ethic, it would be necessary to develop (classroom) activities that engage with this relationship, by not simply reducing mathematics to a remedy for and an answer to the problems we face, and by breaking with many myths about mathematics and its use.

Some questions to start with:

- "How are pupils to be enabled to criticise [and critique] models and modelling, including the formalised techniques that underpin so much the use or abuse of mathematics in society?" (CIEAEM 2000, p. 9)
- How can teacher education contribute to building up reflexive knowledge on mathematics necessary for pursuing this target?

- How do students and teachers balance the didactic fictionality and the reality of social, economic, environmental, etc. phenomena in mathematics education?
- What can we learn from examples of mathematics education practices that engage locally with social, environmental, etc. issues?
- How can we develop learning environments so that students learn to use mathematics as a tool of emancipation to question the social reality they live in?
- How can we develop learning environments so that students can emancipate themselves from mathematics, in order to assert agency over apparently mathematically validated necessities?

2.4 Subtheme 4: Mathematisation of pedagogy

Even when it is not intentionally used as a didactical principle or made an object of reflection, mathematisation does not remain out of school. It enters, for instance, in the form of standardised high-stakes testing and thus changes the “governing assessment dispositive” (Björklund Boistrup, 2017). Sometimes directly, sometimes more indirectly, schools receive ‘support’, and teaching is ‘improved’, by evidence-based recommendations about what works in the classroom, and in education more generally (Biesta, 2007). Randomised control experiments seem to be the gold standard for some policy makers and researchers in education (e.g., Slavin, 2002). Once the impact of evidence-based recommendations is mathematised, interventions can be compared with each other, and moreover, measured against their monetary costs in terms of efficiency, promising policy-makers to find the "biggest bang for the buck", as Jablonka and Bergsten (2017, p. 115) critically capture. However, as Herzog (2011) asserts, “to expect that we would soon be able to control the education system more effectively and efficiently due to the politically motivated strengthening of experimental educational research, is naïve” (p. 134).

Some questions to start with:

- What are the effects of the mathematisation of research on mathematics pedagogic activity in school?
- What are officially stipulated strategies and instructions to implement evidence-based research results in mathematics education?
- How do teachers and students deal with the new regime as it affects mathematics education? How do they enact or resist it?
- What are the effects of the mathematisation of pedagogy on mathematics teacher education?

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3. Introduction (française)

Les intentions de la conférence CIEAEM 69 sont d'interroger le concept de mathématisation qui est communément accepté dans l'éducation mathématique formelle. Un des objectifs de la 69eme conférence de la CIEAEM est de rendre explicite les conditions de la mathématisation dans les domaines sociaux, économiques, écologiques, etc. Le deuxième objectif de cette conférence est de revenir sur les expériences curriculaires qui portent une attention particulière aux relations entre les savoirs mathématiques et les savoirs du quotidien.

Dans cet appel, la mathématisation est utilisée dans un sens très large. Elle peut ainsi inclure l'utilisation de toute forme de mathématiques, par exemple l'interprétation mathématique de notions du monde (incluant des objets mathématiques), ou bien exprimer des idées d'une façon mathématique. Il peut aussi inclure la façon dont on peut rencontrer les mathématiques comme étant utilisées “sur” elles et dans leur contexte, par exemple les mathématiques au cœur de la description d'activités humaines, de la prise de décisions éclairée par les mathématiques.

3.1 Rendre explicite les conditions de la mathématisation dans les domaines sociaux, économiques, écologiques, etc.

La mathématisation – dans cette acception large – est un concept qui est présent dans les préoccupations de la CIEAEM depuis plus d'un demi-siècle. On peut rappeler les textes de 1954 quand Servais décrivait les changements fondamentaux de la société qu'il espérait dans les termes suivants :

> Notre époque marque le début d'une ère mathématique. […] Ce fait, quelque soient les réactions, les opinions et les jugements qu'il puisse provoquer, accroît la responsabilité de tout professeur, qui enseigne les mathématiques et quelque soit son niveau d'enseignement. […] S'il convient d'être fier d'une tradition mathématique, il est aussi important de permettre la mathématisation [du monde] à venir. Autant il est vrai que celui qui consacre sa vie à l'enseignement, accepte une mission de construire un monde nouveau sur le monde passé, autant la responsabilité envers le futur est plus grande que la loyauté au passé. (Servais 1954, p. 89 ; cité de Vanpaemel, De Bock, & Verschaffel, 2011, traduit par nous)

Cette déclaration est appuyée sur un optimisme considérant que en se fondant sur le développement social et technologique de la tradition mathématique, le futur sera plus favorable que le passé. En effet, comme Davis et Hersh ont montré 30 ans plus tard, "les mondes physiques et sociaux ont été mathématisés à une vitesse de plus en plus grande" (1986, p. xv, traduit par nous). Le prolongement de cette mathématisation fait que David et Hersh nous mettent en garde sur le fait que "nous devrions y regarder de plus près, parce que tout ne sera pas bon pour nous" (ibid.). Keitel, Kotzmann et Skovsmose ont repris cet avertissement en décrivant un processus cyclique :

> D'un côté la société devient formalisée et mathématisée sous l'influence des environnements technologiques autoproduits et des structures économiques ; de l'autre côté, les mathématiques sont naturellement une aide majeure pour penser les environnements technologiques et quantifiés. Ainsi, la société a besoin de plus en plus d'aides techno-mathématiques. Dans ce processus, de nombreuses structures de l'activité humaine sont reconnues comme ayant un caractère formel. Donc, on peut utiliser les mathématiques pour contrôler ou changer ces structures. Une caractéristique des technologies et des sciences actuelles est que non seulement l'objet de recherche détermine les moyens mais *a contrario* les moyens déterminent ou créent les finalités. (1993, p. 249, traduit par nous)

La mathématisation des relations sociales, économiques et technologiques en termes de structures formelles est une arme à double tranchant. D'une part, elle a prouvé son efficacité et son efficience en termes de développement de structures de plus en plus complexes. Comme Fisher le faisait remarquer, "plus les mathématiques sont utilisées pour bâtir une réalité, mieux elles peuvent être appliquées pour décrire et manipuler précisément cette réalité" (1993, p. 118, traduit par nous). D'autre part, une fois établi comme un standard ou une (unique) façon de décrire, prédire et prescrire les processus sociaux, économiques écologiques, etc., elle réduit les possibilités de trouver des solutions informelles, non quantifiables et non mathématiques aux problèmes qui se posent (Straehler-Pohl, 2017).

En outre, la mathématisation des relations sociales, économiques et technologiques ne peut être complètement comprise sans prendre en compte un processus parallèle (Gellert & Jablonka, 2007) – la *démathématisation* des pratiques sociales, par exemple, les impôts sont maintenant déduits automatiquement des salaires et plus calculés dans une perspective de participation aux autorités :

> La plus grande réussite des mathématiques, qui est immédiatement intriquée à leur progrès, peut paradoxalement être considérée dans le processus sans fin et double (explicite) de démathématisation et (implicite) de mathématisation d'objets et de techniques socialement produits. (Chevallard, 1989, p. 52, traduit par nous)

Pour Keitel, la technologie fondée sur les mathématiques vue comme une forme de mathématiques implicites "fait disparaître les mathématiques des pratiques sociales ordinaires" (1989, p. 10). En conséquence, la démathématisation (explicite) des pratiques sociales conduit à une dévaluation des savoirs mathématiques embarqués dans ces pratiques. Quel type de savoir est ainsi utile pour que les citoyens puissent faire mieux que simplement obéir aux structures qui semblent "inséparablement connectées à notre organisation sociale" (Fischer, 1993, p. 114, traduit par nous) ? Une menace au fondement démocratique de nos politiques est ainsi rendue manifeste, ce que Skovsmose traduit par la relation entre le savoir technologique et le savoir réflexif :

> Le savoir technologique lui-même est insuffisant pour prédire et analyser les résultats et les conséquences de ses propres productions ; des réflexions construites sur des compétences variées sont nécessaires. La compétence requise pour construire une voiture n'est pas adaptée pour l'évaluation des conséquences sociales de la production de voitures. (1994, p. 99, traduit par nous)

D'un point de vue pédagogique duquel la démocratie et la citoyenneté critique sont considérées, l'objectif ultime de l'éducation, la mathématisation/démathématisation des relations sociales, économiques et du développement technologique peut être considéré comme un point de départ pour des réflexions curriculaires imaginatives. Cependant, que savons-nous vraiment des structures et des effets de la mathématisation et de la démathématisation ? En poussant la réflexion plus loin, est-il même nécessaire de préserver la capacité et la confiance à rejeter "la résolution de problèmes ayant une signification sociale au moyen des mathématiques" (Straehler-Pohl, 2017, p. 49, traduit par nous) ?

3.2 Revenir sur les expériences curriculaires

Le deuxième objectif de la conférence CIEAEM 69 est lié à la pratique où, dans la plupart des pays, les mathématiques scolaires, et particulièrement à l'école élémentaire, sont et ont été historiquement construites comme une discipline dans laquelle le savoir de tous les jours et le savoir scientifique se rencontrent. L'hypothèse forte de ces pratiques, est que les savoirs mathématiques peuvent être utiles dans n'importe quel type de contextes professionnels ou quotidiens. Voyons, par exemple cet ancien manuel allemand pour des élèves de grade 7 dont la couverture montre les mathématiques comme fondamentaux dans les travaux manuels (Figure 1). Des exemples de ce type abondent. Keitel se réfère à un manuel américain de 1937, dans lequel la table des matières des mathématiques est ouvertement liée aux besoins supposés de la communauté, en arguant que "une orientation sociale triviale quoique dogmatique" est souvent la force motrice de la construction du curriculum.

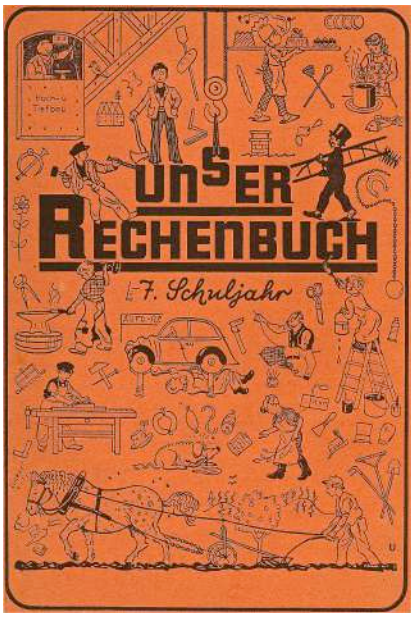

Figure 1. Couverture du livre *unser Rechenbuch*, Baßler et al. (1949).

Des considérations élaborées des relations des mathématiques et de la vie quotidienne ont servi et continuent à servir, de balise essentielle de différentes conceptions des curriculums dans l'éducation mathématique (Jablonka, 2003 ; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). Dans certaines de ces conceptions la mathématisation est considérée comme un principe didactique essentiel pour l'enseignement et l'apprentissage des mathématiques.

Un exemple internationalement influent d'une conception curriculaire reposant explicitement sur la mathématisation est l'exemple de l'éducation des mathématiques réalistes (p. ex. de Lange, 1996 ; Treffers, 1987). EMR distingue entre des mathématisations horizontale et verticale. Une mathématisation horizontale concerne l'activité des élèves dans laquelle ils expriment mathématiquement une situation quotidienne réaliste à partir de laquelle un développement mathématique peut être étudié. Ce peut être interprété comme un pas de côté. Cependant, les situations de la vie quotidienne sont appréciées surtout pour leur potentiel didactique comme point de départ de la mathématisation. Leur propos est illustratif et motivant, et l'authenticité n'est pas le critère principal pour la conception de ces situations de la vie quotidienne. Une fois la formulation mathématique de la situation quotidienne effectuée, le pas suivant est celui de la mathématisation verticale, dans laquelle la structure du savoir mathématique est le point d'intérêt. Les élèves rentrent "profondément" dans les mathématiques, ou atteignent des niveaux plus hauts d'abstraction mathématique.

La modélisation mathématique (p. ex. Blum, Galbraith, Henn, & Niss, 2007 ; Stillman, Blum, & Salett Biembengut, 2015) est une autre orientation pour la construction des curriculums qui est considérée dans le monde entier. Dans la modélisation mathématique, l'authenticité des situations de la vie quotidienne est de première importance. De ces situations émerge un modèle du monde réel qui se traduit à son tour dans un modèle mathématique qui pourra être utilisé pour des calculs ou des procédures mathématiques. Cette traduction est appelée la mathématisation. Dans cette perspective curriculaire, l'éducation mathématique est construite sur une simplification des mathématiques appliquées.

Le deuxième objectif de CIEAEM 69 sera ainsi de réfléchir théoriquement et empiriquement sur les expériences de conceptions curriculaires et en particulier de celles qui portent une attention particulière aux relations entre les savoirs mathématiques et ceux de la vie quotidienne.

Deux points ne devraient pas être oubliés. Tout d'abord, dans une perspective psychologique du développement cognitif, la mathématisation est fortement reliée à l'abstraction et à la décontextualisation. Ces questions ont été substantiellement étudiées par Vergnaud qui décrit le processus d'examen des concepts mathématiques et d'ensemble de problèmes à travers des concepts tels que les invariants opératoires, les théorèmes en action, et les schèmes. Les représentations symboliques et les processus d'instrumentation des élèves représentent un point important dans ce champ (p. ex. Vergnaud, 1999). Il est intéressant de noter que le travail de Piaget, qui est une référence centrale pour les développements théoriques de Vergnaud ont influencé longtemps les discussions au sein de la CIEAEM. Voir par exemple Servais (1968), dans lequel un passage de la mathématisation du monde à une mathématisation d'une situation est visible.

> L'implication réelle des élèves dans un travail mathématique ne peut être seulement assurée par une motivation adéquate à leur niveau ; le plaisir de jouer ou la compétition, l'intérêt pour les applications, la satisfaction et l'appétit de la découverte, l'affirmation de soi, un goût pour les mathématiques elles-mêmes. Pour apprendre les mathématiques activement, il est préférable de présenter aux élèves une situation qui est à mathématiser. Ainsi, la didactique actuelle est fondée, autant que possible, sur des propositions de situations faciles dans leur approche et suffisamment intéressantes et problématiques pour créer et soutenir des investigations des élèves. Ils apprennent par l'expérience à schématiser, à démêler les structures, à définir, à démontrer, à appliquer eux-mêmes plutôt que d'écouter et de mémoriser des résultats déjà prêts. (p. 798, traduit par nous)

Ensuite, la plupart du travail conceptuel qui repose sur la mathématisation comme un principe didactique se réfère explicitement aux écrits de Freudenthal. Dans *Mathematics as an Educational Task,* son point de départ est une analyse de ce que la mathématisation devrait signifier à différents niveaux des mathématiques :

> Aujourd'hui, la plupart d'entre nous serons d'accord pour dire que les étudiants devraient aussi apprendre la mathématisation de questions non mathématiques (ou insuffisamment mathématiques), c'est à dire, d'apprendre à les organiser dans des structures accessibles à des traitements mathématiques. Saisir des formes de l'espace comme des figures participe à la mathématisation de l'espace. Organiser les propriétés d'un parallélogramme de telle façon que l'une d'entre elle apparaisse comme fondement des autres pour déboucher sur une définition du parallélogramme, c'est mathématiser le champ conceptuel du parallélogramme. Organiser les théorèmes géométriques pour les déduire tous de quelques uns, c'est mathématiser (or axiomatiser) la géométrie. Organiser ce système à travers le langage est encore mathématiser ce sujet, ce que l'on appelle maintenant formaliser. (Freudenthal, 1973, p. 133, traduit par nous)

Dans cette citation les concepts de mathématisation horizontale (mathématiser le non mathématique) et verticale (axiomatiser et formaliser) sont déjà élaborés.

4. Sous-thèmes et Questions

Le thème de la conférence, m*athématisation : processus social et didactique,* a pour but d'attirer des contributions fondées sur des expériences et des analyses de diverses natures et d'une large variété. Quatre sous-thèmes, qui représentent quatre directions de pensée et qui seront utilisés pour la composition des groupes de travail, aident à orienter et à catégoriser les contributions.

- Le sous-thème 1 se focalise sur les questions de mathématisation comme principes didactiques. Il rassemblera des recherches sur l'enseignement et l'apprentissage des mathématiques à travers la mathématisation fondées sur des expérimentations dans des classes à différents niveaux ainsi que des développements ou des évolutions curriculaires dans ce domaine.

- Le sous-thème 2, contrairement au sous-thème 1, n'est pas directement lié à l'apprentissage des mathématiques. Il concerne la façon dont la société elle-même est mathématisée dans sa relation aux actualités de mathématisation dans les domaines sociaux, environnementaux, etc. aussi bien en considérant des points de vue locaux que globaux dans ce processus de modélisation.
- Le sous-thème 3 essayera de combiner les sujets des sous-thèmes 1 et 2 pour tirer des réflexions fécondes de ces interactions. L'importance de telles questions ont été mises en exergue dans le manifeste 2000 de la CIEAEM :

> L'éducation mathématique doit permettre la compréhension des processus de “mathématisation” dans la société. […] Comment l'enseignement et l'apprentissage des mathématiques peuvent-il être présentés non seulement comme une introduction à des idées puissantes issues de notre culture, mais aussi comme une critique des idées et de leurs applications ? Enseignons-nous comment les mathématiques sont utilisées dans notre société ? Est-ce que nous comprenons suffisamment dans quelle mesure la société devient de plus en plus “mathématisée” ? (CIEAEM 2000, pp. 8–9)

- Le sous-thème 4 est consacré à l'analyse et à une réflexion des effets de la mathématisation sur la pédagogie. Les enjeux de ce groupe porteront sur les façons dont les standards et les incitations institutionnelles, évaluations et résultats, influencent, impactent ou détériorent les pratiques quotidiennes des professeurs de mathématiques aussi bien que celles des chercheurs dans le domaine de l'éducation mathématique.

Dans la dernière partie de ce document de discussion de la conférence CIEAEM 69, nous développons les quatre sous-thèmes. Les descriptions et les questions sont proposées pour stimuler les contributions et les discussions. Elles fournissent une structure générale du sujet sans empêcher, et plutôt même en encourageant, l'exploration de pistes aux frontières de ces thèmes.

4.1 Sous-thème 1 : Mathématisation comme principe didactique

Le point central de ce sous-thème est constitué des expériences d'enseignement et des recherches portant sur les conceptions de l'éducation mathématique qui croisent les mathématiques et le monde de tous les jours. Les contributions pourront se référer à des concepts déjà bien établis comme *Real Mathematics Education* (Éducation fondée sur les mathématiques de la vie quotidienne) ou la modélisation mathématique, mais pourront aussi les interroger ou explorer de nouvelles voies permettant de connecter les mathématiques et le monde. Nous encourageons les contributeurs de ce sous-thème à analyser les défis et les potentiels que la mathématisation procure en tant que principe didactique, tout comme nous invitons les réflexions critiques de développements historiques ou de politiques d'éducation. Une question qui pourra être aussi traitée concerne les conséquences de la mathématisation comme principe didactique pour l'apprentissage des élèves et la formation de leur identité.

Quelques questions :

- Qu'est-ce qui qualifie un contexte du monde réel comme point de départ et/ou point d'arrivée d'une construction didactique construite sur la mathématisation ?
- Dans quelles mesures les contextes de la vie quotidienne sont-ils pertinents pour l'apprentissage des mathématiques ?
- Quels sont les processus cognitifs, sociaux ou discursifs qui sont présents dans des environnements d'apprentissage construits sur la mathématisation ?
- Est-ce que tous les élèves bénéficient également de ces conceptions de l'éducation mathématique ?
- Quelles organisations matérielles aident à l'apprentissage des élèves dans un contexte d'apprentissage des mathématiques par la mathématisation (p. ex. artefacts, expériences physiques, espaces de travail, etc.).
- Quelle épistémologie des mathématiques se construit à travers ces principes didactiques de mathématisation ?

4.2 Sous-thème 2 : Mathématisation de la société

Le sous-thème 2 étudie les modèles, dans lesquelles les mathématiques sont partiellement ou largement impliquées, et par lesquelles, les processus sociaux, économiques, écologiques, etc. peuvent être décrits, prédits et prescrits. Ces modèles éclairent très souvent des politiques sociales ou environnementales sur des questions vives comme celle des réfugiés, de l'eau, de l'énergie, des changements climatiques (Hauge & Barwell, 2015), de la santé (Hall & Barwell, 2015); ils peuvent aussi être utilisés pour justifier et légitimer des décisions politiques. Ce sous-thème porte sur les récents développements à la frontière entre mathématiques, technologie et mondialisation : *big data*, sécurité, internet des objets, mathématisation des espaces urbains, etc. ; en gardant en mémoire que la mathématisation n'est pas un phénomène naturel que nous ne puissions éviter. Il sera de propos et il pourrait être important de questionner les intentions qui se réalisent (Davis, 1989).

Quelques questions :

- Que savons-nous des modèles mathématiques utilisés ? De quelle façon sont-ils rendus publics ?
- Quelles expériences et quelles pratiques sont facilitées par la mathématisation et n'auraient pas été possibles sans ? Est-ce qu'il existe des expériences et des pratiques qui sont peu probables ou même impossibles dans le cadre d'une mathématisation ?
- En comparant des technologies qui utilisent des modèles mathématiques et des algorithmes différents, quels sont ou pourraient être des effets de bord imprévus ?
- Comment la mathématisation de la société est relayée et réfléchie dans les medias et la culture populaire (p. ex. dans les publicités, journaux, romans, films, documentaires, séries) ?
- Comment les modèles mathématiques influencent les conditions fondamentales de vie de groupes sociaux spécifiques, p. ex. en régulant le bienêtre social, en pourvoyant de l'aide aux réfugiés, ou même en restreignant ou sanctionnant les importations de nourriture et de produits de santé (voir, p. ex., Alshwaikh & Straehler-Pohl, 2017) ?
- Qu'en est-il de la recherche (en éducation mathématique) : comment cet accroissement de la mathématisation dans la société affecte les recherches en éducations ? Quelles sont les implications politiques dans le développement de la recherche en éducation mathématique (Hoyles & Ferrini-Mundy, 2013) ?

4.3 Sous-thème 3 : Interconnecter la mathématisation comme processus social et comme principe didactique

Des voies se sont élevées pour fournir de façon urgente une “éthique des mathématiques pour la vie” (Renert, 2011, p. 25, traduit par nous) et que les “dimensions politiques et sociologiques des relations entre mathématiques, technologie et société sont fondamentales” (Gellert, 2011, p. 19, traduit par nous). Pour une telle éthique, il serait nécessaire de développer des activités (pour la classe) qui prennent en compte ces relations, en ne réduisant pas simplement les mathématiques à un remède et une réponse aux problèmes que nous rencontrons, et en brisant de nombreux mythes relatifs aux mathématiques et à leurs utilisations.

Quelques questions :

- “Comment les élèves peuvent pouvoir critiquer (et être critique) les modèles et la modélisation, incluant les techniques formalisées qui soutiennent l'utilisation ou l'abus d'utilisation des mathématiques dans la société ?” (CIEAEM 2000, p. 9)
- Comment la formation des enseignants peut contribuer à construire des connaissances réflexives sur les mathématiques nécessaires pour atteindre ces buts ?
- Comment les élèves et les professeurs peuvent prendre en compte à la fois la fiction didactique et la réalité des phénomènes sociaux, économiques, environnementaux dans l'éducation mathématique ?
- Que peut-on apprendre des exemples des pratiques de l'éducation mathématique qui ont à voir localement avec les questions sociales, environnementales, etc. ?
- Comment peut-on développer des environnements d'apprentissage de telle façon que les élèves apprennent à utiliser les mathématiques comme un outil d'émancipation permettant de questionner leur réalité sociale ?

- Comment peut-on développer des environnements d'apprentissage de telle façon que les élèves puissent s'émanciper des mathématiques, de façon à revendiquer une position contre des arguments apparemment validés ?

2.4 Sous-thème 4 : Mathématisation de la pédagogie

Même quand elle n'est pas utilisée comme principe didactique ou un objet de réflexion, la mathématisation ne reste pas en dehors de l'école. Elle entre, par exemple, dans les normes des évaluations standardisées et ainsi change la "gouvernance du dispositif d'évaluation" (Björklund Boistrup, 2017). Parfois directement, parfois plus indirectement, les écoles reçoivent des aides et l'enseignement est "amélioré", sur des recommandations à propos de ce qui fonctionne dans la classe, et dans l'éducation plus généralement (Biesta, 2007). Des expérimentations avec contrôles randomisés semblent être le standard pour des politiques et des chercheurs en éducation (p. ex. Slavin, 2002). Une fois l'impact des recommandations construites sur les résultats de recherche, les interventions peuvent être comparées entre elles, et de plus, mesurées en regard de leurs coûts en termes d'efficience, en promettant aux politiques de trouver le "bon coup pour le financement" comme Jablonka et Bergsten (2017, p. 115, traduit par nous) l'ont montré de façon critique. Cependant, comme Herzog (2011) le propose, "attendre que nous soyons bientôt capables de contrôler le système éducatif plus efficacement et de façon efficiente en appuyant les décisions politiques sur les résultats de la recherche, est naïf" (p. 134, traduit par nous).

Quelques questions :

- Quels sont les effets de la mathématisation sur les recherches en mathématiques sur les activités pédagogiques à l'école ?
- Quelles sont les stratégies et instructions officiellement stipulées pour mettre en œuvre des résultats des recherches dans l'éducation mathématique ?
- Comment les professeurs et les élèves tiennent compte de ce nouveau régime dans ce qu'il affecte l'éducation mathématique ? Comment participent-ils ou résistent-ils ?
- Quels sont les effets de la mathématisation de la pédagogie sur la formation des professeurs de mathématiques ?

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PLENARIES

CIEAEM 69

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MATHEMATISATION: SOCIAL PROCESS

& DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL

& PRINCIPE DIDACTIQUE

Multimodality and mathematisation: Different communicational resources in relation to mathematisations within and outside the mathematics classroom

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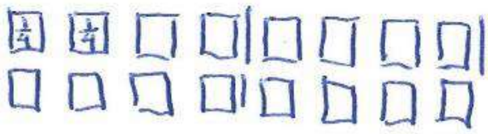
Abstract. Here I will elaborate on mathematisation and the roles of different communicational resources (modes), mainly drawing on a multimodal social semiotic perspective. On one hand, the notion that mathematics may be expressed through a broad range of communicational resources (e.g., symbols, speech, pictures) may seem obvious and indisputable, not the least as part of the learning and teaching of mathematics in classroom practices. On the other hand, I make the point that we have something to learn from specifically researching ways of expressing mathematics. The interest that I am exploring here is how mathematisations, in classroom practice and in society, may be expressed, and what the consequences of different ways of expressing mathematics may be for students, teachers and researchers in mathematics education. From a multimodal social-semiotic perspective, a change of communicational resources from, for example, pictures and speech into mathematical symbols, may, in fact, also affect the "what" that is communicated, such as the mathematical content.

Résumé. Ici, je vais élaborer la mathématisation et le rôle des différentes ressources sémiotiques (modes), en m'appuyant principalement sur une perspective social-sémiotique multimodale. D'une part, la notion selon laquelle les mathématiques peuvent être exprimés au travers d'un large éventail de ressources sémiotiques (par exemple des symboles, des discours et des images) peut sembler évidente et indiscutable, notamment comme élément de l'apprentissage et de l'enseignement des mathématiques dans la pratique en classe. D'autre part, j'estime que nous avons quelque chose à apprendre spécifiquement par la recherche des moyens d'expression mathématique. L'intérêt que j'explore est la façon dont les mathématisations, dans la pratique en classe ainsi que dans la société, peuvent être exprimées, et quelles conséquences peuvent avoir les différentes façons d'expression pour les étudiants, les enseignants et les chercheurs en éducation mathématique. D'un point de vue social-sémiotique multimodal, un changement des ressources sémiotiques, p.ex. de l'image et de la parole en symboles mathématiques affecte, en fait, également le «quoi» de ce qui est communiqué, tel que le contenu mathématique.

1. Introduction

This paper takes its departure from two pieces of data from different research projects. I present them in brief here, in relation to their respective context, and will come back to them throughout the text.

The first data excerpt is from a project on mathematical arguments created in classroom communication (Nordin, 2016).

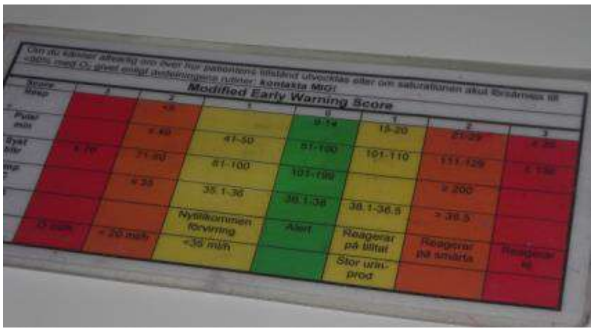


Excerpt 1. Frida solves a school mathematical problem

In excerpt 1, Frida has just finished showing the class how she solved a problem. She has mainly communicated this through the drawing, but also through speech and symbols.

The second set of data excerpts (Excerpts 2 and 3) derives from a project on mathematics as part of workers' workplace activities (Wedege, 2013, see also FitzSimons & Boistrup, 2017).

Excerpt 2. Anita's (a nursing aide) notes



Excerpt 3. A chart on early warning scores for measurements such as blood pressure

In excerpt 2, the nursing aide Anita has written notes from the "controls" she has done on the patients in a semi-emergency department. Excerpt 3 is a photograph of a chart that Anita was expected to follow. In the chart, the colours and numbers indicate which scores, as in too low or too high, require Anita to call on a doctor or a nurse.

In relation to excerpt 1 with Frida, I will address the roles a variety of communicational resources (such as drawings, symbols, speech) may have for mathematisations in the sense of mathematical argumentation in classroom communication. Further on, I will use the context of Anita and excerpts 2 and 3 for addressing multimodality and mathematisation from the perspective of the workplace. I will adopt a multimodal approach for addressing a multiplicity of resources for communication (modes), and a broad understanding of mathematisation.

The objective of this paper is to present reflections on how the roles played by different resources for communication, "interact" with mathematisations in and out of school. The overall question I am posing is: What may mathematisation be about when taking into account mathematical processes and various communicational resources, and what may be gained from adopting such a multimodal approach?

2. 'Multimodality' in mathematics education research

In the mathematics education research literature there are frequent acknowledgements of a broad range of communicational resources. Such an approach is often labelled as 'multimodality.' Multimodality can be described as an approach where a multiplicity of modes are addressed and/or captured, where modes can be seen as equal to communicational resources. It is quite common, when adopting a multimodal approach, to focus on mathematical classroom processes. In such an article there are examples of classroom communications, and what students and teachers are saying is clearly documented. The author also pays attention to other resources, not least when they are regarded as needed for the reader's understanding. An example of this is Krummheuer (2007), where the interest is in how learning mathematics depends on the student's participation in processes of collective argumentation. Krummheuer presents excerpts where the speech is in normal text, and where other resources are written in italics, as in this example: "Marina: *raises her hand animatedly* ten plus . three \" (formatting same as in original). Apart from this, in the article little attention is paid to the different modes. For example, there is no elaborate discussion on how different modes were part of the processes of collective argumentation. In literature such as this, the 'multimodal' approach is adopted mainly as part of the production of transcribed data, and not explicitly as part of the analysis, findings, or discussion. Moreover, modes such as gestures appear often in relation to speech, and interactions always seem to start with speech.

An approach where the researchers pay attention to a multiplicity of modes is also adopted in philosophically positioned research on what mathematics may be about, such as de Freitas and Sinclair (2012), who draw on new materialism. One theme that they present is how "learners' bodies are always in the process of becoming assemblages of diverse and dynamic materialities" (p. 453). In this way they do not distinguish the human body as being distinct from other materialities in the world. Hence a mathematics concept is also material.

Another common theme in the literature is on students' algebraic thinking and learning in relation to a variety of modes (e.g., Radford, 2006). Radford (2014) puts it like this: "cognitive functions such as thinking, memory, and imagination remain directly and indirectly related to the materiality and conceptuality of the world."

When paying attention to modes other than words and symbols, it is quite common to focus on a specific mode in the literature, and this is often gestures and their roles in mathematical communication. In Sabena (2008), the interest is in the roles of gestures in mathematics classroom communication, and how these differ from other resources such as language and symbols. In Arzarello, Paola, Robutto, and Sabena (2009), the interest is in how gestures are related to other modes as part of semiotic bundles. Morgan and Alshwaikh (2012) investigate how students and teachers used hand gestures. The teachers made more differentiation between everyday gestures and the more specific gesturing related to the programming that the task was about.

A multimodal approach is also adopted in research on communication with an interest in a broad range of modes, and with an interest in functions of the modes (e.g., Arzarello & Sabena, 2011; O'Halloran, 2000). For example, Arzarello and Sabena investigate mathematical argument and proof activities while analysing the signs in the communication in which symbols and graphs play a part, as well as the drawings made by students, and their gestures. They also address theoretical aspects in terms of what kind of mathematical theory was adopted.

The interest in this paper is similar to the one in the previous paragraph, in that I present reflections and research where a broad range of modes not only play a part in excerpts, but also are analysed per se. The specific topic in the paper is mathematisation and how it may be related to multimodal aspects. I will give account for mathematisations in the activities with Frida (excerpt 1), and Anita (excerpt 2 and 3), and will discuss the role of different modes in relation to this.

3. Multimodality in this paper

The theory that I adopt in this paper is multimodal social semiotics (Hodge & Kress, 1988; Kress, 2017; van Leeuwen, 2005; see also Boistrup, 2010). In this theory there is an acknowledgement of a broad range of modes, such as speech, drawings, symbols etc., without a 'language-centric' approach (cf. de Freitas & Sinclair, 2012). Avoiding a language-centric approach is, for example, not to take for granted that a communication between a teacher and student in a classroom starts through speech. Rather, the communication may start when the teacher is approaching the student, or changing their body posture in the direction of the student. As I will show, speech may even have a subordinated role, although the mathematics presented may be rather advanced.

The 'social' in social semiotics takes into account the meaning a sign 'has' in a communication in relation to the overall situation. Signs are seen as always socially motivated. In the institutional context, such as in society, signs, and their possible meanings, are in a constant change. In this respect, meaning is always newly created, and affected by institutional decisions and norms (Selander, 2008). Hence, how an utterance is interpreted by the researcher is affected by the overall communication within the context, and by how the other part of the communication (person or people) is/are responding to the utterance. This means that similar utterances can be interpreted as bearing different meanings at different times, and/or in different contexts. The modes and the 'content' of the communication are closely connected and not possible to take apart.

4. Mathematisation in this paper

The way I adopt the term mathematisation, in this paper, is to view it broadly. I draw on Jablonka and Gellert (2007) and adopt the distinction that mathematisation occurs when mathematical processes are becoming increasingly formal. This is similar to Gellert (2017), where mathematisation includes "people's active use of some kind of mathematics, for example by interpreting notions (including mathematical objects) in the world mathematically, or by expressing one's ideas in a mathematical way" […] and "mathematics as being at the core of how a certain activity is described, or how decisions are made on a mathematically informed basis" (p. 1).

The interest I have in this paper is in mathematisation as a didactic principle, both in school (as in the case of Frida solving a problem), and out of school (as in the case of the nursing aide, Anita). To some extent I address mathematisation as a social process, when I address social processes in mathematics education in terms of the consequences of a multimodal approach to the teaching of mathematics, and in the accompanying analysis. I will address how it is possible to capture rather formal mathematics, but with less formal modes, and the relevance of this.

5. A tentative model of multimodal aspects of mathematisation

In line with the Introduction, the objective of this paper is to present reflections on how the roles of different modes 'interact' with mathematisations in and out of school. To frame these reflections, I will use a tentative model. The model is built on mathematisation viewed as mathematical notions becoming more and more formal (Jablonka & Gellert, 2007), and on how Kress (2017) distinguishes between different kinds of multimodal changes (see below).

The model in Figure 1 illuminates how mathematisations may go in different directions, where the arrows represent two extremes (explained below). However, in most cases a mathematisation rather occurs in the space 'between' these extremes.

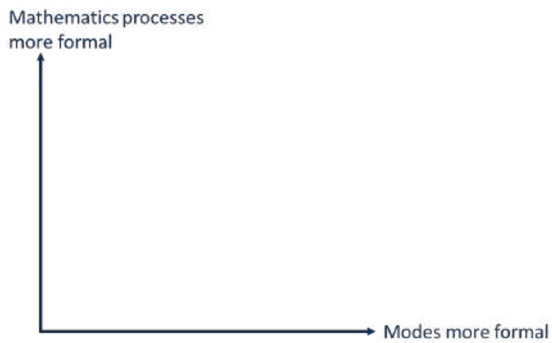


Figure 1. A tentative model of multimodal aspects of mathematisations

The vertical arrow represents a mathematisation which concerns mathematical ideas becoming more and more formal, while the modes remain the same. As an example we can imagine a student who is solving a problem on a test. The problem is described in words and a few numbers. The student solves the problem in a clear and structured way, but without using symbols such as in equations. Instead, the student describes her solution mainly in words, while the procedures are performed in a logical way. In this way, the students' solution is an example of where a problem is transformed into more formal mathematics through a structured solution to the problem. Drawing on Kress (2017), changes such as these can be labelled as *transformations*, where there are (mathematical) changes while keeping the same mode. Furthermore, Kress describes that such transformations entail changes in epistemology, since the meanings (mathematical ideas in this text) are becoming different.

The horizontal arrow represents a mathematisation which concerns the modes as such, where the extreme case is that the ‘same’ mathematical idea is represented in more and more formal modes. What is counted as more formal is contextual, which means that the same mode can be counted as more formal in one context and not in another. An example of this kind of mathematisation could be a fraction in the school mathematics context, such as one third. First it is shown to the students when the teacher cuts an orange in three equally sized pieces. The teacher holds one of the pieces in his hand, and calls it a third. The teacher then draws a circle on the white board, divides it into three equal pieces, colours one, and calls it a third. Finally, the teacher writes $\frac{1}{3}$ on the whiteboard. In this case the mathematical idea of one third was communicated through more and more formal modes, going from the orange, to the circle, and finally symbols. Kress (2017) calls changes of modes, such as in this case, *transductions*. A transduction entails both changes of meaning (epistemology) and changes of ontology. Changes of meaning occur even though many people would consider that it is the same mathematical idea, one third, that is represented through the different modes. However, the meaning of one third is becoming slightly different depending on how it is represented. One consequence is that a third is not the ‘same’ when represented as a part of an orange in comparison to when it is represented in writing with symbols. Teachers and students generally agree that they represent the same number, but from a multimodal perspective they are not strictly the same. The ontological changes concern the way that the world is displayed to us, thought about, and classified. In this case it is about what a representation of a number, such as one third, may be like.

In the above, I have tried to explain the two extreme cases, represented by the arrows; one where mathematisation is about how mathematical ideas may become more formal, even though the mode is the same, and one where the mathematical idea is the ‘same’ but expressed in more formal modes. In most cases, where mathematisations occur, these two kinds are both present and intertwined. This will be illuminated when I come back to the examples with Frida and Anita.

6. Mathematical arguments with younger students

In a project by Nordin (2016, see also Boistrup & Nordin, 2017), arguments in whole class discussions with younger students were investigated. Arguments in this study are based on a reduced version of a model by Toulmin (2003; Toulmin et al., 1979; see also Krummheuer, 1995). An essential part of an argument is that a *claim* is being made. As an example, I am making the claim in this paper that multimodal aspects need to be taken into consideration when investigating mathematisation. In order for a claim to be trustworthy, the claim needs to be based on something, which is called *data.* In this paper I am drawing on the examples of Frida (in this section) and Anita (in the subsequent section), which can be viewed as the data for my claim. In order for it to be regarded as an argument a *warrant* is needed. A warrant explains the link between the data and the claim. In other words, the warrant motivates how the data constitutes a foundation for the claim. The warrant for my claim will be presented below, when I show how the multimodal approach was helpful in understanding and elaborating on mathematisations in the examples of Frida and Anita. An argument may sometimes include a *backing*. A backing backs up the warrant, through for example a theoretical statement. In this paper, I back up my warrant with how I define mathematisation (drawing on Jablonka & Gellert, 2007), and how I explain multimodal social semiotics (e.g., Kress, 2017). I adopt features of the theories for analysing the examples (the data) in order to illuminate my claim about multimodality in relation to mathematisation.

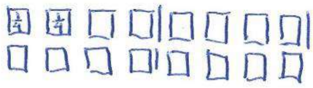
In the lesson where Frida was a student, the students had been working in pairs solving this problem:

> It is a field day and it is sunny and warm. The school will provide food and drinks. Each student is given a quarter of a litre of juice to drink. There are 16 students. How much juice will be needed?

Excerpt 4. Problem about juice

Some of the groups are asked to present their solution to the whole class. In many of the presentations the students use the modes of speech and symbols to explain how they went about the task. However, few of them presented any warrant. Hence, they did not present an argument in the sense of the model by Toulmin. With Frida things went in a different way, when she presented the work by her and her friend.

The claim, that four litres of juice was needed, was communicated at the end of the presentation through the finalised drawing (excerpt 5).



Excerpt 5 (same as excerpt 1). Finalised drawing by Frida

The portions of the sixteen students in excerpt 5 are grouped into four groups. The claim was also communicated by another student, who said four litres.

Earlier during the presentation there were two sets of data present. Frida presented one data set right at the beginning. She started to silently draw a rectangle. She then said "I started to draw all students' mugs," while pointing at the rectangle. She then continued to draw all 16 mugs, which was regarded as one set of data. The other data set was addressed by the teacher, and concerned the fact that one mug contained a quarter of a litre.

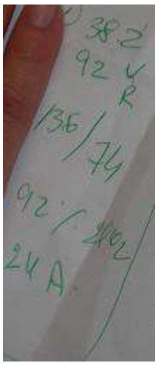
The warrant was constituted by speech and drawings. Frida said "I did it like this", and then she drew one line after four mugs (see excerpt 5), one after 8 mugs, and one after four mugs in the second row. In this way, Frida showed through her drawing of the lines how four mugs together were one litre, leading to the claim that four litres were needed. While the actual drawing of the lines was the warrant, the final drawing was then analysed as the claim.

The presentation by Frida is an example of a mathematisation in terms of the creation of a mathematical argument. Without many words, but through drawings and hand gestures, Frida claimed how many litres of juice that were needed. Moreover, Frida supported her claim when she explained how she drew on the 16 mugs, and she warranted the claim when she showed how four quarters of juice constituted one litre. Early in the project, when not all modes were paid equal attention, it was hard to capture arguments. Adopting a multimodal approach led to more arguments being possible to reconstruct, as in the case with Frida.

7. Mathematics as part of workplace practices

The next example is about Anita, a nursing aide, who works at a semi-emergency department. Anita took part in a project where the interest was in what school mathematics could 'learn' from workplace mathematics (Wedege, 2013, see also FitzSimons & Boistrup, 2017). In the project we strove to understand mathematics as part of workers' workplace activities, instead of using school mathematical notions as our analytical framework. We adopted a multimodal approach, which was complemented with sociological perspectives, for example Bourdieu (see Johansson, 2014), or an institutional perspective (Selander, 2008; see Boistrup and Gustafsson, 2014).

One significant assignment for Anita on a daily basis was to take the 'controls', such as taking patients' blood pressure.



Excerpt 6 (same as excerpt 2). Anita's notes

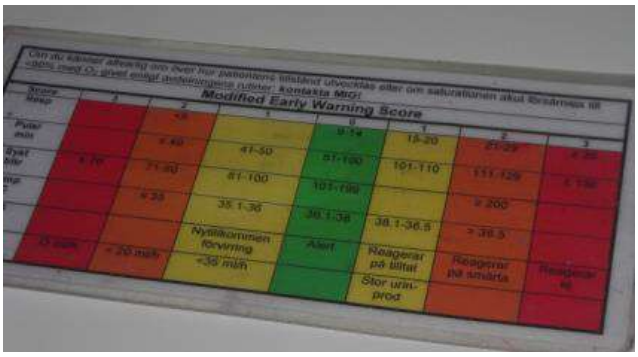
Excerpt 6 is a photo of the paper where Anita took notes when doing the controls. While checking blood pressure was something rather easy for us as researchers to capture, much of the time the main activity that we could see was that Anita seemed to take care of the patients. In a couple of interviews, one of the research team (Johansson, 2014) had the possibility to ask questions in order to learn more about what was taking place. In one instance, the interviewer introduced the idea to Anita that some of what she talked about was related to mathematics. Anita then responded:

> "For example I count the respiratory rate. All people breathe differently and when they are ill even more differently. Such things you don't learn … so I count the respiratory rate of one patient and get 17 … let's say 17 but I have learnt that this patient has 17 because s/he is ill. I can judge that, I can judge that the other has 17 because s/he really doesn't feel well, yet another has 17 because s/he is hyperventilating, and that one has 17 by pretending in order to get more morphine than s/he has already got, and that one has 17 because…" (Johansson, 2014).

Excerpt 7. Anita on mathematics that hardly can be taught at school

In the analysis it became clear to us that in Anita's workplace activities, there were mathematical aspects integrated with other aspects. In relation to excerpt 7, Anita made different judgements in order to discern if the patient was in the need of a doctor's attention or not, and it is clear that there was much more to it than to only count the breaths. It is possible to assume that she paid attention to the depth of the breath, and to the changes over time, and also other relevant aspects.

In her workplace context, Anita had the chart in excerpt 8 as a guideline.



Excerpt 8. A chart on early warning scores

The chart in excerpt 8 specifies which measurements of the controls are counted as warning scores. The scores in the green area are considered to be on the safe side, while the ones in the yellow areas may be problematic, and those in the orange and the red areas even more so.

In the research on mathematics as part of workers' activities, we adopted a multimodal approach in the data collection (videos and pictures) and in the analysis (explained in Boistrup & Gustafsson, 2014). As discussed above, we also performed interviews. These really helped in the analysis of the data from Anita's work.

As I see it, Anita did some kind of implicit mathematisation when she made judgements about the changes of her patients' respiratory rates. Clearly, there was no need for advanced mathematics, but the way it was integrated into her work activity seemed advanced to us researchers. Moreover, what Anita actually did was far from how the workplace is presented in textbooks and much of mathematics education for vocational students in Sweden (e.g., Boistrup & Keogh, 2017). It is also far from the mathematisation present in the chart in excerpt 8. The chart did not take into account aspects like the depth of breaths or the changes of the number of breaths. In relation to Anita's mathematisation, the chart could in reality be seen as representing a *demathematisation* in that it simplifies the work done by the nursing aides, and removes much of the mathematical aspects that were part of Anita's judgements.

8. Multimodality and mathematisation in the two examples

In this section I return to the tentative model in figure 1. In the case of Frida, she started by performing a transduction when she turned the words in the problem into schematic sketches of the mugs. In doing this she interpreted what the problem was about, and hence she also made transformations into more formal school mathematics. When she continued to solve the problem, she also continued to use drawings and some speech, and hence the mathematisation then was mainly about mathematical processes becoming more formal.

In the case of Anita, she took repeated measurements of the patients' respiratory rates while also making them comfortable, and even talking with them. She made a transduction of the controls into symbols in her notes. More implicit were Anita's subsequent judgements on how to value the number of breaths taken by a patient. I regard this to be mathematisation in the sense of mathematics ideas becoming more formal. While the mathematisations made by Anita were rather implicit, the coloured chart was more explicit. Here the measurements were reflected in numbers and colours. As such, this could be interpreted as a mathematisation in the sense of modes becoming more formal. The problem is that some of the relevant mathematical information might be lost here. What I mean is that it could rather frequently occur that a patient's value is within the green spectrum, but other evaluations could reveal the case that the patient, in fact, is getting worse.

9. Conclusions, multimodality and mathematisation

Throughout this paper I have illuminated how mathematisation is not a straight forward term. What I claim is that mathematisation may be understood as formalisations of mathematics ideas and/or formalisations of modes. One situation where such a consideration is relevant is in research processes. If we as researchers are interested (also) in implicit mathematics processes in school (as in the case of Frida) or out of school (as in the case of Anita), much is gained in adopting a multimodal approach. Doing this, it may be possible to capture mathematisations communicated with less formal modes that otherwise might go undetected. Moreover, with the tentative model in figure 1 I elaborated on how different kinds of mathematisations, in terms of mathematical ideas and in terms of modes, often are intertwined. Hence, there is a need for open interpretations in relation to both axes of the model. A model like that in figure 1 may be a possible foundation for a deeper and/or broader understanding of the complexities of contexts in relation to mathematisation.

The same argument could also be made for the teaching of mathematics. If teachers capture only mathematisations made with more and more formal modes, much of the mathematics that students display may be missed. This is relevant also in various assessments. If a student displays a mathematisation when solving a problem, there may be good reasons for acknowledging this in the marking, even though the modes used are not formal in a mathematical sense. In other words, adopting a multimodal approach in research and teaching may enable mathematisations in classrooms and elsewhere to be captured and elaborated on to a greater extent.

In addition to the above, the teacher has a significant role also in advocating more formal modes. Part of learning mathematics is to learn the function of formal mathematical modes. Here also, a multimodal approach as outlined in this paper may be useful. A teacher could capture the mathematisations of students through, for example, words and drawings. The teacher could also capture a similar mathematisation but transducted into more formal modes. In ways such as these, students may have the opportunity to learn more about mathematical modes.

To sum up, my claim is that it is important for us as researchers and teachers to make well-grounded decisions about how we deal with the multimodal communication taking place in all sites where education in mathematics takes place. This is relevant to many papers at CIEAEM69, where different multimodal aspects are present, and I hope that this paper can inspire in the writing of future texts.

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Construire des dispositifs à la frontière des mondes sociaux

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Abstract. Cet article présente une réflexion autour d'activités spécifiques de mathématisation, expérimentées dans l'enseignement professionnel. Ces activités s'ancrent dans les travaux du philosophe Michel Fabre sur la problématisation et son modèle du savoir. J'utilise ce modèle comme grille de lecture des cadres opérationnels qui ont permis de penser les pratiques d'enseignement des mathématiques dans l'enseignement professionnel. J'explique comment j'ai été amenée à conceptualiser la notion de « dispositif didactique frontière » dont je donne un exemple en statistique.

1. Introduction

Le double constat de la montée en puissance de la demande sociétale vis-à-vis des mathématiques et de l'invisibilité croissante de cette discipline, notamment en raison des développements technologiques, n'est pas nouveau (Keitel et al, 1993).

La mathématique peut se retrouver intégrée dans des construits culturels (numeracy, Goos et al, 2013, techno-mathematical literacy, Hoyles et al, 2010) ou au sein d'un ensemble interdisciplinaire (STEM). Parfois elle est entraînée dans un mouvement plus général de recomposition disciplinaire, qui fait émerger de nouvelles « disciplines ». Par exemple, la statistique, elle-même discipline hybride dont le statut par rapport à la mathématique varie selon les pays, se reconstruit en se combinant à d'autres disciplines : elle devient ainsi économétrie lorsqu'elle se rapproche de l'économie. Dans le cas du Big Data dont on parle beaucoup aujourd'hui, il ne s'agit pas seulement de recomposition du paysage disciplinaire. Les techniques mises en œuvre conduisent à réinventer les méthodes statistiques et au-delà, influencent les modes de construction des savoirs par la recherche comme en médecine par exemple. Le développement du Big Data soulève bien d'autres questions, notamment éthiques. En effet, une technologie n'est jamais neutre et toujours en lien avec une conception du monde !

Comment organiser l'enseignement des mathématiques dans cet environnement mouvant ? Quelles activités spécifiques peut-on penser pour la classe afin d'ancrer l'éducation mathématique dans un monde où la mathématique est à la fois omniprésente et de plus en plus invisible ?

Faut-il encore élargir la perspective, penser un cadre global afin d'aider les élèves à combiner des logiques différentes, voire contradictoires, pour problématiser un environnement de plus en plus complexe (Fabre, 2011) ? Quelle place donner aux savoirs mathématiques dans un tel cadre ?

C'est une question sur laquelle je travaille depuis de nombreuses années. Je vais présenter ici quelques éléments du cadre dans lequel sont ancrés mes travaux, le contexte particulier de l'enseignement des mathématiques dans la formation professionnelle, les ingénieries qui ont été mises en place et les résultats obtenus, et je discuterai les apports de ces travaux à la question générale de la mathématisation.

2. Un modèle du savoir en 3 dimensions

La question de la mathématisation nous renvoie à un autre débat, qui n'est pas nouveau non plus, mais toujours prégnant : il porte sur l'introduction de la « réalité » extra-scolaire dans la classe afin de relier les activités scolaires avec le monde en dehors de l'école, et sur la manière de procéder. On sait depuis longtemps qu'importer des contextes extra-mathématiques n'est pas suffisant pour que les élèvent s'approprient les savoirs mathématiques (Adda, 1976). Cela peut même avoir des effets négatifs comme je l'ai constaté lors d'un premier travail mené avec des apprentis vendeurs (Hahn, 1999). J'ai observé que pour calculer un rabais de 20%, ces apprentis multipliaient systématiquement par 0,8, puis soustrayaient le prix net du prix brut, au lieu de le multiplier par 0,2. Bien sûr, la méthode employée n'est pas incorrecte mais elle est plus longue. En fait, il s'agit d'une connaissance enracinée dans les pratiques professionnelles locales et renforcée par les pratiques scolaires. En effet, en magasin, les apprentis ne calculaient jamais le montant d'une réduction mais toujours le prix net et, à l'école, les enseignants leur demandent de calculer le prix net pour s'adapter aux pratiques en milieu de travail. Par conséquent, les élèves ne savaient plus comment calculer le montant d'une réduction directement.

Ainsi une utilisation excessive à l'école de « vrais » problèmes issus de l'environnement professionnel peut nuire au développement de connaissances plus générales. Doit-on en conclure qu'il faut renoncer à utiliser des contextes extra-scolaires et revenir à un enseignement des mathématiques décontextualisé, si tant est qu'il puisse l'être ?

Ou peut-on aborder la question du sens différemment ? Deleuze (1969) nous explique que le sens a trois dimensions : la signification (le rapport du sujet à ses actes), la manifestation (le rapport au concept) et la référence (le rapport au monde). Quel enseignement des mathématiques mettre en œuvre afin de prendre en compte ces trois dimensions ?

Selon Fabre (2011), le savoir a justement trois dimensions : historique, systématique et pratique. Le savoir est inscrit dans une histoire, une culture, il est intégré dans un corps structuré et il a aussi une valeur d'usage. Ainsi le théorème de Thalès renvoie « *à la fois à un procédé d'ingénieur pour mesurer les pyramides, à sa formalisation dans le système d'Euclide et à son engagement dans des problèmes qui n'auraient plus rien à voir avec la mesure des objets inaccessibles* » (Fabre, 2011, p 126).

Quel cadre opérationnel est en mesure de « permettre une circulation » entre les trois dimensions du savoir comme le recommande Fabre (2011, p126) et ainsi d'accéder aux 3 dimensions du sens ? Quelles activités scolaires peut-on penser dans ce cadre ?

L'enseignement professionnel me semblait un terrain favorable à l'exploration de ces questions. Aussi est-ce le terrain que j'ai choisi pour mener mes travaux de recherche, travaux qui ont pris successivement appui sur plusieurs cadres opérationnels utilisées par des chercheurs de ce domaine. C'est ce que je vais exposer dans le paragraphe suivant.

3. L'enseignement professionnel et la mathématique

Le monde professionnel est le plus souvent vu comme une source d'activités mathématiques « réelles » car il s'agit là d'un terrain précis et non d'un vague « reste du monde » (Strässer, 2015). Les situations professionnelles collectées peuvent servir de base à des problèmes scolaires, dans l'enseignement général comme professionnel. Le projet Australien RIUMIT (Rich Interpretation of Using Mathematical Ideas and Techniques) en est un exemple très abouti. Dans le cadre de ce projet, des enseignants ont été envoyés en entreprise afin de rechercher des cas d'utilisation des mathématiques pour illustrer des compétences-clé « UMIT » définies en amont. L'objectif premier était de collecter du matériel pour l'enseignement mais il s'est avéré que le projet a également été source de développement pour les enseignants qui effectuaient la collecte d'information : ils ont été souvent amenés à prendre part à l'activité du professionnel qu'ils observaient (Hogan & Morony, 2000).

L'enseignement professionnel n'est pas seulement une banque de contextes réels. C'est aussi un terrain privilégié pour qui s'intéresse à l'apprentissage des mathématiques (Strässer, 2015). En effet, l'enseignement professionnel peut permettre de mieux comprendre la relation entre cognition « en captivité » (à l'école) et cognition « sauvage » (en dehors de l'école) et de s'interroger sur les dispositifs à mettre en œuvre pour la faciliter (Bakker, 2014 ; Wake, 2014). A mon sens, il est au cœur de la tension entre dimension émancipatrice et dimension immédiatement opérationnelle de l'éducation mathématique.

Néanmoins, la place accordée aux mathématiques est de plus en plus réduite dans l'enseignement professionnel. Les contenus mathématiques sont intégrés dans les enseignements liés au métier choisi et ne sont souvent plus assurés par un spécialiste de la discipline (Hahn, 1998). Cette évolution est causée, au moins en partie, par l'absence de visibilité des mathématiques en entreprise, absence liée au développement des technologies (Strässer, 2000).

Di Sessa et Cobb (2004) proposent une typologie des théories utilisées dans la recherche en éducation. Ils définissent ainsi notamment les « domain specific theories » qui portent sur l'enseignement de savoirs disciplinaires spécifiques, les « framework for action », des cadres opérationnels parfois généralisés à partir des théories développées au sein d'un domaine disciplinaire et les « orienting frameworks », des cadres de pensée paradigmatiques.

Plusieurs cadres opérationnels, des « framework for action » au sens de Di Sessa et Cobb, ont permis de penser les pratiques d'enseignement des mathématiques dans l'enseignement professionnel.

Ces cadres relèvent d'ancrages théoriques différents et, compte tenu de ces ancrages, conceptualisent différemment l'apprentissage des mathématiques. Selon les cadres, développer la connaissance mathématique passe par la maîtrise d'une compétence de modélisation « circulaire » (dont le modèle de Blum & Niss, 1991 est un exemple paradigmatique), la « réinvention » des mathématiques (la Realistic Mathematics Education, RME), le dépassement d'obstacles épistémologiques (la Théorie des Situations Didactiques), l'intégration dans une pratique culturellement située (l'ethnomathématique et la théorie de l'activité historico-culturelle, CHAT). Ces cadres mettent l'accent soit sur le processus de construction individuel du savoir par le sujet, soit sur l'intégration du sujet au sein d'un collectif.

La plupart des cadres théoriques dans lesquels s'inscrivent les travaux de recherche, menés sur ce terrain, font référence à deux formes de mathématisation / classification / dimension, verticale et horizontale. Il n'est en effet plus possible de considérer que seule la seconde dimension doit être développée en formation professionnelle compte tenu de l'évolution rapide des métiers et de l'importance des connaissances mathématiques dans la formation du citoyen. FitzSimons se référant à Noss explique que « *In many work situations there is less reliance on traditional school mathematics skills which can be carried out more efficiently by computers, and greater reliance on ability to think in a mathematical way.* » (FitzSimons, 2015, p 110).

Pour Noss et FitzSimons (2015) qui s'inscrivent dans un cadre socio-culturel, il s'agit de « penser de manière mathématique ». Didacticienne, formée dans la tradition française, mon questionnement est centré sur les savoirs. Quelle place leur est accordée par ces cadres opérationnels ? Il faut d'abord s'entendre sur ce qu'on appelle savoir mathématique. En effet, il est défini de manière très différente dans les cadres opérationnels auxquels font appel les chercheurs en éducation mathématique dans la formation professionnelle. Certains ne le définissent pas du tout.

En utilisant la définition du savoir de Fabre (2011) comme grille de lecture des théories opérationnelles mentionnées ci-dessus, il semble que, bien qu'au sein d'un même cadre opérationnel les approches diffèrent, trois des cadres considèrent avant tout l'une des trois dimensions du savoir selon Fabre : la dimension pratique pour la modélisation, systématique pour la TSD et historique/culturelle pour l'ethnomathématique. La RME et le CHAT semblent permettre une approche plus équilibrée, mais cela varie en fonction des époques et des chercheurs.

Peut-on concevoir un cadre opérationnel qui puisse intégrer les trois dimensions du savoir selon Fabre ? C'est l'objet des recherches que je mène depuis plusieurs années, recherches dont je vais présenter quelques résultats maintenant.

4. Des dispositifs didactiques à la frontière de mondes sociaux

J'ai été conduite à hybrider mes pratiques et à faire évoluer les dispositifs conçus en intégrant des outils issus de différents cadres opérationnels. Je vais ici décrire deux de ces outils qui m'ont paru particulièrement pertinents et illustrer mon propos par quelques résultats de recherche avant de présenter le dispositif général que j'ai été amenée à conceptualiser.

Objets frontières

Un moyen de surmonter la difficulté liée à l'utilisation de problèmes « réels » pourrait être d'utiliser des objets frontières (Star & Griesemer, 1989), c'est-à-dire des artefacts utilisés en entreprise mais qui font aussi sens à l'école. Le recours à des objets frontières facilite la communication entre différentes communautés, ici l'école et l'entreprise (Akkerman et Bakker, 2011). Dans le travail que j'ai mené avec des vendeurs en bijouterie, j'ai observé que, la plupart du temps, pour calculer un prix hors taxe (taxe de 18,6%), ils « soustrayaient » 18,6% au lieu de diviser par 1,186. C'est une erreur très commune et résistante, un obstacle épistémologique (Bachelard, 1970). En se référant à Vergnaud (1990), les apprentis ont utilisé le théorème-en-acte « +x% est l'opération inverse de - x% » associé au concept-en-acte « % est une unité ». Pour aider les élèves à surmonter cet obstacle, j'ai utilisé différents objets frontières. Par exemple des instructions envoyées aux commerçants par leurs syndicats professionnels lorsque le taux de taxe était passé de 22% à 18,6%. Ainsi, j'ai utilisé une affiche remise aux bijoutiers, présentant le changement de taxe comme une remise : « -2,3% sur les prix marqués ». La recherche menée a montré une amélioration, mais seulement à court terme. Par ailleurs j'ai constaté que certains apprentis bijoutiers n'avaient pas identifié la réalité de la situation, alors même qu'ils y étaient confrontés au cours de leur activité professionnelle !

Concept pragmatique

Une autre notion que j'ai intégrée dans mon travail est celle de concept pragmatique. Cette notion a été définie par Pierre Pastré (1998), père de la didactique professionnelle. Elle est inspirée de la notion de concept quotidien de Vygotsky. Les concepts pragmatiques sont des formes de conceptualisation qui organisent l'action efficace. Ils sont de plus faible portée que les concepts quotidiens car limités au domaine professionnel mais, comme les concepts quotidiens, ils sont liés aux concepts scientifiques par une relation dialectique. Pastré affirme que les situations didactiques basées sur des concepts pragmatiques améliorent le développement professionnel. Je donne ci-dessous un exemple de concept pragmatique et explique comment je l'ai utilisé.

Il y a quelques années, une entreprise internationale de soft drinks m'a demandé de concevoir un cours de mathématiques pour ses marchandiseurs. J'ai interviewé des marchandiseurs et je les ai suivis pendant des négociations avec des chefs de rayons. J'ai réalisé que les stratégies des marchandiseurs dépendaient de la façon dont ils conceptualisaient la situation. Ainsi, pour vendre un nouveau produit, un marchandiseur fait d'abord un diagnostic en évaluant la « place », puis il adapte sa technique de négociation. Les chefs de rayon sont généralement réticents à référencer un nouveau produit car il est nécessaire de réorganiser les linéaires. Si le marchandiseur identifie un espace assez grand pour le produit en déplaçant les autres produits sur le linéaire, il a alors de bonnes chances de convaincre le manager. L'expertise est liée à la conscience de cette notion de « place ». L'évaluation de la « place » nécessite d'intégrer de manière implicite des calculs d'aires et de volumes et de géométrie dans l'espace, ce qui n'était pas prévu, ni par l'entreprise, ni par moi. Il est d'ailleurs à noter que le mot « linéaire » utilisé dans le commerce pour nommer les étagères en magasin est déjà une conceptualisation géométrique de ce que les étagères représentent dans ce contexte particulier. Comme cette conceptualisation pragmatique de la « place » est centrale dans le succès de la négociation, j'ai conçu une situation pédagogique basée sur celle-ci. Il a été demandé aux étudiants de préparer une stratégie de négociation entre un marchandiseur et un superviseur de département en utilisant la carte du département et les étagères, des informations sur les produits, sur les ventes des concurrents, etc. Les résultats obtenus ont été positifs mais l'activité était difficile à généraliser en formation initiale. Il fallait donc concevoir un dispositif plus général. C'est ce que je vais décrire maintenant.

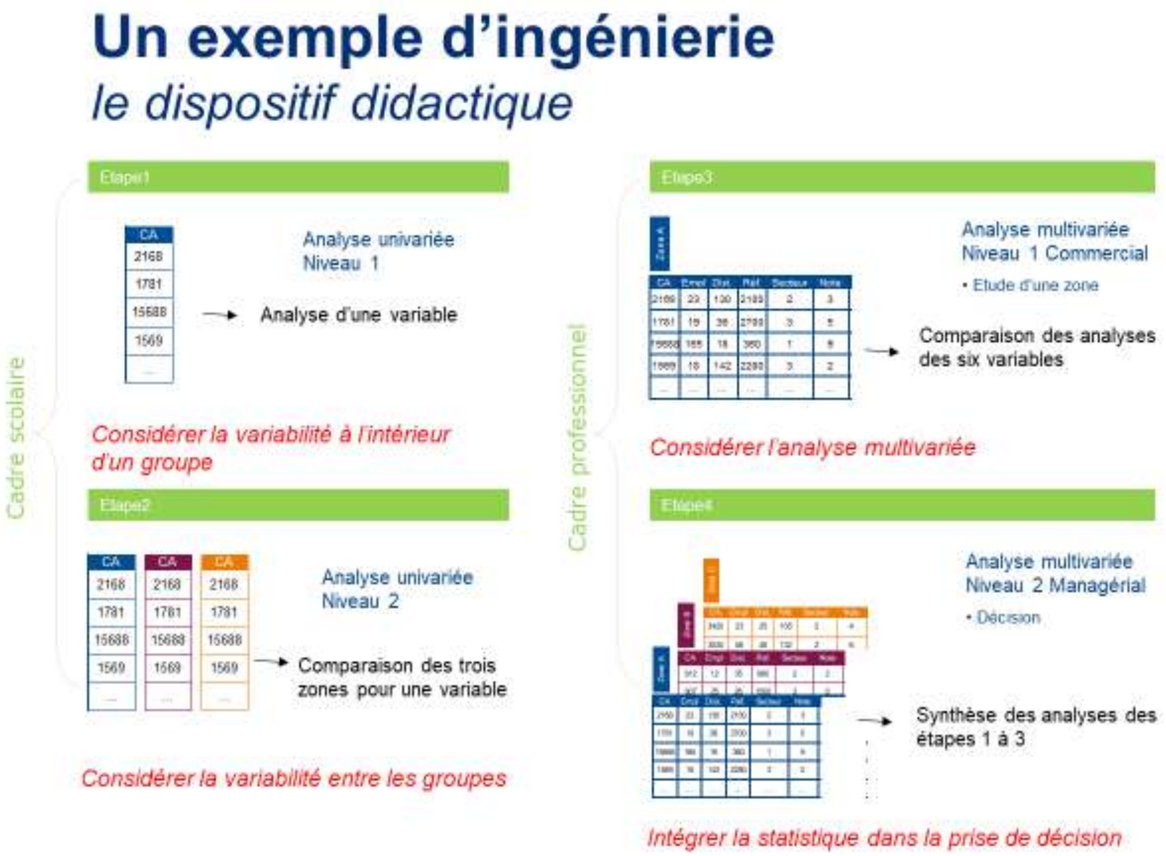
Dispositif didactique frontière

Dans le prolongement de ces recherches et ces expérimentations, j'ai été amenée à conceptualiser la notion de « dispositif didactique frontière » (Hahn, 2016). Il s'agit de dispositifs pédagogiques qui répondent à plusieurs caractéristiques :

1. Ils comportent plusieurs étapes qui doivent entrainer l'élève dans un jeu entre cadre de référence scolaire et cadre de référence professionnel
2. Ils reposent sur un processus de problématisation, c'est-à-dire combinant logique de l'enquête de Dewey et compréhension critique de Bachelard (Fabre, 2005, 2009). Ces dispositifs comprennent nécessairement une première étape de position (faire émerger le questionnement) et construction du problème (élaborer l'énoncé) par l'élève.
3. Ils intègrent des éléments ancrés dans le vécu professionnel de l'élève.
4. Leur structure respecte la logique interne de la discipline

Je vais décrire brièvement un dispositif didactique frontière en statistique, dispositif que j'ai expérimenté avec 36 étudiants en master dans une école de management (Hahn, 2014). La plupart d'entre eux avaient déjà travaillé comme commerciaux et avaient intégré un master afin de devenir directeurs des ventes. Ce dispositif est basé sur un problème authentique situé dans une entreprise réelle et décrivant un événement qui s'est vraiment passé : les étudiants devaient choisir entre trois zones commerciales afin d'obtenir le poste de directeur des ventes. Ils ont reçu un fichier excel de données collectées sur un échantillon de clients (entreprises) dans chaque zone. Pour construire la base de données, j'ai procédé à une analyse a priori (Artigue, 1988) afin d'identifier les obstacles statistiques. A partir de la revue de la littérature, il est apparu notamment que les étudiants ont rarement recours à des indicateurs statistiques (Konold et Pollatsek, 2002) et que, s'ils en calculent, ils n'utilisent pas le bon sens pour répondre aux questions statistiques (Bakker, 2004). Par ailleurs, les élèves ont une stratégie naturelle pour étudier les extrêmes et diviser en sous-groupes

(Hammerman et Rubin, 2004, Noss et al, 2000) et ils ont du mal à passer du local au global et à construire le concept de distribution (Makar et al. Confrey, 2005). Enfin il a été montré qu'il est nécessaire de considérer deux types de variation : à l'intérieur et entre les groupes (Garfield et Ben-Zvi, 2005). Le dispositif que j'ai construit comportait quatre étapes qui sont décrites dans la figure ci-dessous.



Je prévoyais que grâce à ce dispositif, les étudiants devraient passer d'un point de vue local à un point de vue global et que, pour choisir la meilleure zone commerciale, ils devaient relier les connaissances des deux cadres de référence, scolaire et professionnel, et intégrer les outils statistiques dans leur prise de décision. Les résultats de la recherche (Hahn, 2014) ont montré que l'utilisation d'indicateurs statistiques était limitée, diminuait au fur et à mesure que les étudiants progressaient dans l'expérience et dépendait du contexte (la signification de la variable) et pas seulement de la distribution des valeurs de la variable. Ainsi les étudiants utilisaient des pratiques professionnelles pour les variables qui leur étaient familières et avaient recours à des pratiques scolaires pour les autres.

Par ailleurs, j'ai constaté que les étudiants ne résolvaient pas tous le même problème : les procédures mises en œuvre étaient principalement liées à l'expérience personnelle qu'ils avaient associée au problème. Ainsi, en ce qui concerne le calcul de la moyenne pour la variable « note attribuée par le client », j'ai pu identifier trois conceptions différentes associées à trois stratégies différentes : la moyenne était vue comme une formule associée à un calcul, ou comme le centre de l'étendue (appelé faussement moyenne) entrainant une stratégie d'ancrage ou enfin comme un signal conduisant à comparer les distributions.

Les conceptions des étudiants étaient liées à trois formes de rationalité : une rationalité technique (application de techniques qui ne sont pas mises en perspective), une rationalité pragmatique (utilisation de stratégies intuitives pour atteindre un objectif limité à court terme) et une rationalité scientifique (intégration des théories pour éclairer le problème). Bien que la rationalité scientifique soit celle qui est privilégiée par l'école, j'ai constaté que la rationalité pragmatique prévalait généralement. Chaque forme de rationalité semblait liée à une identité dominante : la rationalité technique à une identité d'étudiant, la rationalité pragmatique à une identité de commercial et la rationalité scientifique à une identité de manager.

Abreu (2000) affirme que la résistance à l'utilisation de certaines connaissances spécifiques peut s'expliquer par le fait que certaines identités sont moins valorisées que d'autres. L'identité de commercial semblait effectivement plus valorisée que l'identité d'étudiant. Le défi consistait donc à faire converger les deux identités afin de donner accès à la nouvelle identité de manager. Les quelques étudiants qui ont construit une approche plus managériale du problème semblent questionner et lier les savoirs de différentes origines, commerciaux et statistiques, en s'émancipant des rôles qu'ils avaient précédemment construits.

5. Conclusion

J'ai présenté dans ce texte le cheminement qui m'a conduit à imaginer un dispositif que j'ai appelé « dispositif didactique frontière ». Il s'agit d'un dispositif de problématisation reposant sur un jeu entre cadre scolaire et cadre professionnel et respectant la logique interne de la discipline. Il ne s'agit donc pas de simplement « traduire » le réel mais de favoriser la mise en relation de conceptualisations de différentes origines afin d'établir la circulation entre les 3 dimensions du savoir mathématique, selon le modèle proposé par le philosophe Michel Fabre.

Bien entendu, ce type de dispositif ne peut se substituer aux activités traditionnelles en salle de classe mais, ponctuellement, il peut permettre d'aider les élèves à appréhender la complexité et l'ambiguïté des situations professionnelles en intégrant les dimensions humaines et éthique.

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Mathematisation in environments of Big Data – “implicit mathematics” revisited

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Abstract. On the occasion of this conference, I attempt to begin to explore the continuities and discontinuities in the functioning of the process of the mathematisation of social, economical and technological relations as outlined in the discussion document for the conference. In particular, I will argue that the discourse around Big Data constitutes a shift in perceptions of calculability and programmability. The associated techniques appear as at odds with orthodox practices of mathematisation in the form of constructing mathematical “models” and applying mathematical procedures. This shift can be interpreted as a new form of “veridiction” (in the sense of Foucault), but also as a strategy that abolishes the space for developing categories for identification, analysis and interpretation altogether.

1. Introduction

Three decades ago Davis and Hersh (1986) called attention to the political and ethical dimensions of processes of mathematisation in search for a “philosophy of computation”. In taking Descartes’s *mathesis universalis* as the symbol of a universal and unified method based on the capacity of reasoning by sets of rules for analysis and synthesis – modelled by mathematics, Davis and Hersh were concerned with the impact of applying mathematics in an increasing speed across a range of social practices. In particular, they explored how these applications were effected through the computer, in which they recognised a “marvellous mathematico-logical engine” (p. xii) or “a mathematical instrument par excellence” (p. 11). While they saw an emerging “mathematisation of our intellectual and emotional life” and imagined a society “in the grips of the symbol processors and the number crunchers”, Davis and Hersh did not yet envisage environments of Big Data, when stating that mathematisations intended to capture the human mind will “fall over their own absurdity and pomposity” (Davis & Hersh, 1986, pp. 13-16).

Meanwhile, speedy computations with massive data (often unwittingly contributed) are increasingly used across a range of social practices. Well-known examples include recommendation and filtering systems (such as news feeds, academic repository platforms, search engines, edit quality evaluation for wikis), online dating sites for partner matching, credit screening, high-frequency stock trading, data capturing and profiling for a variety of purposes (insurance policy, biometric recognition, mapping of criminal activity hotspots). The diversity of these examples suggests that for constructing a differentiated theoretical account, a careful analysis of modes of (re-)presentation and decision-making in a range of settings would be needed. There is more than one structuring logic in processes of mathematisation in environments of Big Data.

With Big Data comes a discourse about the capacity of computerized algorithms to complement, augment, replace or bypass human judgement, and eventually overcome imperfections and limitations of human faculties. A philosophy of computation that attends to political and ethical dimensions of processes of mathematisation, as envisaged and developed by Davis and Hersh, would indeed be relevant for explaining why the discourse of calculability by means of computerized algorithms in the context of Big Data does not fall over its own absurdity and pomposity.

On the occasion of this conference, I attempt to begin to explore the continuities and discontinuities in the functioning of the process of “mathematisation-demathematisation” as summarised by Jablonka and Gellert (2007). In particular, I will argue that the discourse around Big Data constitutes a shift in perceptions of calculability and programmability. The associated techniques appear as at odds with orthodox practices of mathematisation understood as constructing suitable mathematical representations, or mathematical “models’, and consciously applying mathematical procedures. This shift can be interpreted as a new form of "veridiction" (in the sense of Foucault). One project to pursue then consists in a critique that seeks to determine "under what conditions and with what effects a veridiction is exercised, that is to say, once again, a type of formulation falling under particular rules of verification and falsification." (Foucault, 1979, p. 36).

2. Point of departure: implicit mathematics in the form of "realized abstractions"

In drawing on Alfred Sohn-Rethel (1970), Keitel et al. (1993) described mathematisation as a process of formalisation that presupposes schemes of action, "real abstractions", that can be transferred across contexts. Reflective awareness of these real abstractions constitutes the basis for "thinking abstractions" that operate in a hypothetical-symbolic problem space and can be articulated in mathematical language. Mathematics so provides a symbolic technology that helps establishing equivalences between real abstractions. Once formalised by means of mathematical language, particular schemes of action can be installed as procedures in a material technology, for example as computerised algorithm. This process results in implicit (invisible) mathematics in the form of "realised abstractions". These are re-specialised and mechanised in the form of black boxes. In a summary of Keitel's et al. (1993) contribution, Jablonka and Gellert (2007) pointed out that the apparent permanent materiality of the computer together with the opacity of the programme, makes black boxes look particularly robust against being creatively changed, in contrast to explicit mathematics, for example as written and manipulated on paper.

Keitel et al. (1993) discuss Luis Mumford's (1967) famous analysis of the mechanical clock as an instrument for the management and control of social interactions. The example of the clock illustrates very clearly how technologies intervene in the very processes of the social practices from which they emerge. With respect to more complex technologies, Mumford assumed that these rely on highly developed schemes of action that manage the division of labour between singular techniques. Still in line with this conception, computer technology, which offers a very general functionality, can then be viewed as providing an organisational infrastructure for this coordination. Implicit mathematics so becomes a general symbolic organizing principle programmed into computers.

Mathematics here appears as a de-specialized symbolic technology that is manipulated according to explicit rules. Technology in general is conceived of as practical rationality directed towards achieving conscious goals. "The point of interference of mathematics within this complex [of technology] is the causal-logical order as a prerequisite of any mathematical modelling; the mathematical model both providing the means of acting and delineating a scheme for action." (Keitel at al., 1993, p. 246). This conception does not appear to apply to mathematisation in environments of Big Data. Mathematisation by means of computerized algorithms entails a change in the role of mathematical modelling as a basis for action.

3. Big Data and "veridiction"

3.1 Big Data

While in common parlance Big Data is associated with surveillance, Wigan and Clarke (2013) found the label used as promotional term in academic computing literature since the 1990s. In the context of public administration, early computer-aided systems were advertised as "automated information systems"; discrepancies between promises, perceptions and functioning in the interaction of social and technical elements were, for example, already discussed by Laudon (1974) or Kling (1978).

In computer science, Big Data refers to a range of strategies and specific techniques for increasing the efficiency in processing massive amounts of heterogeneous data sets and streams at high velocity. While "machine learning" and "artificial intelligence" have been pursued for quite some time, some advancements in mathematics and theoretical informatics have indeed been made possible because of the availability of masses of structured or unstructured data, which also result from increased sales of personal computers, tablet-computers and smartphones as well as improved web-accessibility.

Besides mathematics and informatics, which provide the operational principle of those components that involve computerized algorithms, conditions for the possibility of Big Data include developments in a range of engineering techniques and knowledge domains. These include computer hardware, in particular the miniaturization of storage media and "interface" devices for generating "data" (such as geolocation receivers, radio-frequency identification chips, SIM cards, sensors), and developments in signal technology.

Clearly, the notion of "data" here expands far beyond the conception of data we might imagine as represented in a symbolic language and consciously entered into a computer by a subject; "data" in Big Data rather resembles the conception of monitoring and control systems for complex technical plants, in which the source of the data is objectified. For example, for use in car insurance, "data" might include monitoring of route, speed, journey time, distance and "driving style" composed of some measuring of harsh acceleration, braking or accident deceleration as well as use of lights, seatbelts etc., "engine management" such as temperature, oil and fluid levels and fuel usage.

Technology associated with Big Data is mostly developed in partnerships with government, industry, and academia. Vendors and researchers who based their academic career on its development might tend to form alliances in reiterating a discourse about the creation of potential knowledge. This discourse is not just a marketing strategy. In following Foucault's conception of power as relational and productive, the discourse effects the mutually constitutive relations of procedures of governing conduct on the one hand, and "games through which one sees certain forms of subjectivity, certain object domains, certain types of knowledge come into being" (Foucault, 1973/1994, p. 4). Foucault analysed these as "rituals of truth" about the subject (Foucault, 1977, p. 194) or specific forms of "veridiction" (Foucault, 1979), that is, statements (in social sciences and humanities) that fall under particular rules of verification and falsification. In this spirit, some examples of Big Data applications will be examined in order to illustrate a discontinuity, a departure from the orthodox interpretation of computation and mathematical algorithm, and the underlying notion of mathematical model.

3.2 Example 1: Predicting the Future with Social Media

This example[1] results from a published article entitled "Predicting the Future with Social Media" (Asur & Huberman, 2010). In contrast to many reports about Big Data applications, this publication provides access to the strategy pursued. The study concerned the question of using tweets about new movies prior to or after their release for predicting the box-office revenues generated by the movies. The data consisted in 2.89 million tweets about 24 movies during a "critical period" of three weeks from 1.2 million Twitter users. Relatively simple measures were used and analysed with respect to their suitability for prediction of box office revenue data from the Box Office Mojo website. The models for the movies' "attention or popularity" were:

$$(1)\ Tweet\text{-}rate(mov) = \frac{|tweets(mov)|}{|hours|} \qquad (2)\ PNratio = \frac{|Tweets\ with\ Positive\ Sentiment|}{|Tweets\ with\ Negative\ Sentiment|}$$

$$(3)\ Subjectivity = \frac{|Positive\ and\ Negative\ Tweets|}{|Neutral\ Tweets|}$$

The first model uses the rate of mentions on Twitter, the second one takes the message of the tweets into account by constructing a sentiment analysis classifier by means of supervised machine learning. The training data, that is a sample classified as positive, negative or neutral texts, were produced through people hired through a service providing on-demand online-workers. The trained automatic classifier was then used to predict the sentiments for all the tweets in the critical period for all the movies considered. The third measure was used to test the assumption that stronger sentiments are found in tweets after release.

The study suggests that the intended prediction of box-office revenues worked well and may outperform market-based approaches. In conclusion, the authors state:

> "[…] this method can be extended to a large panoply of topics, ranging from the future rating of products to agenda setting and election outcomes. At a deeper level, this work shows how social media expresses a collective wisdom which, when properly tapped, can yield an extremely powerful and accurate indicator of future outcomes." (Asur & Huberman, 2010, p. 499)

3.3 Example 2: Mathematisation of "Ideological Segregation" in Online News Consumption

This is an example of how Big Data is employed to study the effects of Big Data. Flaxman, Goel, an Rao (2016) report from a study that addressed the effect of filter algorithms, known as Echo Chamber or Filter Bubble, which results from "personalised" searches or use of social media. They intend to study whether

[1] I am indebted to Felix Lensing for providing this example.

online news consumption leads to an "ideological segregation effect". As in the previous example, machine learning algorithms were employed in order to deal with the massive data used in the study.

One machine learning algorithm using a binary classifier on article texts identified 1.9 million stories as news. A second classifier separated descriptive news from opinion pieces. The training data set consisted of pre-categorised articles (news and non-news, descriptive and opinion). Articles were represented as vectors of relative word frequencies (excluding direct citations). Different sets of words, taken from most frequently occurring words in the used corpus of articles, were associated with the difference between categories of texts. An "audience-based approach" was used to construct a measure of an outlet's ideological slant. The political composition of an outlet's readership was inferred from the location of a webpage view via the IP address. The "conservative share" was measured by the fraction of its readership that supported the Republican candidate in the most recent presidential election.

They found, inter alia, that users mostly visiting "left-leaning" news outlets rarely visited substantive news articles from conservative sites, and vice versa for "right-leaning" readers; this was even more pronounced for opinion articles.

3.4 Example 3: Machines Judge Us Better than Humans

Kosinski et al. (2014) studied how some psychometric measures of users might correlate with their use of online social networks as reflected by some features of their Facebook profile, and in their preference (Likes) for websites, based on altogether 350000 U.S. Facebook users. They suggest that in the future, given a "personality profile" for many websites, larger scale studies based on browsing behavior would allow "personality" to be correlated with any other observable information about users.

In a similar study based on 700 Likes, correlations with some user's psychometric measures as well as with a set of simple classifications comprising two religious affiliations (Christians and Muslims), two political views (Democrats and Republicans), two categories for sexual orientation (Gay and Lesbian), gender (Male and Female), ethnic origin (African Americans and Caucasian Americans), relationship status (Single and Relationship), and substance use (Alcohol, Cigarettes and Drugs) was explored. Some "highly predictive Likes" included predictors of high intelligence by liking "Thunderstorms," "The Colbert Report," "Science," and "Curly Fries," and low intelligence by "Sephora," "I Love Being a Mom," "Harley Davidson," and "Lady Antebellum." (Kosinski et al., 2013, p. 5904).

To bypass self-report as a source of data and to automatize judgment, features as a major argument for the studies:

> "Our approach suggests that personality could be measured automatically based on records of online behaviour; thus enlarging the scope of psychological assessment to an unprecedented scale. It may even improve the quality of results as it considers actual behaviour in the increasingly natural digital environment rather than self-reported test answers." (Kosinski et al., 2014, p. 360)

> "Automated assessment based on large samples of behavior may not only be more accurate and less prone to cheating and misrepresentation but may also permit assessment across time to detect trends." (Kosinski et al., 2013, p. 5805).

A further study (Youyou et al., 2015) involved more than 70000 volunteers who were assigned psychometric measures based on a self-report (a questionnaire), which for some sub-samples were compared with computer-based "personality judgments" based on Facebook Likes, and "human personality judgments" from a sample of the participants' Facebook friends. In conclusion, they state:

> "Using several criteria, we show that computers' judgments of people's personalities based on their digital footprints are more accurate and valid than judgments made by their close others or acquaintances (friends, family, spouse, colleagues, etc.). Our findings highlight that people's personalities can be predicted automatically and without involving human social-cognitive skills." (Youyou et al., 2015, p. 1036)

Consistent with this statement, but with some non-committal hedging, in an interview with a German newspaper the first author of three of these studies, Michal Kosinski, told the interviewer:

> "My research shows that we now can make very detailed characterisations about you and your personality by using algorithms that look at your traces on the Internet … What we have seen so far is

that machines judge us better than humans, but I don't say though that I am right. There may well be other findings." (Kosinski, 2016, my translation)

Besides cementing the assumptions, methodological orientation and outcomes of the underlying psychometric approach to personality, the discourse in these studies, including the scientific publications, anthropomorphises the algorithm and "computers" in general. The machines "found" correlations of a mix of some socially available categories or stereotypes (such as Caucasian or African), psychometric measures from online questionnaires invoking "science", and digital traces in the form of Likes that can point to any element from an assemblage of "products, activities, sports, musicians, books, restaurants, or websites" (Youyou et al., 2015, p. 1036).

In the interview, when asked about the correlation of high intelligence with liking curly fries, Kosinski stated:

"It could be, and this I am also making up, that someone once at some university made a joke about curly fries and this joke has then spread amongst mathematicians. Or it was totally different. This is one of those connections only the algorithm sees, not humans." (Kosinski, 2016, my translation)

3.5 From Finding 'Known Unknowns' to Revealing the 'Unknown Unknown'

Donald Rumsfeld (former U.S. Secretary of Defence and congressman) infamously talked about the "unknown unknown" in a statement reported to be made in a press conference in 2002 in response to a question of making decisions in the face of lack of evidence for nuclear weapons held by Iraq.

The phrase has been popularised through a documentary film entitled the "Unknown known" (Morris, 2013) about Rumsfeld's career, in which he comments on the failure of the United States intelligence to anticipate the attack on Pearl Harbor:

"There are known-knowns, the things we know we know. There are known unknowns, the things we know we don't know. There are also that third category of unknown unknowns, the things we don't know we don't know. And you can only know more about those things by imagining what they might be." (Morris, 2013, 7:09 – 7:31).

The missing fourth category, the "unknown known" is used as the title of the documentary film, pointing familiarity and acquaintance, but without grasping. A lack of explicit theory, as it were.

In the context of "data mining", the "unknown unknown" became popular after a talk with the title "Big Data, Small World" given by Kirk Borne (2013), professor at the Department of Computational and Data Sciences at George Mason University (Virginia). In this presentation given at an event organised by TEDx and licenced by the media organization TED (Technology, Entertainment, Design), Borne explains he had used the "unknown unknown" in an earlier presentation to the U.S. Homeland Security Transition Planning Office. About "things you can do with data" he expands:

"You can discover things you know about but new examples; so these are the known knowns. Then you can discover things that are new examples of things you knew about but you never saw that example for; so these are the known unknowns. I also talked about things that we could discover and knew nothing about but would truly surprise us; and I called those things the unknown unknowns" (Borne, 2013, 3:10 – 3:34)

A rather unsystematic inquiry[2] of a sample of statements containing "big data" and "unknown unknown" suggests that there are two senses of the unknown event. On the one hand, there is a positive emotive connotation; the "unknown unknown" is a great idea, novelty discovery, jettison of traditional beliefs, innovation, opportunity, transformation and improvement. On the other hand, the connotation is negative; the "unknown unknown" is a potential fraud, threatening anomaly, detrimental effect, security issue, deterioration, or another unwished event. Incidentally, the "unknown unknown" also features as an ironic avatar name.

Practices associated with uncovering the "unknown unknown" include "data mining", "knowledge discovery", "information discovery", "data dredging", "data fishing" and "data snooping".

[2] I used the first 50 out of about 48000 websites found in 0,56 seconds by Google's query processor.

3.6 Automated "veridiction" and "truth" without a discourse

What may emanate from the examples, is a difference in strategies and techniques. These differences are also reflected in the discourse about proceeding from "known knowns" to identifying "unknown unknowns", the latter being associated with "machine learning". In the context of mathematical modelling one often speaks quite loosely about "data" or "variables". But, as noted above, with so called machine generated data, "data" that are being processed while captured, these notions are fuzzy. Further, the "variable" used by the algorithm at the microscopic level is not what one might think of when one looks at the mathematisation at a macroscopic level. Hence there is some need to problematize and discuss these notions. The same holds for the concept of "algorithm".

The differences in strategies of mathematisation can be captured by looking at contrasting tendencies in the construction of "knowledge". One tendency is to proceed from "data" to classifications and associations; the other to start with some theory and seek to link the "data" to the assumed classifications and associations. In conceptually separating "data" on the one hand, and the mathematical structure of the "model" on the other hand, the examples differ in the extent to which there are principles available for how the mathematical relationships relate to some "theory", and for how the quantification of the "data" is achieved. Principles for quantification include having *a priori* categories considered important and how these are measured, that is, translated into numbers. The internal mathematical structure of the "model" establishes some equivalences needed for mathematical operations.

These considerations lead to distinguishing the four different approaches depicted in Table 1[3]. The names used for the different strategies are used to connote the openness or closure of the process.

A "Definitive Mathematisation" involves a mathematical structure that is underpinned by *a priori* theoretical arguments as well as explicit rules for quantification. This is possible when a theory has become mathematised by being mapped on a mathematical structure and includes hypotheses that explain the conditions for results of measurements across particular contexts. One may, for example, think of applications of classical mechanics. By using the mathematical structure then there can be made theoretical derivations. Mathematics is intrinsic to the character of such a theory. The point is that this strategy of producing "knowledge" works without massive data. In the Big Data discourse, the "known known" features as the least interesting category.

A "Derived Mathematisation" can be thought of as a theory articulated in mathematical language, in which it remains unclear how variables or parameters actually would be or are measured; typical examples may be found in economic theory. In the Big Data discourse this strategy leads to identifying the "known unknowns", that is, "things that are new examples of things you knew about but you never saw that example for" (see section 3.5 above).

Table 1. Strategies of developing mathematics-based "knowledge".

	Internal mathematical syntax of "model" (mapping rule)	
Relation of the "model" to "data" (quantification rule)	Principles available	Principles not available
Principles available	*Definitive Mathematisation* Big Data discourse:	*Ad-hoc Mathematisation* Big Data discourse:

[3] For a discussion of these approaches and for their usefulness in understanding classroom activities in mathematical modelling see Burke, Jablonka, & Olley (2013).

	"Known Known"	"Unknown Known"
Principles not available	*Derived Mathematisation* Big Data discourse: "Known Unknown"	*Originative Mathematisation* Big Data discourse: "Unknown Unknown"

An "Ad-hoc" strategy establishes some rules for quantification and calculation, but without enunciating *a priori* principles on which the mathematical equivalences needed for the calculation are established in terms of their relation to the objects that are being mathematised. One may think of applications of empirical statistics. As this strategy of producing "knowledge" only relies on "data" (as opposed to "theory"), applications in Big Data are numerous. All examples reviewed above include this strategy, with the second and the third example also including some elements of an "Originative Mathematisation". Finding the "unknown known", which is missing in the Big Data discourse (but is the title of the documentary mentioned above), perfectly fits here, as this strategy is occupied with identifying instances or events without aiming at establishing theoretical principles.

"Originative" signifies inventiveness, even though lack of principles for selecting what is important and for how anything is calculated would not be associated with traditional methodological standards. So for this category one cannot find examples other than in Big Data, when "unknown unknowns" are supposed to be revealed through "patterns" in unstructured and heterogeneous large sets of "data", which often are themselves generated by algorithms, the ingredients being what happens to be available. In some applications of machine learning ("unsupervised"), what constitutes the categories in the field of interest is only in the making and so the criteria cannot be available before the mathematisation. For the same reason, there cannot be any theoretical relational space, on which the mathematical model is based. Moreover, in "machine generated data", the quantification rule is embedded in an algorithm.

4. Concluding discussion

In Big Data, often techniques that have been developed in engineering research are repurposed for use in other research areas, public administration or consumers' products. As a consequence, the practice in which these techniques then are used becomes arbitrary vis-à-vis their functioning. Therefore, their suitability cannot be articulated from the standpoint of the practice, in which they have been imported. Some of the studies discussed above draw on common social categories, such as U.S. ethnic categories or conceptions of "left-leaning" or "right-leaning", while others turn to constructions from psychometrics, such as in the example that uses Facebook profiles for "predicting" personality traits. The use on massive data of such classifications tends to immunise and stabilise particular constructions. On the other hand, in practices that have been based on traditional strategies of mathematisation, such as the use of statistics in public administration, computerised algorithms may produce classifications and associations, for which there are no social categories available; hence there are no tools for debate or establishing consensus.

As to specific forms of "veridiction" by means of calculations, Desrosières (1998) offers interesting examples in the history of the role of mathematics and statistics in economic theory. He maps two opposing trends in the emerging sciences about the population. One approach consists in manufacturing new laws from data; it starts with constructing indexes and aggregated data. The other approach starts with enunciating general laws presumed to be true, and then trying to connect these to data in order to measure the parameters. The latter strategy, which I called a "Derived Mathematisation", might also proceed towards using data to decide whether to accept or reject a "theory" (hypothesis, model, law) by means of statistical tests based on probabilistic models. Desrosières's (1998) history shows that different discourses are related to these strategies. Moreover, deviations between observations and generalizations have been interpreted according to various rhetorics.

The absence of a language of "hypothesis, model, law or theory" in the Big Data discourse reflects the view that "more data" have to be taken as inherently better and closer to the "truth". Moreover, there would not arise need for using inferences from samples for patterns at the macro-level, because, as the discourse suggests, there will be ever larger records of data. The "unknown unknowns" are not only extracted without any *a priori* perspective; but what the machine eventually produces will most likely not reflect any conceivable category or relation for which classifications and explanations are available in a discourse. And as the short interview extract (see Example 3, above) suggests, the point is that this does not matter.

In the context of studies of government that draw on Foucault, Rouvroy and Berns (2010) introduce the notion of governmentalité algorithmique (algorithmic government) and suggest that Big Data is a strategy of transforming the form of government established by public and private institutions. Their analysis assumes much of what "data mining" or similar techniques claim to be achieving with respect to "automated knowledge production" as already achieved. Rouvroy and Berns characterise algorithmic government as addressing the subject always indirectly by using often trivial non-interesting data about individuals that seem to appear spontaneously, which leads to a "banalisation of a surveillance" that does not target some persons *a priori*, but by default applies to everyone, and aims at action on actual as well as potential behaviour.

I have intended to show that there is more than one mode of "truth telling" about the subject in Big Data. The machine-aided generation of "unknown knowns" tends to reproduce myths of the culture industry by drawing on common sense cultural categories or stereotypes in producing a mix of other "Ad-hoc" classifications with reference to psychometrics as a "science". The associations between categories produced with this strategy can be illuminated by arbitrary narratives and a range of categories about individuals to which others and oneself could make reference, based on conflicting ideological positions. The automated production of "unknown unknowns", however, embodies the paradox of a knowledge without a discourse, that is, an "opaque evidence" as a basis for action. The Big Data discourse then suggests a new regime that abolishes the space for conflict or political discourse altogether.

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Geometrisation as a didactic challenge

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Abstract. A review of contemporary research related to teaching geometry shows that researchers have focused their attention on computer-based learning. I question the teaching of early geometry in this way. But new techniques have raised a problem that has been hidden so far - how can we start teaching geometry to take into account dynamic geometric reasoning? In this article I discuss some tips on this issue. I also show that a student's natural intuition about movement, which takes place in physical space, may be different than those that are important in mathematical descriptions concerning transformations.

1. Geometry and computers

Geometry as a school subject has been treated distrustfully for a long time. Its importance in the general process of mathematical education has been disturbed by many controversial views on school mathematics in general, and in particular by the so-called "New Mathematics." In addition, the discussion led by mathematicians themselves concerning the place of geometry among other branches of mathematics also influenced such a state of affairs. The sources of this state can be quite distant. Felix Klein reorganized geometry, suggesting that the 'content' of its founding is the concept of a group of transformations. Perhaps this approach – far away from an intuitive access - was one of the reasons that school geometry was slowly abandoned by researchers and educators as an object of interest.

Fortunately, this trend has changed, and for some time it has been possible to observe an interest in didactic problems related to teaching geometry. During conferences, more and more educators present research on school geometry. This is visible in the publication *The Second Handbook of Research on the Psychology of Mathematics Education,* published in 2016. A whole chapter, written by K. Jones and M. Tzekaki, titled 'Research on the teaching and learning of geometry' is devoted to problems related to geometry in school. But a review of research in modern geometry teaching, presented during PME conferences, leads to the surprising conclusion that it is dominated by using computer techniques. Almost every field of study demonstrates the use of computers in teaching geometry. In the light of this survey, you can even say that it is thought to be a remedy for any problems in teaching geometry. As the authors of the chapter write:

> *In general, the emphasis of subsequent geometry education research has increasingly been on the use of technology (especially forms of dynamic geometry software) and how this impacts on geometry teaching and learners' geometrical thinking (especially on the teaching and learning of geometrical reasoning and proving), on teachers' geometric content knowledge, and on teacher development for geometry education.* (Jones, Tzekaki, p.109)

Computer software undoubtedly brings a new aspect to teaching and learning geometry. It supports not only visualization but also construction and reasoning. In general, the essence of using computer programs consists of the ability to perform various transformations of geometric objects They facilitate experimentation – students can check certain assumptions and verify hypotheses. The screen shows what happens to the object when changing certain parameters. Programs allow users to introduce dynamism and observe processes over time, in a specific space. In this way, the new technology has brought to light the two elements that until now were rather in the shadows: spatial reasoning and the dynamism of actions taking place at a certain time. Here are some statements on this subject, derived from the chapter by Jones and Tzekaki:

- *….. spatial reasoning is now seen as a vital component of learners' successful mathematical thinking and problem-solving.*(Sack, Vazquez and Moral, 2010, p. 113)

- *…. spatial reasoning, more than being an important component of human action and thought, is known to be closely connected to geometric thinking and development of geometric knowledge* (Jones, Tzekaki, 2016, p. 113)
- *spatial tasks combining 2-D and 3-D geometric figures supported by relevant technological tools are likely to foster spatial-knowledge development and improve students' spatial reasoning, confirming, thus, the important role of technological environments in the development of spatial thinking.* (Jones, Tzekaki, 2016, p. 113)

Such an approach, consisting of using dynamic computer programs, is often appreciated by mathematicians who want to speak about mathematics education. Celia Hoyles and Richard Noss, in their memoir about Seymour Papert, (Hoyles, Noss, 2017) refer to his lecture in 1986 at the Tenth Conference of the International Group of the Psychology of Mathematics Education (PME10) in London. One of Papert's quoted statements was: "*….there is no doubt that in general much more can be done at an elementary level with dynamic than with algebraic characterizations of curves"* (Hoyles, Noss, 2017, p. 35).

Seymour Papert was convinced that computer programs could be a great boon for teaching geometry. he foresaw curve dynamics in the use of computers, in the possibility of observing the process of creating curves. While the juxtaposition: the curve - algebraic description suggests the use of computers at a slightly higher educational level than traditionally understood in elementary-level education, recently it has become quite common to believe that it is also worth teaching geometry to 6 and 7-year-old children in this way.

However, regarding the introduction of dynamical thinking into geometry, the question arises: is it really a good idea to use computer programs at the very beginning of learning geometry? Methods of mathematization are dependent on the tools available for learners. These need to be tools suitable for their skills, suitable for their own understanding of the world. If we give them tools that are too strong, we can break links with previous experiences and damage their own intuition. Even if we teach them how to use such a tool, the gathered knowledge will be isolated.

I will illustrate my objections about learning geometry at an early stage by using computer programs with a certain episode. 7-year-old students were asked to solve two tasks related to 'spatial orientation'. These were Task A and Task B (Figure 1 and Figure 2). Task A was as follows:

Take a closer look at the picture and draw the indicated routes. Describe aloud your route.

Some proposals of the route in Task A were like this:

- Go along the way from the circus to the school, you will pass a square with a row of trees on the right.
- The way from the bank to the museum, which does not turn left at all.

Task B was: Draw the tram route from their initial stop to the final one. The tram routes are described using arrows that illustrate directional changes.

Figure 1. Picture for Task A

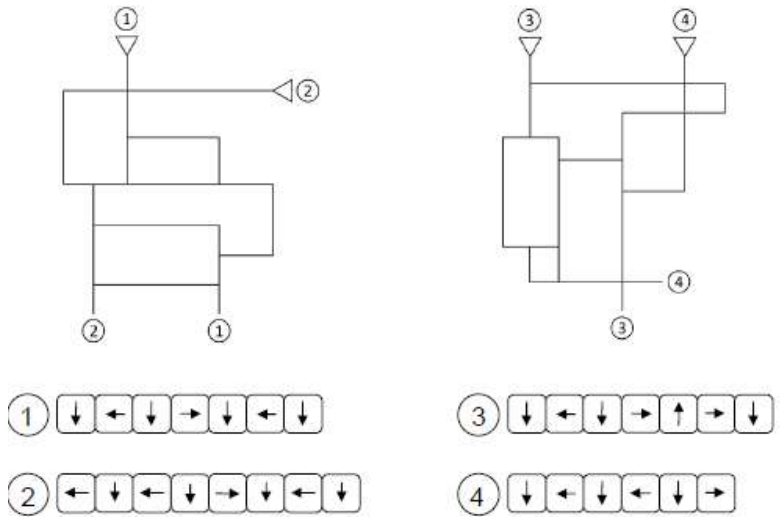


Figure 2. Picture for Task B

Surprisingly, it turned out that these tasks are not equivalent in the opinion of the students, and not only because of the need for a verbal description of the path in Task A. This is the conversation with one of the students – Adrian.

Adrian's work (7-year-old boy)

- Ex: I will read your instructions and you will have to draw a route and tell me how it runs.
- A: Erm…. I do not like this task.
- Ex: What do you dislike in this task?
- A: I just do not like it. And I will not do it.
- Ex: And this second task?
- A: Oh, there are arrows similar to those on a computer keyboard.
- Ex: Good association.
- A: What would I have to do here?
- Ex: In this task you have to draw a route from the starting point to the end of the line, guided by these arrows.
- A: Okay, this can be done.

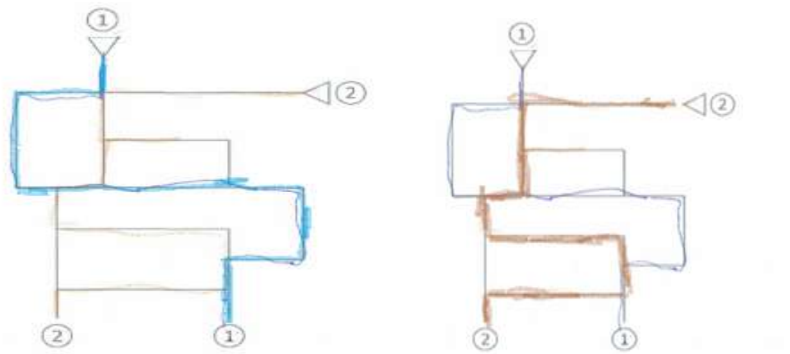


Figure 3. Solution of the first two problems from Task B, performed by Adrian

As you can see, Adrian had no problems solving Task B, although he assessed Task A as too difficult. Of course, changing the working environment always enforces specific actions typical of this environment. But is it not that flat-screen operation minimizes those elements that are important in the "space orientation" area? Is it not that the environment is very simplistic, it is already being automated, therefore does not require the mastering of the process mathematization?

It seems, therefore, that a reliable approach to the problem of the foundation of geometric education is needed. One that does not deal selectively with certain issues but that provides a clear framework for teaching activities. One that also contains dynamic geometric reasoning.

2. Does geometric reasoning require dynamics?

So, let's try to reflect on the role and place of movement in geometry through the lens of students' actions. Firstly, let's look at some student approaches to solutions of geometric tasks. The 14 and 15-year-old students in the groups solved two tasks concerning the cross-section of a cube. Although these tasks were very similar, in one of the groups the reasoning in one task was different than in the second one.

Task 1. Consider the net of a cube. Construct a shape ABIJ [where I and J are the mid-points of EF and GH respectively]. What shape is ABIJ in a cube? (Tasks 1 and 2 are suggested by the article of Fujita, Jones, Kunimune, Kumakura, Matsumoto, 2011).

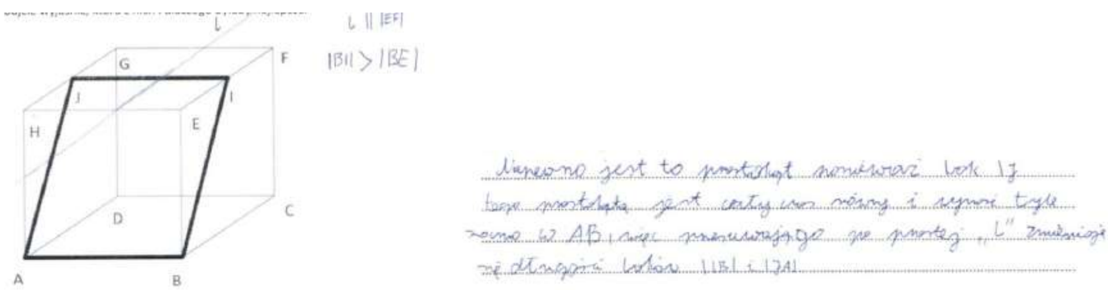


Figure 4. Task 1 and the answer proposed by the students

Answer: Certainly, this is a rectangle, because the side IJ of this figure slides along the straight line L. It doesn't change its length and is equal to |AB|. The sides of |AJ| and |BI| change their length.

The proposed solution is based on movement. This is a description of the motion of the frame, the two sides of BE and AH are made of elastic tape; the other two are rigid. The frame moves across the top of the cube. Despite some improper language or mistakes in the argumentation - the idea itself is interesting.

Task 2. Consider the net of a cube. Construct a shape IBJG [where I and J are the mid-points of AH and CJ respectively]. What shape is IBJG in a cube?

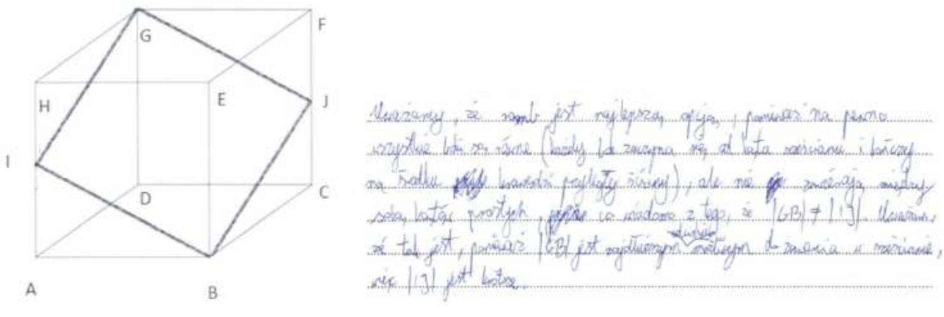
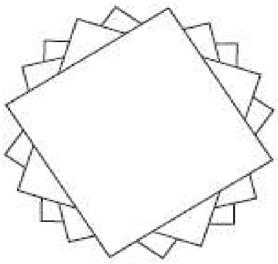


Figure 5. Task 2 and the answer proposed by the students.

Answer: We think that the rhombus is the best option, because all sides are equal (each side starts with a vertex of a cube and ends at the center of the adjacent edge), but do not contain straight angles, as is known from |GB| ≠ |IJ|. We think this is because |GB| is the longest segment possible to find in the cube, so |IJ| is shorter

Here we have another approach – by construction. Each element of the figure was treated separately. The main key to finding the answer lay in the comparison of two different diagonals.

The above tasks require some skills in making argumentations and are suitable for students with a quite large geometric experience. The question arises: is there such dynamic reasoning spontaneously present at a lower educational level? The results of one task (Fig. 6) taken from a large survey conducted by the Institute of Educational Research in Warsaw could deliver the answer. The main aim of the task was to check if children can see a square.



Mom put together some square napkins, as shown in the picture. How many napkins do you see in the picture?

A 20 B 16 C 10 D 5 E 4

Figure 6. The 'napkins' task from the survey for 10-year-old students.

In most sketches made by the students, we can see that they were trying to extract all the squares by matching the protruding corners. They marked pieces related to one square, sometimes by using different symbols for successive squares (Figure 7, 8). Such an approach could be classified as a static one, based on the global concept of the figure.

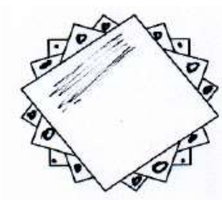


Figure 7.

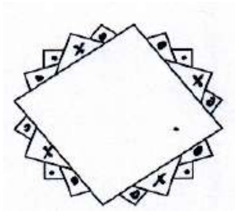


Figure 8.

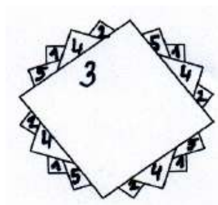


Figure 9.

In some sketches, however, we can find an approach where the student tried to reconstruct the process of laying napkins (Figure 9). In the presented example, the student numbered the tops of the particular squares. Number 1 is located on the square which, according to the student, was placed first.

3. What do theories say about procedural reasoning in geometry?

Let us start from the theory of forming a geometric concept – is there any place for movement here? What about the procedural understanding that is mentioned in theories on creating geometric concepts?

One feature of this approach relates to the distinction between the process and the concept made by Gray-Tall (1994). D. Tall and E. Gray have taken a decisive stand. They considered that geometric concepts are built in a separate way than arithmetic and algebraic concepts, which is illustrated in the following diagram

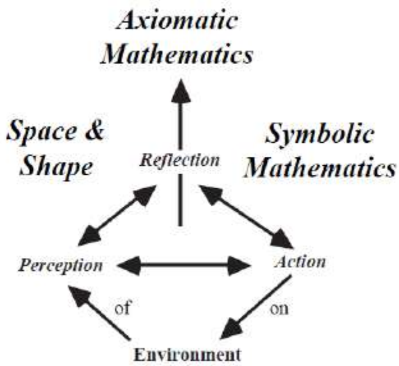


Figure 10. Various types of mathematics from D. Tall (2001)

Geometric concepts 'emerge' from the surrounding world through a specific "geometric sensitivity", a kind of sixth sense. "To notice something" is the first condition for the consciousness to focus on the geometric phenomena (Vopenka, 1989). This *first* cognition is passive and static. Such an attitude is a mathematical specification of what developmental psychology defines as a place of ***visual thinking*** in the development of intelligence.

Adopting such a model means that the creation of the transition between perception and action starts to be treated as a didactic challenge. A group of Czech educators, for many years involved in teaching geometry at all levels of education, have repeatedly emphasized the essence of physical experience (in the three-dimensional world) related to the accumulation of geometric experiences. The author, who mentions the necessity of connecting two approaches - conceptual and procedural - and who deliberately builds such passages in his teaching proposals, is Milan Hejný. *M. Hejný showed (2000) the importance of perceptual transfer in pupils' minds when they are grasping a processually perceived situation conceptually or conceptually perceived situation processually.* ***And it is the latter of the two directions that is much more frequent in geometry than in arithmetic*** (Jirotková, 2016).

4. Implementation of theory into school practice

Two outlined facts: the intrusion (invasion) of computer technology with dynamic educational programs to be used for teaching mathematics and the existing theoretical foundations of the creation of geometric concepts means that an educational researcher must seek answers to the following questions:

- When we assume that the creation of our own mathematics is carried out by mathematizing the phenomena of the world around us, then there is a need to understand how the process of this mathematization proceeds.
- If geometry is not limited to the static representation of objects, then there is a need to trace the process of movement mathematization

Movement analysis is a research area in the field of physics. Looking for elements where one can clearly talk about the relationship between physical motion and its mathematical description, in a natural way we turn our attention to geometric transformations, for example. Geometric transformations are a certain type of transforming one object into another. This is the idea of transformation as a function. In isometrics, it corresponds to the movement of one object into another. If we want to trace the origins of this concept in the realm of physics, the physical movement of an object will be suitable here. Nevertheless, such a transformation happens in a given time and while making a movement we can possibly trace the trajectory of an object. Everyday experience does not offer the possibility of recording consecutive stages of the object's movement.

Starting the investigation on the functioning of movements in geometry (which is some form of reflection on dynamism in geometry), one should answer the initial question:

Do we commit abuses assuming that the mathematization of movement is based on the experiences gained in physical experiments? A mathematician and a physicist look otherwise on the physical movement of the object, regardless of the fact that both use their own experiences in physical reality. Both must, therefore, focus attention on different observed phenomena. Which of them are important for mathematization?

5. Some various tips - how to mathematize movement

The functioning of movement in ancient geometry and, what is more, describing movement was not clearly allowed. Aristotle insisted that mathematical objects are not subject to movement except for those relating to astronomy. In Metaphysics, Book XI, Part 7 we can find: *Physics deals with the things that have a principle of movement in themselves; mathematics is theoretical and is a science that deals with things that are at rest, but its subjects cannot exist apart*.

However, the history of the development of science shows that the descriptions of movement have existed for a long time and the approach has changed over the years. The science historian and philosopher, Ladislav Kvasz (2002), characterizing the early stages of motion description, writes that **Aristotle's** movement theory represents each motion by a pair of "photographs" – a photograph of its initial place and a photograph of its terminal place. Nevertheless, this theory does not describe what happens between these two places (Kvasz, 2002 p. 603). **Galileo** inserted between the initial and the terminal place of Aristotelian motion a trajectory in the form of a *geometric* curve. Thus, he converted motion into a ***geometric flow,*** a continuous sliding of the body along a trajectory. (…) For Galileo, the motion is thus a *series of* ***static*** *photographs* ***put on*** *the thread of time.* ..... **Descartes** turned to dynamic photographs: on each photograph is captured, together with the position of the body, also its instantaneous velocity.

These comments should be interpreted didactically, according to the idea of didactic parallelism.

 ……………………… 

Figure 11. Construction created by a 5-year-old boy. **Figure 12.** Pattern created by a 4-year-old boy.

What we see in works created by 4 and 5-year-old children (Figures 11, 12) can be interpreted in different ways. It is very likely that a mathematician will see the intuitions of a parallel shift in them. This can also refer to Aristotle's idea of motion in which the relationship between the starting position and the final one is important. For the child who created the work, the movement was probably unimportant. What was important was the final effect of the movement, a nice arrangement of one object in relation to the other. So, what can be assessed visually as static layout? Such an arrangement - the relationship - can be taken as a good starting point for creating the notion of isometry. But from this point, it is, undoubtedly, very far to the dynamic vision of the movement.

Consecutive historical stages of motion description show that researchers have drawn attention to the trajectories of the moving body. Perhaps this is another element to which teaching should pay attention. Trajectories with the proper arrangement of physical experience can be observed. But what mathematical importance does their observation have and to what extent is this a natural approach?

Kvasz in his quoted work (2012) clearly highlights the differences between the physical approach to the studied objects and the mathematical approach. This difference manifests itself both in the method of analyzing the phenomenon as well as in its description. In word description, by linguistic reduction, we focus on some properties which are important for any reason. **The linguistic reduction is equal to abstraction,** it is the replacing of reality by its linguistic representation (Kvasz, 2012, p.532). But it is necessary to stress that *abstraction presupposes idealization, therefore it cannot explain it.* (Kvasz, 2012, p. 524).

In the process of idealization, we need to know what to pay attention to. The ideal object must, therefore, exist before we start the process of abstraction, so we must know what we should neglect. Therefore, physical idealization is different from the mathematical ones, because the physical properties of the objects are not the same as the mathematical ones. Physical properties are visible, mathematical ones are hidden as they are related to relations and invariants.

There are, however, authors who see a great advantage in using physics to shape mathematical (geometric) thinking. One of them is Mark Levi, the author of the book: *The Mathematical Mechanics: Using Physical Reasoning to Solve Problems* (2009). He states, inter alia: *Mechanical intuition is a basic attribute of our intellect, as basic as our geometrical imagination, and not to use it is to neglect a powerful tool we possess.* ***Mechanics is geometry with the emphasis on motion and touch***. *In the latter two respects, mechanics gives us an extra dimension of perception. It is that which allows us to view mathematics from a different angle* (p.4).

The following table is also created by him (Levi, 2009, p.7). It juxtaposes the benefits of early mathematical investigation of a problem with physical methods.

Table 2 Relative advantages of the two approaches. Juxtaposes – to present advantages.

Physical approach	**Mathematical approach**
Less or no computation	Universal applicability
Answer is often conceptual	Rigor
Can lead to new discoveries	
Less background is required	
Accessible to precalc. students	

In his opinion, *The main point here is that the physical argument can be a tool of discovery and of intuitive insight—the two steps preceding rigor. As Archimedes wrote, "For of course it is easier to establish a proof if one has in this way previously obtained a conception of the question, than for him to seek it without such a preliminary notion"* (Levi, 2009, p. 3).

6. Searching for a description of rotation

The above observations indicate the necessity of finding appropriate methods for investigating a relatively new phenomenon, which is the problem of dynamic reasoning in geometry. One of the attempts is presented below.

6.1. Methodology

Studies were carried out by different research groups. One of these groups was formed from elected students from Classes V and VI (12 and 13-year-old pupils). Primary school students possess knowledge on axial symmetry (solving tasks on drawing axially - symmetrical figures or plotting image figures in axial symmetry, primarily using square grids). Another group was created by high school students who were familiar with the definition of such transformations as parallel shift, rotate to any angle, axial symmetry and central symmetry. None of these groups had experience using dynamic computer programs related to geometry.

Students worked in pairs. They received a small book (size 10cm x 10cm) with 30 clean sheets and a template of a triangle.

The task was formulated in the following way:

> We would like you to draw a triangle in this book and make animations that change its position.

6.2. Aim of the research

Analysis of students' actions will lead to:

- determining spontaneous behavior related to the representation of movement in a plane,
- capturing those elements that may be included in the isometrics image and those that should be ousted during the development of mathematical concepts,
- diagnosing how the proposed activities can be useful as a starting point to talk with students about geometric transformations,
- determining the role of the teacher - what facts the teacher should be sensitive to in order not to impose ready-made mathematical formulas to students but develop spontaneous knowledge and skills; what to talk about, what to indicate, what to question.

6.3. Some findings from work done by primary school students

Analyzing 'the animations' created by primary school students who decided to create the rotation, it could be clearly noted that they were not focused on trajectories which the rotated figure had designed.

Example 1. Animation prepared by a 12-year-old boy (Figure 13):

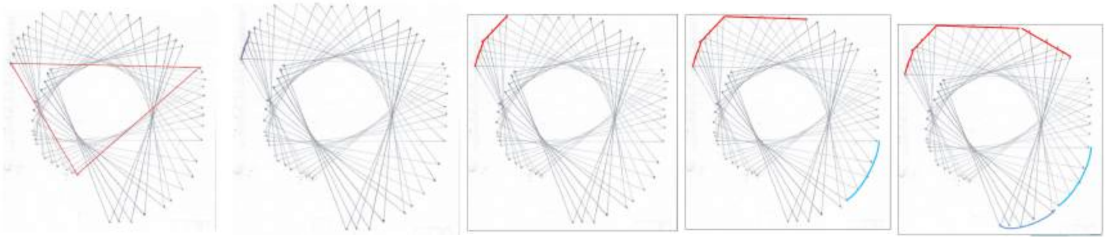


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Figure 13. Rotation created as 'a cartoon' by a 12-year-old boy.

The boy, working each time on a separate sheet, referring the new system only to the outline of the previous triangle's position and to the edge of the card, does not notice any trajectory indicated by the vertices' position. The first picture (Figure 13.a) shows a comprehensive summary presentation of all the positions of the triangle, as proposed by the student. The triangle highlighted in red was the original location. The following figures show the reference point, which is what directs the attention of the boy when making changes. This is not the center of rotation, but on the contrary – one of the vertices of the triangle. This vertex has changed its position. And the trajectory does not need to be circular.

A very striking example of a lack of such awareness can be seen in the work of one of the students (Zuzia, an 11-year-old girl). She worked together with a friend and, at some point, she decided to work independently. Then, she received a separate booklet and devices for drawing – unfortunately, she didn't have a template of the triangle. Undaunted by this, not waiting for a new template (easy to cut from the available materials), she decided to mark the vertices of a triangle on the following pages, using reflections from the previous card. Below, there is the result of her work, showing how she marked the location of the vertices:

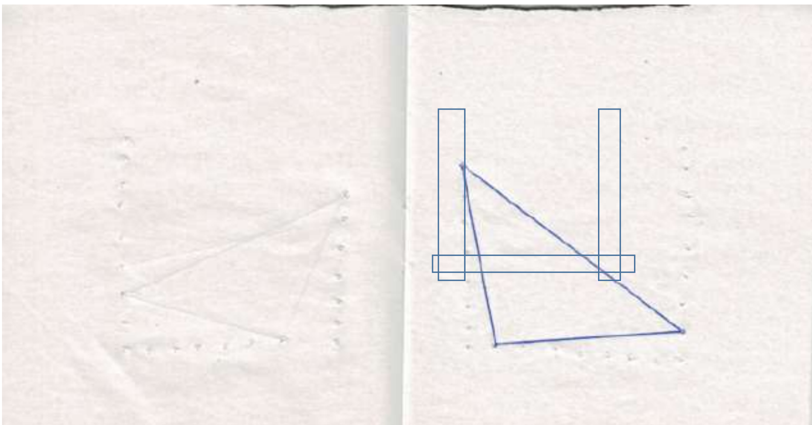

Figure 14. Location of one triangle and marked position of successive vertices, planned by Zuzia.

Zuzia was focused on the trajectory, determined by the successive arrangement of the vertices of the triangle. Regardless of the fact that her intention was to represent rotation, these trajectories formed straight lines, and they were aligned with each other at right angles. It can be assumed that such a layout has suggested to her square-shaped booklets, but not the idea of movement. In addition, she did not take any account of how this might affect the shape of subsequent triangles she drew. What she did is presented in two other figures (Figure 15 and Figure 16)

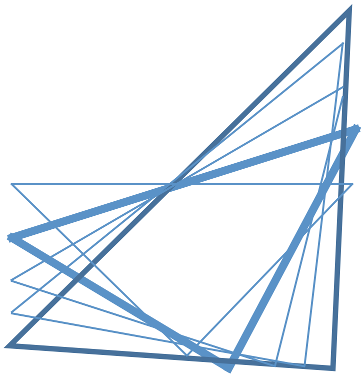

Figure. 15. A summary of a few slides made by Zuzia.

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Figure 16. Changes in the shape of the triangle when creating animation by Zuzia.

6.4. Work done by 16 – and and 17-year-old students – where is the center of rotation?

Another group was formed from 12 high school students (16 years old) who realized the topic "isometries". In these experiments, the students were not explicitly told that they must refer to any school (math) experience.

One of the groups created the following series of images (Figure 17):

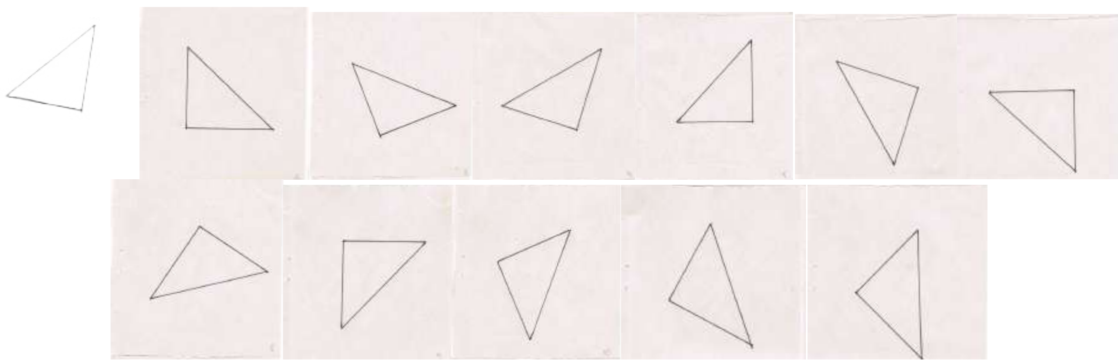

Figure 17. The series of images created by two 16-year-old students.

The combination of successive layouts on one slide shows that their intention was to create rotation.

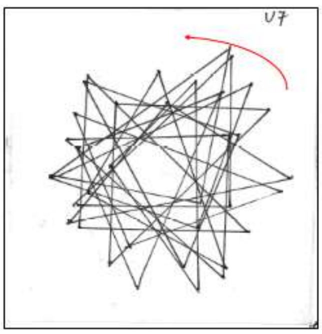

Figure 18. Summary scan of work with the direction of rotation, from Szkoła E. (2016).

Searching for the expected common center of rotation shows that there is no such thing. If the arrangement of each successive pair of triangles is treated as a relation in the rotation, then each center of rotation is in another place, as is evidenced for the pairs 1-2 and 3-4 (Figure 19).

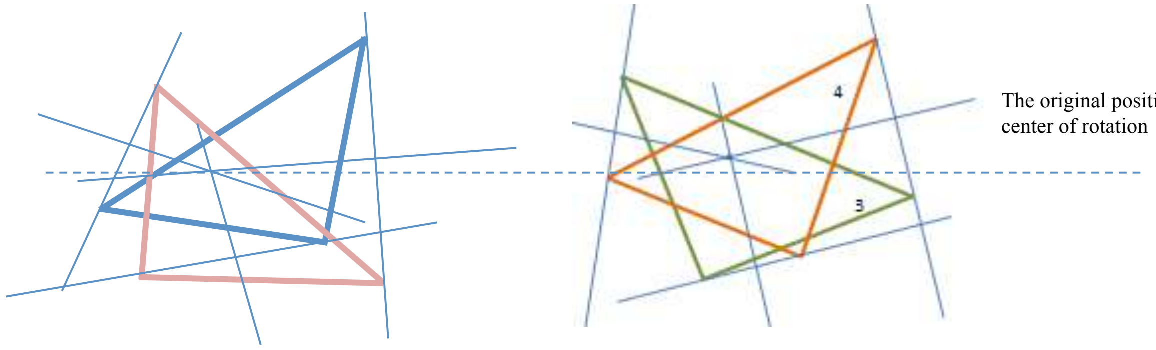

Figure 19. Changing position of the center of rotation.

In addition, the process of creating the animation itself shows that the student did not think about the center of rotation. She did not begin her work from determining the center of rotation. The knowledge that another "picture " will be created by her on the next sheet of the notebook with no possibility of recourse to an earlier position does not affect her need to establish some fixed points of reference. Her first action was the emplacement of the figure (template) in the middle of the paper. Her next action revealed that she intended to make the rotation. She implemented rotation around a point located near the center of the triangle. The work - as you can see on the scan - is very chaotic. The angle of rotation each time is selected quite freely, although the direction of rotation is retained.

6.5. General findings

In general, the approach to creating "a cartoon" with a rotated triangle has the following features:

- When the motion is realized, the most important aspect is to change the position of one side of the triangle in relation to its position on an earlier slide. This change takes place in a specific way - as a change in slope. With this approach, the angle is understood as a measure, not as a geometric figure.
- For students, it is enough 'to skew' a triangle. The existence of a fixed point is pushed aside.
- The presented examples show that student activities deviate from the descriptions of motion included in the geometric rotation. Students in **their actions** (solving problems) do not focus on maintaining invariants but rather think about the change. In such a way they focus on physical features, not mathematical ones.

7. Conclusions

New teaching resources create new challenges, both for teachers who intend to use these resources and for researchers. My point of view is the researcher's point of view. Dynamic computer programs have extracted a significant research problem that has so far existed in hiding. This is a problem of the mathematical awareness of observed motion. As such - this is a multi-faceted problem. One of them is the ability to use a dynamic image in the process of solving a geometric task. Another - the ability to emphasize these aspects of movement, which is described by mathematical (geometric or algebraic) language.

The results of observing students' work confirm that dynamic thinking is not a natural intuitive strategy for them. But, on the other hand, the students showed that they can think dynamically in the geometric environment. It seems that an attempt at shaping a dynamic intuition can give good results because such strategies appear in the children's work. But the matter of how it should be performed in a proper way is still open.

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SEMI-PLENARIES

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MATHEMATISATION: SOCIAL PROCESS

& DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL

& PRINCIPE DIDACTIQUE

De/mathematising the political: Bringing feminist de/post-coloniality to mathematics education

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Abstract. Various interpretations have been given over to the term 'mathematising' in relation to a variety of social thematic contexts in mathematics education. By way of theoretical intervention, we offer the beginnings of a feminist de/postcolonial commentary in response to such social, cultural and political programmes of work, while recognising the important contributions they have made to advancing complex political approaches to mathematics education as praxi. These social thematic approaches have promoted alternative ways of envisioning mathematical activity. Our critique is as much a celebration of these programmes of work that have offered diverse conversations about what it means to de/mathematise, as it is a way of moving these conversations forward in newer, alternative politico-epistemological directions. We argue that feminist de/postcoloniality offers opportunities to centralise ethical, democratic and (geo)political considerations in de/mathematisation activities and events, while bringing concerns about social and economic development, culture, gender, and global (in)justices to bear on mathematics education arguments. We suggest that feminist de/postcoloniality provides theoretical concepts by which we can speak of ontological and epistemic considerations politically in mathematics education.

1. Introduction

Much has been said on the nature of mathematising and demathematising and their necessary underpinnings beyond a conception of 'horizontal and vertical mathematisation' (Gellert & Jablonka, 2007). A need for critically engaging the concept is advocated in the CIEAEM 69 conference theme announcement. We are taking as parallel working definitions of mathematising: first, a process that renders a set of activities increasingly mathematical; as well as, second, the increased political influence of the voice of mathematics and mathematics education (and STEM) in the social domain. Dialectically, demathematising would have the dual reverse effect of diminishing the mathematical effect or qualities in a set of activities, and/or its political diminishment. These multiple, dialectical understandings are constantly at play in the way in which they concomitantly and often contradictorily inform each other. An example of this may be mathematics and mathematics education's increasing technological and economic utilitarianism under global neoliberal governance (Swanson, 2010), which has the dual effect of entrenching its perceived importance while in servitude to specific technoscientific and economic agendas.

In this paper, we attempt to move beyond current understandings of what it means to mathematise in relation to varied social thematic contexts at the service of mathematics education. These social thematic contexts span from realistic mathematics education, to real world activity, critical mathematics education (CME), Ethnomathematics, arts or craft based mathematics, playful contexts, media-based materials, pseudo-contexts based on word problems, and related themes. By way of theoretical intervention, we offer an entree to feminist de/postcoloniality in response to such programmes of work, recognising the important contributions they have made to advancing complex didactic and socio-political approaches to mathematics education as praxis in relation to society, and the way in which they have promoted alternative ways of envisioning mathematical activity. They have encouraged us to think beyond the confines of school, classroom and curriculum as containers of knowledge and knowledge circulation, ways that envision mathematical pedagogy as praxis that relates to people and their histories of the present in complex societies.

Nevertheless, we provide some distinction from these programmes of work by attempting to move these conversations forward in newer, alternative politico-epistemological directions, ones which more centrally consider feminist and de/postcolonial perspectives.

While socio-political projects are still relatively recent in mathematics education (Ernest, Sriraman & Ernest, 2016), there is a need to explore how a feminist de/postcoloniality might offer opportunities to centre ethical, democratic and (geo)political considerations in de/mathematisation activities and events, while bringing concerns about social and economic development, culture, gender, and global or local (in)justices to bear on arguments in relation to mathematics and mathematics education discourses. We suggest that feminist de/postcoloniality may usefully provide theoretical concepts that enable us to speak of ontological and epistemic considerations *politically* in mathematics education in 'glocal' contexts.

2. Mathematising, demathematising and 'the political'

From the time of Gallileo who argued that the book of nature is written in the language of mathematics, to Freudental who coined the word 'mathematising', a dominant conception of mathematics as the 'Queen of the Sciences' has pervaded discourses in the public domain, and these inheritances largely remain in educational contexts where mathematics is taught. Pervasively, in the school setting, 'pseudo' contexts of real-life have often served as exemplars of mathematising as if providing a straight-forward entry to 'the real'. In the many educational and social contexts, mathematics often has been divined as revealer of Truth. Its reification within Enlightenment discourses has perpetuated such dominance (Swanson, 2005) thus giving rise to critical conversations about the potential benefits and dangers of de/mathematising within society in the context of a world structured according to socio-economic, epistemic, embodied and political hierarchies and widespread inequalities of every form (Ernest, Sriraman & Ernest, 2016).

The need to open up diverse meanings and spaces for conversations about the nature of de/mathematising has become ever more urgent in the face of the perpetuation of a widespread singular logic structured around a hyper-pragmatic, ecomomically-informed 'reality' and pervasive neoliberal 'common sense'. In this light, we argue that there is a need to see de/mathematising as a broad processual, interactive and evocative space where discourse, power, and 'the body' come to influence ecologies of knowing and being by way of coming to know the world through mathematics (Swanson, 2013a; Chronaki, 2009, 2010). Mathematising is therefore unavoidably political, and cannot escape such influences and positionings through a call to objectivity and the lure of certainty (Swanson, 2005). In another sense, mathematising often works like religion as a moral axis (Chronaki, 2005).

De/mathematising's necessarily political nature is a condition we purposively embrace rather than attempt to render as neutral, which we argue acts as a political positioning in itself. Forefronting the acknowledgement that mathematising activities are informed by relations of power and cultural-historiographical investments, it is in the understanding of this purposive political act that we bring feminist de/postcolonial perspectives to bear on such conversations. We are not following an expected paradigm of academic engagement by offering 'solutions', but rather attempting to grapple with complexity in problematising the myriad of issues at hand and in opening up alternative conversations about what it means to mathematise and its many effects in contemporary society in this political moment. The effects of de/mathematising social activities can be traced to some degree through the effects of power in which mathematics education discourses and practices cohere, constructing particular 'regimes of truth' (Foucault, 1980), through the evocative power of context (Bernstein, 2000) and in embodied ways. Yet, the ethical implications of power dynamics are often left unattended in the literature, with some attention being given to Levinasian perspectives for example (Maheux, Swanson & Khan, 2012).

3. Ethics of mathematising and democracy

Considering ethics in terms of rights and democracy, many areas of theoretical interest to mathematising as social processes often see the advocacy of mathematics as an automatic good, albeit that the manner and nature of mathematising and pedagogy count. Within these terms, the effects on people's lives and ecology are understated. Here, much advocacy of mathematising from these perspectives leaves fundamental assumptions unquestioned and unquestionable. A critical relationship with democracy for mathematics education (Skovsmose and Valero, 2001) involves an active (re)direction of its intents and purposes. What is seldom asked, however, is the question of whether choosing *not* to participate in experiences of mathematics education or its (re)direction were itself also a critical relationship with mathematics education.

Seldom is the view held that the refusal and disobedience to the evocative power of mathematics is also a democratic action. Swanson & Appelbaum (2012b) argue that mathematics education for democracy and development must take seriously specific acts of refusal that directly confront the construction of inequality common in most development contexts. They argue that globalisation and development discourses, via citizenship and nationalism, construct oppressive relationships with learners and mathematics education. Such relationships are coercive and based on assumptions of the inherent goodness of learning mathematics and of mathematising as a virtue or the right to mathematics education is one and the same as the expectation to do so, for the person and/or society's own good. Seldom is the action of refusal to participate in mathematising activities understood in the light of a refusal to participate in mathematics education's colonising and/or globalising neo-liberal gaze. Bringing Jacques Rancière's (2009) notion of 'radical equality' to mathematics education theory helps to advance the ethical and emancipatory position of intentional disregard for ideological narratives such as the ones produced by mathematics education discourses. Consequently, by reconsidering the assumptions behind mathematics education, one can reframe refusal, disengagement, disobedience or resistance not as deficit or failure but as a critical position of radical equality in relation to arguments on mathematising, access and choice.

4. Feminist de/postcoloniality and mathematics education

The origins of postcolonial studies in the field of science and technology, as Harding explains (1998), can be traced back to the 1940s when a West Indian historian looked at how the immense profits from Caribbean plantations had played such a crucial role in making European industrialisation possible. This early investigation revealed how the British had intentionally destroyed the Indian textile industry in order to create a market for imported British textiles. Postcolonial studies have helped to reveal that imperial control has driven the politics of scientific knowledge historically, likewise postcolonial scholars have undermined the assumption of a single universally-valid scientific and technological tradition by offering evidence of alternate, localised ways of knowing scientifically. Furthermore, they have documented how the modern European 'utopia' of a perfectly coherent account of nature's regularity and order is beginning to take on the character of 'tragedy of the commons' (Lloyd, 1833; Hardin, 1968).

Throughout the years, the categories of 'woman' or 'black' have become the subject of an extensive literature mainly through the accounts of travelers, missionaries and colonial officials. Andrea Cornwall (2005), in her review of postcolonial feminist studies in Africa during the last three decades, explains that efforts to 'read' women range from studies that tend to define women as invisible, weak, and powerless to studies that challenge stereotypical assumptions about women's ability to participate in economics, mathematics and politics. Such representations are often firmly— but tacitly—located in a Western feminist perspective and evoke contradictory images, while their relevance and utility have been increasingly questioned by activists and academics. Postcolonial feminisms differ from the liberal, radical or socialist feminisms as they focus mainly on conceiving gendered and power relations within global political, economic and social programmes. They interrogate the assumption that the liberal pursuit of progress, development and colonialism are distinct and dominant projects. Thus, the question is how the distinctive concerns of postcolonial feminisms call for distinctive approaches to questions of science and technology, and call for a revisiting of children and adults' relation to mathematising via a feminist de/postcolonial lens.

While anticolonial has been touted as having some relevance to mathematics education discourses, there has been little attention given to postcolonial and decolonial thinking in the ways in which it offers a critique of mathematical knowledge and mathematising, informed by colonial relations and the politics of knowledge. Increasing neo-liberalisation of institutions and the global modernisation agenda has set the terms of global economic and social participation, by increasing the monitoring and regulation of individuals, groups and targeted communities. Such measures serve to perpetuate the global neo-colonial project.

The current conception of development, framed as 'economic progress' within the neo-colonial project, has excluded a range of other possible meanings and ways of engagement (Swanson & Appelbaum, 2012). This has been the experience of mathematics education in its increasing standardisation across the globe in assessment regimes, curricula, and pedagogy. This 'standardisation' has been invested in power, suppressing the cultural and localised ways of knowing in majority world contexts or global South via 'development' agendas, while installing the values, codes and epistemic relations of the minority world or global North 'as universal'. Development as a concept presumes a *need* for development on the part of the targeted communities. In this sense, any development programme situates the communities that are ostensibly aided as 'lacking' and in need of assistance. At the same time, political discourses within developing countries often frame the needs of their (often black and/or female) citizens in terms of deficit and economic lack (Swanson & Appelbaum, 2012), blaming their citizens for their own and the country's economic 'failures' (Swanson, 2013b) for which national school mathematics results become the weapon.

Considering the global social imaginary of the current neoliberal world, it may be timely to bring some de/postcolonial theoretical concepts to bear on mathematics education in global development contexts in providing a geo-political focus that more centrally considers the role of the nation-state, the geo-political imaginaries of empire and the broader neocolonial/neoliberal global(ising) condition in respect of mathematics education in global context. Some post/decolonial ideas valuable to critiques and conversations in mathematics education are enscribed around such foci as (for example): centre-periphery discourses, loss and exile, disavowal and dispossession, epistemic violence, epistemic suppression, epistemic racism, abyssal thinking, representation and voice in geo-political context, othering and exoticism, global social and ecological injustices, discourses on dominance and the subaltern, benevolence and salvationist discourses, global/local asymmetrical relations, cultural imperialism; and the problem of 'dividing the world' (East/West; South/North; developing/developed; margins/centre; majority world/minority world) (Swanson, 2010, 2013a). These, and others, offer opportunities to provide frames of reference with which to converse with mathematics education from wider geo-political and global justice-oriented perspectives (Chronaki, 2008; Swanson, 2013a, 2013b).

5. Conclusion

Mathematising and demathematising have been given some attention in relation to social processes of mathematics education via such work programmes as ethnomathematics, critical mathematics education, and realistic mathematics. They have done much to underscore an interpretation of mathematics as being invested in cultural, historical, economic and social norms and values. Critical mathematics education in particular has pushed the conversation forward in considering mathematics in its broader political enterprise, but the theoretical concepts borne from feminist de/postcolonial thought situate conversations on mathematics education in terms of contestations between global political imaginaries, while bringing into play the epistemic and ontological implications of such political considerations. Feminist de/postcolonial thought begins to reverse the symbolic violence of Northern-emanating discourses within the mathematics education field, by introducing the thought of theorists, such as Spivak, Mignolo, and Quijani, that hail from the global South. It opens up the opportunity to consider global ethics and democracy in relation to mathematising activities and discourses. As such, it also brings in the sphere of the geo-political while attending to the local or individual level, (i.e. the glocal), in considering culture, gender, socio-economics and class, amongst other difference discourses, in historical and political contexts and their investments in global social relations of power.

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Mathématisation en contexte d'enseignement : quelques enjeux autour de la résolution d'un problème « réaliste »

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Abstract. This study is part of a collaborative research that aims at informing the work of mathematics pedagogical consultants on classroom problem solving (PS) and teachers support. More specifically, we try to identify key issues mathematics pedagogical consultants face, within their professional practice, about PS in a teaching context, and ways in which they can cope with them. In this presentation, we will focus on some challenges emerging from data analysis related to mathematization of realistic problem solving.

Résumé. Cette étude fait partie d'un projet de recherche collaborative visant à éclairer le travail de conseillers pédagogiques en mathématiques au regard de la résolution de problèmes (RP) et de l'accompagnement des enseignants. Plus spécifiquement, il s'agit d'éclairer les enjeux auxquels ils sont confrontés à l'égard de la RP en contexte d'enseignement, de l'intérieur de leur pratique professionnelle, et les manières de faire permettant d'y faire face. Nous nous attardons, dans cette présentation, aux enjeux, émergeant de l'analyse des données, portant sur la mathématisation de problèmes réalistes.

1. Résolution de problèmes en contexte d'enseignement et travail des conseillers pédagogiques

L'importance accordée à la résolution de problèmes (RP) dans l'enseignement des mathématiques, est confirmée par plusieurs décennies de recherches, qui ont permis d'éclairer la notion de problème (Brousseau, 1983; Lukenbein, 1984-1985; Douady, 1987) et ses caractéristiques (Artigue & Houdement, 2007; Burkhardt & Bell, 2007; Cai & Nie, 2007; D'Ambrosio, 2007; Hino, 2007; Santos-Trigo, 2007; Schoenfeld, 2007; Coppé & Houdement, 2009; Lajoie & Bednarz, 2012, 2016), le processus de RP (e.g. Schoenfeld, 1985, 1994), les différentes fonctions assignées à la RP dans l'enseignement des mathématiques (Lajoie & Bednarz, 2014) ou encore l'exploitation de certains types de problèmes en classe (Buckhardt, 1984; Arsac et al., 1988; Charnay, 1992-93; Tanner & Jones, 1994; Grenier & Payan, 1998, 2003; Adjiage & Rauscher, 2013; Oval-Soto & Oliveira, 2012). Parmi ces travaux, plusieurs mettent en évidence les difficultés que pose la gestion de cette RP en classe, notamment dans la prise en compte des solutions des élèves, dont celles erronées (Oliveira, 2008), la prise en charge de la validation (Barry, 2009; Saboya, 2010), l'instauration d'une culture de recherche dans la classe (Barry, 2009) ou encore l'exploitation mathématique de problèmes complexes (Maheux, 2007). Ces différentes études révèlent la complexité de cette RP en classe pour l'enseignant.

Par ricochet, ces résultats viennent questionner les CP, placés aux premières loges lorsqu'il s'agit d'accompagnement des enseignants. Ces CP sont en effet amenés à soutenir et accompagner ces enseignants au regard de la RP, un élément clé du programme de formation, agissant comme « ressources » dans la mise en œuvre de ce programme (Houle & Pratte, 2003). Or, une analyse historique des documents officiels québécois de 1900 à nos jours met en évidence le caractère de plus en plus ambitieux de ce travail associé à la RP (Lajoie & Bednarz, 2012, 2016) et l'éclairage quasi inexistant fourni aux enseignants pour aborder cette résolution (Lajoie, Bednarz, 2014). Elle souligne ainsi indirectement la présence d'enjeux importants associés à la résolution de problèmes en contexte d'enseignement et à l'accompagnement des enseignants, comme nous avons pu d'ailleurs le constater lors de rencontres nationales avec ces CP. Les difficultés vécues par les enseignants, en lien avec l'exploitation de problèmes en classe et leur évaluation, se répercutent dans les demandes qu'ils adressent aux CP, qui ne sont pas toujours en mesure d'y répondre, ou qui s'interrogent à leur propos. Bien sûr, les CP ont des *manières de faire* pour répondre à ces demandes, mais ils sentent le besoin de se distancer de ces pratiques spontanées, et de se faire une idée plus précise des enjeux, questions qui se posent, afin de supporter leur intervention.

Ainsi la nécessité de clarifier ce que recouvre cette résolution dans un contexte d'enseignement, d'en cerner les enjeux et les approches possibles, constitue un défi de taille, et confirme l'importance d'avancer, sur le plan de la recherche, dans la clarification et la prise en compte de ces enjeux. Nous nous centrons plus particulièrement dans cette présentation sur quelques enjeux, émergeant de notre analyse, associés à la mathématisation.

2. Processus de mathématisation : une première réponse théorique

Dans la perspective de la « Realistic Mathematics Education » (RME) (Freudenthal, 1991), le processus de modélisation, plus large que celui de mathématisation, s'articule sur l'activité informelle des élèves. Ces modèles émergents, qui permettent d'avancer dans la résolution de situations « réelles » (Gravermeijer, 1999), sont appelés, tout au long du processus, à se restructurer et à être revisités par les élèves, et ce en lien avec cette situation de départ ou de nouvelles situations. Streefland (1991, 1993) parle, pour rendre compte de cette activité, du passage du *« model of »* au *« model for »* pour exprimer le fait qu'au début un modèle est créé en lien avec une situation donnée et que ce modèle sera appelé à être généralisé à d'autres situations. Ce processus de modélisation comprend deux phases profondément imbriquées, celles de formulation et de validation (Burkhardt, 1984). Selon Bélair (2004), l'aspect le plus difficile de la phase de formulation, le plus sous-estimé et aussi le plus imprévisible, est celui de la mathématisation. Le courant de la RME réfère à deux types de mathématisation, une mathématisation horizontale où des outils mathématiques sont mobilisés et utilisés pour structurer et résoudre une situation (du monde réel au monde des représentations, symboles) et une mathématisation verticale qui se joue à un niveau purement mathématique (circonscrite au seul monde des symboles) (Treffers, 1987, Freudenthal, 1991). Cette mathématisation progressive, permettant d'aller plus loin sur la généralisation des modèles, part des stratégies disponibles, dont les possibilités de généralisation se négocient dans la classe avec les pairs et l'enseignant. C'est donc à travers les interactions dans la classe que se construit une « culture de modélisation » (Tanner et Jones, 1994), s'articulant sur la production et la validation de construits provisoires développés par les élèves. On perçoit bien à travers ce qui précède le rôle central qu'est appelé à jouer l'enseignant dans ce processus et, en conséquence, le défi auquel est confronté le CP chargé d'accompagner les enseignants au regard de cette activité de modélisation.

3. Quelques repères méthodologiques pour aborder l'analyse des enjeux

Une recherche collaborative (au sens de Desgagné et al., 2001, Bednarz, 2013, 2015) a été mise en place pour explorer, avec des conseillers pédagogiques, les enjeux rencontrés en lien avec la résolution de problèmes en contexte d'enseignement. Il s'agit ici de faire sens avec les CP, de l'intérieur de leur pratique professionnelle, de ces enjeux, en s'appuyant pour cela sur leur expérience du métier, ce que Lessard (2008) nomme une « intelligence du terrain ». Les chercheurs sont également appelés à participer à cette explicitation en puisant à un bagage d'expériences et de connaissances sur le plan didactique, par rapport à l'objet RP, qui peut ici être mis à profit. Plus précisément, 8 CP responsables du dossier mathématiques au primaire, provenant de 5 commissions scolaires différentes, prenant en compte des contextes diversifiés susceptibles d'affecter le travail du CP, participent à ce projet de recherche collaborative qui s'étend sur deux ans (2016-2018). Il prend la forme de rencontres réflexives d'une journée complète (5 la 1ère année, 6 la deuxième année) qui forment le matériau de base de notre analyse. Nous revenons dans cette présentation sur une partie de ces rencontres, la 3ème (22 février 2016), ayant pris forme autour de problèmes sans données numériques, amenés par les uns et les autres : un devoir que le groupe s'était donné lors de la rencontre précédente, dans la perspective de réfléchir au choix de problèmes susceptibles de forcer une analyse et un engagement dans une activité mathématique de la part des élèves.

4. Autour d'un problème réaliste : quelques enjeux émergeant de l'analyse

Le problème des taxes, amené par l'un des CP, va faire ressortir des enjeux fondamentaux à propos de la résolution de problèmes « réalistes » et de leur mathématisation. Ce problème, formulé ainsi par CP4[4]: « Doit-on choisir de calculer la taxe avant ou après un rabais? », est une adaptation d'un problème de Mason (1994)[5]. En puisant ici à diverses ressources- expérimentations conduites en classe par les CP avec des élèves (10-14 ans), expérimentations auprès d'enseignants du primaire lors d'accompagnements par les CP, leur propre engagement face au problème- les discussions vont mettre en lumière à la fois le potentiel d'un tel problème pour la mathématisation mais aussi les entraves possibles, enjeux que soulève le travail autour de ce problème.

Un problème intéressant du point de vue d'une mathématisation progressive

La discussion entre les chercheurs et les CP fait ressortir a priori le potentiel de ce problème du point de vue de sa mathématisation. Cet énoncé met en effet en jeu une double généralisation possible ouvrant sur un processus de mathématisation horizontale et verticale, comme le montrent les propos qui suivent.

CP4 (référant ici au problème de Mason, dont le problème proposé est une adaptation) : on se place dans la peau de la caissière, mais en fait est-ce que ça marche tout le temps, que ce soit 20% de rabais ou euh si je donne un rabais de 60%, est-ce que c'est la même chose? C'est comme on peut le donner, on peut le mettre sans donnée numérique [sous-entendu sans montant de départ sur lequel s'applique la taxe ou le rabais].

C2 : on revient à ton idée de généraliser tout à l'heure, parce que peu importe le montant considéré [cas du problème initial de Mason] ou les taxes considérées [cas du problème énoncé par CP4], on va arriver à la même conclusion

L'analyse montre toutefois qu'un tel processus de mathématisation ne va pas de soi au regard du contexte choisi, un contexte dit « réaliste ».

Emprise du contexte : une entrave possible à la mathématisation

Le contexte va exercer, nous le voyons dans ce qui suit, une forte emprise sur la manière dont des élèves et des enseignants, à qui un tel problème a été proposé, s'engagent dans sa résolution.

CP 4 : Bien ici on peut se placer dans la peau de l'acheteur, du vendeur ou même du gouvernement.

C1 : C'est vrai

CP4 : Quand je l'ai vécu en classe, rapidement les élèves ont demandé « qu'est-ce qu'on doit faire? ». Ils voulaient savoir c'était quoi la règle, puis j'ai été obligé d'aller chercher pour la TPS[6], pour aller trouver que la taxe doit absolument être calculée sur le montant effectivement payé, donc le rabais doit se placer avant. Sinon c'est un peu injuste pour le vendeur qui se trouve à payer une taxe sur le prix qu'il ne nous a pas vendu.

[4] L'abréviation CP reprise dans l'analyse réfère aux conseillers pédagogiques, et C réfère aux chercheurs
[5] Le problème initial, formulé par Mason (1994), est ainsi énoncé : un magasin accorde un rabais de 20% et facture la taxe de vente de 15%. Qu'est-ce que la caissière devrait d'abord calculer, le rabais ou la taxe ?
[6] TPS : taxe sur les produits et services.

CP8 (faisant référence à une expérimentation avec des enseignants) : Face à cette enseignante qui cherche quand même à nous challenger un peu souvent …donc euh j'ai sorti ce problème là en me disant si tu comprends l'idée des propriétés, tu es en mesure de répondre à cette question là. Donc je leur soumets le problème puis là elle voit, elle comprend l'idée des propriétés derrière, que pour le consommateur ça change rien en bout de ligne. Mais effectivement après ça, ça été « bien c'est bien beau, mais on sait que ce n'est pas de même que ça marche » […]

Cette emprise du contexte amène à relativiser le problème posé au regard du point de vue, de la posture de celui qui se pose la question : les enseignants à qui le problème a été donné vont chercher, par exemple, à se positionner comme consommateur, puis comme marchand, en se donnant des exemples, ou encore à comprendre la règle utilisée par le gouvernement. Elle peut aussi, comme le montre ce qui suit, devenir une entrave à la mathématisation, en enlevant toute pertinence à cette mathématisation.

C2 : Mais quelqu'un pourrait te dire c'est quoi l'intérêt de cette question là puisqu'on sait que dans la vraie vie, c'est toujours sur le montant initial [qu'est calculé le rabais]

C1 : bien on se demande si ça fait une différence [….]. Est-ce que ça fait une différence? Est-ce qu'on est en train de se faire avoir comme consommateur?

C2 : Non mais je me fais l'avocat du diable en me disant que dans la classe quelqu'un pourrait dire ça. […]

CP8 : Une fois que tu te prêtes au jeu puis que tu dis admettons « ok on essaie de voir » puis que tu te rends compte que si tu regardes du point de vue du consommateur ça change rien, mais bon l'aspect demeure de dire « ouais dans la vraie vie, on sait qu'on n'a pas le choix »

Un résultat qui surprend, étonne (lorsqu'on accepte de se prêter au jeu) et force à raisonner mathématiquement pour aller plus loin

Un doute quant à la réponse, explicité à travers les propos provenant d'enseignants et de CP, laisse voir le potentiel d'un tel problème au regard d'un engagement dans une validation.

CP4 : parce que je l'ai présenté à des profs de maths, des profs de maths qui sont très matheux là, puis ils étaient pas sûrs.

CP1 : c'est contre instinctif.

CP5 : puis là vraiment je suis en train de douter à savoir « là coudonc, c'est tu un piège cette affaire là, ça reviens tu à la même affaire? ». Je veux dire là, je ne sais même pas mathématiquement, je ne sais même pas si, si…bon je rougis là.

(et plus tard alors que le groupe continue à discuter de ce type de problème)

CP5 : Mais moi ça me prend l'os, je vais rester toute la journée là dessus.

Le caractère contre-intuitif et l'enjeu de la validation

L'objectif est ici double pour ceux qui sont confrontés à un tel problème, essayer de répondre à la question, mais aussi trouver pourquoi une telle réponse est vraisemblable. C'est ici, dans la discussion, que tout l'enjeu de la validation va apparaître au regard du caractère contre-intuitif de cette réponse :

CP5 : ben là je sais pas. J'ai vraiment une conception. Je me dis c'est sûrement une conception erronée que j'ai là. Mais je veux dire, je tomberais, s'il y a un piège, je tomberais dedans.

C : Alors qu'est-ce que tu répondrais toi? Qu'est-ce que la caissière devrait faire?

CP5 : Ben écoute mon intuitif là fait « ah bien oui, je vais aller calculer la taxe après le rabais parce que la taxe va être moins forte ». Mais je sais que c'est un piège, mais je ne sais pas pourquoi.

L'enjeu de la validation, comme nous le verrons dans ce qui suit, n'en est pas uniquement un de *validation pour (se) convaincre* (que la réponse est vraisemblable), en ayant recours pour cela à une exemplification (à l'aide de différents nombres) ou à un passage à l'algèbre qui permet de nous convaincre de l'égalité peu importe le prix de départ ou les pourcentages associés au rabais et à la taxe. Il en est surtout un de *validation pour comprendre* (pourquoi il en est ainsi).

C2 : OK, puis CP5, es-tu convaincu?

CP5 : Bien, j'en reviens pas. Je suis pas convaincu. Ça arrive à la même chose mais je suis pas convaincu. J'essaie d'aller…vraiment j'ai un besoin d'aller comprendre comme il faut les propriétés effectivement […]

C1 : C'est que là tu es convaincu parce que tu l'as essayé sur un montant d'argent, n'est-ce pas?

CP5 : ouais

C1 : OK, puis là tu te demandes si pour d'autres montants d'argent..

CP5 : même pas

C2 : même pas?

CP5 : non

C1 : Tu es sûr que ça marche tout le temps?

CP5 : je suis sûr que ça va marcher tout le temps.

C1 : OK

CP5 : mais je veux comprendre pourquoi en fait. Je comprends intellectuellement la propriété…c'est comme quelque chose…écoute ça traduit réellement comment j'ai appris les maths dans mon enfance admettons. Théoriquement, la propriété elle est là mais pas sûr…hein comme un petit doute pour dire « là il y a quelque chose qui est là mais c'est pas…c'est pas intégré…ça vit pas »

CP4 : Ta tête est pas en accord avec ton ventre [rires].

CP5 : oui exactement, exactement.

CP1 : est-ce qu'on ne fait pas face à deux réactions à ce moment là. C'est se dire comme toi « oui je le sais que ça marche avec tous les nombres, mais comment ça se fait que ça marche ? Ou bien je ne suis pas sûre que ça marche avec tous les nombres »

Se convaincre que cette réponse est vraisemblable mais surtout comprendre pourquoi il en est ainsi, apparaît ainsi un enjeu fort de la modélisation associée à ce problème, dans la mesure où la réponse obtenue est contre intuitive. On retrouve là deux significations essentielles de la preuve mises en évidence dans les analyses épistémologiques menées à son sujet à différentes époques : convaincre versus éclairer (Barbin, 1987-1988).

Valider pour éclairer vient interroger le type de modèle qui permet d'entrer dans une telle compréhension. Le recours à l'exemplification ou à la symbolisation algébrique, ce que CP5 nomme une compréhension intellectuelle de la propriété, ne suffit pas, ou du moins pas toujours, comme le montrent bien les propos précédents.

5. Discussion

Quelques enjeux associés au pilotage d'un tel problème en classe peuvent être anticipés à la lumière de ce qui précède : comment gérer l'emprise d'un contexte réaliste dans une classe, levier possible à toutes sortes d'entrées non nécessairement pertinentes sur le plan mathématique? Comment contrer l'intuition dans l'exploitation du problème? Quels modèles peuvent aider à éclairer le caractère vraisemblable de cette réponse contre-intuitive? Quels sont ces modèles émergents développés par les élèves pour justifier le caractère vraisemblable de cette réponse? Comment tirer partie de ces différents modèles dans le retour sur les solutions des élèves?

Ces différents enjeux reviennent dans d'autres cas. L'emprise du contexte a ainsi été soulignée par plusieurs recherches qui pointent que ce dernier peut constituer un frein à la mise en place d'une activité mathématique (Perrin-Glorian, 1993, Roiné, 2012). Aussi la question des relances face à des solutions des élèves, notamment des solutions erronées, l'exploitation des solutions des élèves lors du retour, au regard notamment de la validation, constituent des moments critiques dans l'instauration d'une culture de modélisation dans la classe.

Ils posent, pour les CP, la question centrale de l'accompagnement des enseignants autour de ces moments clés : comment accompagner des enseignants à piloter des problèmes en classe pour que soient pris en compte ces enjeux et que se développe une culture de modélisation chez les élèves?

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WORKING GROUP A / GROUP DE TRAVAIL A

CIEAEM 69

Berlin (Germany)

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MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

Introduction to Working Group A / Introduction au Group de Travail A

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The group was organized with the aim of giving voice to all the participants through an initial and final video-recording of the possible and accomplished opportunities to reflect about mathematization and modelling of everyday context as a didactical principle. For the nine papers, we choose to limit the presentation to ten minutes, leaving 5 minutes for reaction prepared in advance by another participant of the group, and ten minutes more for a generalized discussion.

Discussions helped to analyse what qualifies a real-world context as a point of departure and/or a point of arrival of a didactic arrangement, the relevance of the authenticity of everyday contexts, which material arrangements support students' learning of mathematics by mathematization, and which epistemologies of mathematics are built into particular didactical principles of mathematization. The group felt that this allocation of time and roles was extremely productive, generating fruitful discussions and great involvement among the members of the group, as was illustrated in the last moments of the final video recording.

Reflecting on what qualifies a real-world context for didactic arrangements, differences among the active roles of students (from Kindergarten to High School), in-service teachers, teacher trainers, experts and researchers were discussed. Five of the nine papers addressed mathematization as a didactical principle with different emphases in the modelling process. Those didactical principles were: the interpretation and application of the theoretical notions of non-Euclidean geometry (experience authored by Bini); giving meaning to three keywords -whole, unit and quantity- (presented by Rottoli); diverse mobilizations in each of the different stages of the modelling cycle of the mathematical and physical epistemological knowledge (research work defended by Mouted); the transference of mathematical knowledge between micro-contexts in successive cycles of modelling (research co-authored by Tsitsos and Stathopolou); and *a priori* analysis of a problem posed in life science context (defended by Yvain). Two papers considered both the didactical and social principle of mathematization at the same time, placing at the core how a contextual situation is explored, modelled and validated jointly by researchers, teachers, students and experts (discussed by Giménez), or aiming to raise critical consciousness within social and cultural relevant contexts for students, teachers, teacher trainers and families (co-authored by Anhalt and Turner). A third paper enhanced the social and didactical principle of decisions made, using mathematically-informed thinking to empower students in everyday situations of risk and probability (presented by Serradó).

The didactical arrangements presented used a variety of different artefacts, physical experiences, materials or learning spaces to promote students' horizontal and vertical mathematization. Although, initially, the physical experience of preparing a fruit salad was designed as an opportunity to construct the notion of fraction, the discussion recognised the potentialities of the project to analyse the difference between discrete and continuous conceptions of number. Meanwhile, the fruit salad was an opportunity for the horizontal mathematization. Analysing the non-Euclidian geometry of an orange was convenient for understanding the mathematical model, as a consequence, deriving the formula for the surface area of a spherical triangle, and providing students opportunities for critical thinking about their interpretation as residents of a giant sphere. The reaction about the usefulness of the orange, as an artefact used in the virtual world to understand the real world, questioned the didactical possibilities of beginning with a discussion that is about the meaning of being a resident of the Earth to promote horizontal mathematization. In spite of understanding the conditions as favouring the devolution of horizontal modelling, *a priori* analysis of a "didactic engineering" activity - using the fictional reality about the growing of a tree - was discussed. Two didactical principles for this horizontal modelling were evidenced: the implementation of a didactical device, and the elaboration of a specific situation. In this case, the specific situation came from a biologic epistemological study.

Moreover, the physical epistemological study of the representation of space-time of Minkowski provided information about the differences of *a priori* analysis, using pen and paper and dynamic geometry implemented when transferring the mathematical results to the reality. The theoretical framework proposed by Mouted to analyse this transference considered three parallel planes: the physical and mathematical epistemological, and the cognitive. However, it was discussed whether it would be necessary to consider a transversal technological plane to understand the transitions between the epistemological and cognitive planes. Tsitsos' work suggested the importance of conjecturing, representing and validating mathematical practices through the use of physical artefacts and dynamic geometry. Those devices allowed understanding the mathematical model involved while abstracting the mathematical concept of variance.

To sum up, the important understanding of the relevance of the authenticity of the contexts that provide the basis for mathematization in realistic situations and modelling means widening the contextual view to the epistemological, cognitive, technological and social perspective. From a contextual view, the authenticity was analysed through the differences when real, virtual real, fictional real situations are the point of departure and/or arrival that builds on mathematization when involved in modelling processes. We also considered the authenticity of the representations, artefacts and theoretical references, which allow for interconnecting the extra-mathematical epistemological plane (biology, physic, historic, cultural, etc.) with the mathematical one through didactical arrangements. Those didactical arrangements (designed, implemented and analysed, *a priori* and/or *a posteriori)* examine the non-neutral intersection between the technological and cognitive plane. The cognitive authenticity should derive from the understanding of the purposes of social and cultural practices for actively engaging the different actors (students, teachers, teacher trainers, researchers, experts, etc.) in mathematization and modelling of everyday contexts.

This wider approach to the relevance of authenticity is a challenge to reflect about: how and when we take into account the extra-mathematical epistemological, technological and cognitive planes that intersect with the modelling process.

A didactic approach in mathematical modeling: raising critical consciousness within social and culturally relevant contexts

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Abstract. This paper reports on a year-long professional development project, Mathematical Modeling in the Middle Grades (M^3), that created unique opportunities for thirty mathematics teachers and teacher leaders in grades 5-8 to explore mathematical modeling through culturally relevant community contexts. The teachers were from rural schools near the U.S.-Mexico international border with diverse student populations. The work of M^3 focused on interconnecting mathematisation as a social process and as a didactic principle in a professional development setting. The goals of M^3 was to build teachers' background knowledge in mathematical modeling and to prepare them for implementing modeling tasks in their classrooms that focus on students' mathematics knowledge and leverage cultural and community contexts. Teachers learned about and utilized the modeling process while exploring local contexts in the communities in which they teach. They implemented modeling tasks which advantageously used their students' background knowledge as part of the solution process resulting in successful student engagement in the mathematical modeling process. The student models yielded creativity, interesting mathematics, and revealed their understanding of real-world community issues, thus raising awareness and critical consciousness.

Résumé. Ce papier annonce sur un projet de développement professionnel d'un an, un Modelage Mathématique en Qualités du Milieu (M3), qui a créé des occasions uniques pour trente enseignants de mathématiques et chefs d'enseignant dans les qualités 5-8 pour explorer le modelage mathématique par les contextes de communauté culturellement pertinents. Les enseignants étaient des écoles rurales près des Etats-Unis-Mexique la frontière internationale avec les populations étudiantes diverses. Le travail de M3 s'est concentré à raccorder mathematisation comme un processus social et comme un principe didactique dans un cadre de développement professionnel. Les buts de M3 étaient de construire la connaissance de base d'enseignants dans le modelage mathématique et les préparer à exécuter des tâches de modelage dans leurs classes qui se concentrent sur la connaissance de mathématiques d'étudiants et exercent une influence culturel et les contextes de communauté. Les enseignants ont appris de et ont utilisé le processus de modelage en explorant des contextes locaux dans les communautés dans lesquelles ils enseignent. Ils ont exécuté des tâches de modelage qui ont utilisé avantageusement la connaissance de base de leurs étudiants dans le cadre du processus de solution ayant pour résultat l'engagement étudiant réussi dans le processus de modelage mathématique. Les modèles étudiants ont produit la créativité, les mathématiques intéressantes et ont révélé leur compréhension d'éditions de communauté de monde réel, en levant ainsi la conscience et la conscience critique.

1. Introduction

Mathematical modeling demands that students apply the mathematics they know and their background knowledge to solve problems in everyday life situations. This is due to the contextual nature of mathematical modeling activities, which can accommodate students' experiential interpretations that lead to their own assumptions and decisions for their models. Moreover, modeling allows students to experience the type of problem solving that mathematicians engage with in the real world and supports investigations of critical social issues. This offers opportunities for leveraging students' cultural competencies through mathematical modeling tasks with the potential for meaningful learning. The process of bringing everyday knowledge and scientific knowledge together through modeling can be especially effective when the situation context is relevant to the students' communities and lived experiences (Gay 2000; Ladson-Billing 1995). These connections make it possible for students to participate in meaningful mathematics learning that is connected to their background (Anhalt, Cortez, & Smith, 2017).

2. Theoretical Perspectives

Mathematical modeling has gained prominence in the K-12 mathematics curriculum in the United States, in part due to emphasis in the Common Core State Standards for Mathematics (CCSSI, 2010) and the Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) report (Garfunkel & Montgomery eds., 2016). Due to its open-ended nature, non-linearity, and multiple solution paths and strategies, mathematical modeling tasks allow for a broader range of students to be successful than in the context of a more standardized mathematics curriculum (Anhalt, 2014; Lesh & Lehrer, 2003). In particular, modeling problems require that students critically analyze situations.

As teachers often have had limited exposure to mathematical modeling in their preparation, there is a pressing need in teacher education for increased learning opportunities nested in real-world contexts that allow critical conversations surrounding social issues. Our work draws on Ladson-Billings (1995) Culturally Relevant Teaching (CRT) framework, which focuses on student academic achievement, cultural identities, and critical consciousness. The M^3 project explicitly connects mathematical modeling to equity-based culturally responsive mathematics teaching. This comprehensive teaching approach addresses mathematics, mathematical thinking, community-based funds of knowledge, and cultural identity to support student mathematics learning and engagement (Aguirre & Zavala, 2013; Bartell, 2013). Teacher professional development with an explicit commitment to culturally responsive mathematics teaching focuses on learning experiences that enable teachers to deepen their understanding of mathematics, make connections to student lived experiences, and help students experience mathematics as an analytical tool to make sense of, critique, and positively transform the world (Aguirre & Zavala, 2013; Gutstein, 2006; Turner et al., 2012; Simic-Muller, 2015).

3. Background on the Mathematical Modeling in the Middle Grades (M^3) Project*

This paper reports on a year-long professional development project, Mathematical Modeling in the Middle Grades (M^3), that created unique opportunities for thirty mathematics teachers and teacher leaders in upper elementary and middle grades to explore mathematical modeling through culturally relevant community contexts. The teachers were from rural schools near the United States-Mexico international border with diverse student populations. The goal of M^3 was to prepare teachers for implementing modeling tasks in their classrooms that focus on students' mathematics knowledge and leverage cultural and community contexts, resulting in successful student-created models, and consequently, raising critical consciousness in real-world community issues. The work of M^3 focuses on interconnecting mathematisation as a social process and as a didactic principle in a professional development setting. The teachers' commitment to learning about mathematical modeling and meaningful ways to teach modeling to diverse student populations was key to the success of the M^3 project. The project consisted of 65 hours of professional development spread over two summer institutes and monthly academic year meetings.

The initial summer institute consisted of establishing a professional learning community in which all ideas were valued, therefore, listening skills and active participation became central. Participation in collaborative groups consisted of a Launch-Explore-Summarize approach (Schroyer 1984) as teachers worked through the cyclic nature of the mathematical modeling process: (a) *launch* a context-rich problem to capture interest; (b) *explore* with background information to create and operate on a mathematical model; and (c) *summarize* to share outcomes that could inform recommendations for future action. We began with simple modeling problems in non-specific contexts that grew to more complex problems as teachers transformed their curriculum to include modeling tasks with local community contexts as settings for problems.

During the academic year, the teachers participated in monthly study groups, which focused on analysis of student work as they implemented mathematical modeling tasks in their classes. Sessions also included opportunities to discuss teaching mathematical modeling, and its challenges and benefits for student learning. The teachers expressed interest in developing tasks with more relevant contexts involving serious issues impacting the community.

During the second summer institute, the exploration of more complex modeling problems increased as we investigated local community issues such as border crossing time, produce imported from Mexico to the U.S., and landfill costs for processing produce waste. These topics impact the lives of the families in these communities on a daily basis. An example of a modeling task based on a local community issue of produce waste and landfill costs is in 'figure 1'.

Figure 1. Local foodbank waste and landfill costs[1]

- Using the data provided, find a way to approximate the waste by Borderlands from 2004-2012.
- What was the cost to the county each year from 2001-2013?
- What are ways that Borderlands Food Bank could reduce the amount of produce being dumped in the landfill?

Background information revealed that produce near expiration date that comes across the international border is donated to local food banks for distribution to low income communities, yet the food bank cannot manage all of the donated produce, so it often must dispose of some produce at the local landfill. Unfortunately, the cost of processing the wasted produce at the land fill must be paid for by the food bank or the local county taxes. The teachers were concerned because food waste and county taxes impact the whole community.

4. Findings

Throughout the project, the teachers expressed enthusiasm for teaching mathematical modeling. Teachers learned and utilized the modeling process and valued carefully created modeling activities. Teachers changed their views of curriculum development by incorporating engaging modeling activities that advantageously use their students' background knowledge as part of the solution process. Our project results show that teachers were able to successfully engage students in the mathematical modeling process as they participated in culturally relevant contextual tasks in mathematical modeling. Their models yielded creativity, interesting mathematics, and revealed their understanding of the situations.

Teachers reported that problems that were explored promoted civic awareness and reflection of a broad range of issues facing local communities. The teachers indicated that they learned that real-life issues can be adapted to create engaging mathematical modeling problems and that different views of the problem can lead to different acceptable solutions. This is an important realization and a feature of modeling tasks that does

[1] http://www.nogalesinternational.com/news/landfill-cap-puts-pressure-on-food-bank/

not occur in traditional word problems (Anhalt 2014). In mathematical modeling, the assumptions are central to the process as they determine in large part the construction of the model. However, even under the same assumptions, the modeling approach may vary, as they learned throughout the project. Discussion on the topic of assumptions was at the heart of balancing the didactic fictionality and the reality of the social phenomenon. The teachers produced quality work in mathematical modeling through posters incorporating their group thinking and solutions. Based on their reflections, the teachers felt that learning about mathematical modeling changed their way of thinking about curriculum in that modeling offered students opportunities to take ownership of the problems, especially when the contexts are socially and culturally relevant.

A subset of teachers took the initiative to make a presentation to their school administration advocating for curriculum that incorporated mathematical modeling with cultural relevance for students. They presented their students' work and learning in mathematical modeling as evidence to support their claims. This event proved to be a compelling testament to the transformation brought about by their participation in the M^3 project.

5. Conclusions

Mathematics teacher education should seek to challenge the status quo by creating professional learning activities that combine mathematical modeling and critical social awareness in ways that help teachers and their students raise civic advocacy, think more critically about what is happening in our communities and take transformative action. Consequently, teachers can increase student learning of rigorous mathematics, nurture students' cultural identities, and instill critical consciousness. Explicitly prioritizing rigorous mathematics investigations that address all three goals has been a challenge, particularly because it means to confront potential implicit bias about specific students and their communities or uncover social, economic, and environmental inequities in local communities and in broader contexts. An emphasis on mathematical modeling has the potential to simultaneously grow teachers' and students' mathematical knowledge and critical consciousness if the experiences include analysis of local community problems that promote activism in students.

*The *Mathematical Modeling in the Middle Grades (M3) Project* was funded by the Arizona Board of Regents, Improving Teaching Quality Award ITQ015-08UA, University of Arizona, United States.

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A non-euclidean clockwork orange: from reality to mathematics and back

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Abstract. This paper presents a teaching experience in non-Euclidean geometry involving the use of artefacts and physical experiences. The teaching practice was set up for a group of 25 4th-year high-school students. The aim of this activity was to encourage the "process of translation from 'reality' to mathematics and back". Further, the students were stimulated to evaluate the authenticity of the proposed context, which was designed for learning about spherical geometry and its value in helping students understand and organise the abstract key facts of the topic. The overall aim was to shift back to the starting point and apply the acquired theoretical knowledge in order to interpret a real-world problem.

Résumé. Cet article présente une expérience pédagogique dans la géométrie non-euclidienne impliquant l'utilisation d'artefacts et d'expériences réelles. La pratique pédagogique a été mise en place pour un groupe de 25 élèves du quatrième année de l'école secondaire. Le but de cette activité était d'encourager le "processus de traduction de la 'réalité' aux mathématiques et retour". En plus, les étudiants ont été stimulés à évaluer l'authenticité du contexte proposé, conçu pour apprendre la géométrie sphérique et sa valeur pour aider les élèves à comprendre et à organiser les faits clés abstrait du sujet. L'objectif général était de revenir au point de départ et d'appliquer les connaissances théoriques acquises afin d'interpréter un problème réel.

1. Introduction and theoretical framework

According to Jablonka and Gellert's reflection on mathematisation and demathematisation (Jablonka & Gellert, 2007), "if classroom talk concentrates on the public language of description in order to help students construct meaning, then the mathematical knowledge of students tends to remain in the public domain of its origin. If, on the other hand, the classroom talk is mainly esoteric, then the individual construction of meaning appears to be more difficult."

In high-school, non-Euclidean geometry is often treated in a particular way: from an epistemological point of view, it is a highly significant subject. However, there is a risk of losing the cognitive ties to reality if the subject matter, as taught in schools, is kept strictly in the abstract realm. Further, the key role of non-Euclidean geometry, considering the interpretation of geometry in real-world settings, is endangered by the demathematisation of such by, for example, Google Maps and GPS navigation.

The overall aim of this teaching experience was to encourage the "process of translation from 'reality' to mathematics and back" called for by Jablonka and Gellert (Jablonka & Gellert, 2007). In addition, the purpose was to evaluate the effectiveness of the authenticity of the proposed context for the learning of spherical geometry. The evaluation included the teaching experience's value in terms of its support for students' understanding and organisation of the key abstract facts about non-Euclidean geometry. Further, the students were questioned about their newly developed ability to shift back and apply the acquired knowledge to interpret the real-world problem that was given to them.

2. Method and activity

This learning experience has been inspired by Lénárt's work on comparative geometry and by the article "Grapefruit Math" by Evelyn Lamb, published in the Scientific American online issue in May 2015. The added value of this specific project is the use of artefacts and physical experiences that effectively support students' learning by means of connecting abstract concepts of thought and discoveries to real world tangible results.

The activity was carried out with a class of 25 18 y.o. students attending the 4th year of Liceo Scientifico. The results were gathered by the teacher through the observation of classroom discussion and the assessment of students' homework.

After an introductional theoretical approach to non-Euclidean geometry, which primarily entailed the discussion on the role of the fifth postulate and the consequences of its negations with reference to the validity of the "classical" theorems about parallels and transversals and the sum of a triangle internal angles, students were asked to bring an orange, three elastic bands and a protractor for the upcoming lesson.

In said lesson they were confronted with the following tasks:

- Draw three points on the orange that do not lay on the same great circle.
- Put two elastic bands, each passing through two of the three points as in 'figure 1', and derive the expression for the area of each lune.

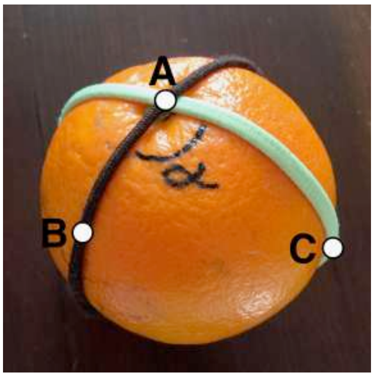

Figure 1. from Elena's work: the construction of the lune

- Add a third elastic band to form a spherical triangle as in 'figure 2'.

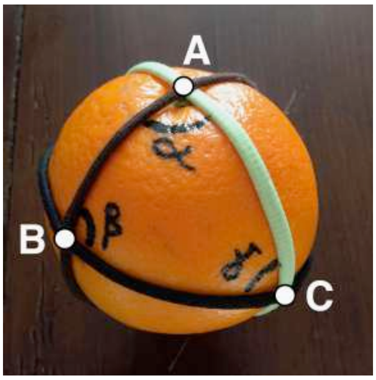

Figure 2. from Elena's work: the construction of the spherical triangle

- Use a protractor to measure the internal angles of the spherical triangle, add them up and contextualise the result.
- Derive the formula for the surface area of the spherical triangle.

The final questions encouraged critical reflection on the subject matter that required from the students to switch from the real-world model (the triangle on the orange) to the mathematical model (the formula for the surface of the triangle). After the students had completed these tasks, they were confronted with those follow-up questions, which aimed to link the mathematical interpretation to their everyday life:

- What happens if the triangle on the sphere gets smaller?

- What does that have to do with your everyday life as a resident of a giant sphere?

The students discussed their results and findings in class. After that, each student summarised her/his conclusions in a written essay that was finally submitted through the school's Moodle platform and assessed by the teacher.

3. Results and conclusion

The key aspect of this learning experience lies within the fact that "the essence of the process is that students are searching for the truth themselves, through experiments with palpable and virtual models, and through discussion or debate with one another or their teacher." (Lénárt, 2009).

The outcomes of this activity proved the efficiency of mathematical modelling as a didactic principle. On the one hand, the students were able to assess the mathematical behaviour of great circles as "straight lines" on the sphere by themselves. Thereby, the theoretical assumption presented in class beforehand gained a contextualised meaning due to personal experience. The overall, shared comment was:

I couldn't believe that the rubber bands stayed put only on great circles!

On the other hand, it clearly showed that by integrating real objects in order to draw in on abstract mathematical concepts, the students were more willing to apply the mathematical concept to real world observations thereafter. Further, this made them get an idea of why – in a small scale - we cannot identify our reality as non-Euclidean, as Chiara neatly pointed out in her paper:

This happens because the surface area of the shrunk spherical triangle is so small compared to the one of the sphere that the curviness of its sides becomes negligible and it is perceived as a Euclidean triangle.

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Le cadre théorique l'ETM étendu : analyse d'une séquence utilisant la relativité restreinte

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Abstract. We want to show how the theoretical frame of the extended MWS allows to analyse the tasks operated during the process of modelling. The frame of the extended MWS allows to show, through the example of a special relativity teaching sequence in a grade 12 class in France, which are the interactions between the cognitive plane and the epistemological planes of the physics or the mathematics. The geneses, the association of geneses and the interaction between the epistemological plans of the mathematics and the physics can be clarified for certain stages of the cycle of modelling.

Resumé. Il s'agit de montrer comment le cadre théorique de l'ETM étendu permet d'analyser les tâches mises en œuvre lors du processus de modélisation. Le cadre de l'ETM étendu permet de montrer, au travers de l'exemple d'une séquence d'enseignement de relativité restreinte en terminale S en France (grade 12), quelles sont les interactions entre le plan cognitif et les plans épistémologiques de la physique ou des mathématiques. Les genèses, l'association de genèses et l'interaction entre les plans épistémologiques des mathématiques et de la physique peuvent être explicitées pour certaines étapes du cycle de modélisation.

1. Présentation du cadre théorique de l'ETM étendu

L'espace de travail mathématique (ETM) a été développé afin de mieux comprendre les enjeux didactiques autour du travail mathématique dans un cadre scolaire (Kuzniak et al., 2016). L'ETM comporte deux niveaux : un de nature cognitive en relation avec l'apprenant et un autre de nature épistémologique en rapport avec les contenus mathématiques étudiés. Le plan épistémologique contient un ensemble de représentamen (signes utilisés), un ensemble d'artéfacts (instruments de dessins ou logiciels) et un ensemble théorique de référence (définitions et propriétés). Le plan cognitif contient un processus de visualisation (représentation de l'espace dans le cas de la géométrie), un processus de construction (fonction des outils utilisés) et un processus discursif (argumentations et preuves). Le travail mathématique résulte d'une articulation entre les plans cognitifs et épistémologiques grâce à une genèse instrumentale (opérationnalisation des artefacts), une genèse sémiotique (basée sur le registre des représentations sémiotiques) et une genèse discursive (présentation du raisonnement mathématique). Les différentes phases du travail mathématique associées à une tâche peuvent être mises en évidence par la représentation de trois plans verticaux sur le diagramme de l'ETM. Les interactions de type sémiotique-instrumentale (sem-ins) conduisent à une démarche de découverte et d'exploration d'un problème scolaire donné. Celles de type instrumentale-discursive (ins-dis) privilégient le raisonnement mathématique en relation avec les preuves expérimentales. Enfin, celles de type sémiotique-discursive (sem-dis) sont caractéristiques de la communication de résultats de type mathématique. Le diagramme des ETM a été adapté (Moutet, 2016) en rajoutant un plan épistémologique correspondant au cadre de rationalité de la physique (figure 1). Il a été choisi de ne garder qu'un seul plan cognitif car les spécificités du plan cognitif des deux disciplines en jeu (physique et mathématiques) n'ont pas été particulièrement étudiées dans le cadre de l'étude exposée dans cet article. Le cadre de l'ETM étendu permet d'analyser finement les interactions entre les différents cadres de rationalité et le plan cognitif de l'élève puis de qualifier la nature du travail réalisé par l'élève ou celui qui lui est demandé.

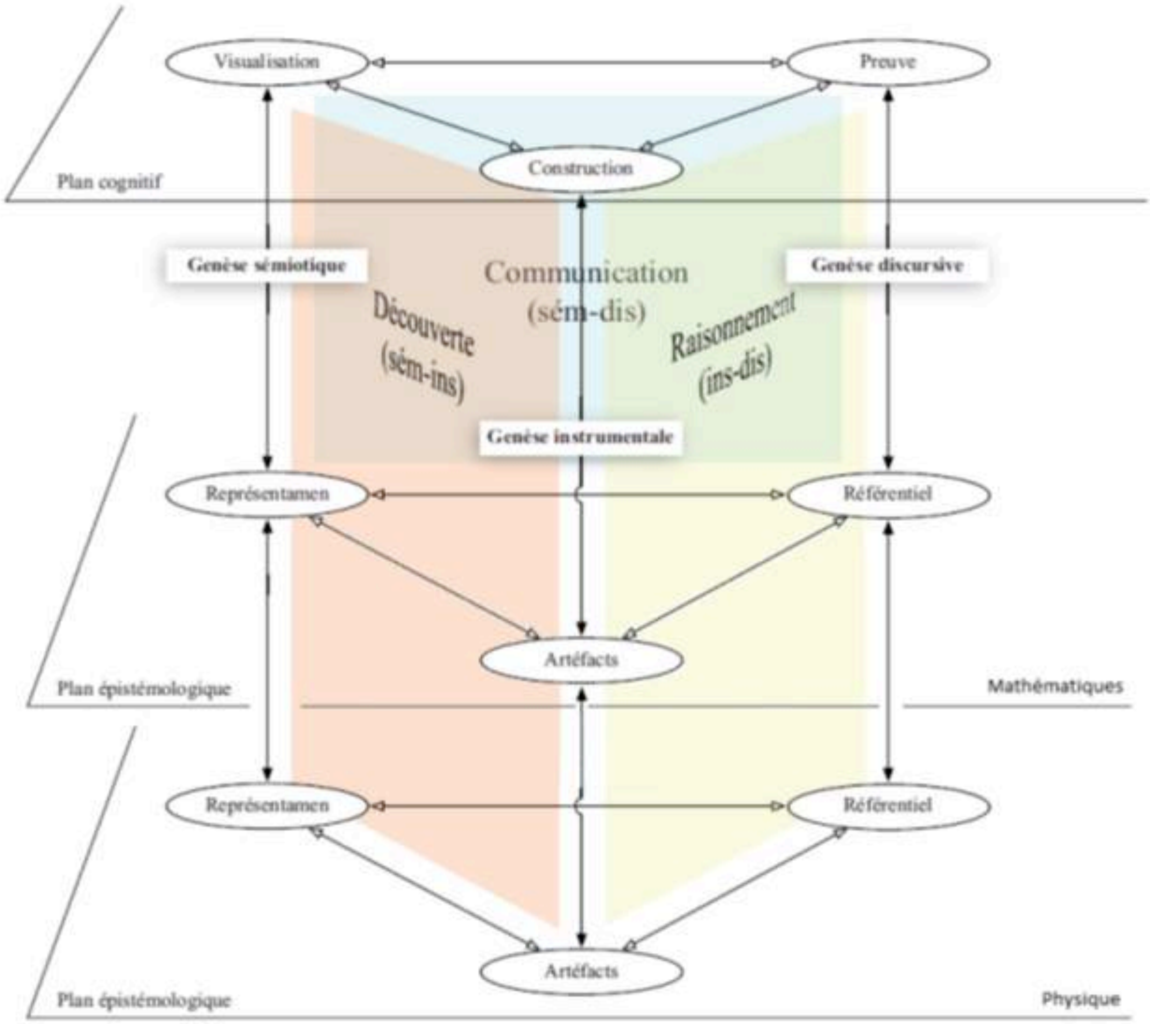

Figure 1. Modèle de l'ETM étendu

2. De la « situation modèle » aux « résultats réels »

Nous nous sommes basés sur le cycle de modélisation (Blum & Leiss, 2005) pour analyser une séquence d'enseignement (Moutet, 2016) portant sur le changement d'ordre chronologique d'événements en fonction du référentiel dans le cadre de la relativité restreinte (de Hosson, 2010). Elle est destinée à des élèves de terminale S (grade 12). Deux référentiels liés à deux observateurs, Armineh et Daniel, sont utilisés. Armineh conduit une voiture se déplaçant à une vitesse proche de la vitesse de la lumière par rapport à Daniel. Ce dernier se trouve sur le bord de la route à côté de trois flashs lumineux S_1, S_2 et S_3 associés à trois événements particuliers, et initialement connus dans le référentiel de Daniel (figure 2). Le diagramme d'espace-temps de Minkowski a été construit par les élèves et utilisé en classe à l'aide d'une activité papier-crayon relativement guidée par l'enseignant. Le logiciel de géométrie dynamique GeoGebra a permis par la suite, au travers d'une activité dans laquelle les élèves étaient en autonomie, de réinvestir le diagramme d'espace-temps de Minkowski. Il s'agit d'utiliser avec GeoGebra une autre genèse instrumentale afin de pouvoir effectuer une analyse *a priori* différente de l'activité papier-crayon à l'aide du modèle de l'ETM étendu. Le diagramme de Minkowski permet de représenter le repère (xOc.t) relatif au référentiel de Daniel et le repère (x'Oc.t') relatif au référentiel d'Armineh. Cette dernière se déplace à la vitesse v de 0,6 fois la vitesse de la lumière dans le vide (on considère que la vitesse de la lumière dans le vide est à peu près égale à celle dans l'air) par rapport à Daniel suivant un axe (Ox). Les droites (Ox) ou (Ox') correspondent à la route dans les référentiels de Daniel ou d'Armineh (figure 3).

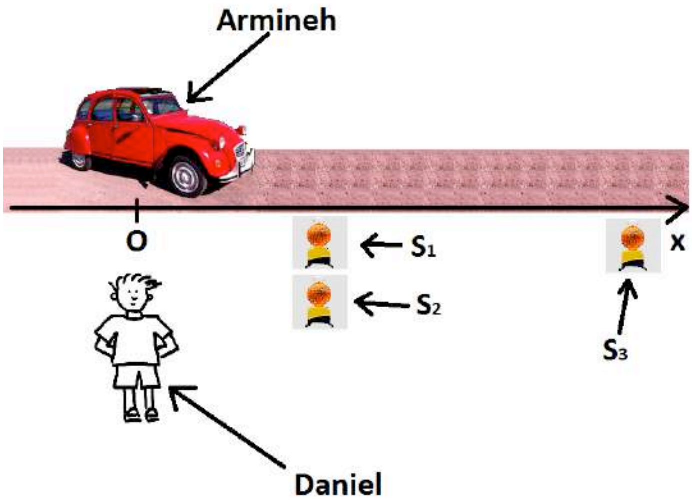

Figure 2. Le "modèle réel" de la situation

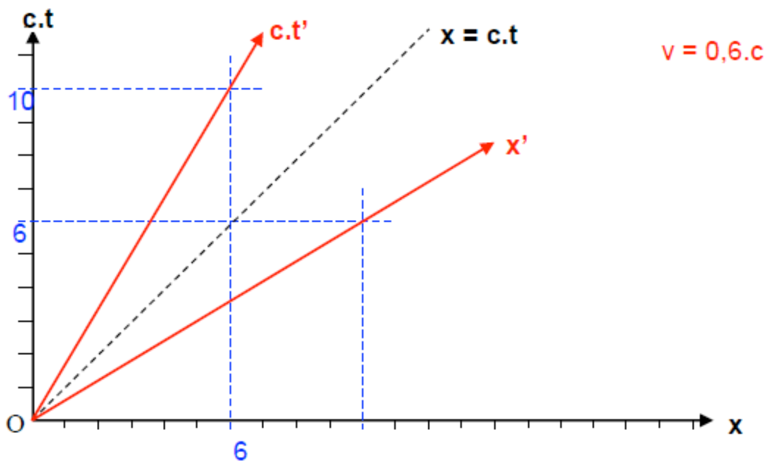

Figure 3. Diagramme de Minkowski pour v = 0,6.c

Le curseur de GeoGebra permet de modifier les conditions expérimentales en changeant la vitesse v, ce qui permet également une genèse sémiotique différente par rapport à l'activité préliminaire papier-crayon. Le plan épistémologique des mathématiques et le plan cognitif sont mobilisés lors de la construction du curseur. Les axes Ox' et Oc.t' sont modifiés en fonction de la vitesse v, ces deux axes se rapprochent de la droite x' = c.t' lorsque la vitesse v se rapproche de c. Le plan épistémologique de la physique est également mobilisé lorsque les élèves concluent sur l'ordre chronologique des événements suivant les deux référentiels (figure 4 et figure 5). L'utilisation du logiciel GeoGebra amène donc aussi une genèse discursive différente par rapport à l'activité papier – crayon. Le modèle de l'ETM étendu permet de réaliser l'analyse *a priori* de chacune des tâches à effectuer par les élèves et il nous a permis également de tester avec succès l'analyse *a posteriori* du travail effectué par quatre élèves.

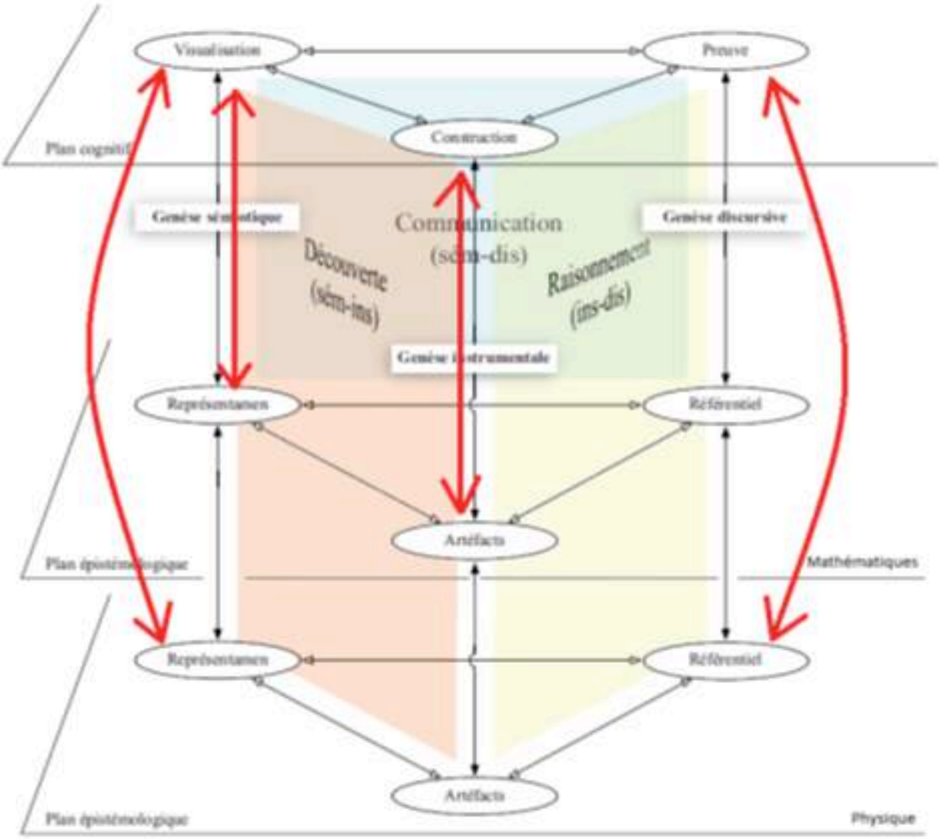

Figure 4. Analyse de l'utilisation du curseur avec GeoGebra

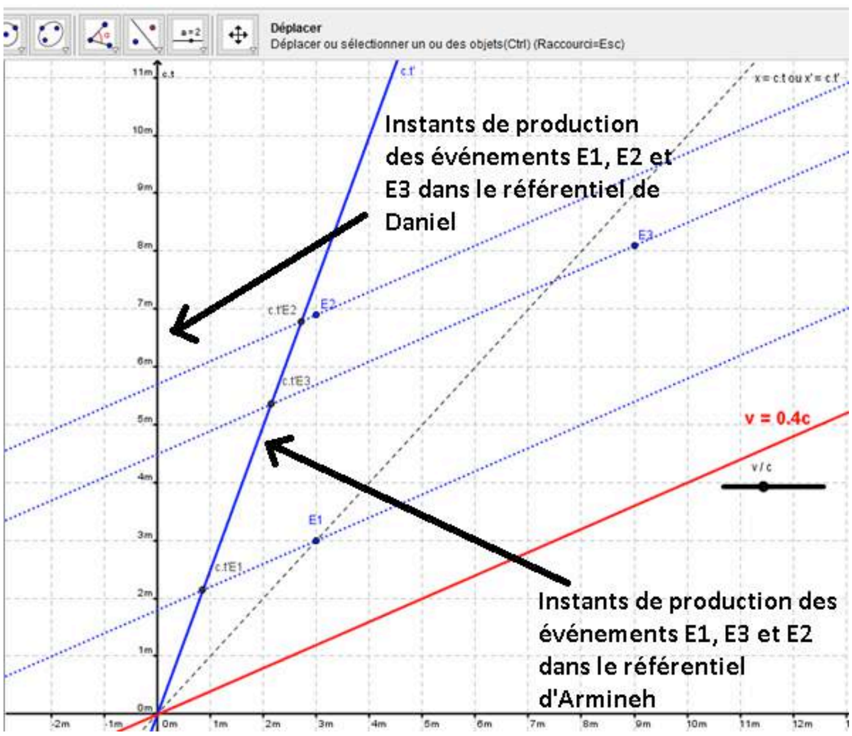

Figure 5. Diagramme de Minkowski avec GeoGebra

3. Conclusion

Le cadre de l'ETM étendu nous a permis d'analyser les tâches associées à certaines étapes du cycle de modélisation. Il permet de prendre en compte la mobilisation des plans épistémologiques des mathématiques et / ou de la physique pour chacune des tâches tout en tenant compte des genèses mobilisées. Le cadre de l'ETM étendu nous a également permis de montrer que le logiciel GeoGebra développe des genèses spécifiques par rapport à une activité papier - crayon. Une nouvelle génèse sémiotique permet une visualisation du changement des coordonnées temporelles des événements en fonction de la vitesse v d'Armineh par rapport à Daniel. Une nouvelle genèse instrumentale correspond à la manipulation du logiciel de géométrie dynamique avec la fonctionnalité curseur permettant de changer simplement les conditions expérimentales. Enfin une nouvelle genèse discursive permet de conclure sur l'ordre chronologique des événements en fonction du référentiel d'étude et de la vitesse v. Nous envisageons, par la suite, d'analyser grâce au modèle de l'ETM étendu ou à une de ses évolutions, les tâches mises en œuvre à chacune des étapes du cycle global de modélisation lors de séquences utilisant la relativité restreinte ou la mécanique classique. Des résultats préliminaires tendent à montrer que les genèses ainsi que les plans épistémologiques des mathématiques et de la physique ne sont pas mobilisés de la même façon en fonction de l'étape du cycle de modélisation. Des études utilisant la chimie sont également envisagés en tenant compte cette fois-ci du plan épistémologique de la chimie à la place de celui de la physique.

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Mathematisation in the universe of fractions: didactic principle

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Abstract. In this presentation that concerns the familiarization of children of primary school with the concept of fraction, we focus on the activity of preparation of Macedonia by the children of two fifth grades. The discussion on the construction of the Whole, results in flexibility for children in giving meaning to the keywords "Whole", "Unit", and "Quantity" in accordance with the specific situations. The flexibility is fostered by the fact that our didactic proposal is a mathematisation process, as manifested by the presence of three insights: (a) the starting point is an elementary act that is simultaneously principle and trace; (b) Euclidean division puts ideas into relation and coordinates them; (c) the new didactic universe of fractions is coherent and consistent, and it safeguards the confidence of the children.

Résumé. Dans cette présentation qui concerne la familiarisation des élèves de l'école primaire avec le concept de fraction, nous nous concentrons sur l'activité de préparation de la Macédonia par les élèves de deux cinquièmes. La discussion sur la construction de l'«Entier» produit flexibilité pour les élèves qui donnent un sens aux mots-clés «Entier», «Unité» et «Quantité» en fonction des situations spécifiques. La flexibilité est favorisée par le fait que notre proposition didactique est un processus de mathematisation, comment la présence de trois renseignements montre: (a) le point de départ est un acte élémentaire qui est à la fois principe et trace ; (b) la division euclidienne met les idées en relation et les coordonne ; (c) le nouvel univers didactique des fractions est cohérent et « consistent » et il sauvegarde la confiance des élèves.

1. Introduction

We are a working group on didactics of mathematics that has structured its specific method we like to mark with EEE: Exploring[2], Enquiring[3], Evaluating[4]. This method is emerging in his own form during our activities on familiarization[5] of children of primary school (starting from grade 3) with the concept of fraction. Our didactic proposal is aimed to introduce the concept of fraction and its fundamental properties by starting from an act of mathematisation and by following a coherent didactic path, which avoids the formation of inhibitions[6].

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[2] Our exploring on the "familiarization" of children with fractions has lasted two school cycles (= 10 years) and has allowed to find a synthesis between the indications obtained both from scientific literature and from teaching practice.

[3] Enquiring is characterized by the constant interaction between two distinct groups of reflection. In the first group, individual educational acts are discussed, before and after their presentation to the classes. This group is gradually enriching by pages of children's exercise books, which become explicit objects of noticing. The second group takes care of the "reflective, philosophical practice". It is the activity of this group that allows us to give completeness to our enquiry.

[4] Evaluating. We have planned different times of evaluation of the effectiveness of our teaching proposal; times managed both by us and by others.

[5] Familiarization differs from the teaching/learning process because it favors the formation of "correct intuitive representations" rather than the learning of formal rules.

[6] Inhibitions make the teaching/learning of fractions persistently unsatisfactory, as it is frequently claimed in

We have already presented our project at various conferences[7], in which motives, objectives, stages of our proposal have already been the object of discussion. During the current school year, the classroom activities related to our enquiring are going to be completed. Our enquiring has begun in 2014 with the two third classes of primary school in Locate di Ponte San Pietro. During the current school year, we are enquiring with two fifth classes in Locate, with one fourth class in Milan, and with one third class in Sondrio.

This is therefore the time for final considerations on the implementation of our project into the classrooms. In this presentation, we choose to emphasize some considerations concerning the features that distinguish our teaching proposal as process of mathematisation. We start from a problematic situation used in the classrooms for assessing the key competencies in giving meaning to the keywords "Whole", "Unit" and "Quantity" in accordance with the specific situations.

1.1 Determination of the "Whole", of the "Unit", and of the "Quantity".

In the scientific literature, the names "Whole", "Unit" and "Quantity" are not always used with univocal meanings. In our approach, we refer their meanings to their "forms" (Whole = n/n, Unit = 1/n, Quantity = m/n) by a process of mathematisation that have involved the activities throughout the entire third class. The first step of this process consists in identifying the act of comparison by a pair of numbers; in this step the Unit is qualified as Common Unit. In the second step the fraction is specified as comparison between the Quantity and a special Quantity, called Whole. It is in this step that the Whole acquires its form n/n; then the Unit and the Quantity respectively take their form 1/n and m/n. These forms are explicitly used in all the activities.

After reporting the teaching situation of preparing the Macedonia, the process of mathematisation will be the heart of our reflection.

2. Problematic situation: La Macedonia (the fruit salad)

This problematic situation has been proposed at the end of school year, in each of the two classes participating in the project. Each class is formed by sixteen children.

For the year-end party, each class prepares the Macedonia. Children bring fruit from home: 6 bananas, 5 apples, 6 pears, 2 bags each with 4 oranges, 1 basket of Kiwi, 1 kg of strawberries, 1 pineapple, 1 bag with 4 lemons, 100 g of sugar.

The class is divided in four groups. Each group must prepare its Macedonia according to the following recipe.

Cut into small pieces the following fruits:

3/2 bananas, 5/4 apples, 1+1/2 pears;1/2 oranges, 1/4 kiwi, 25% strawberries; 1/4 pineapple.

Add 1/4 sugar. Squeeze 1/4 lemons. Mix.

The activity begins with *a discussion to determine the "Whole" for each ingredient.* Consequently also "Unit" and "Quantity" result determined. The teacher leads the discussions within each group.

a. With regard to the bananas, the apples and the pears.

Naturally the children choose the individual fruit as Whole. Children record the Wholes (W_B=1banana; W_A=1 apple; W_P=1 pear).

The Quantities coincide with the fruits brought from home. Regarding these fruits, the Quantities are larger than the Whole. Children record the Quantities (B=6 bananas; A=5 apples; P=6 pears).

The discussion concerning the Unit is most significant in this first part. Thanks to the activities related to the form of the Unit and carried out since the third class, the children identify that U_B=1/2banana, U_A=1/4apple, U_P =1/2pear; then the banana and the pear must be divided into 2 equal parts, while the apple must be divided into 4 equal parts. The fact of obtaining the Unit by partition, echoes the classroom activities with pies, water and so on.

In this part of the activity, the Whole differs from the Quantity, and the Unit is obtained by means of partition. The activities concerning the Unit have been driven by the familiarity with Euclidean division, that

scientific literature.

[7] CIEAEM 66, Lyon (2014), CIEAEM 67, Aosta (2015), XX Congresso UMI, Siena (2015), HPM, Montpellier (2016), ICME-13, Hamburg (2016), SIRD, Milano (2016).

allows children to immediately write 3/2 = 1+1/2 and 5/4=1+1/4, and to identify the Units 1/2 and the Unit 1/4.

b. With regard to the oranges, the kiwis, the strawberries and the lemons.

The Whole is constituted by 1 bag or 1 basket (W_O=1bag=4/4; W_K=1basket=8/8; W_S=1Kg=48/48; W_L=1bag=4/4).

The Unit coincides with the single fruit (U_O=1orange=1/4; U_K=1Kiwi=1/8; U_S=1strawberry=1/48; U_L=1lemon=1/4).

The Quantity is the number of bags or packages (O=2bags; K=1basket; S=1kg; L=1bag).

Here the actual situation promotes the identification of the Whole with the basket, bag, or kg. It is useful to underline that Quantity does not correspond always with the Whole. The fact of identifying the Unit as a single fruit, echoes the classroom activities with Lego, picture cards, eggs packs and so on.

c. With regard to the pineapple.

The Whole and the Quantity coincide (W_{Pi}=Pi=1). The teacher raises the question of a suitable subdivision of the pineapple "To divide the pineapple into four parts is not effective. How can we distribute it in a more fair way?" A convenient way to divide the pineapple is to cut it into 8 slices: 1/4=2/8; an opportunity to discuss equivalent fractions. The teacher cuts the pineapple in 8 slices. The Unit is a slice ($U_{Pi}=1/8 \rightarrow W_{Pi}=8/8$).

Here the partition has a central role. Notably, reference is made to the appropriate choice of the Unit, that had been highlighted in the activities with water and with pies.

d. With regard to the sugar.

The Whole and the Quantity coincide (W_{Su}=Su=1 glass). The Unit is obtained by dividing the Whole into 4 parts ($U_{Su}=1/4 \rightarrow W_{Su}=4/4$).

Children have solved themselves this subdivision problem by taking 4 empty glasses and pouring into each glass the same amount of sugar; a translation of the activities with water.

e. Comments on the activity.

We summarize the properties of names "Whole", "Unit" and "Quantity" with the different ingredients.

Whole and Quantity coincide with the kiwis, strawberries, lemons, pineapple, sugar; Whole and Quantity differ with bananas, apples, pears, oranges.

The Unit is a single fruit with oranges, kiwis, strawberries, lemons; the Unit is obtained by partition with bananas, apples, pears, pineapple, sugar.

For the strawberries it is requested to take the 25%. This request provides an opportunity to bring the discussion on the percentage. "Write the requests for kiwis, strawberries, lemons, and sugar using the percentage".

In this activity, several "subconstructs of the construct" of the concept of fraction[8] come simultaneously into play, creating a rich articulation of meanings of the names. Nevertheless, the children have moved with "lightness" and flexibility, giving quiet, serene answers, and getting adequate results.

3. The process of mathematisation

The flexibility children have shown in similar situations, is fostered by the fact that our didactic proposal is a mathematisation process: *the confidence given by this latter constitutes the background for the search of new meanings*.

What are the elements that, in our opinion, reveal the presence of a mathematisation process?

To answer this question we let be guided by the Wheeler's insights concerning the clues to presence of mathematisation; insights that we summarize in the following way: (a) structuring (b) putting ideas into relation and coordinating them, (c) searching for universality.

a. Structuring. Principle as trace. The trace structures the enquiring.

[8] The five Kieren's "sub-constructs of the construct of rational number" are: part-whole, quotients, measure, ratios, operators. Other possible subconstructs are: proportionality, point on the number line, decimal number, and so on.

The comparison of two quantities is a pair of natural numbers (ratio).

Children start their familiarization with fractions from this act of comparison. Beside the other forms of registration of the initial act, they *are instructed to write* A;B = 9;24: "the comparison between the quantities A and B is the pair of numbers 9;24". Just this writing transforms the initial act into an *elementary act*: it originates and founds the subsequent didactic process.

This didactic choice is the result of our "*reflective, philosophical practice*"; a practice that has led us to a historical and mathematical exploration about the "originary" meaning of the concept of measure.[9] However, in classroom practice this elementary act is lived without any trouble.

"Elementary act" must be understood in Euclidean sense: not only it is the initial act, but above all it implicitly directs and structures the didactic process of fractions; it is simultaneously *principle and trace* of the process of mathematisation. The name "trace", which we understand according to Lévinas,[10] emphasizes how this act becomes a structuring factor. Indeed, in our activities on fractions, it has structured the path towards the Euclidean division. Initially it has guided us, without us realizing it: our teaching proposal contained no explicit idea of Euclidean division, but we were unwittingly pursuing its features. It was a surprise for us when it was pointed out that the sign structure we were disclosing, is the Euclidean division. Euclidean division has emerged through a process of interaction between teachers, children and trace, with the latter that continuously has refocused us.

The elementary act as principle and trace: it is the first clue to presence of mathematisation.

b. *Putting ideas into relation and coordinating them. Euclidean division as icon.*

The mathematisation process takes its comprehensive form in Euclidean division.

"Euclidean division is experienced by the children not as a formula to be memorized, but as the icon[11] of their active process of learning": it is the synthesis concerning the memory of own history of learning and it becomes the core of the universe of fractions.

Euclidean division is the core because (a) all "subconstructs of the construct of rational number", have their roots and find unity in it, (b) and because it engenders a spiral didactic process. This process explores the various contexts keeping Euclidean division in mind and continually comes back to it. [Longoni et al., 2016]

The icon-Euclidean division plays a coordination function and is element of unity in the process of entrusting meanings to the names "Quantity", "Whole" and "Unit". The problematic situation of Macedonia is characterized by the fact that each of these names has different meanings depending on the fruits: the Whole coincides with the single fruit for some ingredients, while coincides with the pack or the basket for others. The Quantity sometimes coincides with the Whole, sometimes not. The Unit is several times obtained by means of partitions, other times it corresponds to the single fruit. The process of signification reflects the multiplicity of faces / sub-constructs of the construct of fraction, but finds unity in the icon - Euclidean division.

Euclidean division puts "the sub-constructs" into relation and coordinates them: it is the second clue to presence of mathematisation.

c. *Universality. The new didactic universe of fractions.*

To build a new didactic universe of fractions, with its own rules and properties.

This aim has guided us since the exploration stage. It differs from the more common choice to take that whole number knowledge could be utilized by children in their construction of initial fraction concepts.[12] This aim has originated from the consideration of ineffectiveness of teaching and learning fractions, as repeatedly reported in the scientific literature.

The new universe is *coherent*. Coherence is referred, rather informally, to the existence of the structuring principle, and to the role of the core, the Euclidean division, with its potentiality for relating and

[9] More details have been presented at HPM 16 in Montpellier.

[10] The binomial "presence / absence in relation to the other and to the originary" characterizes the concept of trace in Lévinas.

[11] The evocative value we ascribe to the word "icon", refers to art history, to which we have recourse to highlight and enhance deeper meanings that this word has in the context of our cultural training.

[12] "But thinking in terms of integers and integer relations often interferes with the acquisition of rational number concepts".

coordinating.

It is a new universe but also *consistent* with what the children have already learned: it interacts with their knowledge. In our didactic experience, for example, the correlation between the concept of fraction (based on the subconstruct "ratio") and the subcontract "division", has occurred thanks to the used manipulative: with these manipulative (egg boxes, picture cards, etc.) the idea of measure as comparison and the idea of division, interact and guide the children to recognize themselves the fraction as measure and division at the same time.

The universe of fractions safeguards the *confidence* of children: it constitutes the background for the search of new meanings. The activity on Macedonia is a rich process of search of meanings, based on the confidence on the forms/icons that hold the universe of fractions.

The third clue to presence of mathematisation is the universality, understood as coherence, consistence and confidence on the new universe.

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Enquiring and mathematising an authentic situation

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Abstract. This paper focuses on analysing the potentialities of using multidisciplinary contexts to enhance the integration of inquiry and mathematical modelling practices. We focus on a particular teaching sequence where modelling becomes an essential tool to analyse several questions emerging from the interplay between Mathematics and History. More concretely, we focus on the design and implementation of a didactic sequence based on an archaeological context with 12-13 years old students. We can see the important role, on the one hand, of the real context that allow students validating their hypothesis and finding answers in their process of inquiry and, on the other hand, of the devices used to facilitate the interaction between the two disciplines.

Résumé. Cet article se concentre sur l'analyse des potentialités de l'utilisation de contextes multidisciplinaires pour intégrer des pratiques de recherche et de modélisation mathématique. Nous nous concentrons sur une séquence d'enseignement particulière où la modélisation devient un outil essentiel pour analyser plusieurs questions émergeant de l'interaction entre les mathématiques et l'histoire. Plus concrètement, nous nous concentrons sur la conception et la mise en œuvre d'une séquence didactique basée sur un contexte archéologique avec des étudiants de 12-13 ans. Nous pouvons voir le rôle important, d'une part, du véritable contournement qui permet aux étudiants de valider leur hypothèse et de trouver des réponses dans leur processus d'ingestion et, d'autre part, des dispositifs utilisés pour faciliter l'interaction entre les deux disciplines.

1. Introduction

In this paper, we assume the importance of mathematical modelling to inquire into the study of authentic extra-mathematical questions in schools. The main aims of our paper are both reflective and experimental to (1) reflect on the use of multidisciplinary contexts for the teaching and learning of mathematics, posing the starting questions but also providing sense to the whole study, and (2) analyse their affordances to enhance the integration of inquiry based learning (IBL) and the development of modelling students' competences. In particular, this paper focuses on the design, implementation and analysis of teaching sequences based upon archaeological contexts, and their role to improve not only mathematisation itself but also improving inquiry open attitude.

In previous research, we have worked on the design of teaching sequences where mathematics and history appear dialectically fostering inquiry and modelling attitudes of students (Sala, Giménez & Font, 2013; Sala et al., 2015). In all these experiences, we found that without integrating a constant interaction with extra-mathematical contexts, most of the interesting questions raised and the answers found by students could not be emerged. Moreover, we insisted on the importance of these interactions to make mathematical knowledge, on the one side, and historical knowledge, on the other side, progress, complement and validate each other.

In the following sections, after introducing the most important theoretical aspects we take into account for the design and analysis of the teaching sequences, we focus on the particular case of one inquiring into the kind of building that is Roman ruins found in Badalona (Baetulo in the Roman time) can correspond to. In this paper, we focus on the following questions: Which is the role of the extra-mathematical context (as starting, intermediate and finish point) along the inquiry project? Which didactic devices were more useful to promote the dialectics between inquiry and modelling, helping students to progress in the mathematical learning through modelling? In addition, how the constant interaction between history and mathematics helped to integrate a rich validation of mathematics to provide them completeness and functionality?

2. Theoretical issues

The investigation that supports this paper takes into account different theoretical aspects. First, the notion of basic competence that the Catalan curricula guidelines (similar to the NRC, 1996) include, particularly we focus on the notion of inquiry competence with the objective of assessing the students' progression related to improve their inquiry skills. These inquiry skills are hardly related to modelling approaches, and sometimes it is difficult to find the differences between both processes (Artigue & Blomhoj, 2013). From our viewpoint, mathematical modelling processes must be placed at the core of mathematical activities and of the didactic sequences to ensure the development of important processes and competences assumed in the Catalan curricula, in particular, to ensure the development of inquiry competences (Sala et al., 2017).

In order to guide the mathematical and didactic design of the teaching sequence, we use of the notion of the study and research paths (SRP) as epistemological and didactic model (Chevallard, 2015; Barquero & Bosch, 2015), to face the problem of moving towards a functional teaching of mathematics and, particularly, where mathematics are conceived as a modelling tool for the study of problematic questions. About their main characteristics of the SRP, the ones that were central for their design and which will be later explained, are the following: (a) the starting point of an SRP is a 'lively' generating question with real interest for the community of study. In our case, the generating question was about: *Which kind of building could the ruins discovered in Badalona correspond to*?; (b) during an SRP, the study of the generating question evolves and opens many other 'derived questions' which traces the likely paths to follow in the open-inquiry project, such as: questions about the geometrical shapes of the ruins, or of the public Roman buildings of the epoch, ways to simulate geometrical models corresponding to the building, ways to estimate the dimensions of the buildings, among others; (c) to enhance mathematical modelling in the SRP, the experimental milieu may be progressively enrich by many elements: questions posed, answers provided, different resources (class discussions, physical experiences, technological artefacts, etc.), but also by the integration in the study of external pre-establishes knowledge or answers that can come from mathematical (what is a circumference? an ellipsis? how to estimate their coefficients?) or from history or archaeology (reports from experts, documented ways how Romans built the buildings, etc.).

3. The historical context and its epistemological and didactic role

The starting situation which was presented to students was the discovery of some Roman ruins in a Badalona's suburb by the archaeologists' team of the city Museum. The archaeological report that was provided explained that these ruins could belong to an ancient Roman building, but *what kind of building could it have been? Could be a theatre? A circus? An amphitheatre?* Being inspired by a real archaeological investigation, the context used to design the didactic sequence was an *authentic* situation (Vos, 2011) in the sense that the situation introduced to the students is not created for any educational purposes even though some elements are included for educational purposes.

Three phases can be distinguished in its implementation. In the first phase, the archaeological question was presented to the students as the generating question of the inquiry. At this stage, they could search for information in the web, they had access to the map of the zone where the ruins were found and to the original curved Roman wall found. Moreover, they had accessible an adaptation of the real report of the experts that discovered the ruins. Some questions emerged after analysing the different resources: *What do the geometrical shape of the partial Roman wall discovered determine?* When the students enquire into the information about the Roman architecture, they found there were few buildings that had a curved part. For instance, theatres were circular, amphitheatres were elliptic or circuses had a part circular and other part quadrangular. So that, the students got easily into a geometrical mathematising process. Some students used arguments, based upon the geographical principles: "*Using Google-Maps we can see the shape of the ruins found. The houses and the streets have also curious shapes. It looks like a circle! Therefore, we think there could be a theatre under the current buildings*". In this moment, the students could formulate their first hypothesis according to the Roman wall found. Then, during one session, the group experimented with a replica of a Roman wall. There, they could try out where could be the centre of geometrical shapes, such as: a circle, an ellipsis and other conics. They reach some conclusions like: "*It is possible that the curb was an ellipsis, but if it is so, it only matches near the focus, the rest does not have the same curvature".*

In the second phase, most of students agreed that the most likely shape was a circle and that the building corresponded to a theatre. Then, the teacher guiding the inquiry let them know about the Vitruvius cannon and make accessible to students the information about how different buildings where design by the roman architect, Vitruvius. Students could then improve their initial hypothesis and design their model of the theatre following the canon. In order to draw it properly, they could use the Geogebra as artefact to simulate geometrical models. In the third phase, the students could validate and contrast their hypothesis and first answers by matching the geometrical model simulation with the real image or map of the area studied where the ruins had been discovered. At the end of this last phase, students could interview an expert of the archaeological team of the Museum who let them know about the procedure experts follow in an archaeological inquiry. Which are the tools that make possible the modelling approach?

4. Devices to promote the dialectics between inquiry and modelling

In a first phase, the students contrast with the real world, using tools to compare and contrast forecasts provided by the mathematical models with real data. The overlapping Google Map of the zone, over Roman imagined map gives opportunities to conjecture that the wall is part of a circle. The adaptation of the experts' report given to students offer information about the different elements: the curved Roman wall that could belong to the front closure of the theatre, the "proscaenum" near the centre where it is possible to find a small street, and two small squares on the opposite points of diameter would be the exit of the "vomitorium". The students make conjectures about geometrical models through such devices, provided by real data, using profits of codisciplinary approach, to find elements coming from the history, to find mathematical (geometrical) results.

During the second phase, the students essay to proof the conjectures by going to the rules done by the Roman architect as Vitruvius. It emerges the need to use representations according the rules of the history. In this stage, the technology had an essential role in order to allow students constructing a geometrical model using the software Geogebra for validating their first hypothesis. This part of task allowed students to interpret their models considering the specific context and verify if their constructions and hypothesis were suitable. The groups must write a report about their findings and inquiry process. Such device, allowed the teacher to follow the progress of inquiry teams, and to assess the global coherence of the work done, and the final results.

In the third phase, the students have done a lot of work, having many proposals about which building corresponds to the ruins after testing them on the map of the zone studied. They value the opportunity to met a specialist of the Museum, who can contribute to their proposals and clarify the questions appeared. During this interview, the students could contrast and validate their results about the model selected.

5. The role of validation to provide completeness and functionality to mathematics

One crucial characteristics of the project is the constant interaction between real context and date with the mathematical tools emerged. At the end of each phase, the students ask to check the validity of the mathematical artefacts used. Such validation moment has a functional role that helps students to raise new questions to continue with the study. For instance, to find if the model build with Geogebra serves to fit other similar ruins, to study Romanization processes, to compare the findings with Greek buildings, etc. This process of inquiry enables them to work differently when facing new other problematic situations. We observed that the way to work (work in collaborative teams, using inquiry and modelling terminology, writing reports involving mathematics and history, defending the arguments and justifications, etc.) empower the ability of organizing mathematical knowledge.

Using Inquiry Based Learning and modelling approach, the students get 'deeper' into the mathematics, or arrive at 'higher' levels of abstraction. There is a didactical intention of "going forward and backward to the real world" in order to promote new questionings. The answers give sense not only to know about but giving sense to the community, not only looking for historical and mathematical legitimation. It also empowers the idea of mathematics to be used for applications. Both, didactical and mathematical aspects of mathematisation, could be empowered in our experiences.

6. Conclusions

Due to the contribution of the historical information, the problem became achievable at the students' level of mathematical knowledge. The methods to find these solutions only from the mathematical point of view are totally beyond the powers of secondary school students'. The study and research paths (SRP) act as a didactic device to facilitate the inclusion of mathematical modelling in educational systems, and more importantly, to explicitly situate mathematical modelling problems in the centre of teaching and learning processes (Barquero, Bosch, & Gascón, 2008). Along the SRP the students were asked to build up their own 'piece of history' through: the formulation of hypotheses based on the analysis of real data and historical information.

Historical events lead to use closed models to justify the use of mathematical objects and no more. Nevertheless, the interaction among disciplines open minds of students as a didactical principle. It gives sense to mathematical models and enriches the idea of mathematical applications.

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Empowering students' decision making in an everyday risk situation

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Abstract. In the last years, the interest in statistics education about the need of empowering citizens to become a risk-literate society has increased. Research in the field has proposed possible common concepts between risk and probability for Elementary and High School; however, there are still no suggestions about possibilities for Middle School. We discuss in this paper the possibilities for an innovative curriculum, which should attend to the need of simultaneously developing the concepts of probability, strategies for decision-making and risk management. Moreover, we present a task in which students have to develop strategies to decide which is the best choice between multiple routes to move injured people from a car accident to the nearest hospital. Those routes have adverse, unwelcome and hazardous circumstances as possible sources of risk. We explore the strategies that students use when confronted with verbal, visual and numerical representation of risk. We report on the insights gained from the retrospective analysis of the implementation of the task with 9 Spanish students of grade 8 (aged 13).

Résumé. Au cours des dernières années, l'intérêt pour l'enseignement des statistiques à propos de la nécessité d'habiliter les citoyens à devenir une société alphabétisée a augmenté. La recherche dans le domaine a propos des concepts communs possibles entre le risqué et la probabilité pour l'école primaire et secondaire ; cependant, il n'y a toujours pas de suggestions sur les possibilités de l'école intermédiaire. Nous discutons dans cet article les possibilités d'un programme novateur, que devrait participer à la nécessité de développer simultanément les concepts de probabilité, des stratégies pour la prise de décisions et la gestion des risques. De plus, nous présentons une tâche dans laquelle les élèves doivent développer des stratégies pour décider quel est le meilleur choix entre plusieurs routes pour déplacer les blesses d'un accident de voiture à l'hôpital le plus proche. Ces routes ont des effets indésirables, et des circonstances dangereuses comme sources possibles de risques. Nous explorer les stratégies que les élèves utilisent lorsqu'ils sont confrontés à des verbale, visuelle et de représentation numérique de risqué. Nous faisons rapport sur les leçons tirées de l'analyse rétrospective de la mise en œuvre de l'équipe avec 9 étudiants espagnols de la 8 année (âgé, 13).

4. Introduction

Over the last years, the interest for policy makers, curriculum developers and researchers in the field of stochastics about the need of empowering citizens to become a risk-literate society has increased. The Canada and USA curricular guidelines suggest empowering students through exploring alternative situations of risk management, which lead them to become confident risk-takers. Meanwhile, many other educational systems, such as Germany, UK or Spain, are looking for ways to embed risk-based decision making into the K-12 school system, through the analysis of the potentialities of decision-making when assigning probabilities (Russel, 2015). Researchers have proposed that the possible common concepts between risk and probability are proportions, conditional probabilities and expected values for Primary Education; and conditional probabilities, Bayes' formula, independence, absolute and relative risk, and distribution of probabilities and frequencies for High School (e.g. Martignon, 2016). Nevertheless, there are no suggestions about the possible curriculum for Middle School (ages 12 to 16).

With the aim of providing continuity in the transitions between Primary Education and High School, this possible curriculum should attend to the need to simultaneously develop the concepts of probability, strategies for decision-making and risk management. From a cognitive point of view, we consider the possibility of decomposing the concept of risk management into smaller pieces of knowledge to analyze the possible changes when making informed decisions in probabilistic contexts. We have called this process systematic *operational risk management* [ORM]. We define operational risk management as a decision-making process to systematically evaluate possible courses of action, identify risk and its benefits, and determine the best course of action for any given situation (Serradó, 2016).

In order to think about the possibilities of this innovative curriculum, we have designed, implemented and retrospectively analyzed a task, which included a systematic ORM.

5. Empowering decision-making in risky situations

In this desirable risk-literate society, citizens should have the ability to interpret risk in its many forms in order to make informed life-decisions (Gal, 2005). It would mean empowering people to think for themselves in a Kantian sense, to estimate risk and make choices in the environments they encounter. This *Bildung* may equip them with strategies for translating information about risks provided by (possible) any environment into formats they could naturally understand and deal with. Once they are confronted with adequate information formats, they can make choices and decisions, often by means of very simple heuristics or presented in percentages or even in probabilities (Martignon, 2016). Those who have been empowered by simple rules for dealing with information can translate these formats into so called natural frequencies. Once they grasp the validity or "diagnosticity" of the features in their decision environment in terms of natural frequencies, they tend to make use of simple heuristics for making comparisons, for estimation and for categorization. And, finally they can make informed decisions in any risky situations.

A second controversy emerges due to the kind of numbers that citizens are able to use when assessing risky environments. Gigerenzer (2002) informs us about one main king of statistical innumeracy when assessing risk: the miscommunication of risk. This miscommunication of risk refers to not knowing how to communicate risk. In order to overcome the misunderstanding due to the miscommunication of risk, it is desirable for learners to experience risk information in several formats, and to appreciate that the provider of information may present information in a manner that suits their own agenda, and not necessarily that of the student, understood as a future risk-literate citizen. Three formats of communication are suggested: verbal, visual and numerical.

6. Methodology

We present a study, exploratory in nature, with the purpose of developing a deeper awareness of the possibilities of a curriculum for Middle School, that simultaneously develops probabilistic concepts, strategies for decision-making and risk management. The participants in this study were 9 grade 8 (aged 13) students in a coastal city in south Spain. The students solved a task designed as an ORM process.

The task consisted in helping an ambulance driver to decide the best route to minimize the time to move injured people from a car accident location to a hospital. The students have to develop strategies to decide which is the best of multiple possible routes. The routes have adverse, unwelcome and hazardous circumstances, such as sources of risk, which could modify the driving time for a particular route. We explore the strategies that students use to decide the best route when they are successively confronted by informed decisions with visual, verbal and numerical communication of risk in four stages. The task was implemented collecting written and verbal data about their responses related to the strategies used to decide the best of multiple routes to drive injured people from a car accident to a nearest hospital with verbal, visual and numerical representation of risk. A retrospective analysis of the data was conducted.

7. Results and discussion

In the first stage of the task, students had a map, as a visual communication, with 13 different unidirectional routes with hazardous situations such as many traffic lights, a school, people walking along the route and works on the road. Moreover, they were provided with verbal communication that the accident took place at night, and that the ambulance, driving with a uniform speed of 50 Km/h, and did not need to stop if a traffic light was red.

The verbal communication of the information made students feel that there were not any risks. In consequence, they reduced their strategies to the algebraic codification of these routes (8 of 9 students), computed the length of most of the routes (8 of 9 students), computed all the routes (1 of 9 students), and used proportional reasoning to compute the time needed in each route (1 of 9 students). Students used basically arithmetic procedures and heuristics, which helped them to compared and decide the best of multiple routes.

In a second stage, with the same visual communication of risk, a map, students were asked to reason about how they would change the route selected if the driver must stop at a traffic light in the case of its being red. We consider that the students misunderstood the description of risk, perceiving the risk of stopping at a red light as certain action, not as a possibility. Seven students of nine did not change their previous nomination of the route. Five of the students added to the previous heuristics by counting the number of traffic lights of every route, selecting the route with the least number of them. Four of the nine students considered when comparing the best route both the number of traffic lights and the length of each route. In consequence, they made informed decisions.

In the third stage, students were asked exclusively to analyse the map and describe possible sources of risk. Pedestrians in the road, works on the road or a college were identified by eight of the nine students as sources of risk due to hazardous situations. Nevertheless, only four of nine students considered the traffic lights as sources the risk due to slowing down the driving.

In the last and fourth stage, students were also provided with numerical information about the possible risks. Those risks were expressed as probabilities. One student did not perceive the visual and numerical risks, considering only the time in absolute terms. Six of the students changed their initial route, combining both the perception of the works on the road as a risk source and the analysis of the numerical of traffic lights and the length of the road. Two of the students used the numerical information about the probabilities, the length of the routes, and the uniform speed to estimate the possible risks and compare two or three routes. Their estimations came from a computation considering the division of length by speed and multiplying by the probability of the hazardous situation in each route. In consequence, they made well informed decisions about the route they consider less risky in terms of the possible time of the journey.

Students perceived and reacted to risk as a feeling for those hazardous situations visually communicated, such as the pedestrians or works on the road. Meanwhile, the traffic lights were not recognised as adverse or unwelcome sources of risk by all the students due to the influence of verbal information, understood in absolute terms. Those students who understood it in absolute terms made a deterministic analysis of the situation. Nevertheless, those students, who considered the uncertainty of the colour of the traffic lights, perceived the sources of risk, analysed the situation and made informed decisions. When we introduced the numerical communication of risk, we observed differences in the strategies used by the students. Those students with a faulty use of probability who understood the risk numerically communicated, developed numerical strategies to compute it and decide in accordance with their computations.

8. Conclusions

We have analyzed the possibilities of innovating Lower Secondary School curriculum through the introduction of a task which aimed to integrate probabilistic knowledge, strategies for decision-making and risk management. The task was designed using systematic operational risk management, considering of four stages of visual, verbal and numerical communication of adverse, unwelcome and hazardous situations as sources of risk. Students perceived and reacted to risk differently in the three formats of communication of risk. The visual communication provided students with risk feelings only. The verbal communication was not only an opportunity to undertake risk analysis; but also, a constriction due to the deterministic perception of the situation. The numerical communication of the risk needed the knowledge of probability to manage risk, compute it and decide accordingly.

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Young children solving multiplicative reasoning problems

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Abstract. This paper describes a study focused on kindergarten children multiplicative reasoning. It addresses two questions: 1) how do children perform when solving multiplication, partitive and quotitive division problems? And 2) what arguments do children present to justify their resolutions? An intervention program comprising 4 sessions was conducted with 12 kindergarten children (5-6-years-old), from a state supported kindergarten, in Viseu, Portugal. Similar Pre- and Post-tests were used to identify changes on children’s understanding during the intervention program. In each test children solved 28 problems (18 additive structure problems; 6 multiplicative structure problems; 4 control problems on geometry) in two different consecutive moments. The intervention comprised 4 partitive division problems, 4 multiplication problems, and 4 quotitive division problems. The problems were presented to the children by the means of a story, and material was available to represent each problem. After each resolution, each child was asked “Why do you think so?”. Results suggest that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems. Results also suggest that children’s multiplicative reasoning can be enhanced when they can experience multiplicative structures problem solving, being able to interact with peers and discuss their ideas, after receiving some prompts from teacher. This study also suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Educational implications of these findings will be discussed.the addresses.

Résumé. Cet article décrit une étude centrée dans le raisonnement multiplicatif des enfants de l’école maternelle. L’étude aborde deux questions: 1) comment les enfants résolvent des problèmes de multiplication et division, partitive et quotitive? Et 2) quels arguments les enfants présentent-ils pour justifier leurs résolutions? Un programme d'intervention comprenant 4 sessions a été organisé avec 12 enfants (5-6 ans), d'une école maternelle publique à Viseu, au Portugal. Des Pré- et Posttests similaires ont été utilisés pour identifier les changements sur la compréhension des enfants pendant le programme d'intervention. Dans chaque test, les enfants ont résolu 28 problèmes (18 problèmes de structure additive, 6 problèmes de structure multiplicative, 4 problèmes de contrôle sur la géométrie) en deux moments consécutifs différents. L'intervention comprenait 4 problèmes de division partitive, 4 problèmes de multiplication et 4 problèmes de division quotitive. Les problèmes ont été présentés aux enfants au moyen d'une histoire, et le matériel était disponible pour représenter chaque problème. Après chaque résolution, chaque enfant a été interrogé "Pourquoi tu as fait comme ça?". Les résultats suggèrent que les jeunes enfants peuvent réussir à résoudre des problèmes de multiplication et de division, en s'appuyant sur leurs connaissances informelles, en présentant des arguments qui montrent qu'ils sont en mesure d'établir le raisonnement correct lors de la résolution des tâches, en articulant correctement toutes les quantités impliquées dans les problèmes donnés. Les résultats suggèrent également que le raisonnement multiplicatif des enfants peut être amélioré lorsqu'ils peuvent résoudre des problèmes multiplicatifs, interagir avec des pairs et discuter leurs idées, après avoir reçu des instructions du professeur. Cette étude suggère encore que le raisonnement additif et multiplicatif, sous leurs formes simplistes, semble être simultanément accessible aux enfants de l'école maternelle, ce qui leur permet d'avoir un sens. Les implications éducatives de ces résultats seront discutées.

1. Framework

Children possess informal knowledge relevant for the learning of mathematical concepts. The mathematical ideas children acquire in kindergarten constitute de basis of future mathematical learning. Thus, the development of the mathematical skills in early age is crucial to the success for future learning (NCTM, 2008). Children can use their informal knowledge to analyze and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996). But they also can know quite a lot about multiplicative reasoning when they start school (Nunes & Bryant, 2010b).

Piagetian theory supported the idea that children first quantify additive relations and can only quantify multiplicative relations much later (see Piaget, 1952). In spite of his undoubted contribution to research, more recently research has been giving evidence of a different position. Thompson (1994), Vergnaud (1983) and Nunes and Bryant (2010a) support the idea that additive and multiplicative reasoning have different origins. Vergnaud (1983), in his theory of conceptual fields, distinguishes the field of additive structures and the field of multiplicative structures, considering them as sets of problems involving operations of the additive or the multiplicative type. Vergnaud (1983) argues that "multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects" (p.128). Nunes and Bryant (2010a) also consider that additive and multiplicative reasoning have different origins, arguing that "Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant." (p.11).

Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: "Joe bought 5 sweets. Each sweet costs 3p. How much did he spend?" Or "Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?" are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes & Bryant, 2010a).

Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school. For that they use the schema of one-to-many correspondence. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993), reported high percentages of success when observing kindergarten children solving multiplicative reasoning problems involving correspondence 2:1, 3:1 and 4:1. Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: "In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?", 60% of the 1st-graders and above 80% of the children of the other grades succeeded. When children were asked to solve a division problem, such as: "There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?", the levels of success for 1st-graders was 80% and above that for the other graders (2nd to 4th-graders).

In Portugal, there is still not much information about kindergarten children understanding of multiplicative reasoning, relying on their informal knowledge.

2. Methods

An intervention program was conducted with 12 kindergarten children (5-6-years-old), from a state supported kindergarten in Viseu, Portugal. Pre- and Post-tests were used to identify changes on children's understanding during the intervention program.

2.1 The intervention program

In the intervention program, the participants were divided into three groups of four elements each, having each the same age and pre-test results conditions. The intervention took place in the pre-test following week and lasts for 3 weeks. Four sessions were planned, organized by level of difficulty, equal to all the groups. In each session children solved 3 problems, and the same kind of problems was explored twice a week. Each group had the opportunity to discuss and solve the same type of problem 4 times, in a total of 12 problems. The tasks presented to the children, during the intervention comprised 4 partitive division problems (for example, "Sara has 10 candies to give to 5 children. She is doing it fairly. How many candies is each child receiving?"), 4 multiplication problems (for example, "Bill has 3 boxes with pencils. Each box has 4 pencils. How many pencils does Bill have in total?"), and 4 quotitive division problems (for example, "The teacher Anna has 12 children in her group. She wants to seat the children in groups in the tables. Each group must have 4 children. How many tables does teacher Anna need?"). The problems presented to the children in the intervention program were similar to those of the multiplicative structure problems given in the tests. All the problems were presented to the group of children by the means of a story, and material was available to represent them. After each answer, each child was asked "Why do you think so?" in order to reach a better understanding of his/her reasoning. No judgments were conducted, and group discussion was stimulated. Video recorder and field notes were used in data collection.

2.2 Pre- and Post-tests

Individual interviews were used in the Pre- and Post-tests, in which children solved a battery of 28 problems (18 additive structure problems; 6 multiplicative structure problems; 4 control problems) in two different moments. The problems presented to the children were selected and adapted from the Vergnaud's classification (see Vergnaud, 1982, 1983).

The problems of both tests were similar. The additive structure problems presented to children in the tests comprised: i) composition of two measures, (for example, "Mary has 8 dolls but only 2 are in the box. How many dolls are outside the box?"); ii) transformation liking two measures, with the starting and element of transformation omitted, (2 for addition, 2 for subtraction), (for example, "There are 5 frogs in the lake. Some more join the group. Now there are 8 frogs. How many frogs came to join the group?"); iii) static relation linking two measures, (2 involving "more than", 2 for "less than"), (for example, "Anna has 4 puppies. John has 2 more than Anna. How many puppies does John have?"). The multiplicative structure problems in the tests comprised: iv) Isomorphism of Measures, selecting the problems of Multiplication, Partitive Division, and Quotitive Division. The control problems included only geometry tasks (geometric regularities and shape with tangram).

The problems were presented to the children by the means of a story, and material was available to represent the problems. After each resolution, each child was asked "Why do you think so?" in order to reach a better understanding of his/her reasoning. Data was registered in video and researcher's field notes.

3. Final remarks

This study explores the effects of a short intervention program focused on multiplicative reasoning on young children solving additive and multiplicative structure problems. Results will be presented in the conference. The intervention was effective as children improved their understanding of multiplicative reasoning problems. Multiplication problems revealed to be easier for children than division ones. Also children's arguments revealed improvements. Young children provided arguments and explanations that sustain the idea that their successful resolutions were not obtained randomly, as they were supported by valid or partially valid explanations.

Previous research on kindergarten children solving multiplicative reasoning problems (see Carpenter, et al., 1993) reports levels of success, but does not refer children's explanations or arguments to give a better insight of children's way of thinking. Also Nunes et al. (2005) reports remarkable success levels when 1st-graders solve multiplication and division problems, but give no reference to their explanations. The study reported here gives evidence that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems.

This study suggests that children's multiplicative reasoning can be enhanced when they can experience problem solving being able to interact with peers and discuss their ideas, after receiving some prompts from teacher. In agreement with Soutinho's (2016) ideas, this study suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Thus, perhaps kindergarten mathematics should include more of these experiences in order to develop children's mathematisation. Educational implications will be discussed.

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Teaching shifts using as tool mathematical modeling

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Abstract. An important challenge for students in the secondary school is to be able to move 'comfortably' in different contexts, transferring mathematical knowledge during transitions between these contexts. In this presentation, we explore possibilities of knowledge transfer during transitions[13] between micro-contexts —as they are determined by digital or tangible material— of mathematical practices. Focusing on the idea of shorter path, students of 7th grade are involved in horizontal and vertical mathematization through realistic situations and algebraic and geometrical representations.

Résumé. Un défi important pour les étudiants dans l'école secondaire est d'être capable de bouger 'confortablement' dans les contextes différents, en transférant la connaissance mathématique pendant les transitions entre ces contextes. Dans cette présentation nous explorons des possibilités de transfert de connaissance pendant les transitions entre les micro-contextes — puisqu'ils sont déterminés par la matière numérique ou tangible — des pratiques mathématiques. En se concentrant sur l'idée de sentier plus court, les étudiants de 7ème qualité sont impliqués dans mathematization horizontal et vertical par les situations réalistes et les représentations algébriques et géométriques.

1. Introduction

We discuss part of a research project that was conducted in order to explore issues of knowledge transfer between micro-contexts of mathematical practices as they are defined by digital tools and tangible materials. The focus of the part we present here concerns the big scientific idea: «The nature selects the most economic mode of development, based on shorter paths and the consumption of minimal energy together with maximizing the work produced». The kick-off activity was based on the 'car road problem'[14]. The tool used for modeling was the software of dynamic geometry, Geometer's Sketchpad, with arrangements of Plexiglas and pins as shown in fig. 3 as hands-on materials. The didactical performance was based on the pedagogical approach: «Following students' intentions and expectations» (Tsitsos & Stathopoulou, 2011).

2. Some theoretical points

Mathematization in formal teaching concerns the mathematical modeling process with starting point situations or objects - real or imaginary- that make sense for students. So mathematization is a bridge through the educational community of the classroom between students' knowledge and capabilities, and mathematical structures socially, historically and culturally defined. We can see the mathematization[15] as a

[13] About Transitions between contexts of mathematical practices see: de Abreu, Bishop, & Presmeg (2006).
[14] The motorway problem: The problem concerns the finding of the shortest path that connects four cities located at the four vertices of a rectangle.
[15] The Freudenthal (1991) and Treffers (1987) distinguish mathematisation in horizontally and vertically. The horizontal mathematization concerns the transition from the real world to the world of symbols. It constitutes a metaphor, a shift of semantic structures. The vertical mathematization concerns the reorganization of these concepts of the mathematical framework and τhe connection of magnitudes involved.

transition process – an initiation or entrance into new worlds, one world of human experience which is subjective and changeable, and another, 'objective world of knowledge', produced by science (Husserl, 1970; Patocka, 2008). The mathematization in this sense is a tool of pedagogical action; students have been taught how to find and work with authentic situations and real-life problems (Rosa & Orey, 2010). Bassanezi (2002, p. 208) describes this process as "ethno-modeling".

The main research pillars of the research presented here are both the activity theory approach and the dynamic dimension of knowledge transfer. The activity theory, based inter alia on the Cole work (1996), Engeström, (1999), influenced by Vygotsky (1978) and the Russian tradition, falls on sociocultural approaches with central idea that learning activity is a collective action mediated by cultural symbols, words and tools (Cole, 1996).

The lesson is conceived as an activity system (Engeström & Cole, 1997, p. 304) that both mediates between subject and object, and integrates the community in which the subject belongs, including the rules they are governed by, and the division of labor. 'Dynamic transfer' (Schwartz, Varma, & Martin, 2008) refers to possibilities for prior knowledge to create new concepts, in contrast with 'similarity transfer': In dynamic transfer, people coordinate situations in order to deal with new concepts; the coordination process is performed in this research with the contribution of different frameworks, and mostly via the digital tool of Dynamic Geometry (Geometer's Sketchpad). Students were focusing on the concept of covariant quantities, in an effort to extend previous work by James Kaput (1996), who conjectured that the difficulties in understanding the concept of regression might have to do with a lack of connection between the representations (numerical, algebraic, analytical, tabular, graphical) and the physical or virtual experience of student; Kaput suspected that the graphical representation of covariance might better precede the algebraic and analytic representation.

3. Methodological issues- the method of the research

The methodology followed is that of action research; specifically, we adopted the model of Kemmis and McTaggart (1988), which consists of a series of actions where the researcher repeats the following cycle of steps: plan, act, observe and reflect upon. The actions were organized into ten (micro) cycles, consisting of activities that belong to the category of Design Experiments (Cobb et al, 2003). Both the collection and analysis of data were based on Grounded Theory (Classer and Strauss, 1967), where the task of the researcher is to understand what happens during the research procedure, and to focus on how the people involved manage their roles; analysis in this approach aims to define categories through which the researcher can highlight a theory implied by the data. The basic analysis unit is the activity.

The setting of the research

The research was conducted in a typical 8th grade class with 24 students. It was developed in six teaching hours of curriculum. The students worked divided into six groups of four people.

4. Design and implementation of the research

The main pillar of the teaching objective was to develop a teaching situation where the students would be able to capture aspects of the abstract mathematical concept of covariance, without explicit direction, while the research objective was to investigate issues in students' transitions between the mirco-contexts of mathematical practices.

A short description of the 10 micro-cycles[16] that were developed during our interventions follows. Students designed possible routes with the shortest distance connecting all four cities (Figure 1 and 2), comparing their measurements while experimenting with the real model (Figure 3), adjusting the virtual model in order to match to the actual; they explored the shortest distance on the interactive environments depicted on the pricing table (Figure 4), and tested possible analytical correlations of the involved magnitudes (Figure 5). Figure 6 is an example of how a student had the chance to explore one hypothesis on his own, by re-designing figure 5.

These student actions moved them towards finding algebraic relationships between the magnitudes

[16] As micro-cycle it is considered a task delimited in time resulting from a community of practice constituted by students and teacher and it has an autonomy.

involved in the problem. Then, based on the students' question, "How could the computer automatically find the shortest path?", we turned their attention to the investigation of the problem in designing parametric figures conforming to geometric conditions (Figure 7). This was the chance to pose a new question: "In the case where the cities are not arranged in a rectangle, how might we calculate the shortest path?". This question led to a geometric generalization (Figure 8), and in this way contributed to further emerging questions and the modification of the real model, and hence, to the empirical determination of the geometric solution. Through the formation of the films and the comparison of the position of the nodes in both the real model and the visual one, students posed additional questions, for example: "How does the film know that this is the shortest path?"Around this phenomenon a discussion was developed about the fact that nature seems to always choose the most economical way to act, selecting symmetry to accomplish this.

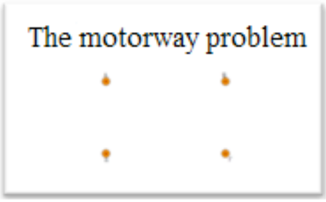


Figure 3.

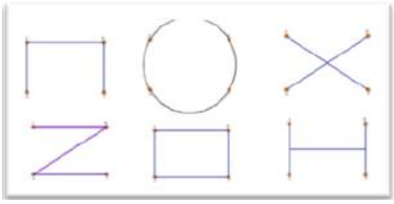
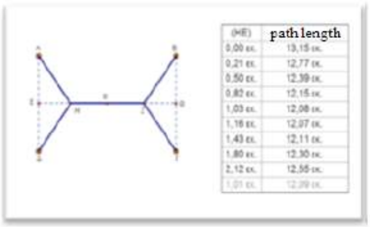


Figure 1.

Figure 2.



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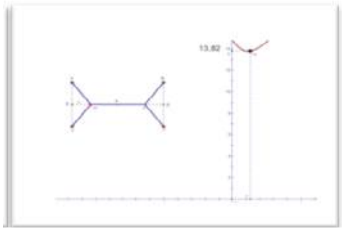

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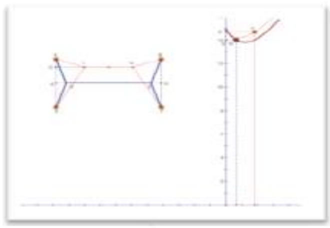

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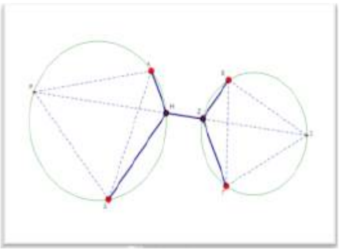

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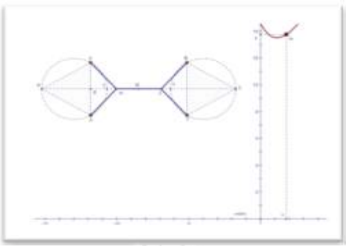

Figure 8.

5. Discussion

The whole procedure constituted a coordination of different cognitive areas and representations that combined different techniques and methods, aiming to emphasize for the students that mathematics does not function in a piecemeal or fragmented way, and is characterized by continuity, structure, affinity, and in connection with other sciences. The action research cycles made evident how the challenging conceptual points that appeared were faced on the one hand by a repositioning of the problem in the Dynamic Geometry framework, and on the other hand by modeling the situation of the problem. Opportunities to make conjectures, refutations and cognitive connections made it possible for students to experience both the repositioning and the modeling. The notion of transitions to different contexts in this research is connected with knowledge transfer. The transfer is considered as the coordination of all these contexts. The crucial point of the transitions is the real experiment (experiment in real situations). In each micro-cycle, we were able to recognize as interconnected both horizontal and vertical mathematization, and even that some very different epistemological approaches and different types of representations were involved. What is also included in our research but cannot be discussed here, because of the limited space, is the study of structures that appeared through the 10 micro-cycles in other thematic areas, e.g., 'How might we represent magnitudes that covary?', 'Which is the role of the minimum on the graph?', 'What does a geometrical construction do?'. We believe that these kinds of questions emerge because of the kind of instructional approach that was enacted within this action research. Both the learning outcomes of the activity and the findings of the research create 'un-mapping regions' of the generalization and abstraction of mathematical concepts related to the digital tools and the tangible materials. Students learn more than the mathematics: they also learn along with the mathematics a belief in mathematics as opening up questions and exciting directions for exploration. Researchers find new pathways for un-mapping both the classroom learning environment and the content they originally thought was being taught.

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Favoriser la dévolution de la mathématisation horizontale aux élèves engagés dans une activité de modélisation

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Résumé. Dans nos travaux, nous étudions les conditions et les contraintes permettant de favoriser la dévolution de la mathématisation horizontale aux élèves de collège et de lycée engagés dans une activité de modélisation. Notre objectif est d'aider les élèves à comprendre la nécessité de faire des choix pour envisager un traitement mathématique d'un problème posé dans un domaine non mathématique. Nous présentons dans cette communication les éléments d'analyse *a priori* d'un problème posé dans un contexte relevant des sciences du vivant, expérimenté en 2016 dans une cinquantaine de classes de l'enseignement secondaire français, ainsi que les tous premiers éléments de notre analyse *a posteriori*. Les premières analyses des travaux d'élèves, en appui sur nos indicateurs de dévolution de la mathématisation horizontale, montrent la faisabilité et la fécondité potentielles d'une telle activité mathématique.

Abstract. In our research, we study the conditions and constraints to favor the devolution of horizontal mathematization to secondary school pupils engaged in a modeling activity. Our goal is to help students understand the need to make choices to consider mathematical treatment of a problem in a non-mathematical field. We present in this paper the elements of *a priori* analysis of a problem posed in a life sciences context, experimented in 2016 in fifty classes of French secondary education, as well as the very first elements of our analysis *a posteriori*. The first analyzes of student work, based on our indicators of devolution of horizontal mathematization, show the potential feasibility and fertility of such a mathematical activity.

9. Introduction

Nous présentons dans cette communication des éléments d'une recherche en cours sur la possibilité de faire la dévolution (au sens de Brousseau 1998) aux élèves du secondaire de la mathématisation *horizontale*. Nous suivons en cela Treffers (1978) qui distingue deux types de mathématisation en jeu dans une activité de modélisation en mathématiques : la mathématisation *horizontale* qui relève de la modélisation mathématique d'un fragment de réalité non mathématique et la mathématisation *verticale* qui relève du traitement mathématique d'un problème mathématique.

Dans ce texte, nous nous intéressons à la mathématisation horizontale la première étape de la modélisation qui consiste à opérer des choix permettant d'envisager un traitement mathématique du problème. Cette étape du processus de modélisation n'est généralement peu voire pas travaillée dans les classes, alors qu'elle est essentielle dans le travail de modélisation des chercheurs travaillant dans le domaine des mathématiques appliquées aux sciences du vivant.

La question didactique que nous mettons à l'étude est celle de la possibilité d'implémenter en classe des situations didactiques permettant de faire vivre cette étape aux élèves tout en prenant en compte les conditions et les contraintes qui pèsent sur les enseignants. Ce questionnement est né des observations conduites depuis plus de dix ans au sein du groupe ResCo (Résolution Collaborative de problèmes) de l'IREM de Montpellier, qui vise à mettre les élèves en position de chercheur (Yvain et Gardes, 2014)

Pour conduire cette étude, nous nous sommes placée dans le cadre de situations de résolution de problèmes posés dans un contexte relevant des sciences du vivant et pouvant se résoudre via une modélisation mathématique.

L'objectif de notre recherche est le développement d'une ingénierie didactique permettant de favoriser ce processus de dévolution. Pour cela, conformément à la méthodologie générale de l'ingénierie didactique

(Artigue, 1988), nous conduisons une étude épistémologique visant à identifier précisément ce que l'on va transposer des savoirs savants d'une part, des pratiques de références des experts du domaine d'autre part. Cette étape de notre recherche est en cours : plusieurs entretiens avec des chercheurs en mathématiques appliquées aux sciences du vivant et des chercheurs en biologie utilisant des mathématiques dans leurs travaux sont en cours d'analyse (Yvain, à paraître). Dans ce qui suit, nous décrivons tout d'abord brièvement les modalités de travail du groupe Resco. Nous présentons ensuite les critères de choix, l'énoncé et l'analyse *a priori* d'une situation mathématique propice à permettre le développement d'une situation didactique pouvant être utilisée dans le cadre du dispositif ResCo. Pour finir, nous présentons les premiers éléments de notre analyse *a posteriori*.

10. Un dispositif et des situations pour favoriser la dévolution de la mathématisation horizontale

Le groupe ResCo existe depuis le début des années 2000. La résolution collaborative repose sur des échanges entre des classes, par groupe de trois, qui travaillent sur le même problème de recherche, pendant cinq semaines. Toutes les classes du secondaire de la 6ème à la Terminale17 sont concernées.

Ce dispositif comporte l'élaboration chaque année d'une nouvelle situation, un stage de formation de deux jours pour former et accompagner les enseignants qui souhaitent engager leurs classes, et la mise en œuvre dans les classes engagées pendant cinq semaines consécutives.

Les problèmes élaborés par le groupe puis proposés pour une session de résolution collaborative sont posés en dehors du cadre mathématique ; ils sont choisis de sorte que plusieurs modèles mathématiques sont envisageables. Néanmoins, étant donné que la mathématisation de problèmes réels (tels que l'on peut les rencontrer au niveau de la recherche) est généralement beaucoup trop complexe pour être prise en charge au niveau de l'enseignement secondaire, les situations proposées ne sont pas directement issues de la réalité mais relèvent ou évoquent de façon réaliste des situations du monde réel. C'est la raison pour laquelle elles sont qualifiées de «fictions réalistes» (Aldon et al., 2014). Nous renvoyons à l'atelier de S. Modeste pour plus de détails quant aux spécificités de ce dispositif didactique.

Lors de la première séance, les élèves découvrent le problème et préparent des questions qu'ils adresseront aux deux classes avec lesquelles leur classe est associée. L'objectif est de faire émerger un questionnement sur les différents choix possibles permettant un traitement mathématique du problème. La place de cette phase de questions dans le dispositif vise à déclencher le processus de mathématisation horizontale. Lors de la phase des questions, les élèves commencent à choisir des fragments de la réalité fictionnelle sur lesquels ils se questionnent. Dans la phase des réponses, ils commencent à faire des choix de mathématisation horizontale.

Nous illustrons ce qui précède dans le cas de la situation mise en œuvre 2016.

11. Un exemple de mise en œuvre: la croissance des arbres

3.1 La situation choisie – motivations – éléments d'analyse a priori

Pour favoriser un travail de mathématisation horizontale, nous avons élaboré une situation permettant:

- d'amener les élèves à se questionner sur le système à modéliser - C_0

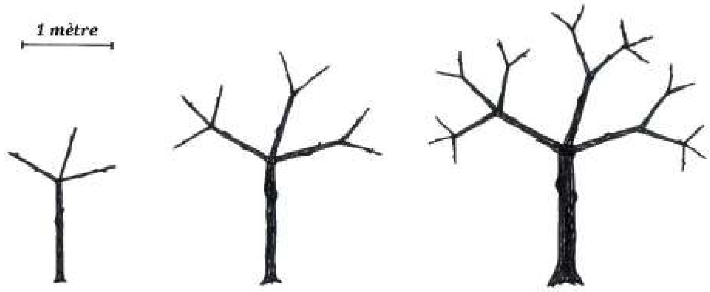
De faire prendre conscience aux élèves:

- de la nécessité de recourir à une modélisation pour répondre au problème - C_1 :
- de la nécessité de faire des choix lors de la modélisation - C_2
- de l'importance de la question posée dans la construction du modèle - C_3
- que le travail de modélisation nécessite un travail mathématique au sein du modèle choisi pour apporter des réponses aux questions posées - C_4

La situation proposée aux élèves est la suivante : il s'agit de prédire la croissance d'un arbre à partir d'informations sur ses premières années de croissance données par des schémas avec une échelle donnée. En voici l'énoncé.

[17] (De 11 à 18 ans)

L'arbre :
Des botanistes du Jardin des Plantes ont rapporté un arbre exotique inconnu, dont on vient de découvrir l'espèce. Pour étudier cette nouvelle espèce, les botanistes ont réalisé les croquis de l'arbre chaque année depuis 2013.



Les botanistes veulent construire une serre pour protéger l'arbre. Ils estiment qu'il aura atteint sa maturité en 2023. Pour les aider, prévoyez comment sera l'arbre en 2023.

Figure 1. ***Énoncé de la fiction réaliste « l'arbre » (échelle modifiée)***

Cette proposition, conçue comme une transposition d'un problème de modélisation de la croissance des végétaux en sciences de la vie (Varenne, 2007) répond à nos critères dans la mesure où

- L'échelle de temps lent de croissance de l'arbre et la trop grande complexité du système motivent le besoin de prévision et la nécessité de produire un modèle.
- Plusieurs hypothèses sur la croissance de l'arbre peuvent être faites ce qui donne lieu à différentes modélisations. Il est donc nécessaire de faire des choix lors du processus de modélisation.
- Les choix dépendront des données du problème (schémas de l'arbre) et des outils mathématiques à disposition des élèves (ou la conception qu'ils en ont).

Pour favoriser la dévolution aux élèves de la mathématisation horizontale, les valeurs des variables didactiques choisies et leur motivation sont les suivantes :

- Des schémas en 2D (et non pas en 3D) pour proposer un cadre suffisamment réaliste tout en permettant une activité de modélisation de la 6ème à la terminale.
- Le nombre de schémas : nous en avons proposé trois.
- La forme de l'arbre (symétrique versus asymétrique): nous avons choisi une croissance asymétrique pour favoriser le questionnement des élèves autour de la prévision de la croissance de l'arbre.
- Le nombre de nouvelles branches apparaissant chaque année : nous avons choisi de faire apparaitre deux ou trois nouvelles branches pour amener rapidement les élèves à faire des choix au niveau de la croissance de l'arbre.
- Les longueurs des troncs et branches : elles ont été choisies pour questionner un éventuel choix d'un modèle de croissance régulière.
- La donnée d'une échelle : afin de permettre des mesures et une prise d'information instrumentée sur les dessins

Pour la phase de l'élaboration des questions, nous avons défini les trois indicateurs de la dévolution aux élèves de la mathématisation horizontale suivants :

La question

- porte sur l'identification de grandeurs pertinentes pour permettre un traitement mathématique (Q_{C0})
- montre la recherche d'un modèle permettant de traiter la situation proposée (Q_{C1})
- porte sur la pertinence d'éléments de contexte à prendre en compte (Q_{C2})

Pour la phase de l'élaboration des réponses, nous avons défini les cinq indicateurs de la dévolution aux élèves de la mathématisation horizontale suivants :

La réponse montre :
- la recherche d'un modèle permettant de traiter la situation proposée (R_{C0})
- des choix de grandeurs pertinentes pour permettre un traitement mathématique (R_{C1})
- des choix d'éléments de contexte (R_{C2})
- l'analyse par les élèves de la pertinence de question reçue au regard de la question posée (R_{C3})
- un premier travail mathématique pour répondre à la question reçue. (R_{C4})

3.2 Premiers éléments d'analyse a posteriori

Les données recueillies lors de l'observation de l'intégralité du dispositif dans certaines classes engagées dans la résolution collaborative en 2016 sont en cours d'analyse à la lumière du questionnement sur la dévolution de la mathématisation horizontale aux élèves. Nous donnons ici des exemples (issus des travaux d'élèves recueillis sur la plateforme d'échanges du dispositif ResCo) qui permettent d'illustrer la manière dont on va utiliser les indicateurs de la dévolution de la mathématisation aux élèves.

Questions type Q_{C0} la question porte sur l'identification de grandeurs pertinentes pour permettre un traitement mathématique	Réponses associées	Indicateurs
Quelle est la circonférence des branches de l'arbre en novembre 2013-2014 et 2015 ?	on mesure le diamètre sur le dessin, et on trouve 1mm la 1ère année, on convertit en grandeur réelle grâce à l'échelle : 1 m est représenté par 2,8 cm, donc 1 mm représente 3,57 cm. On multiplie par pi pour obtenir la circonférence, ce qui donne : en 2013 : 11,21 cm en 2014 : 22,43 cm en 2015 : 33,64 cm	R_{C4}
Le nombre de branches augmente-t-il chaque année?	2 ou 3 branches de plus par branche, mais on n'a pas trouvé de modèle qui définisse le nombre de branches supplémentaires à chaque fois.	R_{C0}
Y a-t-il un rapport entre les angles formés par les branches ?	Les angles ont l'air de varier entre 60 et 80°	R_{C4} R_{C1}
Quand on compte les branches, doit-on compter toutes les branches (avec celles du milieu) ou seulement les nouvelles branches ?	On pense qu'il ne faut que les nouvelles branches.	R_{C1}
Combien d'étapes d'évolution jusqu'en 2023 ?	2015-2016 2016-2017 2017-2018 2018-2019 2019-2020 2020-2021 2021-2022 2022-2023 On a compté qu'il y avait 8 étapes	R_{C4}

Figure 2. ***Exemples de questions recueillies relevant de l'indicateur QC0, réponses et indicateurs associées.***

Questions type Q_{C1} La question montre la recherche d'un modèle pour traiter la situation	Réponses associées	Indicateurs
Est-ce que l'arbre grandit proportionnellement ?	Non, l'arbre ne grandit pas proportionnellement car nous avons fait des mesures : Années Hauteur max. de l'arbre (en cm, sur le schéma) 2013 : 4,7 2014 : 7,1	R_{C4}

	2015 : 8,6 De 2013 à 2014, il a grandi de 2,4 cm (sur le schéma) et seulement de 1,5 ensuite de 2014 à 2015.	
Doit-on travailler en 2D ? ou en 3D ?	Plutôt en 3d pour plus de réalisme	R_{C0}
Les branches grandissent au cours du temps ainsi que le tronc, peut-on estimer la pousse de chaque branche ?	Toutes les branches d'une même génération grandissent de la même façon/même proportion. Et les anciennes générations poussent moins vite que les nouvelles	R_{C0}
Le nombre de branches évolue-t-il comme la suite de Fibonacci ?	Non il suffit de regarder le nombre de branches sur les trois premiers dessins pour se rendre compte que le nombre de branches à une année n'est pas la somme du nombre de branche des deux précédentes.	R_{C4}

Figure 3. **Exemples de questions recueillies relevant de l'indicateur Q_{C1}, réponses et indicateurs associées.**

Question type Q_{C2} La question porte sur la pertinence d'éléments de contexte	Réponses associées	Indicateurs
Taille-t-on l'arbre?	Il faut supposer que non pour se simplifier la tâche	R_{C0} - R_{C2}
La présence de la serre va-t-elle avoir une influence sur la croissance?	Nous ne pensons pas que la serre influencera la croissance.	R_{C1}
Quels sont les apports d'eau dont requiert l'arbre afin d'optimiser sa croissance?	Nous pensons que cela n'a pas d'importance.	R_{C3}
Le matériau de la serre ou sa couleur influencent-ils la pousse de l'arbre (favorisation de la photosynthèse, maintien d'une température optimale,...).	La couleur n'aura pas d'influence si elle laisse passer la lumière naturelle, pareil pour le matériau tant que la température intérieure *correspond à celle du milieu d'origine.*	R_{C2}
L'environnement du Jardin des Plantes est-il semblable à l'environnement initial de l'arbre ?	Nous ne pouvons pas savoir.	R_{C3}

Figure 4. **Exemples de questions recueillies relevant de l'indicateur Q_{C2} réponses et indicateurs associées**

Nous donnons également quelques exemples ci-dessous de questions-réponses ne relevant pas de nos indicateurs de dévolution de la mathématisation horizontale :

Questions	Réponses
Qu'est-ce qu'un botaniste ?	C'est un spécialiste des végétaux.
Que veut dire le mot serre ?	C'est un abri pour arbres en verre
Est-ce qu'il y a des animaux sur l'arbre ? Si oui, vont-ils faire des nids ?	Dans la serre il n'y aura pas d'animaux ni d'insecte.

Figure 5. **Exemples de questions - réponses ne relevant pas de nos indicateurs de dévolution**

Nos premières analyses confortent notre hypothèse selon laquelle le dispositif de questions-réponses est de nature à favoriser la dévolution aux élèves de la mathématisation horizontale.

12. Conclusions et perspectives

En nous appuyant sur les indicateurs de dévolution de la mathématisation horizontale, nos premiers résultats montrent qu'il est possible de laisser les élèves prendre en charge la mathématisation horizontale dans une activité de modélisation sous deux conditions au moins :

- L'élaboration d'une situation spécifique nécessitant de faire des choix pour engager un processus de modélisation ;
- La mise en place d'un dispositif didactique proposant un temps de travail dédié à la mathématisation horizontale

Ces conditions sont en relation étroite avec notre objectif de transposition de pratiques des chercheurs dont le travail relève de la modélisation. Les indicateurs de la dévolution que nous avons retenus s'appuient en effet sur les premiers résultats des analyses en cours d'entretiens effectués auprès de chercheurs engagés dans des recherches en mathématiques appliquées aux sciences du vivant en vue de mieux cerner leurs pratiques dans un processus de modélisation (Yvain, à paraître).

Comme de nombreux travaux le montrent le rôle de l'enseignant est essentiel pour que la richesse potentielle d'un dispositif didactique soit actualisée dans la classe. Aussi, parallèlement à notre étude présentée dans cette communication, nous nous questionnons sur les conditions et les contraintes prendre en compte pour envisager la dévolution aux enseignants (via la formation) de l'enjeu de dévolution de la mathématisation horizontale aux élèves.

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WORKING GROUP B / GROUP DE TRAVAIL B

CIEAEM 69

Berlin (Germany)

July, 15 - 19 2017

MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

"Quaderni di Ricerca in Didattica (Mathematics)", n. 27, Supplemento n.2, 2017
G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Mathematisation: A general mathematical model as a point of departure of a didactic arrangement

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Abstract. This paper is aligned with sub-theme 1 of CIEAEM 69 in that it concerns mathematisation as a didactic principle. The mathematisation of this paper is about drawing on general mathematical structures when students in grades 3, 8 and 9 solve equations where negative numbers may occur. The context of the paper is two research projects conducted with teachers. In this paper we present a specific didactic principle in the form of a part-whole model which can visualize relationships between numbers within equations. Through the adoption of this model we explored how this specific mathematical model may work as a mediating tool for the students when solving equations, even when negative numbers are involved. This approach is based on general mathematical structures and not on real-world contexts or empirical material. In this sense, it challenges an arithmetic teaching tradition where mathematics is introduced using specific numbers and sometimes real-world tasks, and thereafter gradually shifting the teaching to the abstract and the general. In both projects sub-studies have been carried out using semi-structured interviews and a paper-pen-test selectively combined with interviews. The findings indicate that a part-whole model, presented below, is working as a tool for students while facilitating the solving of equations. During CIEAEM 69 we would like to present results from the sub-studies.
Résumé. Ce papier est des alliés du sous-thème 1 de CIEAEM 69 dans lequel il concerne mathematisation comme un principe didactique. Le mathematisation de ce papier s'agit de comprendre des structures mathématiques générales quand les étudiants dans les classes 3, 8 et 9 résolvent des équations où les nombres négatifs peuvent se produire. Le contexte du papier est deux projets de recherche conduits avec les enseignants. Dans ce journal nous présentons un principe didactique spécifique dans la forme d'un modèle partiellement entier qui peut visualiser des rapports entre les nombres dans les équations. Par le biais de l'adoption de ce modèle nous avons exploré comment ce modèle mathématique spécifique peut travailler comme un outil de médiation pour les étudiants en résolvant des équations, même quand les nombres négatifs sont impliqués. Cette approche est basée sur les structures mathématiques générales et pas sur les contextes de monde réel ou la matière empirique. Dans ce sens, il défie une arithmétique la tradition enseignante où les mathématiques sont introduites en utilisant des nombres spécifiques et quelquefois des tâches de monde réel et en déplaçant par la suite progressivement l'enseignement au résumé et au général. Dans les deux projets les sous-études ont été réalisées en utilisant des interviews semi-structurées et une épreuve du stylo en papier sélectivement combinée avec les interviews. Les conclusions indiquent qu'un modèle partiellement entier, présenté ci-dessous, travaille comme un outil pour les étudiants en facilitant la solution d'équations. Pendant CIEAEM 69 nous voudrions présenter des résultats des sous-études.

1. Background

Results from international tests such as TIMSS 2011 illuminate that challenges occur when Swedish students calculate subtraction tasks even without negative numbers appearing (Skolverket, 2012).
Swedish research (e.g., Kilhamn, 2011) as well as elsewhere (e.g., Ball, 1993) highlight that different challenges appear when subtraction tasks with negative numbers are present in teaching. Ball (1993) discusses the importance of bringing negative numbers into the students' context. In this respect, it appears

that it is an advantage for both students and teachers to be aware of the mathematical issues that historically have been a challenge for humanity. Students may perceive negative numbers simply as positive numbers with a subtraction sign in front of them. Moreover, negative numbers are difficult to visualize as quantity of an amount (Kilhamn, 2011).

Mathematics teaching based on an algebraic teaching tradition, and Davydov's curriculum, which was constructed and designed in Russia in the late 1950s, is based on the idea that even young students need to distinguish general mathematical structures, but not based on rules and strategies of knowing how to solve tasks. Instead it allows students to explore relationships, for example, in equations in order to find missing numbers (Davydov, 2008; Kieran, 2004; Schmittau & Morris, 2004; Slovin & Venenciano, 2008; van Oers, 2001). According to Davydov's curriculum, equations can be described as relationships with a part-whole model. This relationship is visualized by a diagram (see Figure 1).

The interest in the two projects, of which this paper is one part, is to explore whether and how the part-whole model is fruitful when solving equations also when the minuend assumes a lower value than the subtrahend (e.g., 4 – 7 = ?); in other words, when the whole assumes a lower value than one or more of its parts. To our knowledge, Davydov's curriculum and the part-whole model has previously only been explored regarding natural numbers. As far as we are concerned, it is not possible to empirically demonstrate negative numbers as quantities. Consequently, students need to handle the part-whole model abstractly and generalize the model mathematically. The diagram (shown below) can be used in order to visualize the part-whole relationships in equations.

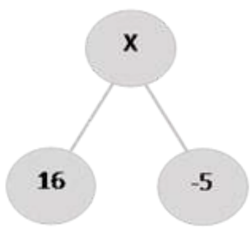
Following Davydov's curriculum, students initially handle various equations on the basis of quantities. After a while graphical diagrams are created, and further on formulas such as: a + b = c; b + a = c; c – b = a; c – a = b (Davydov, 2008).

[…] it is only the use of the letter formulas that produces an abstraction of the mathematical relation. But the letter formulas record only the results of real or mental actions with objects, while a graphical representation […], being a visible quantity (a length), enables the children to perform real transformations whose results can be not merely imagined but also observed. (Davydov, 2008, p. 151)

A consequence of the quotation above is that mathematics tools may mediate new knowledge development, but both students and teachers need to differentiate between the tools themselves and the mathematical content that is intended for students to be aware of (Kinard & Kozulin, 2008).

2. Methodology

Within our two respective research projects, a number of sub-studies have been conducted. The findings of these sub-studies will serve the planning of lessons where the model addressed in this paper will be adopted. The projects are conducted with researchers and teachers in collaboration. In this paper the focus is on the initial sub-studies. Semi-structured interviews were conducted with students in fifteen pairs in grades 3, 8 and 9. The interviews were audio- and video recorded and the data was transcribed and analyzed based on qualitative analysis. A further study has been performed with some students with the intention to analyze the quality of the questions for an upcoming pre-test (before designing a lesson). The study was conducted using interviews or a paper-pen test, or using a combination of these. Also a test was conducted concerning students' ability to find the missing number in two different ways: one with classical equations, and another with a corresponding relationship, visualized by the diagram concerning the part-whole model (see Figure 1).



16 = x – (-5)

Figure 1. A relationship expressed through the "diagram" concerning the part-whole model, and through the corresponding equation.

3. Tentative findings

In the pre-test, when the students (in grades 3 and 8) encountered the equation 8 – 5 = x, and were expected to solve it and write it in its other three forms (8 – 3 = 5; 3 + 5 = 8; 5 + 3 = 8), most of the students managed to do this correctly. However, the students seldom expressed that the equations have relationships to each other, nor that the numbers in each equation have relationships to one another. Instead, they tried to rearrange the numbers in positions "not used before" in each equation. A consequence of this reasoning resulted in expressions like: 3 – 5 = 8. Another common answer from the students in grade 8 was: 5 – 8 = 3.
However, a few students did express the relationships between the numbers in an equation. For example, when the students were supposed to find the missing number in the equation 7 – x = 2, they rearranged the equation to 7 – 2 = x to make an easier calculation.
A subsequent study was conducted in order to explore whether the part-whole model (Figure 1) could give access to finding the missing numbers. Students in grade 8 managed to do this successfully in 83% of the tasks where the part-whole model was adopted, compared to 43% when solving classical equations. The relations and numbers were exactly the same in the two versions of the tasks. For example, two out of eleven students in grade 8 found the missing number solving the equation 16 = x – (-5) algebraically, while seven of eight students found the missing number solving the corresponding task with the part-whole model as a mediating tool. Also in grade 3 the part-whole model seemed to be helpful to the students.
In our different sub-studies, when the students described relationships between quantities from a picture without any number values, most of the students chose to attribute specific numbers to the quantities instead of describing the relationships based on general mathematical symbols.
Findings also indicate that students in grade 3 expressed "-2" (negative two) as a "minus-number," as a "take-away-number" and as a "take-away-two." Students also expressed that "it has to be a number in front of the minus two." For an equation such as: 7 + x = 5 the students' solution was "-2". When asking the students about their solution they explained their solution as replacing the addition of negative two with subtraction of two, and formulated the equation as 7 – 2 = 5, "you need to take away two". A corresponding finding in grade 8 indicates that these students did express that there is no difference between the two equations: (14) + (15) = x and (-14) – 15 = x.
At the conference we will present more elaborated findings based on a deeper analysis.

4. Discussion

Previous research by Kilhamn (2011) shows that students face challenges when the minuend assumes a lower value than the subtrahend in different tasks. The students in our sub-studies do not exhibit such corresponding struggles when solving equations through the part-whole model. Using the part-whole model does not rely on rules in the sense of procedures, a property shown also in Davydov's curriculum (Davydov, 2008; Kieran, 2004; Schmittau & Morris, 2004; Slovin & Venenciano, 2008; van Oers, 2001). Instead, the students need to analyze the relationships within the equations to find the parts and the whole and thereafter choose an appropriate strategy (with respect to their mathematical development).
According to the findings there are indications that the students need to discern several aspects regarding equations. For example, the students in both grades 3 and 8 need to discern the relationships between different forms of an equation (e.g., x = 8 – 5; x + 5 = 8). The students in grade 3 also need to discern that negative numbers exist and that it is possible to operate with them. When the students solve equations with the part-whole model as a didactic tool, they work on an abstract and a theoretical level, not connected to their everyday contexts. Still, the students in this study solved the equations, including when negative numbers were present. Our intention is to further explore whether and how the part-whole model, despite – or maybe owing to – its absence of context, is useful as a mediating tool when solving equations even when negative numbers are present.

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Touchscreen devices and task design to improve plane transformation in high school classroom

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Abstract. Finding ways to improve learning or teaching of geometry with technological resources is still a challenge in mathematics education. In this paper we illustrate some strategies used by students to solve tasks on GeoGebra with touchscreen and we reflect about the design of 4 tasks to explore plane transformation in geometry classrooms. The designed tasks were fruitful to make emerge concepts related with plane transformation and to help students solve them making composition among some of them. The study highlights that the decision on the nature of the task is related with the type of touchscreen devices used. This intertwined process is challenging for both teaching and the design of the research.

Résumé. Trouver de façons d'améliorer l'apprentissage de la géométrie avec de ressources technologiques est encore un défi dans l'éducation de mathématique. Dans cet article, nous voulons exemplifier des stratégies utilisées par les élèves pour résoudre des activités sur le GEOGebra touche et nous réfléchirons sur le projet de 4 activités pour exploiter les transformations dans le plan de classe de géometrie. Les acitivités projetés ont réussi, en faisant surgir des concepts relatifs aux transformations dans le plan et ont aidé les élèves a resoudre en faisant la composition entre eles. L'étude montre que la décision sur la nature de l'activité est liée a un certain type de dispositif écrain tactile utilisé. Ce processus entrelacé est un défi pour l'apprentissage et le design de recherche.

1. Introduction

As we have had a first major shift (cognitive and epistemological) and improved teaching by passing from paper and pencil environments to dynamic geometry environment (DGE) with drag and drop activities (e.g. Cabri Géomètre, Sketchpad, etc.), now we have a further shift and improvement with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, Sketchometry etc.) and to the variety of simultaneous fingers' actions they allow. The evolution of digital technology makes available different practices in the classroom, specifically related to the way users can interact with the screen: from the drag and drop actions with the mouse to the tap, drag, and flick with one or more fingers on the screen of multi-touch devices and from the one-to-one interactions of the former to the multiple simultaneous interactions that the latter makes possible. These different technological features allow designing different tasks, which can change the cognitive processes of users and deeply modify their mathematical inquiries.

The way we deal and interact with touchscreen devices (TD) is providing new insights and challenges in mathematics learning and instruction (Arzarello et al., 2014). For instance, rotating and other kinds of gyrating movements on screen often take place, due the freedom of handling a touchscreen device. In this paper we discuss previous results from a research project[1] that investigates aspects of geometric learning during the process of solving tasks dealing with dynamic geometric environment with touchscreen. In CIEAEM67 we illustrated strategies used by Brazilian High School students applying rotation concept to solve task on GeoGebra with touch. In CIEAEM69 we will provide reflection on how task designing can improve specific cognitive process that occur when students learning plane transformation using touch

[1]Supported by Faperj and CNPq.

devices (Subtheme 3).

2. Interaction on screen and performing plane transformation

We assume that touchscreen manipulation on a mobile device is not cognitively the same as mouse clicks, those we often do in dynamic geometry environment (Arzarello et al., 2014), for instance, due to the simultaneity of motion in different elements (points, sides, angles, areas etc.) from one picture (Bairral et al., 2015). This particular feature was observed by one of the students in our research. According to him, "*in a very complex figure, moving several elements at the same time can become a bit difficult*".

Mobile touchscreen devices provide more freedom in manipulation, that particular way of rotation may serve as an important function of grounding mathematical ideas in bodily form and they may also communicate spatial and relational concepts (Boncoddo et al., 2013) in the field of plane transformation. In general, users manipulate the screen using mainly one or two fingers and, sometimes, when working in pairs they also can share fingers or hands to manipulate some shape. Users also can interact with the device in three different ways: *with* the device itself (gyrating it in different positions etc.), and interact *on* or *from* the screen. In this sensorial process, motion and manipulation on screen take an important cognitive role and, in their movement into existence, in which they become objects of thought and consciousness, geometric concepts are endowed with particular determinations; they have to be actualized in sensuous multimodal and material activity (Radford, 2014).

As Ng and Sinclair (2015) pointed out, transformations do not appear explicitly in many curricula until later elementary or middle school. In Brazil, even in High School, plane transformations do not appear in current official curricula. Based on previous research (Bairral et al., 2015) we identified students who, even without previous instruction concerning rotation and reflection, applied these concepts naturally, sometimes even doing composition between them. Besides alternative kinds of rotation applied by students to solve the geometric tasks, justifications to analyze students performing rotation or other plane transformations in TD are the following (Bairral et al., 2017):

1) Rotation and other gyrating movements on screen are often applied due to the various alternatives of handling touchscreen devices (Kruger et al., 2005; Tang et al., 2010).
a) Rotation and other plane transformations have remained unaddressed in Brazilian
b) Touchscreen devices provide possibilities of gyrating movements on screen, or with the device itself, which might result in new insights on embodied cognition.
c) Rotation and other plane transformations are concepts that involve intrinsically embodied motions.

3. Methodological aspects of the study

We are conducting teaching experiments with High School students (15-17 years old) at *Instituto de Educação Rangel Pestana* (Nova Iguaçu, Rio de Janeiro, Brazil). All of them had no previous experience with DGE and had no lesson concerning plane transformation. Each session was 2 hours long and in each one the students worked alone or in pairs. The analysis process was mainly based on the (1) videotapes of students working on the software, (2) written answers for each task and (3) the use of one shift in which he or she could write down and describe the function of each device icon. We observed all the students' manipulations *(Arzarello et al., 2014)* on the screen and identified the type of actions (tap, double tap, hold, drag, drag to approach, flick, free and rotate).

4. Some tasks for improve plane transformation using GeoGebra touch

In this section we illustrate three tasks elaborated for improve plane transformation in touchscreen device.

Task 1.1: For introduction and familiarization with Geometric Constructer device (30 minute)[2]
Use the software commands (construct, measure, etc.) to understand their functions, them draw the triangle using the commands on the iPad; write your remarks. Before exploring the software write down two observations:

[2] Links where to find the software and this activity:
a) with PChttp://www.auemath.aichi-edu.ac.jp/teacher/iijima/GChtml5/GChtml/server_e/gc_00026-test.htm
b) with I-pad:2012/10/10 16:39 482434 gc_00026-test.htm

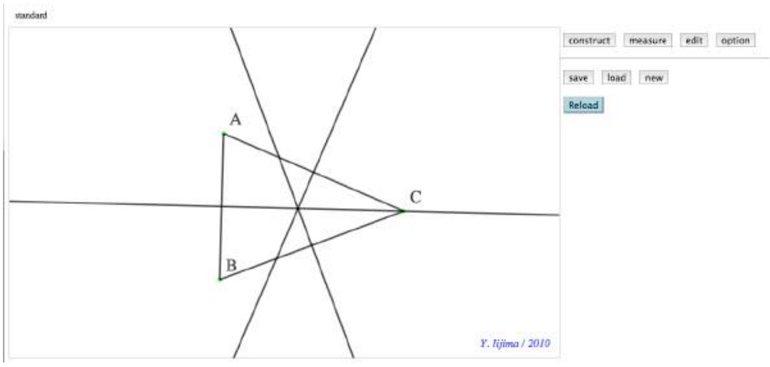


Figure 1. Screen from GC

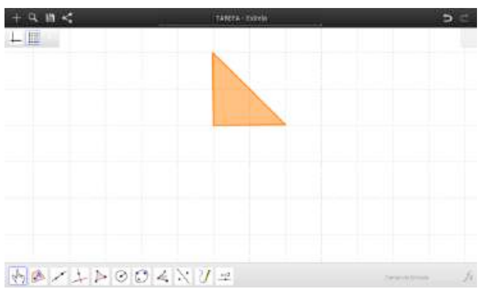
Conceptually, in order to rotate one shape we need to determine before in each point (the center of rotation) and with the use of two fingers the decision could have not been done beforehand. This type of action was not explicit for students exploring task 1.1. We became intrigued and we are investigating new conceptual aspects for the way we deal with rotation and other gyrating movements (with two fingers in movement, one fixed finger and the other in motion etc.).

Task 2.2 (design 1): Stair task
Using only triangle rectangle and isosceles construct the following picture.

Now, write to a friend and tell him or her how you constructed the picture.

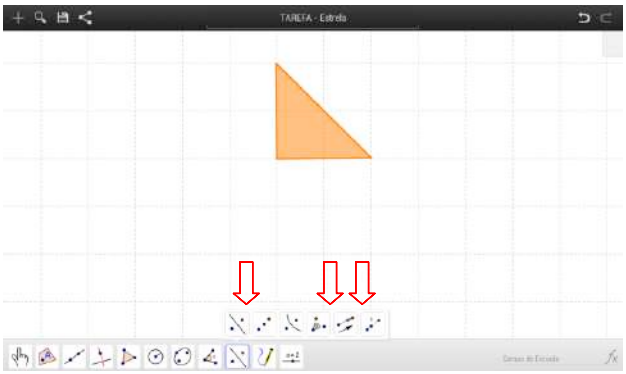
When solving task 2.2, which involved the concept of rotation and using a device with a single touch, we observed that students used their fingers – no more than two (Tang et al., 2010) – in a similar way to what students did when dealing with software Geometric Constructer in task 1.1 which did not apply the referred concept. Although the task 1.1 had been designed (without a specific geometric concept) for free exploration and to know the software, the students made a lot of interesting gyrating movements. After observing such way of manipulation, we elaborated a set of tasks (see task 4.5 below), for which students have to apply the concept of rotation and other plane transformation.

Task 4.5[3] *(design 2 from task 2.2): Stair task*[4]
Open the file "Stair task". Only the following triangle will appear:



Selecting the tool  will open a bar with 6 options:

[3] Access https://drive.google.com/file/d/0B6zQPvF8JeJcbzNsU0dMbUh2bE0/view to see the video recorded by Adriano solving task 4.5 as discussed in Bairral et al. (2015).
[4] This version restricts the use of icon.



Elaborate a strategy to construct the following picture using only the tools 

The iterative task design was mainly based on two strategies: task that generated new (or reformulated) task (for example, task 2.2 became 4.5) and students' answer that inspires new task

5. *Final remarks*

The type of task has an important role in the growth of the mathematical thinking. For researchers it also bears influence on the findings. The way in how a multi-touch-screen is used allows alterations on the task design in a substantial way. The kind of task needs to be strongly interconnected with the choice of the device and its features and artifacts mediators. In current analyses, we are checking whether the students use one and the same sequence in their reasoning, or if their strategies emerge naturally and without the traditional linearity taught in Brazilian schools (reflection/symmetry → rotation → translation).

In terms of promoting new ways to discover and to think mathematically, it doesn't make sense to propose, for instance, task 2.2 using only pencil and paper. The possibility of to make different constructions, to do simultaneous movements and adjusting by touch on screen seems to be a powerful resource for changing tasks as well as the nature of the geometric understanding concerning plane transformations using TD.

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The role of empirical evidence in the construction process of validations of geometric conjectures

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Abstract. In this document, we present the experience of the implementation of a geometry activity designed under the guidelines of the ACODESA methodology and the principles of task design. This work is part of an experimentation carried out with students coursing a Master of Educational Mathematics. The students were proposed a task for them to conjecture and then, validate. Here we show the case of a student who, during the construction process of a mathematical validation, was supported by empirical evidence to understand the conjecture under discussion, even after it had been validated.

Résumé. Dans le texte qui suit, nous présenterons l'expérience de l'application d'une activité de contenu géométrique conçue suivant les principes de la méthodologie ACODESA et les principes de la conception des tâches. Dans ce document nous exposons partie d'une expérimentation menée avec des étudiants d'une maîtrise en Didactique des mathématiques. Les étudiants ont été proposés une tâche pour qu'ils conjecturent et valident. Ici, nous montrons le cas d'un étudiant qui, pendant du processus de construction d'une validation mathématique, utilisa preuves empiriques pour comprendre la conjecture qui été en cours de discussion, même après avoir été validée.

1. Background and research problem

When reviewing the first great precedent based on the deductive method, Euclid's Elements, we observed that the role the figures accompanying the demonstrations of the different propositions play is that of being an aid to understand the chain of logical deductions and not that of being visual evidence. This is detailed by Szabó (1960), who seeks to explain the historic moment in which pre-Euclidean mathematics, practical and empirical, became a deductive system based on definitions and axioms. The transformation of mathematics into a deductive system led to a decline in the importance that visualization had when related to mathematical discovery and as a tool to persuade others. In pre-Euclidean mathematics visualization played a key role in demonstrations; verifications or refutations of any assertion related to geometry consisted in making facts concrete and visible (Szabó, 1960). This prompted us to reflect upon the role that empirical evidence should have in the validation process of a conjecture, considering that validation is the process through which a student justifies and provides reasons to explain why he or she thinks that a conjecture is true or false. The research question we seek to answer in this work is: How does empirical evidence influence students during the construction process of validations of geometric conjectures?

2. Theoretical references

In this research, we consider the justification created by a student as a process we expect to evolve towards mathematical demonstration. In this process, empirical evidence plays a significant role as a medium to verify, persuade and persuade oneself, as stated by De Villiers (2010). According to De Villiers, encouraging students to follow their intuition to create a validation might help them better understand what they seek to justify. It might also help them discover knowledge or mathematical relationships, yet unknown to them, in which empirical evidence is a resource to understand both the conjecture and its validation more completely.

To distinguish how a validation created by a student evolves, we used the typology of levels and types of proof developed by Balacheff (1987), who categorized the students' procedures in two levels of proof:

pragmatic and intellectual. In the first level are the proofs that resort to action and concrete examples: naïve empiricism, crucial experience and generic example. The second level hosts the proofs supported by the formulation of mathematical properties brought into play and the relationship between them: the thought experiment and calculation on statements.

Another theoretical reference used in the research are the guidelines of scientific debate in mathematics class (Alibert & Thomas, 1991; Legrand, 1993, 2001). Scientific debate in mathematics considers that rational arguments—justifications based on the theoretical corpus of mathematics—should prevail. During the development of the debate, the teacher's role is to promote the expression of ideas and allow clarifying different points of view so that students defend their assertions, as long as they consider them to be more reasonable than those expressed and justified by their peers. The students themselves must lead the consensus of the matter under debate.

3. Method

Students of a Master of Educational Mathematics participated in the study for two sessions of two hours each. The data collection was done using the students' work sheets and two video cameras, which recorded an overall view of the classroom and specific moments. We also video recorded the dialogs produced by the students during all the task.

The task was designed following the principles of the ACODESA methodology (Hitt, 2007), which allows promoting processes of conjecture, argumentation and validation in the classroom (Hitt, 2011; Hitt, Saboya, & Cortés, 2016) through its five stages:

1. **Individual work**. The student develops the task individually using paper and pencil.
2. **Teamwork**. The students work in teams of two or three members.
3. **Scientific debate**. The students debate—as stated by Legrand (1993)—on the proposals of solution presented by each team.
4. **Self-reflection**. Each student individually reconstructs the solution to the problem using paper and pencil.
5. **Institutionalization**. The teacher introduces the institutional solution to the problem. To do so, the teacher summarizes and incorporates the contributions that helped to find the solution in the previous stages.

Besides the ACODESA methodology, we followed the recommendations by Prusak, Hershkowitz, & Schwarz (2013) to design tasks that promote argument production in the classroom: creating multiple situations and collaborative situations, involving socio-cognitive conflict, providing tools for checking hypotheses, reflecting and evaluating the created solutions.

The students were proposed a task in which they had to conjecture and justify the relationship between the areas in the triangles formed when tracing the diagonals of any parallelogram. As a tool to verify hypotheses, they were given grids as the ones shown in 'figure 1'.

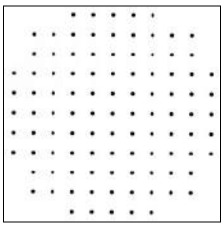


Figure 1. Tool for verifying hypotheses.

4. Result discussion

The conjectures formulated by the students in the individual work stage focused on the four triangles that have no diagonal as side (figure 2). From this, some students conjectured that the four triangles would always have the same area (figure 2a), while others stated only opposite triangles would have the same area (figure 2b). All the students justified their conjectures.

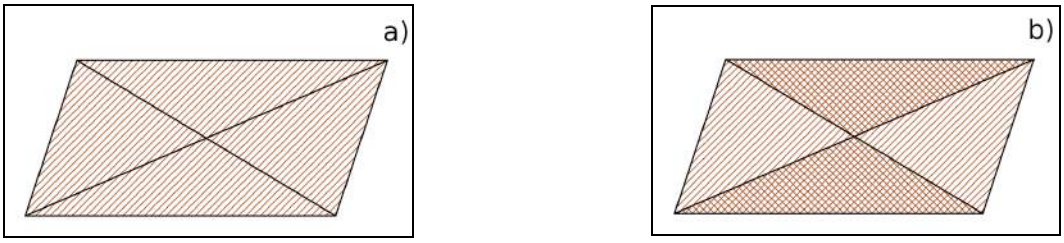
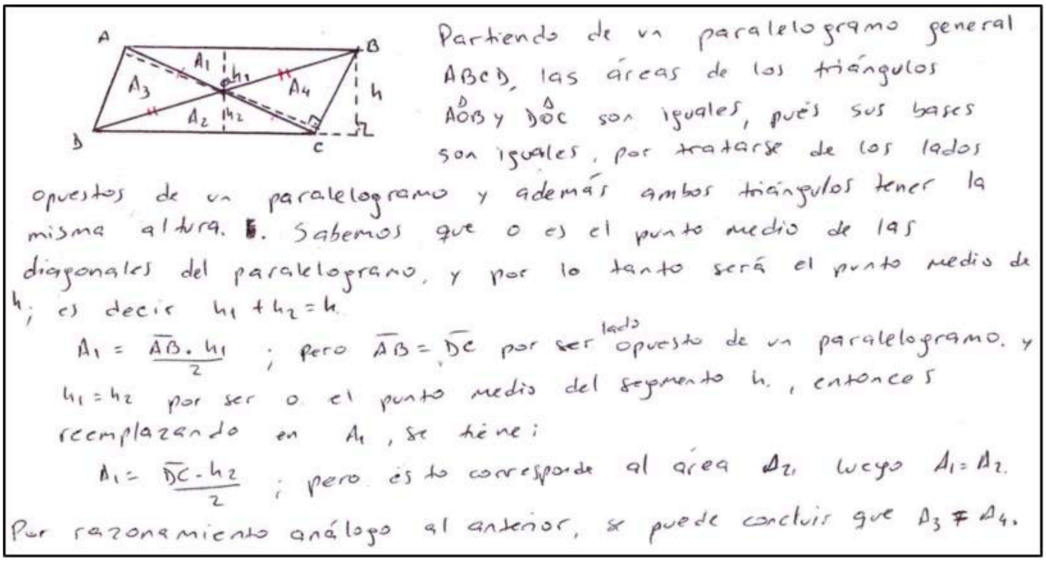


Figure 2. Representation of the conjectures in a general parallelogram; Four triangles of equal areas (a) and Opposite triangles of equal area (b).

In the teamwork stage (three members per team), the students presented to the others the arguments they used to justify their conjectures, which prompted discussions. As a result, the consensus of solution was led by those students who showed more persuasion and quality in the arguments used to validate their conjectures. Although some teams agreed and then validated that the four triangles would always have the same area, regardless the type of parallelogram, others did not fully accept the conjecture since some students stated they saw no equality of areas in the four triangles. They could not visualize how the loss of base in the adjacent triangles was compensated with the gain of height.

An example of this is Daniel who, in the stage of individual work, correctly conjectured and validated that only opposite triangles would have equal areas (figure 3).



Partiendo de un paralelogramo general ABCD, las áreas de los triángulos $\overset{\Delta}{AOB}$ y $\overset{\Delta}{DOC}$ son iguales, pués sus bases son iguales, por tratarse de los lados opuestos de un paralelogramo y además ambos triángulos tener la misma altura. ∎. Sabemos que O es el punto medio de las diagonales del paralelograno, y por lo tanto será el punto medio de h; es decir $h_1 + h_2 = h$.

$A_1 = \frac{\overline{AB} \cdot h_1}{2}$; pero $\overline{AB} = \overline{DC}$ por ser lado opuesto de un paralelogramo, y $h_1 = h_2$ por ser O el punto medio del segmento h, entonces reemplazando en A_1, se tiene:

$A_1 = \frac{\overline{DC} \cdot h_2}{2}$; pero ésto corresponde al área A_2, luego $A_1 = A_2$.

Por razonamiento análogo al anterior, se puede concluir que $A_3 \neq A_4$.

From a general parallelogram $ABCD$, the areas of triangles AOB and DOC are equal since their bases are equal, given that they are opposite sides of a parallelogram and both triangles have the same height. We know that O is the median of the diagonals of the parallelogram; then, it is the median of h, that is

$$h_1 + h_2 = h.$$

$A_1 = \frac{AB \cdot h_1}{2}$; but $\overline{AB} = \overline{DC}$ because they are opposite sides of a parallelogram, and $h_1 = h_2$ since O is the median of segment h; then, substituting in A_1, we have that:

$A_1 = \frac{AB \cdot h_2}{2}$; but this corresponds to area A_2, then $A_1 = A_2$.

After a similar reasoning, we conclude that $A_3 = A_4$.

Figure 3. Conjecture and validation created by Daniel in the stage of individual work

During the teamwork stage, Daniel persuaded his teammates it was impossible for the four triangles to have equal areas. To do so, Daniel based his statement on empirical evidence by relying on a particular verification on the grids (figure 4). Daniel verified his conjecture by counting the points inside the triangles formed on the grid. Using this method, the student persuaded his teammates to think that only the opposite triangles had equal areas since they have the same number of points inside.

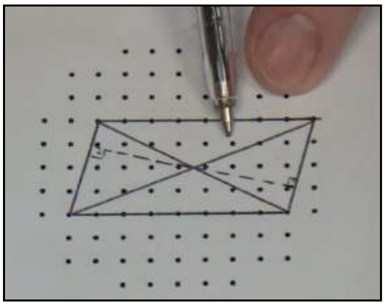
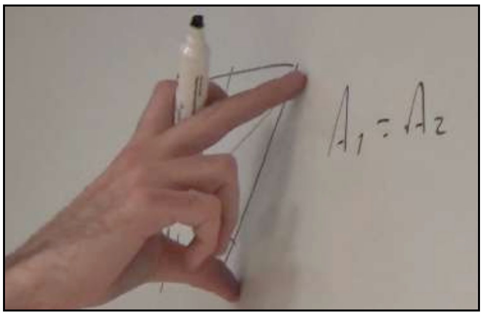


Figure 4. Empirical evidence created by Daniel.

In the debate stage, after all the students agreed on the validation to justify equal areas of all the triangles, Daniel expressed he did not see how all the areas could be equal. He said to the class that he failed to understand how the loss of length in the base was compensated by the gain of height in the same triangle.

Daniel: Then, here in this figure [rhomboid in figure 5] I don't see how what's lost from the base here [segments pointed at in figure 5] is compensated in height [height corresponding to the segments pointed at].



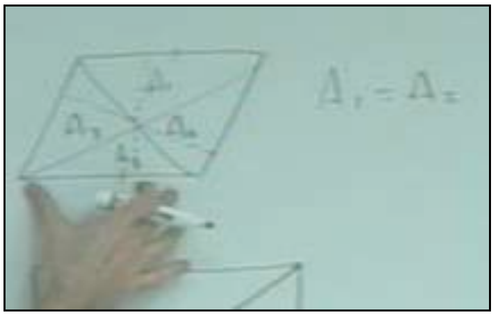
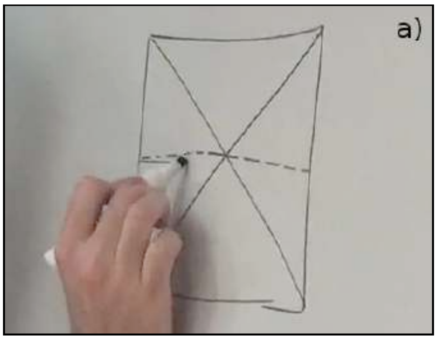


Figure 5. Segments pointed at by Daniel.

After this, another student suggested Daniel to work on another parallelogram, which helped Daniel to clarify his question. He used a rectangle as example (empirical evidence) to visualize and explain how the loss of base is compensated with the gain in height.

Daniel: When tracing the height here, let's say [figure 6a], I see that this, here, is half of the base of this triangle [figure 6b]. And then, there is the compensation, say, that what is lost in the base is gained in height.



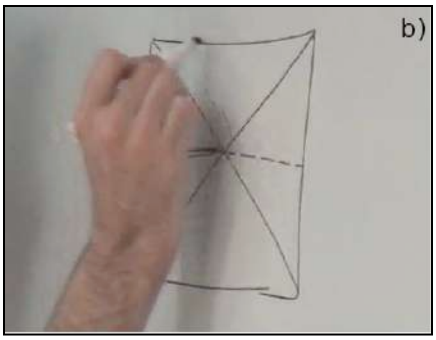


Figure 6. The case of the heights (a) and base (b) in the rectangle.

5. Conclusions and final remarks

During the development of the task and in the first stages of the ACODESA methodology, we observed the creation of an environment for the students to debate about their arguments and agree on what they considered to be the best solution. In addition, we observed validations supported by empirical evidence (verified on the grids) and justifications that were closer to intellectual proofs, as defined by Balacheff (1987). Some students expressed they did not understand why the four triangles had equal areas despite having understood and accepted the demonstration all the class had agreed on. However, the use of other parallelograms as examples (empirical evidence) helped them to better understand the conjecture under discussion. We observed that, for some students, accepting the conjecture—even after having validated it—did not occur after accepting or understanding the demonstration, but after verifying it in particular cases, as Daniel did. The empirical evidence used by the student helped him to understand the equality of the areas in the four triangles. Finally, when designing the task following the ACODESA methodology, an environment of social interaction was created in the stages of teamwork and debate. In these stages, the students themselves constructed the solution to the problem after using both empirical evidence and properties of mathematical relationships to create their responses.

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L'enseignement de la statistique en France et au Brésil

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Abstract. In this article, we present a comparative study on the teaching of statistics in Brazil and in France, on the middle school to high school transition, focusing on the analysis of the work carried out in middle school on the first notions of descriptive statistics and their representations, considering them as precursors for the introduction of inferential statistics in high school programs in France and university programs in Brazil. Based on theoretical constructions of the Anthropological Theory of Didactic (ATD), and more specifically, using the hierarchy of levels of co-determination, we show the existence of different habitats for the introduction and development of statistics in the curricula of the two countries.
Résumé. Cet article présente une étude comparative entre le Brésil et la France sur la transition collège-lycée en statistique. Il se concentre sur l'analyse du travail mené dans le collège sur les premières notions de statistique descriptive et ses représentations, en les considérant comme des précurseurs pour l'introduction de la statistique inférentielle dans les programmes de lycée en France et universitaires au Brésil. Basé sur des constructions théoriques de la théorie anthropologique du didactique, et plus spécifiquement, sur la hiérarchie des niveaux de co-détermination, il montre l'existence d'habitats différents pour l'introduction et le développement des statistiques dans les curricula des deux pays.

1. Introduction

Nous présentons ici, une étude comparative qui se situe dans un projet plus vaste d'étude de la transition de l'enseignement élémentaire a l'enseignement supérieur en mathématiques en France et au Brésil. Des nombreuses recherches sur la transition secondaire-supérieur ont été déjà développées, comme en témoignent les récentes synthèses (Artigue et al., 2007) ou (Gueudet, 2008). Ces synthèses montrent la diversité des contextes dans lesquels ces travaux ont été menés et la nécessité de bien comprendre l'influence de ces caractéristiques contextuelles pour penser l'action didactique. Par ailleurs, les études comparatives qui se sont développées dans la dernière décennie ont bien mis en évidence l'intérêt de telles comparaisons pour identifier et comprendre les effets de ces caractéristiques contextuelles et culturelles (voir, par exemple Clarke et al., 2007 et Leung et al., 2006).

Dans l'étude rapportée ici, nous nous intéressons plus particulièrement au domaine de la statistique au niveau lycée (étudiants de 15 à 17ans) et à ses précurseurs développés dans l'enseignement de la statistique au collège (étudiants de 11 à 14ans). Ce choix est motivé par le rôle joué par ce domaine de l'enseignement élémentaire à l'enseignement supérieur mises en évidence par les recherches didactiques (Gattuso & Vermette, 2013) et par les orientations curriculaires au Brésil et le programmes en France. Nous utilisons le contraste entre les contextes français et brésiliens pour une meilleure compréhension des problèmes de transition dans ce domaine dans les deux pays et pour penser le développement des ressources éducatives susceptibles d'aider à surmonter les difficultés rencontrées.

D'un point de vue théorique, cette étude s'appuie sur la théorie anthropologique du didactique (TAD dans la suite) développée par Chevallard (1992, 2002), et plus particulièrement l'accent est mis sur la notion de praxéologie et l'hiérarchie des niveaux de co-détermination.

2. Méthodologie

En articulant le cadre théorique et les objectifs, la méthodologie du projet combine plusieurs approches: (1) une approche institutionnelle centrée sur la transition entre le collège e le lycée, exploitant les documents curriculaires et les outils d'évaluation à l'échelle régionale et nationale, (2) une approche des relations personnelles aux statistiques développées par les étudiants, (3) une approche des continuités et discontinuités entre les pratiques d'enseignement dans les institutions dans les deux pays. Les comparaisons sont donc à la fois internes à chaque pays et croisées entre les deux pays.

Par le biais de ces différentes approches, notre intention est d'identifier et d'analyser les similitudes et différences entre les deux contextes, et les effets de la transition entre le collège et le lycée sur le contenu étudié, en prenant en compte les conditions et contraintes intervenant aux différents niveaux de la hiérarchie de co-détermination et à leurs interactions.

Dans cette contribution, nous nous limitons à une dimension de cette recherche, celle concernant l'analyse des relations institutionnelles aux notions de la statistique dans les deux pays. Pour cela, au-delà des paramètres et/ou programmes de l'enseignement du collège et du lycée, nous utilisons deux sources de données: pour le Brésil, l'évaluation annuelle des étudiants du collège et lycée de l'etat de São Paulo – SARESP (évaluation régionale) et l'évaluation qui assure la sélection des étudiants à l'entrée de l'université (l'évaluation nationale – ENEM), pour la France, le baccalauréat qui donne accès à l'enseignement supérieur. Dans les deux cas, les données ont été recueillies et analysées sur les cinq dernières années pour permettre de repérer des régularités mais aussi mettre en évidence d'éventuelles évolutions. Cette analyse sur le long terme, connaissant l'influence des évaluations sur l'enseignement, devrait aussi nous donner une idée plus précise de l'activité statistique développée par des étudiants dans la résolution tâches de telles évaluations, et nous permettre de distinguer entre tâches routinières et tâches nécessitant adaptation et créativité, ce qui n'est pas sans influence sur la compréhension des questions de transition (Castela, 2008).

3. Les systèmes éducatifs français et brésiliens

Nous présentons brièvement ci-après les deux systèmes éducatifs et la façon dont la transition collège - lycée y est organisée (caractéristiques situées aux niveaux supérieurs de l'échelle de co-détermination).

Au Brésil, la structure globale de l'éducation comporte un enseignement fondamental, avec deux étapes (5 puis 4 ans) et un enseignement moyen (3 ans) correspondant au lycée français, mais sans filières spécifiques. L'enseignement fondamental et l'enseignement moyen sont obligatoires. Il existe des paramètres nationaux qui définissent des orientations pour l'enseignement, mais pas de programme national, les élèves peuvent suivre leurs cours pendant la journée ou le soir et, dans ce dernier cas, ils ont moins d'heures d'étude. Par ailleurs, la formation des enseignants varie fortement d'un état à un autre. L'entrée à l'université est actuellement basée sur l'examen national ENEM, mais il existe aussi des évaluations sélectives appelées «vestibular», organisées par les universités elles-mêmes. Beaucoup de jeunes fréquentent des cours spéciaux privés pour préparer ces examens.

En France, la structure globale est similaire, avec 5 années d'école primaire, quatre années de l'enseignement secondaire et trois années de lycée. L'éducation est obligatoire jusqu'à l'âge de 16 ans. En entrant au lycée, il y a une séparation entre l'enseignement général, technologique et professionnel. Dans l'enseignement général auquel nous nous intéressons plus particulièrement ici, l'enseignement se différencie aussi du fait des options en seconde, mais surtout des séries en première, et des enseignements de spécialité en terminale. Trois séries existent en première: littéraire (L), sciences économiques et sociales (ES), sciences (S), et pour la série S trois spécialités en terminale : sciences mathématiques, sciences physiques et sciences de la vie et de la terre. Les programmes de mathématiques diffèrent de la classe selon la série choisie ainsi que les horaires. Il y a une évaluation nationale à la fin du secondaire, le baccalauréat, qui donne accès à l'enseignement supérieur. Le taux de réussite est d'environ 85%.

4. Statistique dans les systèmes éducatifs français et brésiliens

La 'figure 1' résume ce qui concerne la statistique développée au deuxième étape de l'enseignement fondamental (étudiants de 11 à 14 ans) et à l'enseignement moyen (étudiants de 15 à 17 ans) dans les paramètres nationaux au Brésil. La statistique est introduite au premier cycle de l'enseignement fondamental en tant qu'outil pour la collecte et l'organisation des données dans des tableaux et des graphiques et l'étude des relations entre des événements, ce qui rend possible les prévisions sur l'observation de la fréquence de leur apparition. Dans le deuxième étape la statistique fait partie du domaine « traitement de l'information » en tant que complément au travail déjà initier et est utilisée comme outil dans l'enseignement pour établir des liens entre les mathématiques et d'autres domaines de contenu et les thèmes transversaux (cf. figure 1). Le niveau d'enseignement des notions à développer est de la responsabilité de l'enseignant qui doit considérer le développement et les intérêts des étudiants de chaque classe. Les capacités attendues sont la construction et l'analyse des différents processus de résolution de situations-problèmes et trouver des solutions pour construire des arguments plausibles.

Niveau et contenu	Domaines	Capacités attendues
Enseignement fondamental : Premières années: comprendre la collecte et l'organisation des données dans des tableaux et des graphiques, pour établir des relations entre les événements. Dernières années: revisiter les connaissances développées dans le premières années, formuler des questions pertinentes pour un ensemble d'informations, développer des conjectures, communiquer de façon convaincante l'information, interpréter des diagrammes et des organigrammes. **Enseignement moyen :** Comprendre la relation entre aperçu statistique, représentation graphique et les données primitives ; exercer la critique dans la discussion des résultats ; construire des arguments rationnels basés sur l'information et les commentaires.	**Traitement des donnés** Les notions de statistique descriptive et ses représentations .	Reconnaître et utiliser, sous forme orale et écrite, les symboles, les codes et la nomenclature du langage scientifique. Lire, articuler et interpréter les symboles et les codes dans différentes langages et systèmes de représentations. Identifier dans une situation problème donnée les informations ou des variables pertinentes et élaborer des stratégies de résolution possibles. Reconnaître, utiliser, interpréter et proposer des modèles pour des situations problématiques, des phénomènes et des systèmes naturels et technologiques.

Figure 1. Paramètres brésiliennes pour le développement des notions de la statistique au deuxième etape de l'enseignement fondamental et à l'enseignement moyen.

Les notions de statistique descriptive développées au enseigment fondamental sont reprises dans l'enseignement moyen au Brésil. Elles s'inscrivent alors en tant qu'objets mathématiques dans le domaine du traitement de l'information et sont reliées à de nouveaux objets de mathématiques et des autres sciences.

En France, la statistique est introduite au collège et son développement est regulier jusqu'au fin de lycée, c'est-à-dire, la statistique descriptive est développée au collège et la statistique inferentielle est introduite au lycée. La situation est donc de ce point de vue très différent à celle du Brésil comme nous pouvons remarquer dans la 'figure 2'. On a donc un habitat pour la statistique très différent de celui du Brésil où nous ne développons que la statístique descriptive. Dans la 'figure 2', nous ne présentons que les donnés pour le quatrième et le seconde.

Série et contenus por le quatrième au collège, série, filière et contenu pour le seconde au lycée	Domaines	Capacités attendues
Quatrième Effectives cumulées, fréquences cumulées ; moyennes ponderées ; iniciations à l'usage des tableurs-grapheurs ; valeur approchée de la moyenne d'une série statistique regroupée en intervales. **Seconde** Resume numérique par une ou plusieurs mesures de tendance centrale(moyenne,	**Collège (quatrième)** Statistique Descriptive; **Lycée (seconde)** Statistique; Inferentielle	**Collège** S'engager dans une démarche de résolution de problèmes; utiliser des outils mathématiques pour résoudre des problèmes concrets; appréhender différents systèmes de représentations; tenir compte d'éléments divers pour modifier son jugement; utiliser l'oral et l'écrit pour expliciter des démarches, argumenter des raisonnements. **Seconde** Utiliser les propriétés de linearité de la moyenne d'une série statistique;

médiane, classe modale, moyenne élargiée) et une mesure de dispersion. Définition de la distribuition des fréquences d'une série prenant um petit nombre de valeurs et dela fréquence d'um événement. Simulation et fluctuation d'échantillonage.		Calculer la moyenne d'une série à partir des moyennes de sous-groupes et de distribution de fréquences; concevoir et mettre em oeuvre des simulations simples à partir d'échantillonsde chiffres au hasard.

Figure 2. Orientations françaises pour le développement de la statistique au collège et au lycée.

Pour le lycée, il y a encore les filières pour les deux annés succédant à la classe de seconde, à savoir : série économique et sociale (ES), série littéraire (L) et série scientifique (S). Pour les deux annés des différents séries il y a un programme pour l'enseignement de la statistique. Nous avons remarqué que le programme brésilien du lycée et plus proche de celui de la classe littéraire.

5. Conclusion

La comparaison France-Brésil montre donc des habitats différents pour l'enseignement de la statistique dans les deux pays et des relations différentes aussi entre le collège et le lycée. Pour comprendre ces différences et leur impact sur la transition entre le collège et le lycée, il nous semble nécessaire de revenir aux conditions et contraintes qui ont façonné ces choix curriculaires et leur évolution. Le nombre de pages réduit de cette contribution ne nous permet pas de développer ici cette analyse, pas plus qu'il ne nous permet de rentrer, en nous appuyant sur les données recueillies dans les détails de l'analyse praxéologique. Nous présenterons, si cette contribution est retenue, une synthèse des résultats obtenus selon ces deux dimensions au colloque.

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Algèbre élémentaire et les rapports personnels d'un groupe d'étudiants de l'État de São Paulo - Brésil

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Abstract. This paper presents the research that aims at identifying both existing institutional relationships and students' expected real personal relationships and understanding the difficulties for students who complete basic education of the teaching-learning process of algebra. Notions associated to the Anthropological Theory of Didactics by Chevallard are considered for the central theoretical framework. The first results show confinement within the arithmetical frame by the students, even those who have started higher education.

Résumé. Cet article présente les résultats d'une recherche qui vise à identifier les rapports institutionnels existants et les rapports personnels réels et comprendre les diffficultés des étudiants terminant l'éducation de base par rapport à l'enseignement et l'apprentissage de l'algèbre. Les notions associées à la Théorie Anthropologique du Didactique développée par Chevallard sont considérées pour le cadre théorique central. Les premiers résultats montrent le confinement des étudiants dans le cadre arithmétique, même ceux ayant commencé l'enseignement supérieur.

1. Le contexte de la recherche

Nous présentons dans ce travail une recherche sur l'enseignement et l'apprentissage de l'algèbre dans l'éducation basique au Brésil. Nous considérons pour cela la transition entre les trois étapes scolaires, comprenant l'enseignement basique obligatoire au Brésil à savoir: l'école élémentaire (les élèves de 6 à 10 ans), le collège (les étudiants de 11 à 14 ans) et le lycée (les étudiants de 15 à 17 ans).

La problématique de cette recherche est apparue en classe de quatrième année de l'école primaire avec des élèves de neuf et dix ans. L'un des chercheurs a posé une question aux élèves qui de son point de vue représente un grand « défi » pour les étudiants de troisième année de l'école primaire. Ce chercheur, dont la condition est d'être enseignant dans cette classe, a prétexté que les élèves n'avaient pas les moyens nécessaires pour résoudre le problème, car ils n'avaient pas les connaissances requises concernant les notions et les techniques algébriques pour celui-ci. Toutefois, il a été surpris par les réponses de ses élèves ainsi que par l'utilisation des techniques de résolution.

À partir de cette expérience nous avons décidé de mener une recherche basée sur la question suivante: Quelles sont les connaissances, les techniques et les stratégies utilisées par les élèves pour résoudre les problèmes algébriques proposés?

Ainsi, nous avons émis l'objectif suivant : identifier les rapports institutionnels existants, les rapports personnels attendus ainsi que les marques de ces derniers sur les rapports personnels réels des étudiants afin de mieux comprendre les difficultés rencontrées, en particulier, celles des élèves terminant l'enseignement secondaire comprenant l'apprentissage de l'algèbre.

Nous croyons également que l'étude de l'évolution historique de l'algèbre peut révéler des éléments permettant de comprendre les difficultés et les défis auxquels sont confrontés les éducateurs et les chercheurs dans l'enseignement et l'apprentissage de ce domaine des mathématiques. Pour mieux comprendre ces difficultés nous avons étudié l'évolution historique de l'algèbre, selon Robinet (1989) et Radford (1991).

Cette étude nous a permis de remarquer que les difficultés rencontrées dans l'histoire s'approchent beaucoup des difficultés de nos étudiants. Cela nous a amené à réfléchir sur les difficultés rencontrées par les étudiants de l'enseignement supérieur lorsqu'ils ont besoin d'appliquer leurs connaissances de l'algèbre élémentaire et que celles-ci ne sont pas disponibles. Cela conduit à une grande perte d'intérêt pour les cours dont les mathématiques sont un outil important pour le développement.

Ainsi, nous avons choisi la théorie anthropologique du didactique pour développer la recherche et les notions de cadre et changement de cadre selon la définition de Douady (1992) et des niveaux de connaissances attendues des élèves selon la définition de Robert (1998).

2. Cadre théorique de la recherche

La recherche est basée sur des éléments de la Théorie Anthropologique du Didactique (TAD) de Chevallard (1992, 1998). La principale raison pour nous de prendre la décision d'utiliser la théorie mentionnée ci-dessus est parce qu'elle place l'étude de l'activité des mathématiques dans l'ensemble des activités humaines et des institutions sociales (Chevallard, 1998).

Comme la théorie situe l'activité d'étude des mathématiques au sein des institutions sociales, il semble important de considérer les notions de rapports institutionnels et personnels définies par Chevallard (1998). Pour définir ces rapports, Chevallard (1992) introduit la notion d'objet qui est définie comme toute entité, matériel ou immatériel, qui existe pour au moins un individu, ce qui l'a amené à considérer que tout est objet. Un autre élément clé de la théorie est la notion d'institution, qui selon l'auteur, sont des dispositifs sociaux qui permettent et imposent différentes positions des personnes qui peuvent occuper différentes positions dans l'institution.

Alors pour Chevallard (1992) l'univers cognitif d'un individu particulier, c'est-à-dire, l'ensemble de ses rapports personnels à la connaissance est un autre élément qui compose la structure de la TAD, puisque Chevallard (1998) définit que toutes les interactions possibles, que ce soit la manipulation, l'utilisation, etc. d'un objet particulier, correspond donc à un rapport personnel avec cet objet. Cette notion nous aide à l'identification des rapports personnels réels des étudiants, de 10 à 18 ans, en ce qui concerne le domaine de l'algèbre, par la confrontation avec les rapports institutionnels existants.

Pour Chevallard (1992, 1998) la position qu'un objet donné occupe dans une institution est ce qui détermine le rapport institutionnel de l'objet avec l'institution analysée. Ainsi, nous avons analysé les documents officiels et les manuels, pour les manuels nous avons choisi ceux qui sont utilisés actuellement dans les classes de l'éducation de base (étudiants de 6 à 17 ans) à São Paulo. Nous remarquons que, dans les écoles de São Paulo, les manuels peuvent être considérés comme des rapports institutionnels existants, parce que, en général, les enseignants suivent ces documents. Déjà, les documents, qui guident le système d'enseignement de l'État de São Paulo, sont considérés comme des rapports institutionnels attendus parce que, en général, dans ces documents ne sont présentés que des directives générales sans des exemples de la façon de les travailler. Ceux-ci sont développés dans deux autres document appelés "cahier de l'enseignant et cahier de l'élève".

Nous soulignons également que dans la TAD, toutes les activités humaines sont organisées par praxéologie, ce qui rend cette notion un autre principe structurant de la théorie. Une praxéologie consiste en: types de tâches et techniques qui forment le bloc du savoir-faire et le discours technologique-théorique didactique qui forment le bloc du savoir. Les praxéologies ici sont identifiées à travers l'analyse des manuels, parce qu'à partir des types de tâches et des techniques développés dans ces manuels, nous avons identifié les embryions de technologie et nous avons pu considérer la théorie qui les justifient.

Outre la TAD nous avons utilisé comme cadre théorique de référence les idées de Douady (1992), en particulier les notions de cadre et de changement de cadre, car à partir de ces notions sera possible d'identifier la nécessité de changement du cadre arithmétique au cadre de l'algèbre dans l'analyse des tâches proposées dans les documents analysés.

Une autre théorie du support, qui nous avons utilisé, est celle de Robert (1998), plus particulièrement la notion de niveaux de connaissances attendus des étudiants, à savoir: le niveau technique, le niveau mobilisable et le niveau disponible. Notant qu'il n'y a pas de hiérarchie entre eux et ce qui est voulu est que les élèves atteignent toujours le niveau disponible pour les concepts et notions d'algèbre développées pendant leur scolarité. Nous remarquons encore que la différence entre le niveau mobilisable et le niveau disponible est l'explicitation des connaissances à utiliser dans les tâches où le niveau de connaissance attendu est le niveau mobilisable et la non explicitation des connaissances nécessaires pour résoudre les tâches où le niveau de connaissance attendu est le niveau disponible. Le choix de l'analyse des types de tâches proposées en utilisant le niveau de connaissances attendues des étudiants est un outil nous permettant de reconnaitre si les types de tâches proposées dépassent la répétition des techniques algébriques sans être possible d'appliquer ces techniques pour résoudre les tâches qui impliquent que l'étudiant lui-même trouve quelle est la technique la plus appropriée, en particulier quand il est question de situations du quotidien.

Compte tenu du scénario présenté des éléments théoriques que nous avons utilisés dans notre recherche, nous abordons par la suite la méthodologie utilisée pour la recherche.

3. Méthodologie de la recherche

Comme déjà indiqué dans le cadre théorique, il s'agit d'une recherche qualitative basée sur la technique de la recherche documentaire selon Lüdke et André (1986), car nous l'avons commencé par l'étude des documents officiels pour identifier les rapports institutionnels et les rapports personnels attendus.

Les documents analysés pour l'identification des rapports institutionnels, en ce qui concerne les notions du domaine de l'algèbre, étaient: les cahiers de professeur pour les années scolaires équivalents aux cinquièmes et quatrièmes années au collège (étudiants de 12 et 13 ans), qui correspond aux rapports institutionnels existants. Les lignes directrices curriculaires sur le contenu à développer avec les étudiants de cette étape scolaire de l'État de São Paulo sont considéré comme les rapports institutionnels attendus et le rapport pédagogique des écoles de cet état sont les rapports personnels attendus. Nous notons que le rapport pédagogique est publié chaque année après l'évaluation à grande échelle nommé Système d'évaluation du rendement scolaire de l'État de São Paulo (SARESP).

La partie expérimentale de la recherche, qui nous a permis d'analyser les difficultés des étudiants, correspond à un test diagnostic qui a été appliqué à un groupe de cinquante-six élèves de l'éducation de base, âgés entre 10 et 18 ans, répartis comme suit: vingt-six étudiants avec l'âge de 10 ans, vingt-cinq à l'âge de 15 ans et cinq étudiants de 18 ans d'âge, tous dans les phases de transitivité pour les étapes éducatives brésiliennes, c'est à dire, le passage de l'école élémentaire au collège, du collège au lycée et du lycée à l'université.

Les tâches proposées dans le test de diagnostic ont été identifiés grâce à l'analyse des rapports institutionnels et des rapports personnels attendus des étudiants. Pour ces tâches nous avons analysé les techniques, les technologies, les théories, les cadres et les changements et les niveaux de connaissances attendus des étudiants, pour comprendre les rapports institutionnels existants et les marques de ces derniers sur les rapports personnels des étudiants.

4. Résultats des analyses

Dans l'analyse des rapports institutionnels existants, qui ont été analysées par l'intermédiaire de cahier de l'enseignant, nous avons observé que les praxéologies développées dans le matériel de classe sont présentés au moyen d'exemples, dont beaucoup d'entre eux peuvent être résolus en n'utilisant que l'arithmétique, en laissant peu de place pour le développement de l'algèbre, en particulier, lorsque nous considérons les tâches contextualisées.

Dans l'analyse des manuels, nous avons aussi remarqué que les types de tâches proposées favorisent l'arithmétique, même si les exemples sont développés au moyen d'équations et de systèmes d'équations, ce qui indique la nécessité d'un travail qui montre l'importance de l'algèbre en tant qu'outil pour résoudre les tâches pour lesquelles les étudiants ont déjà d'autres techniques pour les exécuter.

L'analyse du test de diagnostic (en annexe), qui nous a permis de commencer notre étude sur l'identification des véritables rapports personnelles des étudiants, appliqué a un groupe de cinquante-six étudiants entre 10 et 18 ans, tend à montrer que, peu importe la phase de transitivité, car ils utilisent tous les techniques, qui ne nécessitent pas de l'algèbre, pour résoudre les tâches du test. La plus grande différence est le nombre d'étudiants capables de résoudre les tâches proposées, parce que les étudiants du «collège» et du «lycée» ont plus de compétences arithmétiques que ceux de l'école élémentaire.

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Annexe: Questions du test diagnostic

UNIVERSITÉ ANHANGUERA
DOCTORAT EN ÉDUCATION MATHÉMATIQUE

1. Une usine produit des poussettes pour bébés et des tricycles. Aujourd'hui, les travailleurs ont produit 11 unités et pour les assembler, ils ont utilisé 40 roues. Combien des tricycles ont été produits?

2. Trouvez deux nombres dont la somme est de 20 et le produit entre eux est 96.

3. 54 oranges ont été réparties entre Kátia, André et Cláudia et on sait qu'André a reçu deux fois plus que Kátia et Claudia a reçu le triple de ce qu'a reçu André. Combien d'oranges chacun a reçu?

Expressing and justifying pattern generalization algebraically

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Abstract. The main objective in this paper is on learning more about younger students' emergence of the ability to express and justify pattern generalization algebraically, particularly in relation to what aspects students need to discern to be able to express and justify pattern generalization algebraically. This forms a point of departure for discussing the meaning of making algebraic generalizations in the early grades. The findings constitute a foundation for a project on classroom teaching and learning in mathematics, carried out as a collaboration between researchers and teachers.

Résumé. L'objectif principal dans ce document est d'apprendre plus sur les façons d'expérimenter la généralisation des schémas par les élèves plus jeunes et ce à propos de quels aspects les étudiants doivent discerner pour pouvoir exprimer et justifier les généralisations de formes algébriquement. Ceci comme un point de départ dans les discussions concernant la signification de faire des généralisations algébriques dans les premières années. Les résultats constituent une base pour un projet de développement de cours en collaboration entre chercheurs et enseignants.

1. Background

Researchers (e.g., Greer, 2008; Usiskin, 1988) advocate alternative approaches to the teaching of algebra, since the literature reveals that teaching today often focuses on learning a number of procedures rather than creating the conditions for enabling students to develop abilities such as reasoning algebraically, making algebraic generalizations, and using algebraic representations. Furthermore, Usiskin (1988) and Greer (2008) highlight how teaching which does not go beyond the practicing of manipulative skills, instead of developing understanding, can prevent students from using algebra as a powerful tool for solving mathematical problems. Included here are processes like describing and analysing relationships, characterizing and understanding mathematical structures and ideas (e.g., Davydov, 2008; Kaput, 2007; Kieran, 2006; 2004; Radford, 2010, 2014). In relation to enabling younger students to develop algebraic understanding, Mason (1996), Radford (2006), and Warren (2006) all suggest the use of mathematical patterns as an introduction**.**

2. Generalizations in relation to mathematical patterns

In research regarding mathematical patterns and generalizations, there are different descriptions of the meaning of making generalizations. Radford (2006), for example, highlights how generalization is about different layers of consciousness; to perceive the pattern's mathematical structure; to perceive the commonality of the pattern; to generalize a local commonality to all the parts of the sequence; as well as being able to express the general. In a more recent article, Radford (2011) stresses the ability to generalize in relation to being able to perceive both the pattern's spatial and numerical regularity, where the spatial structure is about, for example, how matches may be positioned in patterns (see figure 1). This entails distinguishing how both the numerical and the spatial structures belong together, including what is equal and what separates them, and then to abstract this commonality into all elements of the sequence (Radford, 2011). In relation to patterns and generalization, Mason, Burton, and Stacey (2010) highlight how students need to be able to discern an underlying general structure to be able to express a generality algebraically. Mulligan and Mitchelmore (2009), in turn, address pupils' ability to structure based on the pattern's vertical, horizontal and spatial structures.

Radford (2006) and Venenciano and Dougherty (2014) highlight the different strategies students use when making pattern generalization. Radford separated different generalization strategies in relation to how advanced they are. First of all, he made a distinction between the so-called "naive induction" versus generalizations. In the "naive induction" strategy, the students use a "trial and error strategy," which can be described as a guessing strategy and thus, according to Radford (2006), is not a generalization strategy at all. A generalization strategy is about discerning and using a general commonality of a pattern (Radford, 2006). The strategies that are counted as generalization strategies consist of both arithmetic and algebraic ones. The difference between those, according to Radford (2006), is that an arithmetic strategy does not make it possible to predict any term in a pattern as an algebraic strategy could. In other words, a generalization like "It constantly increases with two matches" is seen as an arithmetic generalization, since it only supports the prediction of the "next" positions in the sequence and does not make it possible to predict any term in the pattern.

The algebraic strategy is divided into three different strategies: factual, contextual, and symbolic. They are all categorized as algebraic since the students using these strategies are expressing a commonality that can be applied to all terms in the pattern, and thus used to predict the number of elements in any term in the pattern (Radford, 2006). Here there is not only an increase in the number of elements between the terms perceived, but the number of elements of each term is, rather, related to the position of the term in a pattern sequence (such as "the n:th term") and to all elements in the visible pattern (Radford, 2006).

The difference between those strategies is about *how* the generalization is expressed. In a factual generalization, the indeterminacy remains unnamed, and the "generality rests on *actions* performed on numbers; *actions* are made up of words, gestures and perceptual activity" (Radford, 2006, p.16). The generalization is here based on actions in relation to facts on *a* local term, 'If it's *term 1*, I *did one row*', and is then put in relation to the other terms in the sequence '*term 2*, it's *two'*, *term 3*, it´s *three*'.

Contextual and symbolic generalizations address a more mathematical level of generalization. In the contextual generalization, on the contrary to a factual generalization, the indeterminate is "made linguistically explicit: it is *named"* (Radford, 2006, p. 16). The generalization is, in other words, symbolized by words 'you *double* the *terms number'*. The difference between a contextual generalization and a symbolic generalization is that a symbolic one is based on algebraic symbols, such as '$2 \cdot x$' instead of words 'you *double* the *terms number'*.

Venenciano and Dougherty (2014) highlight another kind of strategy as algebraic. It is a measuring strategy where, for example, two squares are used as a measurement unit (see figure 2) and this puts the number of measurement units in relation to where the term is positioned in the pattern sequence.

> From a measurement approach […] one may view the unit of measure as a composite of the two squares, that which is iterated with each successive figure. This […] approach enables one to apply the notion of defining a unit and consider a scale factor to solve the problem.
>
> (Venenciano & Dougherty, 2014, p. 23)

It is argued that the teaching of algebra should give the students the opportunity to use algebra as a tool for characterizing and understanding mathematical structures (e.g., Greer, 2008; Usiskin, 1988). Additionally, a focus on making generalizations in relation to mathematical patterns is advocated in the early grades (e.g., Radford, 2006, 2011). Distinctions in relation to different types of algebraic generalisations (e.g., Radford, 2006), opens up for a broader understanding in relation to generalizations. What is lacking is descriptions of aspects which students simultaneously need to discern and take into consideration in order to be able to express and justify pattern generalization algebraically. Hence, the main object of this paper is neither about what an algebraic generalization is, nor which strategies students may use. The aim of this paper is to describe the emergence of the ability to express and justify pattern generalization algebraically. The research questions for this paper are: "What are students' qualitatively different ways of seeing pattern generalization?" and "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?"

3. Theoretical framework

Variation theory (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) has been used as a theoretical framework in this study. Learning in a Variation theoretical perspective is considered to arise in the relationship between the one who is learning and what is to be learned (Marton & Booth, 1997, see also Marton, 2015). Variation theory provides theoretical tools for the analysis of the conditions of qualitatively different ways of seeing specific knowledge, and what aspects that are critical to discern in order to be able to see this knowledge in a more powerful way. Variation in relation to a Variation theoretical perspective refers to a meaningful, conscious, directed and systematic variation of content. Critical aspects are aspects that the students need to discern to be able to develop this specific knowledge (Marton, 2015). In this paper we are exploring students' quality different ways of seeing pattern generalization. In a Variation theoretical perspective 'ways of seeing' are seen in relation to what aspects the students are discerning and focusing upon in relation to a demarcated knowledge (Marton, 2015).

4. Methodology considerations

This study is included in a more extensive practice-based research project, in which Learning study (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) is used as a research approach. This paper does not present the final results of the Learning study, but rather the analysis of semi-structured interviews, conducted initially in the study.

The semi-structured interviews

One of the first steps in a Learning study is the mapping of the students' current perceptions of a specific knowledge. In this research project semi-structured interviews were chosen as a mapping tool to grasp the students' qualitatively different ways of seeing pattern generalization and to identify aspects that students need to discern to be able to express and justify pattern generalization algebraically.

The semi-structured interviews were performed with eight of the students from the overall project. The students were 9-10 years old, and both girls and boys were interviewed. The idea was that this selection of students would cover much of the diversity that existed within the group (Marton & Booth, 1997). The students were selected in relation to their previous results in mathematics and were supposed to represent students with different performances in the subject of mathematics. The selected students were divided into pairs and were then, in the interview situation, presented with three different pattern tasks which they were asked to solve together. While the students were working with the tasks, the interviewer asked question such as "Can you tell me how you´re thinking?" and "Can you show me how you are looking at it (pointing at the pattern) when you´re saying this?". The aim was trying to explore the students' ways of seeing pattern generalization in the process of solving tasks where making pattern generalization were required. The idea was not about how pattern generalization may be defined by the students and thus was the interviewer not supposed to ask any direct questions about pattern generalization per se.

Analysis

The data in this paper consists of transcriptions of the interviews. In the analysis, Variation theoretical tools were used (critical aspects and variation of content), in order to try to distinguish qualitative dimensions of the variations in different ways of seeing pattern generalization and in relation to identifying critical aspects of the ability to express and justify pattern generalization algebraically. In the analysis, there was an interplay between the data and previous research (e.g., Radford 2010). The process of the analysis was as follows:

1. *Reading of compiled interviews*. The transcribed interviews were compiled in a running document without markings for which student said what. The document was then read several times without making any markings on the document. The aim was to try to understand what different students were saying in relation to what other students were saying.
2. *Analysis of what the students talked about.* The transcripts were read again, this time with the intention of marking those excerpts where the students talked about pattern generalization. The excerpts of the transcriptions where the students did not talk about pattern generalization were identified and removed.
3. *Analysis of how the students were talking about pattern generalization.* The excerpts where the students talked about pattern generalization were repeatedly read through a so-called comparative reading (Marton, 1995). The aim was to distinguish between the dimensions of variations of students' ways of seeing pattern generalization that were realized through students' expressions.
4. *Categorization of the students' ways of seeing pattern generalization.* Different excerpts of the students' expressions were marked with the aim to identifying qualitatively different ways of seeing pattern generalization. Those excerpts were analysed, in relation to what the students emphasized and what they seemed to discern and focus upon in relation to pattern generalization.
5. *Identifying critical aspects regarding the ability to express and justify a pattern generalization algebraically.* In the identifying process the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does this generalization work to predict any figure in the pattern?" (Radford, 2006)

5. Findings

The findings consist of two parts. Part one answers the research question "What are the students' qualitatively different ways of seeing pattern generalization?". It consists of four categories. This result is in relation to point 1-4 in the analysis.

Part two answers the research question "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?". It consists of identified critical aspects regarding the ability to express and justify a pattern generalization algebraically. This result is in relation to point 5 in the analysis.

Part one - Students' qualitatively different ways of seeing pattern generalization

In the following, there are descriptions of the categories which contain student's expressions. Each category is summarized in relation to which aspects of pattern generalization that the students seemed to discern and focus upon.

...as some kind of grouping structures

In this category, the students emphasize the grouping of quantities in the sense of using a structure as a strategy to see how the pattern is built. Students, for example, undertake groupings based on the number of elements in a term (see figure 1): "... Term 1 has three (matches) and (pointing to term 2) has six (matches) ...".

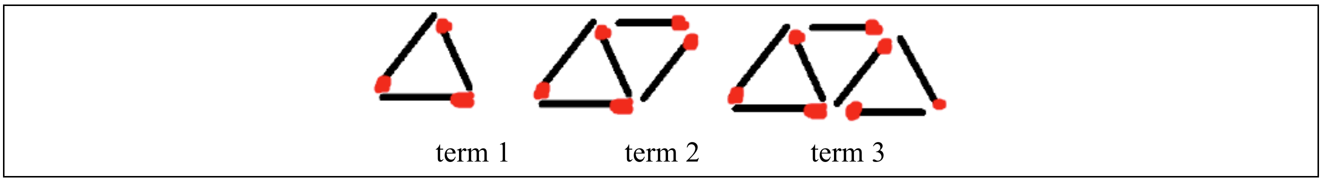

Figure 1. Matches

The students additionally grouped by adding together the number of elements in the visible terms in the pattern (Pattern 1): "... if all the matches up to term 3 is fifteen, then if you take this three, plus this three (the student is talking about the terms 1-3), it is term 6, then it is thirty matches, fifteen plus fifteen is thirty (here the student is talking about the number of matches in terms 1-3)". A characteristic of this category is that the grouping is used rather as a statement, not to predict the number of elements in a specific term.

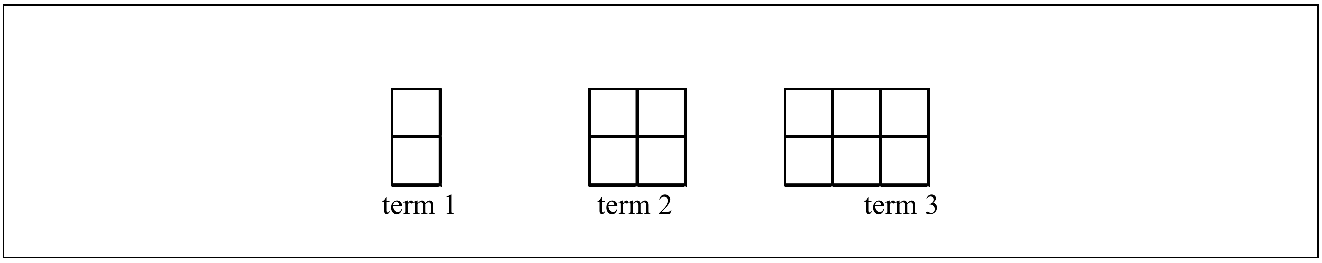

Figure 2. Squares

The students in this category seemed to discern and focus upon the following aspect: that there is a structure to follow that involves grouping objects.

...as additive constant structures

In this category, the students emphasize the adjacent terms in the sequence and the number of elements or units by which the pattern is growing. Students calculate the difference between two terms in a given sequence and distinguish this difference as being the same between all the terms. They conclude that the growth of the pattern is according to an additive structure. Based on the pattern in figure 2, a student expresses how to create the next term in the sequence: "... always add two (squares)." Other students use the column of two squares in the pattern as an integral unit, which they use as a rate of growth of the pattern: "... you only add one of those (pointing to a column of two squares)." The students see the growth of the pattern as "jumps" in the addition table: "... here are two, here are four, six, and the next eight and then it's ten."

The students in this category seemed to discern and focus upon the following aspect: the additive structure of the pattern and what constitutes the so called expansion unit (mathematical) of the pattern.

...as one dimensional relational structures

In this category the students emphasize one dimension of the pattern generalization; the number of elements or units in relation to the position of the term. The following student uses a column of two squares as a unit in relation to figure 2: "When it is term 1, I make one line (shows as a column) when it is term 2 it is two, if it is term 3 it is three columns". Another student expresses the connection between the term and the number of units (pattern 2): "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns ".

The students in this category seemed to discern and focus upon the following aspect: the relationship between the position of the term and its units.

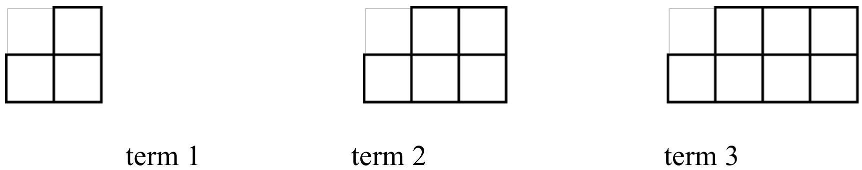

Figure 3. Squares in other way

...as two dimensional relational structures
In this category the students emphasize two dimensions of the pattern generalization; the relationship between the position of the term and the number of its elements or units and use it to predict a non-visible term in the pattern sequence. The following student expresses it, in relation to figure 2, like this: "Look, number 1 it is two, number 2 then it is four, number 3 is six, as it doubles everything… then you have to double forty-eight". In one of the tasks the students are supposed to determine the number of squares in term 46 (figure 3): "... is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)... then you can add forty and forty, its eighty and three plus three is six, then its forty-six and then we take one (the one that the student suggested should be put away) then there will be forty-seven ... no, it is eighty-seven."
The students in this category seemed to discern and focus upon the following aspects: what is the relationship between the figures number and its units and use this relationship to predict any figure in the pattern and what constitutes the constant in the pattern.

Part two - Critical aspects regarding being able to express and justify pattern generalization algebraically

The critical aspects were interpreted in relation to the categories' descriptions and which aspects the students seemed to discern and focus upon. In the identifying process, the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does the aspect enable the student to predict any figure in the pattern?" (Radford, 2006).
The categories, *as some kind of grouping structures* and *as additive constant structures,* do not encompass critical aspects in relation to *algebraic* pattern generalization. The aspect *that there is a structure to follow that involves grouping objects* is not an algebraic aspect in terms of making it possible to predict the number of elements in any term in the pattern. The structure in this case concerns 'only' the grouping of quantities. The aspects *the additive structure of the pattern* and *what constitutes the expansion unit (mathematical) of the pattern* are of general character. However, this kind of generalization only works to predict the adjacent terms, since one cannot say the number of squares of any term in the pattern, such as the thousandth.
It is primarily the categories *as one dimensional relational structure* and *as two or more dimensional relational structures* that we see as encompassing aspects of algebraic character in terms of making it possible to predict the number of elements in any term in the pattern. The aspects we identified as critical aspects in relation to be able to *express and justify pattern generalization algebraically* are:

- to discern the relationship between the term's position and its units
- to discern the relationship between the term's position and its units and to use this relationship to predict any term in the pattern
- to discern what constitutes the constant in the pattern.

Here is a short description of why we consider these aspects to be critical. By discerning *the relationship between the figures number and its units* it is possible for the students to use this relationship to predict the number of squares of any term in the pattern. "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns." In other words, the discerned relationship is the same throughout the whole pattern, and it doesn´t matter if you are talking about term 5 or if you are talking about term 1 000. When the students discern *the relationship between the term's position* and *its units and use this relationship to predict any term in the pattern*, the students both discern the relationship "number 1 it is two" and transform this relationship "as it doubles everything" so it can be used to predict any term in the pattern. Regarding the aspect what *constitutes the constant in the pattern*, the students discern that this unit, the constant, is the same through the whole pattern "is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)?". In other words, the constant is containing the same number of units in any term if the constant is one unit in term 1 it is also one unit in term 1 000.

6. Concluding discussion

The aim of this paper was to describe the emergence of the ability to express and justify pattern generalization algebraically. In the following we will put the categorization and identified aspects of this paper in relation to other research, and mainly Radford's distinction between arithmetic and algebraic generalization strategies. The difference is, according to Radford (2006), that an arithmetic strategy does not make it possible to predict any term in a pattern, which would otherwise be the case with an algebraic strategy. The main contribution of this paper, in relation to Radford's categories, lies in the specification of critical aspects regarding what students need to discern in their learning of how to express and justify pattern generalization algebraically.

In the category *pattern generalization as additive constant structures* the students seem to discern *the additive structure of the pattern* and/or *what constitutes the expansion unit (i.e., mathematical unit) of the pattern*. Students calculate the difference between two terms in a given sequence and identify this difference as being the same between all the terms, concluding that the growth of the pattern is according to an additive structure. There might be a qualitative difference between the expressions "... always add two (squares)," where the students talk about the growth of the pattern, and "... you only add one of those (pointing to a column of two squares)," where the students use the columns of two squares in the pattern as an integral unit. This latter can be put in relation to the position of the term, which can be used to predict any term in the pattern (Moss & London McNab, 2011; Radford, 2006), since it can be seen as the beginning of using an expansion unit as a measurement unit (Venenciano & Dougherty, 2014). However, if the generalization stops at discerning only *the additive structure of the pattern* and/or *what constitutes the so called expansion unit (mathematical) of the pattern,* this is not enough to express and justify pattern generalization algebraically.

In relation to students in the early grades, we want to highlight how factual generalization and contextual strategies (Radford, 2006) can be seen as a starting point regarding developing an understanding of the meaning of algebraic notations. In other words, we consider Radford´s factual strategies and contextual strategies as indicating the emergence of being able to make symbolic generalizations. The difference between those strategies is about *how* the generalization is expressed. In the category *as one dimensional relational structures* a student expresses the following: "When it is *term 1*, I make *one line* (shows as a column) when it is *term 2* it is *two*, if it is *term 3* it is *three columns.*" In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a factual strategy, although the indeterminacy is unnamed, and the generalization here is symbolized by actions. We would equate this, in relation to our findings on critical aspects, as the student is discerning *the relationship between the term's position and its units.*

Contextual generalizations address a more mathematical level of generalization, the indeterminate is "made linguistically explicit: it is *named"* (Radford, 2006, p. 16). In the expression "Look, number 1 it is two, number 2 then it is four, number 3 is six, as *it doubles everything*… then *you have to double* forty-eight", the generalization is symbolized by words '*as it doubles everything'*. In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a contextual generalization, although the indeterminacy is named, and the generalization here is symbolized by words. We would equate this, in relation to the findings on critical aspects, as the student is discerning *the relationship between the term's position and its units and to use this relationship to predict any term in the pattern.* Finally, our point is that this kind of generalization can later be transformed into a symbolic generalization, while drawing on the students' more informal way of describing it.

Acknowledgements

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Production and evolution of functional-spontaneous representations through the communication process

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Abstract. The mathematical modeling process starts with the proposition of non-routine tasks where students use or construct mathematical models for their solution. During this process, students develop functional-spontaneous representations that emerge naturally while they solve the task. The class organization has an important role in the development and evolution of these representations. The research aim is studying the factors influencing the communication process during mathematical modeling activities. Using a qualitative methodology, it is described a non-routine task related with the co-variation between variables with high school students. During the class, we used a methodology that promotes a scientific debate and self-reflection named ACODESA. Results show that the individual characteristics of each student are factors that can promote or limit the learning process in a teamwork organization.

Résumé. Le processus de modelage mathématique commence avec la proposition de tâches non-de routine où les étudiants utilisent ou construisent des modèles mathématiques pour leur solution. Pendant ce processus, les étudiants développent des représentations fonctionnelles et spontanées qui émergent naturellement pendant qu'ils résolvent la tâche. L'organisation de classe a un rôle important dans le développement et l'évolution de ces représentations. Le but de recherche étudie les facteurs influençant le processus de communication pendant les activités de modelage mathématiques. En utilisant une méthodologie qualitative il est décrit une tâche non-de routine rattachée avec la co-variation entre les variables avec les étudiants de lycée. Pendant la classe nous avons utilisé une méthodologie qui promeut une discussion scientifique et une réflexion de soi appelée ACODESA. Les résultats montrent que les caractéristiques individuelles de chaque étudiant sont des facteurs qui peuvent promouvoir ou limiter le processus d'apprentissage dans une organisation de travail d'équipe.

1. Introduction

The information that exists in the actual society is created and disseminating faster than in the past. The maximization of the information had impacted the education and its goals. Schools' purpose is to educate students with the criteria to understand what they read and produce, students with competences for solving real problems using innovative ideas. The traditional curriculums no longer use traditional practices as the memorization or repetition. Those old practices had not shown any benefit in the learning process.

Now, mathematics are tools that can be used to solve real-life problems and this is a way schools could promote teaching and learning. According to that, several countries had included strategies that emphasized mathematics and applications in order to promote a diversify thinking among students. One of those strategies is mathematical modeling.

Mathematical modeling is defined as the cyclic process where a teacher proposes non-routine tasks based in real context and where students use or develop a mathematical model that solve the problem (Niss, Blum & Galbraith, 2007, Rodríguez & Quiroz, 2015). There are three main characteristics in mathematical modeling: the first one is the teachers' role, as the designer of problems that need to be related to the students' interest. Those problems need to be clear and with instructions easy to understand, but demanding a complex solution. Those tasks usually accept several ways of solving, and also accept diverse right solutions.

Besides the role as a designer, teachers guide the mathematical modeling process without saying the right answer or the correct way to solve the situation (Hitt & González, 2015).

The second characteristic of the mathematical modeling process, is the student's role. Students are the main actors that develop solutions of the problem. According with diSessa et al. (1991), in the first approach to the problem, students produce representations that are spontaneous and non-institutional. Institutional Representations (IR) are the kind of representations, which are usually accepted and used by the actors of the teaching system: books, computer screens and teachers.

According with Hitt (2003, 2006), the spontaneous representations are cognitive structures that emerge when the student tries to understand and solve a non-routine task. In more recent studies, those representations have been named Functional-Spontaneous Representations (FSR). While solving the task, students need to make a refinement of their FSR through a communication process. The importance of communication is the third characteristic of the mathematical modeling process. During mathematical modeling teacher must promote the work in teams and also a group debate. The evolution of the FSR is related with the transformation and coherent integration of the external representations associated with the FSR in this process of communication in the mathematical classroom (Leontiev, 1975).

Our research is based in the study of Hitt & González (2015) about ACODESA methodology (Collaborative learning, scientific debate and self-reflection). This metodology promotes the evolution of representations during the solve of non routine tasks and promotes the diversify thinking in students. ACODESA methodology distinguish five stages:

- Individual work where the students facing a non-routine task and construct FSR and produce external representations (verbal and diagrams).
- Teamwork, where students work in teams in order to solve the same task. They make refinements of external representations linked to FSR through a process of argumentation and validation.
- Debate scientific: The entire class discusses different forms of representations to solve the task at hand.
- Self-reflection: Individually, students solve the same activity in home. It allows students to reconstruct what was made in groups.
- Institutionalization: teacher introduces the topic taking into account the students' results and using IR.

Through the use of those theoretical elements, the objective of the research is:

- Describe how the communication process can promote the evolution and refinement of Functional-Spontaneous Representations in a mathematical modeling process.

2. Methodology

The research is based in a qualitative paradigm, specifically in a case study. The sample was conforming by high school students between 14-15 years old. They were in the 9th grade at the moment at the moment of the research. There were chosen three teams of four students each. Those teams had shown different ways of work and to communicate in previous sessions. All sessions were video recorded. The non-routine task was chosen from a set of five activities designed to promote learning of covariation between variables. The activity chosen is the first of the set, and as the student first approach, it is demanded to make a first representation through a design or a diagram where they described the phenomena that is studied. Besides, it is demanded to write an explication individually using words.

During the second moment of the activity, students are organized in teamwork in order to compare their ideas and express a diagram as a social construction of the phenomena. When all teams had designed the diagram, they explain their work to the whole group and make comparisons of the solving. Finally, each team decides if they change the diagram or not.

3. Expected conclusions

Research results showed that students develop different FSR when they initially solve the task. Each student produced an initial diagram where they explain the phenomena that were analyzed. In the diagrams are shown several mathematical concepts as: angle, hypotenuse, distance, sides, parallel lines, and perpendicular lines. ACODESA methodology allowed the implementation of the mathematical modelling cycle during the lesson. Using ACODESA methodology, the first stage, individual work, showed a diversity of procedures to solve the problem. During the others stages of the ACODESA (teamwork, debate in whole group), the students FSR changed, but the changes were different in each team. The communication process of each team influenced in the FSR evolution into IR. Some teams arrived to IR during the teamwork and some others needed to wait until the whole group discussion. The analyze recognized that some of the teams work in a homogeneous way, it means that the students promoted the participation of all the team members and the ideas were listened carefully and with clarity. Nevertheless, in others teams, there was a student that played the role of leader, and his/her ideas were the ideas that the others students followed. The other member's ideas were ignored or simply, not accepted. Another important factor was related with the discussion process. In some teams, this process were open and all students can explain their ideas. In those teams, the change in the FSR was bigger and the representations became almost IR. In the other hand, the teams were the FSR were similar between the students, the discussion process was poor and without reflection.

As a preliminary conclusion, the study showed that the production and evolution of FSR could be affected depending to the team where the student is involved. Because of that, teachers need to take into consideration the students in each team, and also the promotion of methodologies where can be used different forms of organization as teams, group and individual work. ACODESA may be an interesting way to promote those types of organization and also combine the use of mathematical modelling.

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Inquiring the role of visual-representations in inclusive educational activities concerning fractions

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Abstract. Since in Italy there aren't special classes for students with special needs, inclusive educational activities play an essential role in math education. This research focuses on MLD students (students with mathematical learning disabilities or difficulties) and, more in general, dyscalculic students or students with low achievement in math. In order to design inclusive educational activities, this research takes into account both some results of cognitive science and of math education. More in details, the research aims to interpret some research results of cognitive science concerning the MLD students' learning of fractions to define educational hypothesis upon which it can design inclusive educational activities to support teaching and learning of fractions in primary school.

Résumé. Comme en Italie il n'y a pas de cours spéciaux pour les étudiants ayant des besoins spéciaux, les activités éducatives inclusives jouent un rôle essentiel dans l'enseignement des mathématiques. Cette recherche se concentre sur les étudiants ayant des difficultés d'apprentissage en mathématique (appelés, dans la littérature anglaise, MLD : Mathematical Learning difficulties or disabilities) et, plus généralement, des étudiants ayant des trouble d'apprentissage en particulier étudiants dyscalculiques. Afin de concevoir des activités éducatives inclusives, cette recherche tient compte à la fois des résultats des sciences cognitives et des didactiques des mathématiques. Plus en détail, la recherche vise à interpréter certains résultats de recherche des sciences cognitive concernant l'apprentissage des fractions par des élèves ayant des troubles d'apprentissage, pour définir des hypothèses éducatives sur lesquelles on peut concevoir des activités éducatives inclusives pour soutenir l'enseignement et l'apprentissage des fractions à l'école primaire.

3. Introduction

Even if there isn't consensus on definition and identification of MLD students and the inclusivity (Ianes *et al.*, 2013) is not a construct used consistently across different fields (education, society...) or in different countries, in this research work, we considered as MLD students, dyscalculic students, students with difficulties in math and students with low achievement in math. Since in Italy there aren't special classes for students with special needs, we consider "inclusive educational activities" those developed in the context of the class, which meet the needs of all students of the class. Moreover, because in Italy the percentage of children with learning difficulties has increased in the last years, from 0.7% in 2010/2011 to 2,1% in 2014/2015 (among these, the 16% are diagnosed dyscalculic in primary school and the 25% are diagnosed dyscalculic in Upper school), the "inclusive educational activities" are strongly needed in order to design effective teaching and learning of mathematics above all, in primary school. In particular, this research focuses on teaching and learning of fractions. Two main reasons guide my choice: the first is that fractions play an important role in theories of numerical development. As matter of fact, according to Siegler, "Algebra proficiency is more closely related to conceptual knowledge of fractions than to conceptual knowledge of whole numbers" (Siegler, R. S., *et al.*, 2013, p.6). The second one is because fractions are one of the main difficulties detected at the national (INVALSI) and international level (OECD-PISA).

As in math education (Mulligan *et al.*, 2013; Fandillo, P. 2007), cognitive psychology have also been very active in investigating the phenomena of (difficulties in) understanding mathematics, included fractions, even if the different interested fields of research have not yet reached sufficiently common grounds for conducting scientific and interdisciplinary studies. In this paper, I consider some results from research in cognitive psychology about the role of representations in understanding fractions (Marzocco et al. 2014), in order to set up important design decisions during the processing of an educational experiment built around learning fractions in primary school (Robotti et al., 2015). In particular, these results help me to define research hypothesis, in order to design inclusive educational activities about fractions, which face to the needs of MLD students of the classes. To this aim, this research considers UDL (Universal Learning Design, http://www.udlcenter.org/) framework, based on cognitive neurosciences, for designing learning experiences that work across a large spectrum of learners and for making flexible the design of curriculum in order to meet the students' diversity in the same class (Robotti, 2016).

4. Some results of cognitive science research about the learning of fractions with MLD students and their interpretation in mathematics education

In this session, I consider some research results related to cognitive science in order to interpret them through the lens of math education and define research hypothesis with the aim to design inclusive educational activities about fractions.

Cognitive neuroscience shown that accurate representation of fraction magnitudes emerges as crucial both to conceptual understanding of fractions (as part of a whole) and to the arithmetic of fractions (Siegler, R. S., *et al.*, 2013). Moreover, children of 6- and 7-year- olds use 1/2 as a reference point when matching non-verbal representations of fractions. When asked which of two partially filled rectangles match a third, are more accurate when the two options are on opposite sides of 1/2. For example, when matching 3/8, participants are more accurate when the options are sets equivalent to 3/8 and 5/8 than ones equivalent to 3/8 and 1/8. These researches underline that symbolic fraction knowledge develops later than non-symbolic knowledge, but the fraction 1/2 again is prominent in early understanding.

What cognitive research says about MLD students? At this regard, we refer the cognitive science research developed by Mazzocco and colleagues in 2013. This research considered three kinds of students: MLD students (considered dyscalculic students), students with low achievement in math (LA) and students with typically achievement (TA). The research first seems to confirm that children with MLD, relative to their LA and TA peers, were less accurate on symbolic magnitude comparison tasks involving pairs of fractions. Also MLD children have an improvement over the school time (from the 4th to the 8th grade - 9/10 years to 13/14 years), like the other groups, even if the rate of improvement grows up more slowly than that of their schoolmates.

From an educational point of view, we can infer that, even for MLD students, there could be improvement both in the appropriation of meaning of fraction (here considered as part of a whole) and of its arithmetic manipulation.

As introduced before, the fraction "one-half" plays a very central role in processing fraction magnitude. Therefore, the Marzocco's premise in her study design and analysis was that magnitude comparisons of visual-representations of "one-half" are easier to correctly resolve than are fractions items that do not include a visual-representation of one-half. The researchers evaluated rate of growth on the two types of items (one-half, non-half) as a function of MLD status. The results show that, at study entry (Grade 4), children in the TA group had higher rates of accuracy on the one-half items than their LA or MLD peers, that this pattern also emerged for the non-half items, and that children with LA had higher rates of accuracy than their MLD peers on the one-half and non-half items. Students with LA or TA reach and maintain ceiling performance on one-half items over time, whereas children with MLD do not. For non-half items, the TA group is growing significantly faster than the MLD group.

From an educational point of view, this result could lead to the hypothesis that the assessment of accuracy and of the use of an effective strategies that concern ½ at the conclusion of the 4th and 5th (i.e. towards the 10-11 years) could be a relatively efficient way to identify children who may have learning difficulties on fractions and that, therefore, need a further educational support about fractions.

Moreover, Marzocco and colleagues evaluated both rate of growth and accuracy rate about effects of item format (format of representation) on one-half items. Having established that children with MLD (and, at Grade 4, also children with LA) have difficulty comparing fractions, and that even one-half items pose a challenge for children with MLD, the researchers examined whether performance is facilitated (or hindered) by any of the representational formats, across the TA, LA, and MLD groups. The representations considered are: visual representations as part of the whole, symbolic representation as Arabic numbers and incongruent visual representations (see 'figure 1')

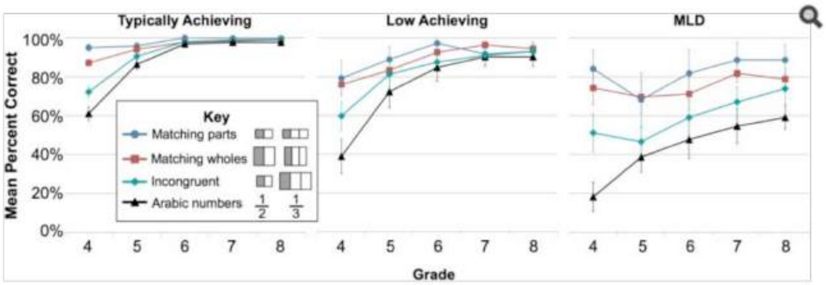

Figure 1. How performance is facilitated (or hindered) by the representational formats (Marzocco et al., 2014, p.12).

When visual models included matching "wholes", the LA group grew faster than the TA group, consistent with the notion of a general "catching up" after Grade 4. Children with MLD showed a faster rate of growth than the TA group, for the Arabic number format; Children with MLD grew faster than the LA or TA group on the spatially misleading format, presumably because of their markedly low initial performance levels on these formats. Rates of growth did not differ between the MLD and LA group, on the Arabic number notations, although accuracy rates did.
We observe that all the groups have performances more correct with visual representations rather than with Arabic representation. Nevertheless, if for TA group the two kinds of representation allow students to the same success after Grade 6 and for LA group after Grade 8, for MLD group the different representations of fraction never allow students to the same success. As matter of fact, we observe that visual representations allow MLD students to compare fractions in better way (with more success and accuracy) than Arabic representations.
From an educational point of view, this result could lead to the hypothesis that MLD students, as LA and also TA students, can benefit from the use of visual models to support effectively learning on fractions and solve problems involving fractions.
However, we can observe that in the MLD group, the performances on the inconsistent visual representations have always a smaller percentage of correctness (during the different school grades) than the congruent visual representations. This suggests that MLD students use visual model exclusively by referring to the perceptual strategies rather than to the meaning of part/whole.
Therefore, from an educational point of view, the teacher needs to pay attention to the use of these representations: the visual representation doesn't should be used through purely perceptual aspects but by strengthening ties with the meaning of part/whole or part of a unit of measure (as we will can see in 'figure 2'). The educational hypothesis that can be defined at this regard, is that teaching should bring out the character of "necessity" that have the solution strategies not purely perceptual. This allows MLD students to overcome the idea that perceptive strategies can be always the most effective, when comparison between fractions is required by visual representations.

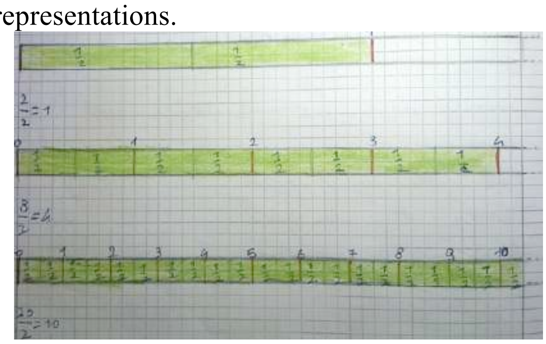

Figure 2. Comparison of "one-half" drawn on three strips of squared paper having different units of measure (30 squares in the first one, 10 in the second one and 4 in the third one).

3. Conclusion

Research in cognitive science suggest that MLD students show a "limited knowledge of "one half," until 8^{th} Grade. From an educational point of view, this means that the time needed for dyscalculic children for

processing of this fraction as an effective tool in order to process the other fractions must be greater.
Moreover, cognitive science suggest that visual model leads to better performances than the symbolic one since the 9/10 years, and this gap is expected to decline from the 10/11 years except for MLD students. This means, from an educational point of view, that visual representation is an effective approach to fractions for all students but, for MLD students, remains the most effective strategy for a longer time. Moreover, in the use of visual model to compare fractions, MLD students prefer the perceptual strategy, which isn't the most effective strategy above all with inconsistent visual representations. This means that teaching, mediated by visual representations, should support the construction of meanings (for example the meaning part/whole) making sure that the perceptual aspect doesn't dominate on the development conceptual (for example, showing that the unity fraction depends on the chosen unit of measure). To this aim, teaching may make evident the need of more "sophisticated strategies". Based on these hypotheses, it was designed and implemented an inclusive educational sequence about fractions for primary school described in Robotti *et al* (2015).

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Students' awareness regarding vector "subtraction" through a dialog with the teacher

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Abstract. This article presents an interpretative microanalysis of the production process of meanings that students and an expert teacher carry out together in a physics class regarding vector subtraction. This is a qualitative study supported by the theory of objectification, which defines learning—objectification—as awareness. Data collection was done through video recordings of the lessons taught by the physics high-school teacher. The results show both the two different representations made by the teacher to report vector subtraction and the student's difficulty to integrate those two representations.

Résumé. Cet article présente une microanalyse interprétative du processus de production des significations que les étudiants et un enseignant expert exercent ensemble dans une classe de physique concernant la soustraction des vecteurs. Il s'agit d'une étude qualitative soutenue par la théorie de l'objectivation, qui définit l'apprentissage-objectivation - comme conscience. La collecte de données a été effectuée par des enregistrements vidéo des leçons enseignées par le professeur de lycée de physique. Les résultats montrent à la fois les deux représentations différentes faites par l'enseignant pour signaler la soustraction du vecteur et la difficulté de l'élève à intégrer ces deux représentations.

5. Background and research problem

The analysis of school practices has led to the need for stressing the role that history and culture play in the development of a subject's education. Among the diverse sociocultural research approaches in mathematics education is the theory of objectification (TO) (Radford, 2014a; 2016). From a semiotic approach, the TO focuses on teaching-learning problems in terms that are different from the ones in the individualistic educational theories revolving around the student. Therefore, in this article we seek to answer the following question: How are the meanings regarding vector "subtraction" produced in a space of joint action of students and an expert teacher?

6. Theoretical framework

This research is supported by the TO (Radford, 2014a) that conceptualizes teaching-learning in terms of a joint activity of students and teachers. Then, the concept of activity or labor is the key conceptual category of the TO (Radford, 2014a). The notion of knowledge in the TO is based on the dialectical materialism; it is not something individuals possess, acquire or construct, but the mere possibility of ways of doing and thinking [on systems of ideas] (Radford, 2014b). Hence, "The only manner in which knowledge can acquire cultural determinations is through specific *activities* [italics in the original]" (Radford, 2014b, p. 7). Learning—objectification—is then defined as the awareness of the [scientific] systems of ideas, that is, the ways of expression, action, and reflection, historical and culturally constituted. However, meanings in the classroom are produced through a social and bodily (language, gestures) process that is symbolically mediated and carried out in a space of joint action, which "is a space of relations and embodied reciprocated tunings occurring in the concrete space of interaction." (Radford & Roth, 2011, p. 231).

3. Method

The results of this article are part of a wider ongoing research that analyzes the practice of two teachers: one expert and one novice. This is a qualitative study performed through a case study. The pilot study was done by video recording the lessons of two physics (expert and novice) teachers in original teaching configurations. This article reports results of the expert teacher's lessons. We video recorded 10 of the teacher's sessions (16 hours) during which he taught Newtonian dynamics. To do so, two cameras were used and field notes were written down. For the objectives of this article, a portion of a lesson in which the teacher talked about vector subtraction was selected and excerpts of the discussion were transcribed.

4. Analysis of the interaction between students and teacher when producing meaning

Here, we present excerpts of the process during which the students and the teacher produce the meanings regarding vector subtraction, in a graphical environment particularly. The data analysis focused on the way in which the teacher promotes participation and awareness on the concept discussed. The analysis is structured in two sections: 1) includes three excerpts dealing with vector subtraction as "sum of additive inverses"; 2) contains two excerpts which graphically show the subtraction of two vectors.

4.1 Vector subtraction by the sum of additive inverse of a vector

The first excerpt begins when the teacher explained the subtraction of natural numbers 5–3 using the analogy of a leaping frog on a number line in which the result [2] is the point the frog reaches after having jumped 5 [units] forward and 3 [units] back.

Teacher: The result [*of 5–3*] seen as a vector, seen as an arrow, would be—starting from the origin—where I reached. Then, I can think about 2 in two ways: as a point or as a small arrow that goes from zero to two. Now picture that the frog can leap in two dimensions. I want to take (2,2) from (3,1) [*writes: (3,1)–(2,2)*]. This means that [*the frog*] leaps 3 along the *x* axis and 1 along the *y* axis.

In the excerpt above, this is the moment when the activity starts, giving rise to concrete determinations related to vector subtraction. Before the teacher presented the situation of vector subtraction, knowledge was a mere possibility to the students. First, the teacher is observed to point at the result [2] on the number line as geometric object: a vector of magnitude 2 and a direction "that goes from zero to two." Then, this is when the concept of vector acquires a concrete determination. Although the teacher introduces the notion of vector subtraction, he does so abruptly and sets to work on a mathematical object that has been barely represented. The teacher makes an analogy between the subtraction of integer numbers and a vector subtraction. A moment later, the teacher graphically explains what happens when subtracting (2,2) and states that (2,2) would really be (–2,–2) even though he does not explain why "one (2,2) is really the other (–2,–2)". Then, he writes on the board: (3,1) – (2,2) = (3,1)+(-2,-2)= (1,-1). It can be said that the teacher defines the vectors as: A = (3,1), B = (2,2) and –B = (–2, –2), as the additive inverse of B and continues with the explanation.

Teacher: This is the leap (–2,–2) [*see figure 1a*] and the result [*of the subtraction*] goes from the origin to where [*(–2,–2)*] reaches [*see figure 1b*]. Then, what the first vector [*(3,1)*] had, then I actually had the second vector, which was this one [*draws vector (2,2), starting at (3,1); see figure 1c*], but I flipped it to the other side with the minus sign [to obtain (–2,–2); *he makes a gesture, see figure 1d*] and added the inverse. (…).

S1: Teacher, can't we make it just like in a Cartesian plane? (…)

Teacher: Why don't we see what you propose? (…).

When introducing the "vector subtraction", the teacher is, in reality, making the operation A + (–B). He even sets to work with –B from the beginning and clarifies that he is really adding inverse of B when he says: "then, I had actually the second vector, which was this one, but I flipped it to the other side with the minus sign and added the inverse" (see figures 1c and 1d).

On the other hand, when trying to graphically address the meaning of the vector subtraction, the teacher is observed to place the vector (–2,–2) starting at the point (3,1) and not at the origin of the Cartesian plane [Figure 1a]. This is a common way of adding vectors [graphically], which is used in courses and is known as "*head to tail addition method*" [figure 1b]. S1 is not familiar with this method and expresses the question. In that moment, there is no explanation as to why the vector sum is made in such way.

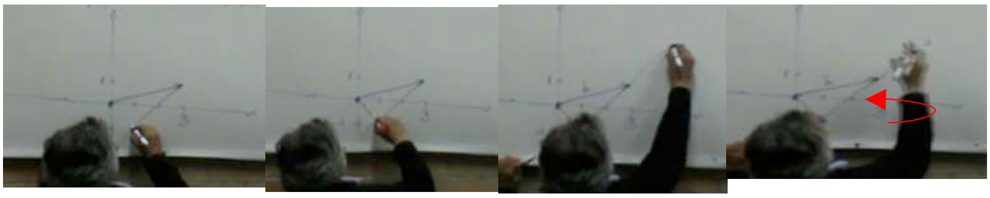

Figure 1. Graphic visualization process of the vector subtraction in four moments: from left to right (1a, 1b, 1c, and 1d).

S1: I was saying we should do it by coordinates [*marks points (3,1) and (–2,–2) in a system of coordinate axes*] (…) and then we get that it is (1,–1), but the result is a line, so I didn't know if that was correct. [*The result*] is something really twisted [*see figure 2a*]. (…).

Teacher: Then, the point that you placed here [*he means (–2,–2)*] is a point, but it is a vector, too. I mean, I can take the arrow that goes from zero to this point. (…) now the problem is how do I add this [*(3,1)*] to this [*(–2,–2)*], how do I add these two vectors? (…) Didn't you see [*referring to a previous simulation*] that we could grab this vector [*(–2,–2)*], this arrow, and take it to the edge of this other one [*makes a gesture to simulate he moves vector (–2,–2); see figure 2b*] without changing the length or the direction? So, if I move it [*referring to (–2,–2)*] it is this one [*see figure 2c; S1 says: "Oh!"*]. And the sum, which was it? It was the vector that, starting from the origin, reached the end of the second arrow. That means what you [*addressing S1*] propose is essentially the same.

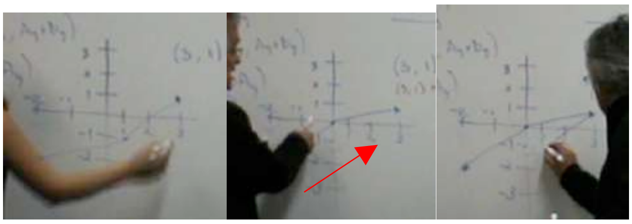

Figure 2. From left to right: process of vector subtraction by S1 (2a) and by the teacher (2b and 2c).

When placing vector (–2,–2) at the origin of the Cartesian plane, S1 is not aware that the differences is the teacher is following the graphical method to sum the vectors while she is using the arithmetic method (placing both vectors starting from the same origin). Still, S1 fails to graphically visualize the result. A moment later, the teacher introduces another representation for vector subtraction in which the additive inverse of B is no longer used.

4.2 Graphical subtraction of two vectors

Teacher: Now, I'm going to make the next [*different*] representation; let's see if it's useful for you. (…) What would this subtraction mean? [*(3,1) – (2,2)*] (…) [*it means:*] What do we have to add to (2,2) to get (3,1)? (…) I'm going to put any two vectors [*writes the vectors on the board; see figure 3a*]. (…) I put them starting from the origin [*as the student had done before*]. What does A – B mean? (…) how much is B missing to become A. However, what B is missing to become A is just the vector that starts at the end of B and reaches A [*draws the vector; see figure 3b*]. Because if I add this [*points at A – B*] to this [*points at B*], I get this [*points at A*]. Then, this is A minus B [*see figure 3c*]. Because it is the vector that added to B results in A. (…)

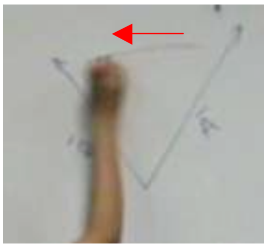

Figure 3. From left to right: representation of vectors A and B that will be subtracted (3a), representation of the subtraction A – B (3b), graphical representation of A – B (3c), and response by E1 to A – B (3d).

Unlike in the previous representation for vector subtraction, the teacher now (graphically) defines the vector subtraction as a process in which not only does he place both vectors starting from the same origin (see figure 3a) but also defines the result as: "the vector that starts at the end of B and reaches A." This work is far to be clear to the student. After the matter is discussed with the class and there seems to be an understanding, the teacher asks student S1 again about the result of A – B.

Teacher: Let's see, this is A and this, B [*he writes the same vectors in figure 3a on the board*], who is A minus B? Including line and direction.

S1: Okay, A minus B goes this way [*see figure 3d*].

The response by S1 shows how difficult it is for her to be aware of the process of subtracting two vectors (in two dimensions). It must be said that S1's response was considered since the teacher addressed mainly to her. However, the rest of the students were present to take part in the discussion regarding the meanings of the task.

5. Conclusions

The aim of this article was to analyze the way in which meanings regarding the subtraction of two vectors are produced in a task involving a teacher and students. Then, we observe from the results how meanings are produced in an original teaching-learning situation. In this situation, the activity prompts a dialog (including language, gestures, and signs) between the teacher and the students, so that the mathematical objects acquire definite determinations. On the other hand, we observed the complexity that awareness of mathematical objects poises and that the interaction between students and teacher is fundamental to that awareness. However, it must be stressed that although the task allows mathematical objects to acquire concrete determinations, it is hard for the students to be aware of such objects. One of the difficulties we observed was the teacher's change of register. This is an indication of the care a teacher should have when working with a concept in different representation registers.

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MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

Considering potential impacts of a high-stakes test on pre-service teacher mathematical knowledge and beliefs

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Abstract. Communities expect their educators to be literate and numerate and governments and regulatory bodies take actions to provide assurances that this is the case. Australia, like other countries, has developed a test to measure pre-service teacher literacy and numeracy. Students completing an initial teacher qualification are expected to achieve the required standard before they can graduate and, in some states, register with their teacher registration body. However, the potential impacts of the test on pre-service teacher mathematical knowledge and beliefs have not been thoroughly explored. These potential impacts need to be considered as they can influence how the pre-service teacher engages with mathematics. This paper aims to start that conversation.

Résumé. Les communautés s'attendent à ce que leurs éducateurs soient *'literate and numerate'*, et les gouvernements et les organismes de réglementation prennent des mesures pour donner l'assurance que tel est le cas. L'Australie, comme d'autres pays, a développé un test pour mesurer *'literacy and numeracy'* des enseignants pré-service. Les étudiants qui terminent une qualification initiale des enseignants devraient atteindre la norme requise avant de pouvoir s'inscrire et, dans certains états, s'inscrire auprès de leur organisme d'inscription aux enseignants. Cependant, les impacts potentiels du test sur les connaissances et les croyances mathématiques des enseignants avant service n'ont pas été complètement explorés. Ces impacts potentiels doivent être considérés car ils peuvent influencer la façon dont l'enseignant pré-service s'engage dans les mathématiques. Cet article a pour but de commencer cette conversation.

1. Introduction

Since 2016, the Australian Government Department of Education and Training (DET) has provided the Literacy and Numeracy Test for Initial Teacher Education (LANTITE) to assess the skills of pre-service teachers. The LANTITE is acknowledged as focusing on personal numeracy and does not access subject-specific knowledge needed for teaching mathematics (DET, 2017). The numeracy component includes "identifying mathematical information and meaning in activities and texts … using and applying mathematical knowledge and problem solving processes … (and) … interpreting, evaluating and communicating, and representing mathematics" (DET, 2017, p. 3).

Meeting the test standard is a requirement to graduate from teacher education courses (DET, 2017) and, as a result, the test is high-stakes for pre-service teachers. High stakes tests on mathematical skills and knowledge can impact on how the pre-service teacher engages with mathematics (Meaney & Lange, 2010). This can occur through the pre-service teachers' perceptions of mathematics (Ernest, 1989; Grigutsch, Raatz, & Törner, 1998; Meaney & Lange, 2010), the pre-service teachers' mathematical knowledge (Ball, 1990), the pre-service teachers' self-efficacy and their self concept (Palmer, 2009; Parker, Marsh, Ciarrochi, Marshall, & Abduljabbar, 2014), the feelings they have towards mathematics (Bates, Latham, & Kim, 2013; Chinn, 2012), their mathematical empowerment or disempowerment (Ernest, 2002), and their beliefs about how mathematics is learned (Boaler, 2015). A single test also provides "only one aspect of the story" (Walshaw, 2011, p. 93).

2. Perceptions of mathematics

Mathematics can be perceived in many ways and these often relate to ideas around what is involved in mathematics and how it is used. Ernest (1989) considered mathematics in terms of three philosophical approaches – instrumentalist, Platonist, and problem-solving. Instrumentalist was described as viewing mathematics as rules and facts that were unrelated but utilitarian; Platonist was described as "a static but unified body of certain knowledge" (p. 100); and problem-solving was considered as a malleable and dynamic field created by people, influenced by culture, and always expanding. Grigutsch, Raatz, and Törner (1998) described a two-layered frame, where four aspects – schema, formalism, process, and application – fed into either a static view of mathematics (the two aspects of schema and formalism) or a dynamic view of mathematics (the aspect of process). The dynamic view of mathematics was proposed to be most likely to lead to application of mathematics.

Meaney and Lange (2010) found that pre-service teachers taking a mathematics test may consider it as an issue of "performance rather than competence" (p. 406) and emphasise procedural knowledge over conceptual knowledge. This could result in pre-service teachers moving towards an instrumentalist (Ernest, 1989) or static (Grigutsch et al., 1998) view of mathematics. These views of mathematics may lead to a focus on the importance of correct performances of the set rules (Ernest, 1989) or a focus on getting a correct result (Benz, 2012). Viewing 'the rules' as external and created by others who 'know' mathematics (Ernest, 1989) may also lead to a belief that there are 'maths-able' people and 'maths-un-able' people, which would effect what mathematical knowledge is seen to be, the actions the teacher should undertake, and how mathematics is learned (Ernest, 1989). Likewise, a focus on the importance of a correct result ignores the learning opportunities provided from mistakes (Boaler, 2014).

3. Mathematical knowledge

Several potential consequences related to mathematical knowledge may eventuate from pre-service teachers completing a high-stakes test addressing mathematical skills and knowledge. As stated above, if a pre-service teacher focuses on procedures in an attempt to meet the standard, this could lead to their focusing on procedures and rules as the required mathematical knowledge and use an approach that conveys these to their future students (Ernest, 1989). A narrow focus such as this would discount the understandings about mathematics and mathematical teaching such as outlined by Selling, Garcia and Ball (2016) in their mathematical work of teaching framework.

Lange and Meaney (2011) proposed that a test of specific, non-subject-matter mathematics may also create un-unhelpful separation of mathematical knowledge and mathematical pedagogical knowledge. In 1987, Shulman discussed the knowledge base required for teaching and the pedagogical processes that use that knowledge base. Content knowledge was listed as an aspect of the knowledge base and comprehension of that knowledge was listed as as aspect of the pedagogical process, with the latter requiring teachers "to understand what they teach and, when possible, to understand it in several ways. They should understand how a given idea relates to other ideas within the same subject area and to ideas in other subjects as well" (p. 14). A focus on procedures would run contrary to these ideas.

Alternatively, those who met the required standard in the high-stakes test by having focused on procedures may have a procedural view of mathematics reinforced by their success. They may continue with their focus on procedures and rules during their course, to the potential detriment of their engagement with mathematics in their course and the mathematical learnings provided to the children they will teach. The teacher educators in their course could struggle to change this focus towards "the need for conceptual understanding … gained through active engagement" (Lange & Meaney, 2011, p. 444). This is supported by Chinnappan and Forrester's (2014) study of the fraction knowledge of pre-service teachers. Their findings indicated procedural knowledge could impede the development of knowledge "necessary for quality mathematics teaching" (p. 894).

4. Self-beliefs and feelings related to mathematics

Completing a high-stakes test of mathematical skills and knowledge may impact on self-beliefs. Two constructs within self-beliefs that have been linked to mathematics knowledge are self-efficacy and self-concept. Parker, Marsh, Ciarrochi, Marshall, and Abduljabbar (2014) viewed self-efficacy as a description of capabilities (I can/cannot do this) and self-concept as an evaluation against external standards (I am/am not good at this), with self-efficacy guided by previous experiences and self-concept guided by comparisons with their other skills and the skills of their peers. They proposed that both were related to mathematical achievement, including high-stakes tests. Their research confirmed the relationships between self-efficacy and self-concept with current and future achievement. Parker et al. proposed that this demonstrated the importance of providing opportunities "develop appropriately positive assessments of their competence" (p. 43). Even though a test of mathematics is assessing a subset of mathematical skills and knowledge, it could have longer term impact on the pre-service teacher self-concept and self-efficacy.

A high-stakes mathematics assessment may provide either a positive or negative impact on the pre-service teacher, depending on how they position themselves and mathematics. Appelbaum (2008) proposed that mathematics could be viewed as an object. The 'object' of mathematics would include all interactions with mathematics, as well as how others are viewed to interact with mathematics, and smaller components that make up mathematics (Appelbaum, 2009). He stated that these connections to mathematics as an object may be positive or negative. The individual pre-service teacher's positive or negative perception of mathematics may contribute to the impact of a high-stakes mathematics assessment, and then flow through to creating further positive or negative conceptions, perceptions, and connections. Therefore, identifying *how* the pre-service teacher positions mathematics and their relationships to it would be highly beneficial in terms of the impact of a high-stakes assessment.

Pre-service teachers may express a dislike of mathematics (Bates, Latham, & Kim, 2013) or an inability to do mathematics and see themselves as not being a maths-person (Palmer 2009). These may become self-beliefs (Kimball & Smith, 2013) and lead to mathematics anxiety (Palmer, 2009) or withdrawing from and not attempting mathematical tasks that they believe they will fail (Chinn, 2012). A high-stakes test that assesses mathematical skills and knowledge may exacerbate these feelings towards mathematics and lead to the supposition that "only some people can be 'math people'" (Boaler, 2013, para. 5). This is in contrast to Boaler's (2016) claim that mathematics teachers should enact the belief that all students can do mathematics.

5. Beliefs about how mathematics is learned

Procedural views of mathematics and anxiety when engaging in mathematics may lead pre-service teachers to specific beliefs about how children learn mathematics. These beliefs may result in a focus on procedures and on mathematics only occurring in the mathematics classroom. Focusing on procedures can lead to an instructor approach (Ernest, 1989), the use of memorisation and tests (Boaler, 2015), and ignorance of the learning opportunities mistakes present (Boaler, 2014). A focus on mathematics as occurring only in the mathematics classroom may empower children within a narrow mathematical domain (Ernest, 2002). This overlooks social empowerment, gained from extending mathematics beyond the classroom to use it in everyday life (Ernest, 2002), and epistemological empowerment, where individuals have "a personal sense of power over the creation and validation of (mathematical) knowledge" (Ernest, 2002, p. 1).

6. Discussion and Conclusion

It may be that the phrase attributed to the Hippocratic oath, first do no harm (Sokol, 2013), should be the focus. A high-stakes assessment may not be "an equitable and quality experience in mathematics" (Walshaw, 2010, p. 98). Finding out how the individual engages with mathematics and sees themselves in terms of mathematics needs to be considered as this will enable the impact of a high-stakes assessment to be monitored and addressed (Appelbaum, 2008; Ernest, 1989; Grigutsch et al., 1998). The points raised in this paper will hopefully start a conversation that encourages consideration of unintentional consequences of actions undertaken, consequences that may impact on the pre-service teacher completing the high-stakes test and on their future students through its potential impact on the pre-service teacher's sense of self (Walshaw, 2010) in terms of mathematics, their "mathematical identities" (Walshaw, 2010, p. 96). Teaching mathematics requires more than a knowledge of mathematics and a focus on mathematics skills and knowledge could be harmful. As Grootenboer and Marshman (2016) state, not considering attitudes, beliefs, and feelings could result in mathematics education that is "irrational, unsustainable and unjust" (p. 124). Mathematics education should make learners want to be involved and engaged with the activities - this will enable mathematics to become available to all, rather than the few (Boaler 2016).

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Embodied, arts-infused, historico-cultural mathematics (out-of-doors) as a counter-narrative to hegemonic scientism and mathematisation

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Abstract. In this paper, the author explores aspects of the nature of hegemonic narratives of scientism and mathematisation, and offer a lived counter-narrative through an alternative pedagogy of mathematics. This pedagogy is one of embodiment of mathematical patterns and relationships, experienced viscerally through whole body movement and voice, using multiple senses and natural materials. It draws on historical and pluricultural traditions of mathematics, on artistic modalities, and aims to engage emotions including pleasure and delight. It is situated in the out-of-doors – in learning gardens (or parks, woods, fields, beaches) – where the living world is palpably present, and where human mechanisms of control are less prominent than in typical school buildings and classrooms. The paper begins by characterizing aspects of hegemonic scientism/ mathematisation and ways that they play out in unquestioned routines and rituals of stereotypical mathematics classrooms, particularly at the secondary level. A brief account follows of counter-hegemonic pedagogies as the author has experimented with them in a Canadian university teacher education program, in collaboration with students, mathematical artists, environmental educators and fellow researchers. Finally, suggestions are given for some further directions for intended research, with an invitation for others to collaborate or take up related research strands.

Résumé. Dans cet article, l'auteur explore les aspects de la nature des récits hégémoniques du scientisme et de la mathématisation, et offre un contre-récit à travers d'une pédagogie alternative des mathématiques. Cette pédagogie est une qui utilise la corporalisation des relations mathématiques, via des expériences viscérales par le mouvement du corps et par la vocalisation, en utilisant de multiples sens et des matériaux naturels. Il s'appuie sur les traditions historiques et pluriculturelles des mathématiques, sur les modalités artistiques et vise à susciter des émotions, y compris le plaisir. Il est situé à l'extérieur - dans des jardins (ou des parcs, des bois, des champs, des plages) - où le monde vivant est manifestement présent et où les mécanismes de contrôle humains sont moins importants que dans les écoles et les salles de classe. L'article commence par caractériser les aspects du scientisme / mathématisation hégémonique et le rôle qu'ils jouent dans des routines et des rituels de plusiers classes de mathématiques, en particulier au niveau secondaire. On décrit les pédagogies contre-hégémoniques avec lesquel l'auteur a expérimenté dans un programme de formation des enseignants universitaires canadiens, en collaboration avec des étudiants, des artistes mathématiques, des éducateurs en environnement et d'autres chercheurs. Enfin, des suggestions sont données pour de nouvelles orientations pour la recherche envisagée, avec une invitation pour que d'autres collaborent ou adoptent des domaines de recherche connexes.

When we, as critical mathematics educators, confront the excessive mathematisation of society and education, our arguments are often congruent with critiques of the scientization of contemporary societies and educational systems. In some ways, mathematisation is a pervasive aspect of scientization that uses numbers as a blunt instrument to quantify and calculate that which cannot meaningfully be counted. By a reductionist strategy of an unquestioning quantification of everything, fundamental epistemological questions may be stifled, and alternative ways of knowing quashed.

Heesoon Bai writes (in her foreword to Hyslop-Margison & Naseem 2007) that scientism comprises "an ideology that believes that science (and its rationalist foundation in modern epistemology) has an undeniable primacy over all other ways of seeing and understanding life and the world, including more humanistic, mythical, spiritual, and artistic interpretations." (p. vii). While Bai has no complaint against science per se, and is grateful for the valuable forms of knowledge science makes possible, it is the *hegemony* of positivistic science and the extension of its ways of knowing to all areas of life and society that she critiques.

We might take up our critique of mathematisation and education in a similar mode: that we are not quarreling with mathematical ways of knowing in themselves, and that we appreciate the valuable and often beautiful insights that these offer, but we have problems with the extension of mathematical models to every aspect of life and society, including those aspects that would be better or differently served by alternative epistemological approaches. It is the hegemony of quantification and its erasure of heterogeneous, potentially fruitful non-quantifiable ways of knowing and understanding that are objectionable, even dangerous.

There is a current pop culture mantra about goal-setting (in personal life, in business, etc.) that claims, for any desired result, that "it has to be measurable". That is to say, only quantifiable things count, and only what has been mathematised can be tracked, ranked, even believed. I have seen this mantra of the sacredness of 'measurability' taken up by all kinds of organizations, including NGOs, environmental and social activist groups, and at all levels of education, in many situations where quantification is entirely inappropriate. Through this neoliberal nostrum of requiring 'measurability' for everything, the hegemony of mathematisation and scientization has come to affect nearly every realm of contemporary society. It has the pernicious effect of distorting values and a sense of community, promoting spurious or dangerous rankings, and suppressing Indigenous and traditional knowledges and wisdoms, and artistic, poetic, holistic and spiritual approaches.

Hegemonic scientism and mathematisation carry with them implicit foundational assumptions for pedagogy, based on Enlightenment/ Modernist values. These values are based in Platonic and Cartesian axiom of mind-body separation, and the identification of scientific mind with a 'clean', decontextualized, sterile white laboratory, and of mathematical mind with universal, context-free esoteric knowledge, and other-worldly 'pure' mental cognition. These values, so familiar as to be invisible within science and mathematics education, are based in fear and loathing of their 'opposites': contextualized place-based knowledge, living (and dying) things, 'dirt', bodies, emotions, the senses. Without stretching the comparison very far, these reviled qualities can be seen to be associated with the Other as foil to the clean, white, male idealized scientist – so that hegemonic scientism implicitly reviles women, people of colour, Indigenous and colonized subjects, local and traditional knowledges, those living close to the land and valuing particular places that have meaning for them, and so on.

Environmental educators in particular have drawn attention to the colonizing, racist, sexist, anti-ecological pedagogies founded upon these assumptions, and the problematic they pose for a more holistic, anti-oppressive education for sustainability (see, for a few examples, Williams 2008; Hauk 2011; Bonnett 2013). They make the case for education outside the walls of the traditional classroom and in more natural settings; for the use of other 'geometries of liberation' (from Hauk) that allow ways of thinking beyond the Cartesian grid; for inclusion of the whole person (body, mind, emotions, senses, spirituality) in learning; for appreciation of patterns in the complex, living, growing world; and for a sense of continuity with our ancestors, our cultures, and the living world.

If we take these ideas seriously, there is an urgency in revisiting many of the normally-unquestioned assumptions and practices around so-called traditional mathematics education pedagogy. Why do we so often hold classes in rather barren, sterile classrooms, with learners sitting static and silent in rows of desks facing the teacher/lecturer? Why are emotions, sensory ways of learning, aesthetic pleasures, bodily movement and social interactions banned from so many mathematics classrooms, and increasingly so as learners become more grown up and mathematically sophisticated? Why are 'wrong answers' feared? Why does so much of assessment ride on individual, timed written tests? So much of what we have learned to take for granted is a reflection of Platonic/Cartesian values systems, and a scientistic domination of education by quantification and ranking of students' academic grades.

For the past seven years, a group of us at the University of British Columbia in Vancouver, Canada, has been experimenting with more integrative ways of teaching and learning mathematics and other school subjects, in the setting of the Orchard Garden, a student-led teaching and learning garden on campus. Our group has endeavoured to work in truly collaborative ways across disciplines and generations, to develop ways to teach and learn mathematics in a school garden (or other outdoor places), with the garden as co-teacher, and in dialogue with the history of mathematics and mathematical arts.

Our experiential mathematics pedagogies include: measuring sky and earth with our bodies, to understand seasonal cycles and growing things through angle, distance and estimation; using six-month pinhole cameras to chart the path of the sun through the sky from summer to winter solstice; building mathematically-interesting artistic objects (e.g. Platonic solids invariant-volume windsocks) and structures (e.g. hyperboloid arbourways) at both small and large scale, in collaboration with mathematical artists-in-residence, to understand these forms in fully-embodied ways; drawing-to-learn to inquire into the nature of angle and line in living and human-made forms; growing and foraging natural fibre materials and exploring geometries through making small and large-scale twine, spun fibres, braids and weavings; and developing narrative, poetry and mini-operas through close collaborative observation of geometries and ecologies in the garden. I will offer examples (and potentially a later hands-on workshop) to introduce these pedagogical experiments and the rationale for using them to establish a counter-narrative to oppressive scientism and mathematisation.

These pedagogical initiatives are only the early starting points for ideas and praxis of embodied, arts-infused, historico-cultural mathematics in the out-of-doors. An important part of the ongoing work in this area will be the discussion and possibly future collaborations and new directions that may come from these beginnings.

Figure 1. Students and conference participants building hyperboloid arbourway with mathematical artist-in-residence George Hart, UBC Orchard Garden. (Photo by author)

Figure 2. Student six-month pinhole camera images, made with guidance from mathematical artist-in-residence Nick Sayers. (Photo by author).

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How to deal with the modelling of epidemics? Some ideas and examples to be implemented in the classroom!

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Abstract. The general objective of this work is to deal with different approaches for the representation of an epidemic considering the state of the individuals in the population, Susceptible, Infected or Recovered (SIR), which generate models that students can explore with the computer using the contents acquired in mathematics subjects. Specifically, the purposes of the tasks designed focus on the identification and analyses of the variables and parameters involved in an epidemic through three modelling methodologies: i) Simple systems of ordinary differential equations reflecting the SIR representation; ii) Systems of difference equations and matrix representations by means of discretization of the time variable in the SIR formulation; and iii) Computational simulations from the use of the agent-based model "Virus" already implemented in the NetLogo platform. Exploration and comparison of the dynamics of Ebola and AIDS epidemics show the students the potential of modelling in this context and allow them to link with real applications. The assessment given by students of this activity was positive showing their interest in the topic.

Résumé. L'objectif général de ce travail est de faire face à différentes approches pour la représentation d'une épidémie compte tenu de l'état des individus dans la population, Sensible, Infecté ou Récupéré (SIR), qui génèrent des modèles que les élèves peuvent explorer avec l'ordinateur en utilisant les contenus Acquis en matières mathématiques. Plus précisément, les objectifs des tâches ont été axés sur l'identification et l'analyse des variables et paramètres impliqués dans une épidémie à travers trois méthodologies de modélisation: i) Systèmes simples d'équations différentielles ordinaires reflétant la représentation SIR; Ii) Systèmes d'équations de différence et représentations de matrice au moyen de la discrétisation de la variable de temps dans la formulation de SIR; Et iii) Des simulations informatiques issues de l'utilisation du modèle "Virus" basé sur l'agent déjà implanté dans la plate-forme NetLogo. L'exploration et la comparaison de la dynamique des éboles d'Ebola et du SIDA montrent aux étudiants le potentiel de la modélisation dans ce contexte et leur permettent de se lier à des applications réelles. L'évaluation donnée par les étudiants de cette activité a été positive montrant leur intérêt pour le sujet.

1. Introduction

In biology in general, and in epidemiology in particular, the real systems are extremely complex, so the models for studying them must inevitably include simplified idealizations. Thus, it is indispensable to know how these models work, the assumptions they make, the possibilities they offer and their limitations, in order to be critical and rigorous with their applications.

The first application of mathematics to epidemiology can be found in 1760, when D. Bernoulli published a treatise on smallpox, an infectious disease caused by the variola virus. In 1926, A.G. McKendrick, who studied medicine at the University of Glasgow but also studied mathematics, published an article on the "Applications of mathematics to medical problems". He introduced a continuous-time mathematical model for epidemics that took into account infection and recovery aspects. W.O. Kermack began to collaborate with McKendrick on the mathematical modelling of epidemics and they published together a series of "Contributions to the mathematical theory of epidemics" (Bacaër, 2011). It was considered that a population of size N (large enough) was made up of diverse groups of persons represented by three variables: $S(t)$, the portion of the population that was susceptible to infection; $I(t)$, the portion of the same population that was currently infected; and $R(t)$, the remaining portion of the population that had recovered from infection (although the R sometimes could mean removed, not recovered, and if the disease was fatal then this third state meant death). Hence, a mechanistic and deterministic representation of a dynamic of an epidemic, assuming homogeneous mixing of the contacts (interactions between individuals are instantaneous) and conservation of the total population, was produced, and a SIR model was generated, which is still the building block for most of the more complex models used nowadays in epidemiology. In the review of mathematical modelling of infectious disease dynamics of Siettos & Russo (2013) the reader can find an interesting presentation of diverse methodologies with references to successful real applications.

In recent decades emerging and re-emerging epidemics such as AIDS cause death to millions of people each year. Modelling is one of the tools used by international health institutions to tackle those epidemics, playing an important part in efforts that focus on predicting, assessing, and controlling outbreaks. In the summer of 2014, Ebola was spreading in Africa. The Centre for Disease Control constructed a modelling tool called *EbolaResponse* to provide estimates of the potential number of future cases, tracking patients through the following states: susceptible to disease, infected, incubating, infectious, and recovered (https://www.cdc.gov/vhf/ebola/index.html), releasing one type of SIR type model to the public.

Agent-based models (ABMs, or also called individual-based models) are computational and stochastic models to simulate the actions and interactions of autonomous agents (individuals) in order to evaluate the effects on the system as a whole (Railsback & Grimm, 2012). In an epidemiological context, ABMs are also being effectively used (Siettos & Russo, 2013).

Continuing in the line of combining different modelling approaches in teaching and learning in a life science context (Ginovart, 2014), a set of modelling activities to work with students in the classroom has been designed in order to deal with epidemics. The general objective of this work is to deal with diverse approaches for the representation of a SIR epidemic, generating diverse models that undergraduate students would be able to explore in the computer lab using the contents acquired in mathematics subjects. Specifically, the purposes of the tasks designed focused on the identification and analyses of the different variables and parameters involved in an epidemic, by means of three modelling methodologies: i) Simple systems of ordinary differential equations based on the SIR model, some of which will be solved by hand and others with the help of mathematical software; ii) The discretization of the time variable in the formulation of the SIR model that will generate a system of difference equations, with their corresponding matrix formalization, obtaining an approximation to the problem from fairly simple calculations that can be assisted by appropriate software; and iii) The use of an ABM to generate a set of individual-based simulations of the behaviour of a population developing in a spatial domain with infected people.

The outcomes of these models were analysed and discussed by the students, comparing their advantages and disadvantages for the representation of real systems.

2. Material and methods

The participants in this study were a group of forty third-year students of a Bachelor's degree in the field of Biosystems Engineering at the Universitat Politècnica de Catalunya (Barcelona, Spain). The prior coursework for these students was related to the following compulsory subjects: Mathematics I and II, Physics I and II, Chemistry I and II, General Biology, Microbiology, and Statistics, among others. This previous preparation guarantees a good knowledge of some biological concepts and basic mathematical and statistical concepts, as well as some control of computer tools for calculation and resolution. The use of Maple in the previous mathematics subjects provides the students with sufficient skills for the resolution of the ordinary differential equations of a SIR model and the calculations involved in the matrix representation of this model.

NetLogo is a free access multi-agent programmable modelling environment, which has a library with simulators ready to be used, with an extensive documentation about their main features and how to use them. Among them there is the ABM called "Virus" (Wilensky, 1998), the simulator chosen and employed for this activity. "Virus" simulates the transmission and perpetuation of a virus in a human population (https://ccl.northwestern.edu/netlogo/models/Virus). Figure 1 shows a screenshot of the friendly interface of this simulator with the input parameters and outcomes generated.

Students' responses regarding analyses and modelling of the dynamics epidemics with the distinct methodologies were collected via outputs of the mathematical software Maple, screenshots of NetLogo, and questionnaires, as well as face-to face dialogues during the sessions in the computer lab. The students' perceptions of the set of tasks conducted were explicitly questioned and collected at the end.

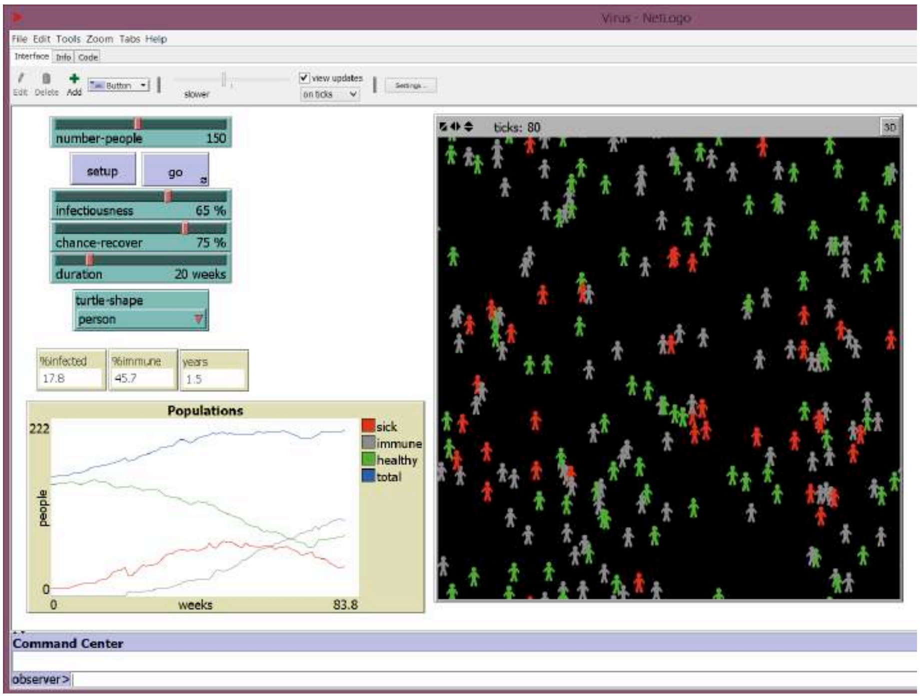

Figure 1. Screenshot of the interface of the "Virus" simulator in NetLogo platform.

3. Results and discussion

The results accomplished with these three distinct modelling approaches and the possibilities offered for each type of model to characterize the various dynamics of the population were assessed by the students, firstly in a mathematical context solving equations with the help of Maple software (Figure 2), and secondly, in a computational framework carrying out individual-based simulations with "Virus" to inspect the behaviour of a population developing in a spatial domain with healthy people (Susceptible), sick people (Infected), and immune people (Recovered) as Figure 1 shows. Students were trained to see how different values for the parameters of the model might approximate the dynamics of real-life viruses.

The use of the ABM "Virus", already implemented in the library of NetLogo, facilitated the development of this activity because the level of skills in programming was not an obstacle. In addition, the documentation provided by this model (Wilensky, 1998) introduced the students to the relevant effects that the parameter values implicated have on the evolution of epidemics, on the dynamics of the S, R and I subgroups of people. The noteworthy parameters to distinguish between different kinds of the epidemics tested and managed by the students were: "Infectiousness" that determines how great the chance is that virus transmission will occur when an infected person and susceptible person meet, "Duration" that determines the time before an infected person either dies or recovers, and "Chance-recover" that controls the likelihood that an infection will end in recovery/immunity (or death if it is zero). For instance, the famous Ebola virus has a very short duration, a very high infectiousness value, and an extremely low recovery rate, but the HIV virus, which causes AIDS, has an extremely long duration, an extremely low recovery rate, but an extremely low infectiousness value. Taking into account these parameters, the students were able to take the role of a public health agent to propose and test, with the help of this simulator, actions to combat a virus with the characteristics described for the Ebola virus or for HIV virus.

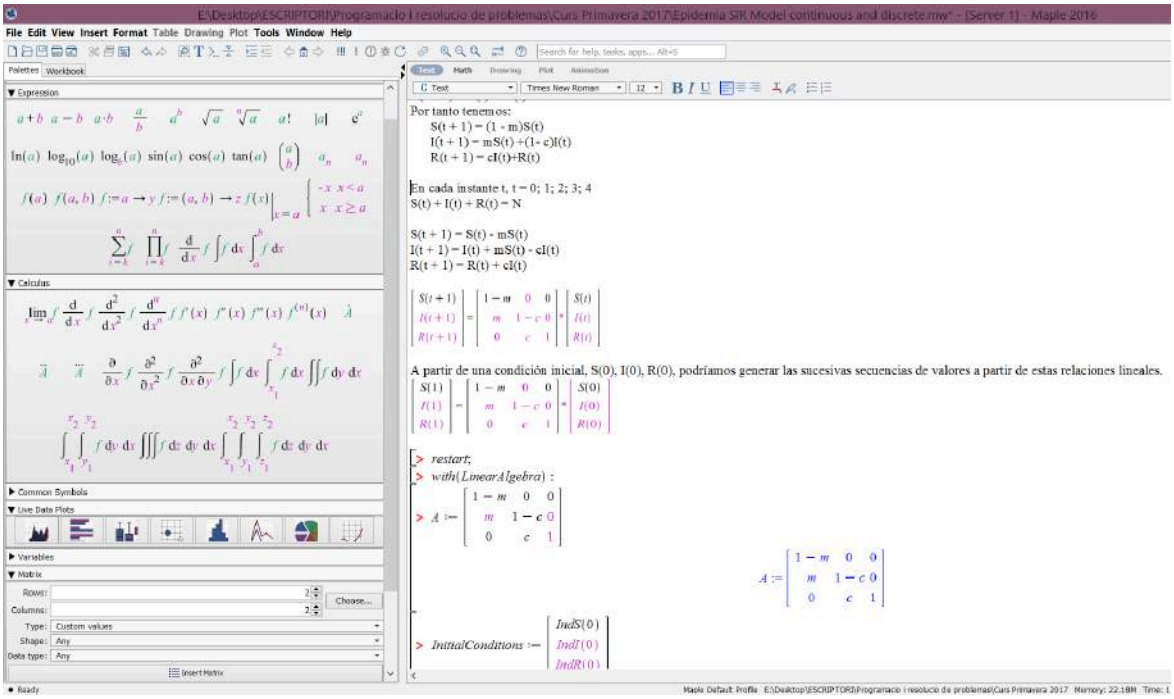

Figure 2. Screenshot of the calculations performed with Maple to resolve a SIR model.

The assessment given by students of this activity was positive and showed their interest and enthusiasm in a topic relevant to their biological studies. These three perspectives of modelling epidemics enriched the process of connecting mathematics with the investigation of life and real systems in our society, and contributed to building up reflexive knowledge on this issue in the classroom.

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Optimisation as a didactic principle

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Optimisation problems are classic problems in mathematics and the real world. Since the 1980s the landscape of solving optimisation problems has fundamentally changed in the era of high dimensional computing capacities as can be used today. Numerical approaches cap analytical ones since then. This shift recasts currently processes in industry as well as modelling of nature, climate change and so forth. In order to allow students to understand how mathematics and specifically optimisation is used and needed today to solve complex application problems, such as landing a spaceship on the moon, controlling robots to place objects precisely or to run a smart farm, mathematicians and mathematics educators need to work together. Inviting mathematics classes from schools to the university to learn about this, is one way of making this knowledge and these new approaches accessible to students and teachers. Principles of this approach and how these can be made accessible to students are discussed in this paper.

Mathematical Modelling (e.g., Blum, Galbraith, Henn, & Niss 2007, Stillman, Blum, & Salett Biembengut 2015) has been discussed in mathematics education as an important component of mathematisation for a long time. Realistic Mathematics Education (e.g., de Lange 1996, Treffers 1987) has conceptualised and examined mathematisation as a didactic principle for nearly half a century now, based on fundamental thoughts of Freudenthal (Freudenthal 1973). Introducing students to 'mathematizing unmathematical matters' (ibid., p. 133) was and is a key concern of this approach. Meanwhile the 'mathematisation-of-the-world', e.g. in modern Information Technology and other high-end technologies, has extended modeling and included simulation in engeneering and industry. High performance computing made this possible, but the widespread trial and error approaches that followed had high costs as an implication. Limiting financial resources in the industrial and economic world led to yet another turn and gave in recent years mathematicians back a stronger voice and more prominent roles in industry. Optimisation – as a mathematizing principle – became an indispensable component, being more rapid, fruitful and efficient in solving problems than mere simulation.

The threefold approach of Modeling-Simulation-Optimisation (MSO) (see Wets 1976), as used meanwhile in mathematics applications in Engineering, Information Technology as well as Natural, Economic and Social Sciences, is to our knowledge not widely discussed in mathematics education so far, neither has this approach been actively made visible to students in schools and their teachers. In contrast, traditional views of mathematics as an abstract discipline seem still to be prominent in schooling. New groundbreaking mathematical methods and ideas have hence not been popularized so far. This seems odd as challenging problems as how to use our limited natural ressource on Earth sensibly or building smart farms seem to be important global issues. Why not also approach these challenges together with students in mathematical ways? Finding more sophisticated solutions of a wide range of discrete, continuous or stochastic problems are of a global and societal interest. Even though progressive developments in mathematics require more complex cycles than even the MSO approach offers, making essential elements of such a cycle accessible to students is possible. We will indicate how on the following pages, based on our research and experience in conceptualising optimisation, linked to application methods and widespread cooperation with industry in the last decade, as well as our recent outreach initiatives to schools, offering math fairs to students and teachers at the university.

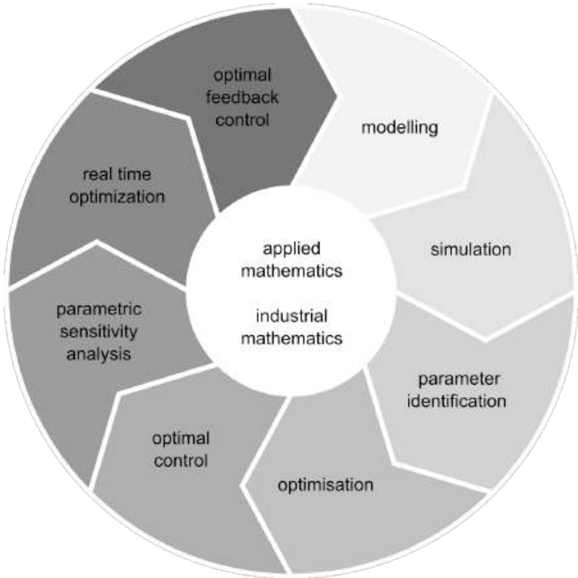

Figure 1. Extended MSO Cycle

As the cycle indicates mathematisation of 'unmathematical matters' is complex, modeling and simulation are only two parts of the whole process. Allowing students to focus on other key elements in this process we chose four components for our recently organised math fair at the University of Bremen: 1. Parameter identification, 2. Nonlinear optimisation, 3. Optimal control, 4. Optimal feedback control.

1. Parameter identification is the focus of the first station. The relevance and meaning of parameters is introduced in the context of the longterm human interest in astronomy. It is then applied to a LEGO Mindstorms vehicle, whose hardware and software parameters are set by the students so that it follows a given path. Students investigate and experience how parameters like the distance of the wheels, the speed of the car etc. determine if the vehicle can follow the path or not. This allows students to practically understand the relevance of parameters and their significance in optimisation problems. 2. In the context of a skiing problem – how to find the lowest point in a valley while avoiding trees using only local information – the theme of nonlinear optimisation is introduced. Mathematically the given problem is an optimisation problem with constraints. The mathematical conceptualisation of the skiing problem is essential at this point and introduces students to fundamental ideas of numerical solutions. The software WORHP Lab, developed by the working group Optimisation and Optimal Control at the University of Bremen, then allows the students to model, visualize and solve the given constraint problem. 3. Optimal control is experienced and thought through at the third station, where a parking manoeuver of an autonomous car is discussed. The students then use WORHP Lab to calculate the optimal trajectory for an industrial robot and experience how balancing a table tennis ball is impossible manually while perfectly easy using WORHP Lab. Sending the results to the real robot the students understand how mathematisation results in time dependent optimisation. Last but not least students get introduced to problems of feedback control at station 4. Given a dog's problem of traversing a river with a current in a most direct way, students get introduced to central ideas of feedback control. This allows students to successfully maneuver and land on the moon in a flight simulator. Besides playing, conceptualising and mathematising the situation supports students to understand how and why feedback control is a key element of optimisation.

Having students work in teams, providing hands on and theoretical activities at the same time, as well as letting them first only experience one station, but demanding a presentation of their station and the mathematisation in this context, allows students and teachers to get a sound first sense of optimisation. Both, teachers and us as a university team, are impressed how optimisation as a didactic principle can be successfully performed in this way. Further research and reflection is necessary to better understand this didactical optimisation experiment. We are looking forward to this and are planning next steps, together as mathematicians and mathematics educators, in sync with schools and collaboratively with international colleagues.

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Questioning the use of secondary school mathematics

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Abstract. This contribution presents a literature survey to defend the provocative claim that, regarding the everyday applicability of the contents taught, secondary school mathematics is widely useless. Although mathematical qualification in lower secondary school is officially considered a prerequisite for the full participation of learners in later vocational and private occupations, the individual usefulness of secondary school mathematics is questionable. Present examples for the application of these mathematical contents are usually not addressing realistic situations. It is argued that this mismatch cannot be overcome by the design of better application tasks, but that this mismatch is the symptom of a general overestimation of the changing role of mathematics in everyday life. Mathematisation is thus problematized as located in a field of idealistic myths. After a brief discussion of who benefits from this overestimation, ways to overcome the situation are outlined.

Résumé. Cette contribution présente une enquête de littérature pour défendre la revendication provocante que, concernant l'applicabilité quotidienne des contenus enseignés, les mathématiques d'école secondaire sont largement inutiles. Bien que la qualification mathématique dans l'école secondaire inférieure soit officiellement considérée un préalable pour la pleine participation d'apprentis dans les occupations professionnelles et privées dernières, l'utilité individuelle de mathématiques d'école secondaire est discutable. De présents exemples pour l'application de ces contenus mathématiques d'habitude n'adressent pas des situations réalistes. Il est soutenu que cette discordance ne peut pas être surmontée par le design de meilleures tâches d'application, mais que cette discordance est le symptôme d'une surestimation générale du rôle changeant de mathématiques dans la vie quotidienne. Mathematisation est ainsi problematized comme localisé dans un champ de mythes idéalistes. Après une discussion brève de ce qui profite de cette surestimation, les façons de surmonter la situation sont exposées.

1. Content-centred education

Given that mathematics education, especially in secondary schools, is the source of hardship for a wide proportion of school students and serves as a gatekeeper for further opportunities in life, mathematics education requires a solid legitimation. Although mathematics education is increasingly claimed to develop meta-mathematical competences such as modelling, argumentation and problem-solving (e.g., KMK, 2003), or demanded to engage critically with social applications of mathematics (e.g., Skovsmose, 1994), official curricula are still organised around contents. Rather than elaborating on meta-mathematical competences or critical agency and allowing teachers to choose contents which fit to these aims, curricula usually still follow the traditional approach to provide a list of contents that have to be taught, learned and tested (e.g. KMK, 2003). For example, the contents of the German curriculum for the 8th to 10th year of schooling comprise the Thales theorem, quadratic, exponential and trigonometric functions and elementary probability theory. The vast body of research in mathematics education focussing on approaches to teach specific contents more efficiently illustrates the dominance of the belief that the contents are of central relevance to the learning of mathematics; and the claim that these contents would be relevant for the mastery of vocational and private situations is widely used to defend the dominance of content learning both by scholars (criticised by Lundin, 2011, and Pais, 2013) and by students (discussed in Kollosche, 2017).

2. Usefulness in decline

The proclaimed everyday usefulness of the contents of secondary school mathematics is confronted by two objections. Firstly, research in the field of situated cognition provides evidence that, on the one hand, mathematical competences required in recurring out-of-school situations are usually acquired 'in practice', and that, on the other hand, mathematical competences acquired in school are rarely applied in out-of-school situations (Lave, 1988). Consequently, it is doubtful whether secondary school mathematics is necessary or even helpful to cope with out-of-school situations. Secondly, the mathematical demands in workplaces and in critical engagements with socially relevant mathematics tends not to match the contents being taught in lower secondary school. In the German case, Heymann (1996/2010), himself a passionate defender of mathematics education, admitted that most of the contemporary *contents* of secondary school mathematics are useless for the wide majority of learners, and that in later occupations learners will usually use only a small part of the mathematics that they have learnt in school. Borovik (2016) points out that developments in automatization and digitalisation have severely limited the usefulness of secondary school mathematics. One the one hand, technological developments allow to have reoccurring mathematical tasks performed by computers or to altogether substitute them by automatized processes. On the other hand, the mathematics needed to produce and maintain this technology exceeds secondary school mathematics by far. OECD's Programme for the International Assessment of Adult Competencies (PIAAC) analysed the statements of adults concerning their use of mathematics in work (OECD, 2013). Although it stays unclear how language, culture and occupational fields may influence the worker's identification of a certain procedure as 'mathematical', the results indicate, firstly, that workers in technologically leading countries, such as Norway, Germany, the Netherlands and Japan, use less 'numeracy skills' than workers in other countries (p. 144), secondly, that young workers use less 'numeracy skills' than aged workers (p. 153), and thirdly, that the extent to which numeracy skills are used depends much weaker on educational attainment than the use of reading and writing, ICT or problem-solving skills (p. 156). All these findings support the assumption that, in the course of technological progress, knowledge of mathematics becomes less and less important in the workplace. The same argument could be brought forward concerning the use of mathematics in private life where technology is also increasingly used to perform mathematical tasks.

3. The myth of usefulness

Despite its fading usefulness, secondary school mathematics is still presented as relevant for everyday life. The concept of mathematical literacy as utilised in OECD's Programme for International Student Assessment (PISA) is only one representation of this phenomenon. Several studies criticised that, in spite of their alleged realistic nature, the supposed applications of mathematics do not contribute to coping with realistic problems but obscure mathematical contents in a realistic disguise which then has to be unpacked by students (Dowling, 1998; Meyerhöfer, 2005). On a naïve basis, it could be argued that better examples for the practical use of secondary school mathematics have to be developed. Following a deconstructive approach based on Foucault, one might however ask in how far this mismatch between the assumed and described relevance of secondary school mathematics is productive in the socio-politics of mathematics education (cf. Kollosche, 2016). Dowling (1998) argues that applications in school mathematics create the myths that mathematics was omnipotent and everywhere relevant, thus creating the ideological conditions to use mathematics as a social technique of power. Lundin (2012) claims that secondary mathematics is a part of the broader 'standard critique' in mathematics education, which constantly criticises the presumably deficient *status quo* of mathematics education and formulates goals for improvement, thus both legitimising school mathematics as the institution it is idealistically supposed to become, and mathematics education research as the institution which will ensure this evolution. Kollosche (2017) proposes that both teachers and students have an interest in assuming the everyday relevance of mathematics against all contradictory experience, as this assumption provides meaning to their obligatory engagement with mathematics. All in all, the myth that secondary school mathematics was useful for the mastery of everyday work and private life constitutes a win-win situation for everybody who is affected by the institution of school mathematics: Students and teachers experience their daily work as more meaningful, researchers can legitimise the need for their income- and prestige-generating activities in mathematics education, and decision-makers can use mathematics as a presumable flawless argumentative and organisational tool.

4. Hope for secondary mathematics education

Alternative philosophies of mathematics education are possible and have been proposed. Heymann (1996/2010) suggests that after an elementary education in mathematics, which might be completed with covering today's contents of the first seven years of schooling, students should be given the choice to learn more advanced mathematical contents or to use their existing mathematical abilities to develop further meta-mathematical competences and discuss social applications of mathematics critically. Courses in advanced mathematics could be opened for students who want to continue their study of advanced mathematics later and would secure the education of mathematical expertise for higher education and specialised jobs. Fischer (2003) sees the goal of mathematics education for all the other students in enabling them to critically interact with mathematically further educated experts and mathematically organised procedures in society. Vohns (2017) provides a recent example of how Skovsmose's (1994) and Fischer's (2003) programs might materialise in the mathematics classroom when he proposes classwork on the critical examination of the mathematical modelling of measures for poverty which are applied in policy and social science. Taking these approaches seriously would widely agitate the traditional list of contents in secondary mathematics education and might lead to an inclusion of approachable forms of a wider range of socially and technically applied mathematical theories, including non-Euclidian geometries, graph theory, cryptography and inferential statistics. However, before such alternative approaches towards mathematics education can be explored in school, a critical mass of concerned protagonists will have to break through the win-win situation around the myth of the relevance of today's contents of secondary school mathematics.

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The dialectics of mathematization and dematematization

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Abstract. Many scholars have argued that mathematization as a social process can be only investigated if its antagonist, the process of dematematization, is also taken into account (e.g. Keitel, 1989; Gellert & Jablonka, 2006; Skosmose, 2014). Although these authors posit the relation between mathematization and dematematization at the centre of the discussion about the interrelations between mathematics and the social in modern societies, a systematic conceptualization of this relation is still due. In shifting the conceptual focus from the content of the processes of mathematization and dematematization to their form, I will attempt to qualify the relation between mathematization and dematematization as a dialectics. I will conclude by suggesting to confront this formal conception with empirical processes of mathematization in order to unfold its analytical potential.

2. Introduction

Since the 1980ies, all investigations of the multifaceted interrelations between mathematics and society were grounded in two related theses: 1) The number of mathematizations which are 'colonising' the social and material world is steadily growing; and 2) "[W]e should observe these developments critically, as they could do damage at all of us" (Davis & Hersh, 1986, p. 17). In the late 1980ies, Chevallard (1989) gave the first of these two founding theses an unexpected twist:

> "No modern society can live without mathematics. [...] In contradistinction to societies as organized bodies, all but a few of their members can and do live a gentle, contented life *without any mathematics whatsoever.*" (Chevallard, 1989, p. 49)

That is to say, the expanding *mathematization* of the social does not require more mathematical skills of the individuals which participate in mathematized practices, but quite the contrary is the case: The mathematization of the social implies a *dematematization of its members*. While "society as a machinery is more and more mathematised, our daily life is more and more dematematised" (ibid., p. 52). Therefore, many scholars argued that the social process of mathematization can be only adequately understood if the "reverse" (Gellert & Jablonka, 2007, p. 1) process is also considered: the accompanying dematematization of the social (e.g. Keitel, 1989; Keitel, Kotzmann & Skosmose, 1993, Gellert & Jablonka, 2007; Straehler-Pohl, 2017). Although it is commonly agreed that mathematization and dematematization *relate* to each other, there are subtle differences about the *nature* of this relation among the scholars that contributed to this discussion (see *Table 1*).[1] In the next section, I will argue for a characterization of the relation between mathematization and dematematization as a *dialectics* (II.). Subsequently, I will conclude by suggesting to confront this formal conception of a dialectical relationship between mathematization and dematematization with empirical processes of mathematization in order to unfold its analytical potential (III.).

Table 1. Overview of the terms indicating the relation between mathematization and dematematization

Authors	Relation between mathematization and dematematization

[1] Due to the limited space of this paper, I unfortunately am unable to provide a close reconstruction of the different conceptual nuances as they are indicated by the different terms in Table 1. Instead, I will focus on those aspects that I view as central to my endeavour of conceptualizing the relation as a dialectical one.

Chevallard (1989)	„contradistinction" (p. 49), „paradoxical" (p. 52), „dialectic" (p. 52)
Keitel (1989)	„contradiction" (p. 9), „paradoxically" (p. 11), „dilemma" (p. 11)
Keitel, Kotzmann & Skovsmose (1991)	"contradictory" (p. 250), "paradox" (p. 251), "parallel" (p. 251)
Gellert & Jablonka (2007)	"reverse" (p. 1) "dialectically" (p. 1), "paradox" (p. 9)
Skovsmose (2014)	"accompanied" (p. 442)
Straehler-Pohl (2017)	"reinforce" (p. 40), "apparently antagonistic" (p. 41), "dialectical" (p. 41)

2. The Dialectics of Mathematization and Demathematization

Gellert & Jablonka (2007) define mathematization purely formal as "a process in which something is being rendered more mathematical than it has been before" (p. 1). Conceptualized this way, the process of mathematization is universal and specific at the same time: It is *universal* because the quality of the particular 'something' (=*X*) which is to be mathematized is not limited at all – in principle, anything can be handled "by means of mathematical insight and techniques" (Skovsmose, 2014, p. 442); however, it is also *highly specific* because, as soon as we decide to mathematize a certain thing, process, practice, etc., we limit ourselves to *one* particular form of seeing the world and thus also to one particular form of seeing these things, processes, and practices. Whenever we are debating about concrete premises and decisions during a process of mathematization of a particular *X*, we should not belie us about the fact that we have always already chosen the *mathematization as such.* That is, we have excluded all other possible forms of seeing the world. At a more general level, this means that "choice is always meta-choice; it involves a choice to choose or not" (Žižek, 2003, p. 66).[2] So, whenever we are faced with the choice to choose between different premises that will lead to different mathematical models, we also choose the *choice to mathematize as such.* To put it another way, we automatically position ourselves on the side of mathematization and we are thus implicitly actualizing the primordial distinction which is the one between mathematization and non-mathematization.

In opposition to mathematization, demathematization indicates that the mathematics which is brought into action within the process is usually "operating beneath the surface of the practice" (Skovsmose, 2014).[3] As soon as mathematical models which aim at regulating a social practice are materialized in different forms of technology, mathematics seems to disappear from the (visible) surface of social practices (Keitel, Kotzmann & Skosmose, 1993). This 'disappearance' of mathematics from the visible surface obviously does not mean that mathematics is factually absent in the concomitant technologies, but, instead, it merely refers to the observation that mathematics brought into action quasi-automatically switches its "mode of presence" (Chevallard, 1989, p. 49) from an explicit towards its implicit form:

> „Implicit mathematics is formerly explicit mathematics that has become 'embodied', 'crystallized' or 'frozen' in objects of all kinds – mathematical and non-mathematical, material and non-material –, for the production of which it has been used and 'consumed'" (Chevallard, 1989, p. 50).

The explicit mathematics which is utilized to construct a certain technology becomes 'crystallized' in the process of mathematization. In this way it gets out of sight of the individuals using this technology at a later point in time. Therefore, every mathematization entails a demathematization, while the phenomenon of demathematization can actually be conceptualized as the *inner negation of mathematization*. That is to say, "an entity [here: mathematization] is negated, passes over into its opposite [here: demathematization], as a result of the development of its own potential" (Žižek, 2002, p. 180). Although demathematization is a process which is opposed to the process of mathematization, we should not treat the two processes as indifferent to one another: "Quite generally, what is distinct in an opposition confronts not only *an* other but *its* other (Hegel, 1991, p. 187). Demathematization is not *an other* of mathematization, but it is precisely *its*

2 Here, we must note that the only free choice or political act is then 'not to' choose the choice as such which can be put into the formula "I prefer not to […]" (e.g. I prefer not to … choose between these and that premises in the modelling process, but boycott the choice as such).

3 For empirical examples see Skovsemose (2014) or Keitel, Kotzmann & Skosmose (1993).

other and thus mathematization and demathematization can be interpreted as "two sides of the same coin" (Straehler-Pohl, 2017, p. 41).

However, with respect to our primordial distinction (mathematization/non-mathematization), demathematization is not a cross of the limit to the side of non-mathematization (as the absence of mathematics in opposition to its presence), but, instead, we fully remain on the side of mathematization. In other words, demathematization is located fully *inside* the coordinates of mathematization and should be merely interpreted as a transition to another state – the implicit mode of presence. At this point, we can conceptualize the dialectics between mathematization and demathematization: The two distinctions that constitute their dialectical relationship are 1) the distinction between mathematization and demathematization and 2) the distinction between mathematization and non-mathematization, whereby these two distinctions are at the same time bound together *via* self-referential closure. To grasp the social process of mathematization in its *form*, we must conceive this process as the unity of the difference of a particular mathematization and its demathematization *as the two distinguished sides of the form* (content). While the form of mathematization as such is itself only one side of the primordial distinction (mathematization/non-mathematization) and thereby immanently referring to its other: the side of non-mathematization. Hence, we actually have to negate the negation itself, that is, we have to move to the negation of the negation (the side of non-mathematization), to really imagine a new *form* of social practice that is beyond *its mathematized form*. Therefore, I argue that it is very important to distinguish between *the other* of mathematization at the level of content, which is demathematization as being simply the implicit mode of presence (in opposition to its explicit mode), and *the Other* of mathematization at the level of form, which is non-mathematization as being the absence of mathematization as such (in opposition to its presence).

If we do not assert this Other (with a capital 'O'), we will be caught in an endless self-referential loop: One mathematization that is regulating a social practice can only be replaced by another mathematization, while the primordial distinction (mathematization/non-mathematization) functions as the blind spot stabelizing the whole endeavour. Every problem which is caused by the implementation of a certain mathematization can be only solved *inside* the realm of mathematization, that is, the solution will always be a new, 'better' or more sophisticated mathematization and the mere oscillation between mathematization and demathematization as two modes of presence constantly reproduces the presence of mathematization as such, the unity of this difference, or simply: *the form of mathematization.*

3. Concluding Remarks

This line of argumentation allows us to draw two conclusions: *Firstly*, every particular mathematization entails its own demathematization, which renders invisible that it depends in its own constitution on an extra-mathematical, subjective act, which is in its most general form: the distinction between mathematization and non-mathematization.[4] S*econdly*, the dialectical dynamic of mathematization transforms social practices beyond the level of our consciousness because the technological materializations of mathematical models function as black boxes in a radical sense: We are not only blind for their functioning, but we are apparently often, and this is way more frightening, blind for our own blindness, that is, we do not experience them at all. Since the number of mathematizations that surround us is nowadays "growing exponentially" (Ernest, 2001, p. 287), the primordial distinction between mathematization and non-mathematization seems to fall more and more into oblivion. However, we cannot limit ourselves to the discussion of the problem of how we can make "*implicit mathematics explicit"* (Keitel, 1989, p. 12) because then we would focus on the 'hidden' content of the mathematizations only. On the contrary, we also have to deal with another, maybe even more important, question: Why does mathematics brought into action assumes quasi-automatically this peculiar two-sided form? And moreover: How did this form historically came into being?

In this paper, I rudimentarily outlined a form analysis of the process of mathematization by conceptualizing the vivid dynamics between mathematization and demathematization as a dialectics. I conclude by suggesting that this purely formal conception of a dialectical relationship between mathematization and demathematization could form a point of departure to investigate empirical processes of mathematization in contemporary late-modern societies, since it is unquestionably an important task for mathematics education

[4] In mathematics, the first one to prove that it is not possible for any formal system to bootstrap its own conditions of possibility was Gödel (1931) in his famous paper *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*.

research to analyze these empirical processes *in content and form*.

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Mathematisation as a ruled practice: Questioning the production of knowledge of school practices under a normative Wittgensteinian perspective.

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Abstract. This is a discussion about the mathematisation processes of cultural practices, regarded as didactic strategies necessary for the production of significant mathematical knowledge in school contexts. We consider initially there is a connection between any symbolic production and the purposes of this production. Moreover, the language is understood as a game that in the use of the words under certain rules establish definitions and meanings that must be given to things. Therefore, we discuss on the mathematisation under what is called "a normative perspective", which emphases the performative character of the mathematical meanings in a symbolic game whose rules do not reside in the individuals, but in the set of social and cultural practices from which the subjects participate.

Résumé. Il s'agit d'une discussion sur les processus de mathématisation des pratiques culturelles, considérées comme des stratégies didactiques importantes à la production de connaissances mathématiques significatives dans les contextes scolaires. D'abord nous considérons qu'il existe un lien entre toute production symbolique et les objectifs de cette production. En plus, le langage est compris comme un jeu qui, dans l'utilisation des mots sous certaines règles, établit des définitions et des significations qui doivent être données aux choses. Par conséquent, nous discutons de la mathématisation sous ce qu'on appelle "une perspective normative" en mettant en évidence le caractère performatif des significations mathématiques dans un jeu symbolique dont les règles ne résident pas chez les individus, mais dans l'ensemble des pratiques sociales et culturelles dont les sujets participent.

1. Language and practice as ruled activities

Our theoretical storyline premise is that there is a connection between any symbolic production and the purposes of this production. In addition, the language is no longer understood as a mediator and communicative agent of certain cognitive operations; but as a game that when moving the use of the words under certain rules establish definitions and meanings that must be given to things. Thus, by thinking about the mathematisation under a normative perspective, we seek to put the performative character of the mathematical meanings produced by the school practices in a symbolic game whose rules do not reside in the individuals, but in the set of social and cultural scenarios from which the subjects participate.

According to Luria, "our intellectual operations involve such verbal and logical systems; they comprise the basic network of codes along which the connections in discursive human thought are channeled" (Luria, 1976, p. 101). If theoretical thought develops, the system of codes will consequently become more and more complex by including not only words (more precisely, meanings, which have a complex conceptual structure) and sentences (whose logical and grammatical structure allows them to function as the basic apparatus of judgment) but also more complex verbal and logical "devices" that makes it possible to perform different operations of thinking without reliance on direct experience.

Distinctively, for Wittgenstein, speaking a language is part of an activity guided by rules, a form of life (Wittgenstein, 2014, § 23); following a rule is part of our practices under certain conditions. And as "forms of life" our philosopher understands all our habits, ways, lifestyles, actions, institutions in which our activities are based and intertwined with language.

For this purpose, Wittgenstein coins the term "language games" not only to establish the ruled character of linguistic activities, but also to understand how people interact according to the forms of life and practices they carry out. Thereby, cooking, farming, or business, as well as explaining, imagining, describing, questioning, reporting, are all practices, language games, and they can take place within and across different domains or subfields.

The analysis of "language games" by Wittgenstein, in particular about what the philosopher exposes through his work *Philosophical Investigations*, does not seek an essence of language, or a "prescriptive" philosophy that identifies the problem of the essence, or simply to deny this path to point the correct one (Vilela, 2010). Wittgenstein (2014, § 90) proposes a grammatical reflection, that is, a reflection that aims to remove misunderstandings concerning the use of words, caused, among other things, by certain analogies among the forms of expression in various areas of our language, or even among different *language games*.

What's the meaning of a word? Wittgenstein would tell us this question is wrong since it suggests just one and definitive answer. It depends on which language games are in use and their set of activities enmeshed (Wittgenstein, 2009, § 96). For our philosopher, the structure of a Language is the structure of a reality. Hence, choosing the more adequate meaning is the result of following the rule regarding the system of reference, which works as a horizon of intelligibility (Régnier et al, 2016)

This point of view also implies that human activities are complex ruled, dynamic, and interchangeable games, and the world of culture is no longer a system of structures, but the variable result of interchanges among different activities.

For Wittgenstein, practice is a priority conceived according to our actions, forms of life, and language accordance (Bloor, 2001). Here, it is noteworthy the understanding given by Theodore Schatzki to Wittgenstein's words. For him, practices are, first of all, organized nexuses of activity; open-ended sets of doings and sayings organized by understandings, rules, and *teleoaffectivities* (Schatzki, 1996, my emphasis).

Moreover, the actions that compose a practice are either bodily doings and sayings or actions that these doings and sayings constitute. By 'bodily doings and sayings' Schatzki means actions that people directly perform bodily and not by doing something else. To say that actions are 'constituted' by doings and sayings is to say that the performance of doings and sayings amounts, in the circumstances involved, to the carrying out of actions (Schatzki, 2001). According to Miguel (2014) we always practice the language with the whole body and not just with culturally ruled vibration sounds emitted by our vocal cords. In this sense, to stage or perform a practice is the same as staging or performing a ruled language game; that is, both endeavors involve disciplining the body to make it follow the rules of that game.

Let us see, for example, the ways in which female workers involved in the world of school kitchen (lunch ladies), run bodily actions to prepare a specific dish or food for children (Ogliari & Bello, 2016). Dealing with the control of quantities, measurements and proportions (a pinch of salt, a tablespoon of oil, five kilos of rice for 130 children), they perform their job by calculating the quantities of ingredients to obtain certain flavors, textures, as well as servings that satisfy certain number of people. One way or another, they follow certain algorithms. This does not take away the possibility of our lunch ladies to come up with other processes. However, they have to know other recipes, to combine other ways to prepare, sometimes rehearsing, trying; making it up and following, unequivocally, another set of a ruled and normative language game; that is, a new algorithm that will lead them to the preparation of servings with certain flavors and textures. The symbolic production that guides their performance in the kitchen is guided not only by the rules and understandings contained by the procedures, but heavily by the teleoaffective component in question.

In that sense, we can also consider mathematics as practices, as language games or at least as a set of rules that govern our ways of doing and sayings in composing practices. By considering mathematics as a family of activities with a family of purposes (Wittgenstein, 2009, p. 273), Wittgenstein offers us an understanding of "Mathematisation" or mathematics in action, [...], that is, a heterogeneous and dynamic set of ruled symbolic stagings.

Many contemporary readings (Gottschalk, 2004, 2007; Shanker, 1987) of Wittgenstein's reflections about mathematics have pointed out that the originality of these reflections has been, primarily, their contribution to the emergence of a normative conception of mathematics, which is not compatible with logicist, intuitionistic, formalistic, or anthropological conceptions such as Ethnomathematics (D'Ambrosio, 1985). In addition, we can see *a priori* the numbers or algorithms as being invariably mathematical objects but they are, first and foremost, signs whose meanings are assigned in relation to performances and actions guided by rules and purposes.

2. Mathematisation as a didactic game.

To pay attention to the different language games, to the different uses a word can have, and recognize the different meanings assigned to that word claims, in our view, a reflection on the language. For Vilela (2010), Wittgenstein's philosophy is like a therapy that consists of traversing the various uses of a concept in the linguistic practice, undoing exclusive footage, inserting them in the plurality of uses. And this is what we propose to do with the notion of mathematisation.

From this perspective, when we put in check the uniqueness of meanings, we put in evidence the implications of these analyses for the teaching of mathematics; especially when it comes to subvert some linguistic "misunderstandings" that reinforce certain meanings over others in the field of school educational activity. The mathematisation as a didactic-pedagogical principle enforces a set of rules regarding the practice of teaching, among them that of "teaching" to solve problems, to understand and propose mathematical wordings, and almost always assign meanings strictly formal when describing, even mathematically, everyday activities. Thus, it is pertinent to remember that it is in the use of the contents, methodologies, the conduct of teachers and students, and especially in their purposes, that the meanings will arise as forms of production of the "mathematical" knowledge. In other words, questions that enunciate different contexts of use, related to different scenarios in mathematical texts and contexts, advocate the mathematics as unique, so that all the existing relations in the different contexts in which mathematics is needed only makes sense by its formalization. Thus, the mathematisation, even if related to different contexts of human activity, is taken as a "practice in itself", that is, a practice with universal characteristics that is worthy by itself. Acting this way is an acculturating strategy that typifies speeches and produces their own language games.

Lave (1996) gives us proof that to investigate the relationship that people have with mathematics in a given situation converges to the understanding of the existence of certain combinations and transformations of the relations of the so-called mathematical entities (whether of numbering, form, or measure) in the current activity, which ultimately lead to an apparent "disappearance of mathematics" in overcoming an impasse on a daily basis and not a relationship with pre-established images by formal or even scholar mathematics. Gottschalk (2008) states that all educational trends within constructivism assume that the mathematical objects preexist, "whether in the empirical, mental, or in the social intersubjectivity."

The condition of non-existence of universality of meanings, as well as these meanings just make sense regarding practices, helps us to take the problem of non-transfer of learning. If we suppose that our lunch ladies successfully performed a practice of measurement in the kitchen, we could ask: Do they have the ability to measure in any situation? Or would it be more prudent to say that they have learned to measure from the use of spoons, hands, "to the naked eye"? Out of the kitchen, could they measure anything with the measuring instruments similar to those used at the school? Or on the contrary, could they directly apply what they have learned at the school to improve their recipes? Each practice is governed by a different set of rules.

It is noteworthy that according to Wittgensteinian's understanding of practice, there is no distinction between theoretical and empirical approaches. The normative condition of language imposes a normative condition of knowledge. Hence, practicality and knowledge are both constitutive of the unique process.

The mathematisation as a didactic principle that highlight the democratization of teaching mathematics should not load only tracks of educational trends that have established themselves in the educational discourse. It should consider that the symbolic production is ruled by the practices constituted, whose circulation is inherent in our different forms of life.

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Ethnomathematical study on folk dances: "mathematisation" of the garbs

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Abstract. This proposal for oral presentation is included in a doctoral project in Mathematics Education. Part of this project aims to analyze and understand the mathematical structure inherent in various elements of folk dances characteristic of Northern Portugal and Galicia (Spain), specifically choreography, accessories, and music. We expect to develop an ethnomathematical study on elements of folklore, within a process of mathematization built on cultural practices. Regarding the accessories, two folk groups' garbs were photographed, in order to study the geometric patterns present on them.

Résumé. Cette proposition pour la présentation orale est incluse dans un projet pour un doctorat dans l'Éducation de Mathématiques. La partie de ce projet a l'intention d'analyser et comprendre la structure mathématique inhérente dans de divers éléments de caractéristique de danses folklorique du Portugal du Nord et de la Galice (Espagne), spécifiquement la chorégraphie, les accessoires et la musique. Nous nous attendons développer une étude d'ethnomathematical sur les éléments de folklore, dans un processus de mathematization a tiré parti des pratiques culturelles. Concernant les accessoires, les costumes de deux groupes folkloriques étaient photographiés, pour étudier le présent de dessins géométrique sur eux.

1. Ethnomathematical study on folk dances: "mathematisation" of the garbs

Mathematical activity is a human activity and so it constitutes a cultural activity (Gerdes, 2007a). Therefore, mathematics must be understood as a knowledge that all cultures produce, not necessarily equally to each other (Bishop, 1988). D'Ambrósio (2001, 2002, 2008) conceived the word "ethnomathematics" to designate mathematics practiced by distinct cultural groups, identified by common goals and traditions. Bishop (1986, 1988) determined the existence of six basic universal activities (counting, locating, measuring, designing, playing, and explaining) through which mathematics as a cultural product has developed not only in our culture but in all cultures. According to Bishop (1986), since these are universal activities, mathematics exists in some form, to some extent and with more-or-less significance for individuals within all cultures. In this sense, we agree with Bassanezi (2002) when he argues that each cultural group has its own way of mathematising the reality, and we should not ignore it in the educational field.

In this line of thought, Gerdes (1988, 2007b, 2013) uses an ethnomathematical approach to mathematise an old tradition of culture *Cokwe*, from the Northeast of Angola, specifically their drawings (composed only by points forming a grid and lines involving the points), known in the local language by *sona* (singular, *lusona*). These drawings are usually executed in the sand and serve to illustrate stories, legends and riddles (Gerdes, 2013). The mathematical potential of *sona* was object of continuous research by Paulus Gerdes. In this regard, Gerdes (2007b) analyzes a particular category of *sona*, monolinear, whose elements are *mirror-generated curves*, and describes some of its basic properties. In a multiphase process, these curves originate matrices. In a different sense, Barton (2008) ensured that the location of an object in two dimensions is, according to the dominant mathematical approach, determined by using the *Cartesian Coordinate System* or the *Polar Coordinate System*. However, in languages *Tahitian* and *Maori*, the location of an object is carried out with reference not to one but to two origins - the speaker and the interlocutor - and, consequently, the amplitude of two angles – one for each origin (Barton, 2008). Another aspect we consider important to invoke here is the study of symmetry from a cultural point of view, which has been widely explored and systematized by Washburn and Crowe (1988), appearing as an element present in artifacts all over the world.

In the previous investigations, ethnomathematics appears as a methodology to "mathematise" cultural practices, by recognizing and presenting the mathematics presented there. In this line, the doctoral project that motivates this proposal aims to analyze and understand the mathematical structure inherent in various elements of folk dances characteristic of Northern Portugal and Galicia. During the project, bibliographic collection about choreographic folklore of Northern Portugal and Galicia will be carried out and two folk dances will be studied. In particular, we intend to study three elements that constitute folk dances, specifically the choreography, the accessories, and the music. We will study the curves inherent to the movement of the dancers, the mutual locations of the elements of the pairs and the symmetry, not only in the consecutive positions of the pairs but also in their own garbs. Therefore, the research strategy to be used is a study with ethnographic characteristics, because it will be a descriptive study of the culture of a community or of some of its fundamental aspects (Baztán, 1995), which are, in this case, folk dances. Ethnography is an attempt to describe the culture or certain aspects of it (Bogdan & Biklen, 1994). The data collection will be carried out in a natural environment, through several methods, and complemented by information obtained through the direct contact of the researcher with this environment (Bogdan & Biklen, 1994).

2. Study on folk groups' garbs

Since this study started with the accessories, two folk groups' garbs were photographed and geometric patterns present on them were studied.

One of the folk groups was the *"Grupo Folclórico de Vila Verde"*, from the district of Braga, Portugal. The garbs that these dancers wear reflect the socio-economic reality of the region. The *"Traje de Encosta, Festa ou Domingueiro"* (figure 1) was used in the wedding day or in festivity days, being later left for the shroud. The *"Traje de Noivos"* (figure 2) is the continuation of the previous garb, now in the ceremonial version. The *"Traje de Ribeira, Feira ou Lavradeira"* (figure 3) was used in the fairs. The *"Traje de Trabalho Rural ou de Uso Comum"* (Figure 4) was used for agricultural work.

Figure 1. *Sunday Garb*

Figure 2. *Wedding garb.*

Figure 3. *Fair garb.*

As we can observe in the previous figures, the totality of the garbs of the dancers of *"Grupo Folclórico de Vila Verde"* displays reflection symmetry of vertical axis (Crowe, 2004). However, there are exceptions that deliberately break with this symmetry. For example, the linen jacket belonging to the *"Traje masculino de Encosta, Festa ou Domingueiro"* (figure 5) has a red embroidered motif in the central part that cancels the isometry. Also, the sweetheart handkerchief placed on the left side of the *"Traje feminino de Encosta, Festa ou Domingueiro"* (figure 6) makes it impossible for the vertical axis reflection to leave the whole invariant.

Figure 4. *Day by day garb.*

Figure 5. *Example of vertical reflection symmetry breaking.*

Figure 6. *Example of vertical reflection symmetry breaking.*

Figure 7. *Symmetrical arrangement of gold pieces.*

As we can observe in the previous figures, the totality of the garbs of the dancers of *"Grupo Folclórico de Vila Verde"* displays reflection symmetry of vertical axis (Crowe, 2004). However, there are exceptions that deliberately break with this symmetry. For example, the linen jacket belonging to the *"Traje masculino de Encosta, Festa ou Domingueiro"* (figure 5) has a red embroidered motif in the central part that cancels the isometry. Also, the sweetheart handkerchief placed on the left side of the *"Traje feminino de Encosta, Festa ou Domingueiro"* (figure 6) makes it impossible for the vertical axis reflection to leave the whole invariant.

The reflection of vertical axis present in garbs is also visible in the disposition of the gold pieces in filigree abundant in the *"Traje feminino de Noivos"* (figure 7). See how the pieces are carefully arranged on the black coat on a search for symmetry. However, as they are all different filigrees, this goal is not achieved.

The other folk group studied was the *"Agrupación Folclórica Cantigas e Agarimos"*, from Santiago de Compostela, Spain.

Some pieces of the garbs that these dancers wear also displays reflection symmetry of vertical axis (fig. 8).

Figure 8. *Example of vertical reflection symmetry.*

Figure 9. *Decorative shape repeated in centre and in each of the cuffs.*

However, there are clothes in these garbs that exhibit distinct patterns. For example, in a man's dress jacket, the same decorative shape is repeated in the central part of the jacket and in each of the cuffs (figure 9). Note that the three black bands (a wide band, a band with squares and a narrow band) appear precisely on these three parts of the jacket, keeping the same order from the top to the bottom on the wrists and from the left side to the right side in the centre.

Another example of pattern visible in these garbs is the continuity of decorative motifs that appears along the outline of women's dress coats (figure 10 and figure 11).

Figure 10. *Continuity of the decorative motif along the outline of the coat.*

Figure 11. *Continuity of the decorative motif along the outline of the coat.*

In figure 10, there is a repetition of a symmetrical motif, which causes the coat, viewed in a global way, to exhibit symmetry of vertical axis reflection. The motif that is repeated (figure 12), forming kind of a frieze, presents four reflection symmetries and four rotation symmetries. However, in figure 11, repetition of an asymmetric motif occurs, the repetition of which does not allow the coat to exhibit vertical axis reflection symmetry. The motif that appears repeated (figure 13) has no symmetries.

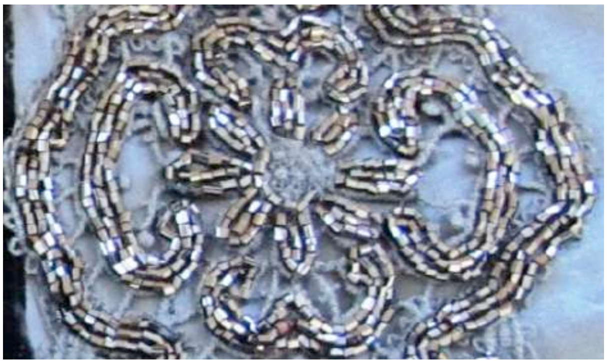

Figure 12. *Decorative motif.*

Figure 13. *Decorative motif.*

3. Brief conclusions

In the scope of the doctoral project, the bibliographical collection on choreographic folklore, as well as the beginning of the study of the folk dances, in particular accessories, allowed to study the symmetry present in the garbs of two folk groups. Briefly, this study will be extended to other groups. Nevertheless, the research initiated has already made possible to study the interrelationships between mathematical ideas and other elements and cultural constituents (Gerdes, 2007a). Symmetry is until now the salient mathematical idea, which is used both in the formation sense and in the intentional and episodic break. The vertical axis symmetry is the most frequent in the costumes of *"Grupo Folclórico de Vila Verde"*. However, it is also visible in *"Agrupación Folclórica Cantigas e Agarimos"*. There are also distinct patterns in these garbs: the repetition of decorative shapes in the central part of jackets and in each of the cuffs, and the continuity of decorative motifs that appears along the outline of coats.

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Mathematization of Selected Real-Life Aspects by Applying Dynamical Systems

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Abstract. Drawing in on what mathematics is, many reasons which justify its teaching become evident. Especially in the context of education, it is important to consider mathematics not only from the scientific perspective, but also include facets that explore mathematics from further perspectives. Throughout history, mathematics has occupied a predominant place in the school curricula. Mathematics has achieved this status for cultural and social reasons rather than because of its self-value. What I want to do in this paper is to develop an updated version of Felix Klein's book: *"Elementary mathematics from a superior point of view*" – a milestone in the didactics of mathematics, published in 1908 by the University of Göttingen. This paper aims to provide novel ideas by describing selected social, economic, and ecological procedures that might solve mathematics-related problems of a person. Therefore, I present several case studies situated in real-life situations which apply dynamical systems (Romero, 2013) as mathematical models.

1. Introduction

Throughout the following sections, from the point of view of mathematical modelling, I try to answer the following questions:

a. What are the mathematical developments of the 21st century that mathematics teachers at all levels, in particular secondary school teachers, should know about and how can they be made accessible?
b. How can we use other mathematical models that are not normally used in daily classroom teaching?

Currently, there are certain issues in the area of mathematical knowledge which relate to the profession of mathematics education. These are problems related to the general question of how to unite and structure curricula in a way that the teaching of the different subjects becomes correlated. This is important due to the fact that the majority of subjects are disconnected from the real world as well as from the sciences themselves. Because of this, students might not conceive the true usefulness of mathematics necessary for their general education. In many countries, e.g. Spain (BOE 5 and BOJA 171, 2007), with the exception of few educational communities, "a tradition has been generated in the way of organizing curricula in mathematics, reducing teaching to a work based on algorithms that does not allow to students to understand the role of mathematics in society" (Aravena & Caamaño, 2007, p. 8). This tradition is deeply rooted in the school system and has shown to be detrimental to achieve more elaborated aims in the learning processes of our students. Furthermore, it enlarges the disparities of achievement among students.

There are extra-mathematical situations (Romero et al., 2015) in the teaching of mathematics in the secondary school curricula which are primarily constructed in order to motivate students to develop mathematical techniques. This is a long-term process. These situations are not always motivated by problems (intra- or extra-mathematical) that can be solved. Conversely, modelling work that entails the investigation of possible models can lead to the increase and consolidation of an exceptional practice.

Since a few years, researchers in mathematics education have focused their attention on the design of activities based on the mathematical modelling of real-life situations. They do so with the aim to guarantee a gain of competences within the learning process of our students as well as our educators.

2. "News" Arguments

"Man cannot discover new oceans unless he has the courage to lose sight of the shore." André Gide

It is necessary to modify the curricula innovatively by the integration of new aspects, new models, and new creative themes (Izquierdo et al., 2004). There are many mathematical domains (Romero, 2011) which remain almost unexplored in primary school education (or in secondary schools), but are organized in an original and creative way that would enrich the classroom activities. These are, among others, graph theory and optimization, dynamic systems, fractals, topology, information processing, code theory and cryptography, and modelling.

One of our main aims is to challenge and provoke our students in order to make them use mathematical models (Duperret, 2009); for example the use of models which are designed for the description and analysis of aspects close to their daily life, so that they consolidate with their mathematical knowledge. Further, this aims to establish and implant solid cognitive principles for the future conception of such. Also, it is important to situate the descriptions, analyses, and dilations of the practical contexts in their everyday living environments as "experimental mathematics".

Throughout the development of the sciences, the various concepts as well as the use of mathematical theories were always considered. The mathematical results obtained are useful for the study in several areas. Presumably, without mathematics, societies would not have reached the levels of development they now incorporate.

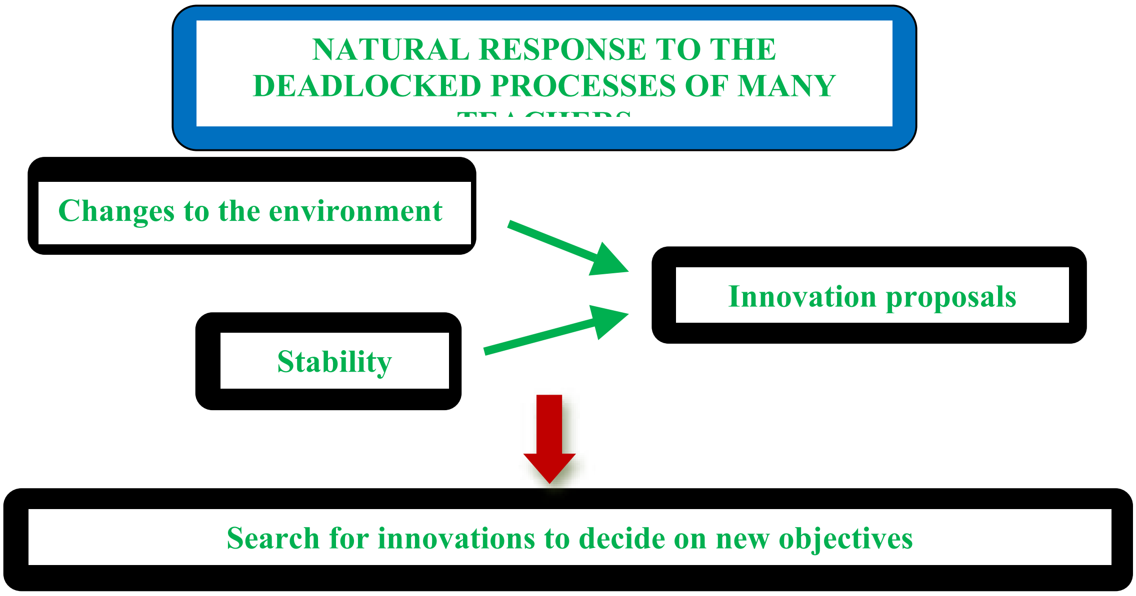


Figure1. New arguments against immobility

3. Dynamical Systems: Ideas

a) In the 17th century, Newton, in his study on gravitation, discovered that Differential Calculus (DC) is a mathematical tool that can be applied to study various other physical phenomena. He successfully applied DC in order to make precise calculations on the orbits of the planets around the sun. The theory of Differential and Integral Calculus has been used to explain a large number of real-life events.

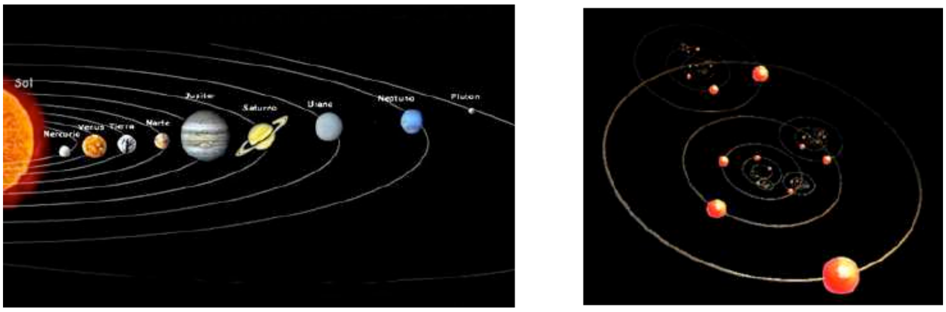


Figure 2. Planets

b) In the 18th to19th century, Euler, D'Alembert, Lagrange, Jacobi, Legendre, Hamilton, Fourier, and many others extensively developed theories on the movement of the planets Heat, Fluid, etc. This clearly illustrates the extend to which Differential Equations (ODE's) have contributed to the development of Physics.

Figure 3. Orbits

c) In the 19th to 20th century, ODE´s that were made within practible studies have shown to have clear predictable results; however, accurately determine those was not always possible.

3.1. What is a Dynamical System (DS)?

It was Henri Poincaré who started the qualitative study of the differential equations in Mathematics, but it was only about 40 years ago that the dynamical systems were established as an independent field of research studies. This was, for example, thanks to the outstanding work not only of mathematicians, but also of engineers among which we can mention Smale, V. Arnold, Lyapunov, etc. Drawing in on the concept of dynamical systems, we could say that they are a rather new area of mathematics which belong to th research area of deterministic systems (Romero, 2013). This entails that they include the consideration of situations that depend on any given parameter, e.g. time, which vary according to established laws. Those considerations then allow us to reconstruct the past and to predict the future. Theoretically worded, we could say that a dynamical system (**Alligood** et al., 2009) is a way of describing the path through time through all points of a given space.

To explain what is meant by the term dynamical system (DS), we need to consider two steps:

1. We must choose a base space; for example, that could be the earth, the sun, the planetary system, a car, a company, a sphere. These base spaces are called topological spaces; we will denote them by *D*.
2. We must define a function in this base space:

$f\colon D \rightarrow D\,/\,x$ any one is associated with $f(x)$

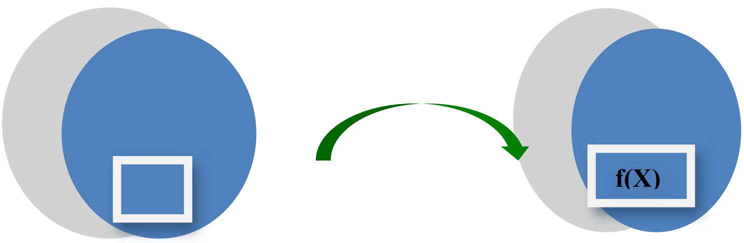
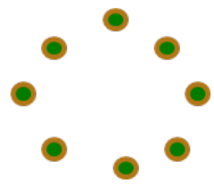


Figure 4. Function in the base space

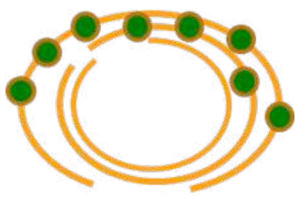
In case that it is possible to repeat the process $f(x) \rightarrow f(f(x))$, an orbit of x, which are the points x, $f(x)$, $f(f(x))$, $f(f((x)))$ is generated.
Each point of D has an orbit; consequently, the study of these orbits is the Dynamical System in D due to f. This may give rise to different DS's which depends on the behaviour of their orbits; for example, *periodic orbit, quasiperiodic orbit*, or *other types.* They are used in order to study how the points are attracted to other sets:

- Single point orbit to Fixed point:

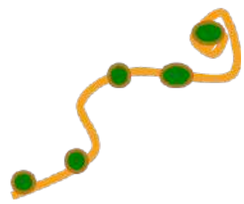
- Periodic Orbit to Finite number of points that, by the repetition of the process, return to the initial position with a finite number of times.



- Quasiperiodic orbit: the process where the points move near a periodic orbit is repeated.



- Another type: the points are attracted to another set



3.2 Case Studies

The activities that I am going to present next stem from a Spanish national project denominated ESTALMAT (**ES**timulate to the **TAL**ent **MAT**hematic).
They refer to mathematical models of processes that evolve with time, i.e. discrete DS's. They can be contrasted to differential equations, which are models to describe situations where changes occur at specific times rather than continuously.

In the next section, I present selected mathematical models which display processes that evolve over time, that is, discrete DS's that often involve an iteration process.

***3.2.1. Case 1**, related to economy: THE BANKS – Working with the concept of BANK CREDIT*

Suppose my son, Enrique, in the middle of the current economic crisis, receives the offer to lend money from a bank. The interest rate charged by the bank is 0.5% per month. Enrique's actual repayment capacity is a maximum of 200 € per month. How much money do we want the bank to provide, reasonably?

A. Ideas for the solution

A. Ideas for the solution

a.1) A naïve answer could be: EVERYTHING YOU CAN GET!
a.2) Let's analyze the question to understand what we are asked for:
Let's call $\boldsymbol{D_0}$ the amount of money we want to borrow from the bank, and ***Dn*** our bank credit after **n** months. The calculation that we come up with is the following:
* After one month, we would owe the bank

$$D_1 = D_0 + 0.005\, D_0 -200 = (1.005)\, D_0 -200$$

* After the second month, our debt would be

$$D_2 = (1{,}005)\, D_1 -200 = (1.005)\, (D_0 + 0.005\, D_0 -200) -200) = (1.005)\, 2\, D0-200\, (1.005) -200 = (1.005)\, 2\, D_0 -200\, (1.005 + 1)$$

a.3) Following this process, in how much debt would we be after 6 months? And after a year?
a.4) What is the general formula for n-months as a function of $\boldsymbol{D_0}$?

$$D_n = 1{,}005^n D_0 - 200\left(1{,}005^{n-1} + 1{,}005^{n-2} + \dots\dots\dots + 1\right) = 1{,}005^n D_0 - 200.\frac{1.1{,}005^n - 1}{1{,}005 - 1}$$

$$D_n = 1{,}005^n D_0 - \frac{200(1{,}005^n - 1)}{1{,}005 - 1}$$

$$D_n = 1{,}005^n D_0 - \frac{200(1{,}005^n - 1)}{0{,}005}$$

Once the above situations are resolved, a reasonable task for our students would be to confront them with the following scenario: An arbitrary amount of money is presented in several situations in order to sufficiently illustrate the different behaviors of the dynamical system in question.
One concrete example that allows us to introduce the concept of DYNAMICAL SYSTEMS in a natural way is the following:

• *If we loan more than 40,000 €, Enrique will have to pay back a monthly increasing debt.*

• *If we loan less than 40,000 €, Enrique will have to pay back a monthly decreasing debt.*

• *What if you loan 40,000 €?*

These three types of behaviour illustrate the concept of a *fixed point or an equilibrium point* of a dynamical system that does not change over time. Hence, we can set up the following equation:

$$D_n = 1{,}005 D_n - 200 \Rightarrow D_n = \frac{200}{0{,}005} = 40.000$$

B) Conclusion

To be familiar with the laws that govern the dynamical system allows us to predict the future. After the students fathomed this out, they were able to answer the question of how much money they would recommend Enrique to loan from the bank.
The answer further entailed expressions with a sequence of values:
Dn with **n** ⍰ **Z** such that the value of ***Dn*** is determined by the previous values $\boldsymbol{D_{n-1}}$, $\boldsymbol{D_{n-2}}$, $\boldsymbol{D_{n-3}}$, ..., . We call these equations equations in differences and thereby give rise to discrete dynamical systems. In this context, the term discrete refers to the parameter time, which could be every month, every year, every hour, etc. The equations in differences, i.e. discrete dynamical systems, more simplistic or more complex systems arise through the iteration of functions.
For example, they appear in:
A) The concepts of recurrence
B) General terms

C) Geometric progression
With the formula of addition, we explain how we obtain the expression of the sum of the terms of any geometric progression of ratio r as well as the initial term a_0 :
$a_0, a_1, a_2, a_3, ..., a_n,.......$
finite or not with
$a_i = a_{i-1}.r$ for i ≥1
where $a_0 + a_1 + a_2 + a_3 + ... + a_n$ is the sum of the n terms of the geometric progression

$$\sum_{i=0}^{i=n} a_i = \frac{a_0 r^n - a_0}{r-1}$$

3.2.2. Case 2, related to the EVOLUTION OF THE POPULATION OF A SPECIES: Malthus growth model.

An Italian mathematician of the XIII century, Fibonacci, was the first to describe the growth of populations with a mathematical formula, today known as the Fibonacci sequence. Because of its general validity, it is still highly valued by the research application fields of the natural sciences.
In the above sections, we already discussed selected contributions of the seventeenth century where mathematicians proposed first attempts of mathematical models that aimed to be applicable to other sciences. Descartes, for example, applied them to physiology; however, these showed to contain a large number of errors and are therefore inaccurate.
In the eighteenth century, Malthus, another mathematician, set up an equation that showed the interdependence of the world population and global food resources. Later, this model became known as the 'Malthusian Catastrophe' because the calculations clearly revealed that the food resources are unsustainable in order to maintain the world population. Consequently, Malthus predicted famines and wars. Malthus presented his ideas in the form of differential equations including the necessary parameters in order to determine the exact points in time where the quantity of food would not be enough to subsist the entire population. Due to its accuracy, many people recognized this model as valid; however, the mathematically forecasted famine did not occur for several reasons. First, the British society of the eighteenth century underwent a significant demographic transition. Second, in the upper and middle classes of society, birth control was established; therefore, the birth rate transitioned from an exponential growth to a logistic growth. Third, due to agricultural development, food resources grew faster than expected.
In alliance to the Malthusian model, in 1838, Verhulst set up a model to describe a population's process of change. In that model, the growth rate is exponential at the beginning (as in the case of the model of Malthus) but alters its growth rate at a certain point in time; i.e. when the members of a society start to compete against each other over highly limited goods. As a result, the growth rate of the population in question decreases (García Rodríguez, E. et als, 2015).

3.2.2.1. Suppose we want to study the evolutionary process of a determined species concerning its growth rate from the moment when the number of individuals equals x_0. We further decided to measure time in years and denote it by x_k. The growth factor of said population between two consecutive years is 0,3. It is our aim to identify the number of individuals in year k.
1. Establish the relationship between x_k and xk-1.
2. Obtain a formula to obtain x_k as a function of k.
3. What types of behavior does the model have?

A. Ideas for the solution

1. Establish the relationship between x_k and x_{k-1}

$$x_1 = x_0 + 0.3x_0 = 1.3x_0$$
$$x_2 = x_1 + 0.3x_1 = 1.3x_1 = 1.3(1.3x_0) = 1.3^2 x_0$$
$$x_k = x_{k-1} + 0.3x_{k-1} = 1.3x_{k-1}$$

2. Obtain a formula to obtain x_k as a function of k

$$x_k = x_{k-1} + 0.3x_{k-1} = 1.3x_{k-1} = 1.3(1.3^{k-1}x_0) = 1.3^k x_0 \Rightarrow x_k = 1.3^k x_0$$

B) Conclusion

Which kinds of behavior can you extract from the equation?

On the basis of the mathematical expression we can say:
1) The mathematical model is reasonable in the early stages. That is, the model serves as far as it serves.
2) The evolution of the species itself allows us to state that the model is not valid.

In short, we can presume that our students, while accomplishing the task, consider three aspects of the GROWTH MODEL OF MALTHUS:

1. Is the model viable? When would it be?
2. In which aspects would the model differ in case that growth is proportional to the existing population with constant ***d***?
3. Would the model serve to study a population that extinguished, i.e. where the decrease of a population was proportional to the existing population?

3.2.2.2. Practical example

A bacterial culture is supposed to follow the growth of Malthus. Initially, there are 1000 bacteria; after one hour, there are 1250 bacteria.
a) How many will be there after 4 hours?
b) Using a table, find out how long it will take until there are 5000 bacteria.

A. Ideas for the solution

Applying the Malthus model, we come up with the equation $x_k = d^k x_0$, where $x_0 = 1000$.
After one hour, we have $x_1 = 1250$. Therefore, we can deduce:

$$1250 = d^1 1000 \Rightarrow d = \frac{1250}{1000} \Rightarrow d = 1.25$$

The model is governed for the equation $x_k = 1.25^k 1000$

a) After four hours, the number of bacteria would increase as followed:

$$x_4 = (1.25)^4 1000 = (2.4414).1000 = 2441,4$$

b) A table that shows how much time would have to pass until the bacterial culture grows to 5000 bacteria is being provided below:

k=1	***d^1=1,25***
k=2	*d^2=1,5625*
k=3	*d^3=1,9531*
k=4	*d^4=2,4414*
k=5	*d^5=3,0517*
k=6	*d^6=3,8146...*
k=7	*d^7=4,7683*
k=8	*d^8= 5,9604*

$$5000 = (1.25)^k 1000 \Rightarrow (1.25)^k = \frac{5000}{1000} = 5 \Rightarrow (1.25)^k = 5$$
$$k \log_e(1.25) = \log_e 5 \Rightarrow k = \frac{\log_e 5}{\log_e 1.25} = \frac{1.6094}{0.2231} = 7.2134$$

*3.2.3. **Case 3,** related to a MODEL OF ECOLOGY*

Suppose the following example: In Spain, we have 1250 individuals of a protected species of birds. Experts believe that the existing bird population decreases by 7% each year; either by natural causes, or by poachers. There is also a captive breeding programme which increases the bird population by 5 individuals each year.
Questions that we could present to our students are:

a. Write the relation of recurrence that relates the existing population in year k, x_k, with which there was year $\boldsymbol{k-1}$, $\boldsymbol{x_{k-1}}$.

b. Determine a formula that allows to obtain $\boldsymbol{x_k}$ as a function of $\boldsymbol{k}$.

c. Suppose the conditions do not change, is this species in danger of extinction? (It is established that a species is in danger of extinction when the number of its individuals is less than 100.)

A. Ideas for the solution

We would ask our students to make the necessary calculations.
Prior to discussing the results, we would ask the students to sketch a graph that visualizes the model adequately with respect to the following questions:

1. Note that the issues we have resolved in the previous exercise can be addressed as follows: We take a cartesian representation system for each point $\boldsymbol{(x,y)}$ where $\boldsymbol{x}$ is the number of individuals at time $\boldsymbol{k-1}$, and $\boldsymbol{y}$ is the number of individuals at time $\boldsymbol{k}$.
2. Check whether the question raised above responds to the dynamic system that is expressed by the equation

$x_k = 0.93\, x_{k-1} + 5$

3. Draw the dynamic growth curve.
4. Starting from the amount of birds in this year $\boldsymbol{(x)}$ and with the help of graphic representation, give the amount of birds that will be there next year as well as in two years.
5. How large would the population of birds need to be this year ($\boldsymbol{x}$), so that there would be an increase of the population in the following year?

How large would the population of birds need to be this year ($\boldsymbol{x}$), so that there would be a decrease of the population in the following year?

6. **B. Graphic visualization of the dynamic behavior of the system of Case III.**

One way to represent the case is to start from a plane where the axis OX is $\boldsymbol{x_{k-1}}$ and the axis OY is $\boldsymbol{x_k}$:
$x_k = 0.93x_{k-1} + 5$
We take the graph $\boldsymbol{y = x}$ as an assistant.

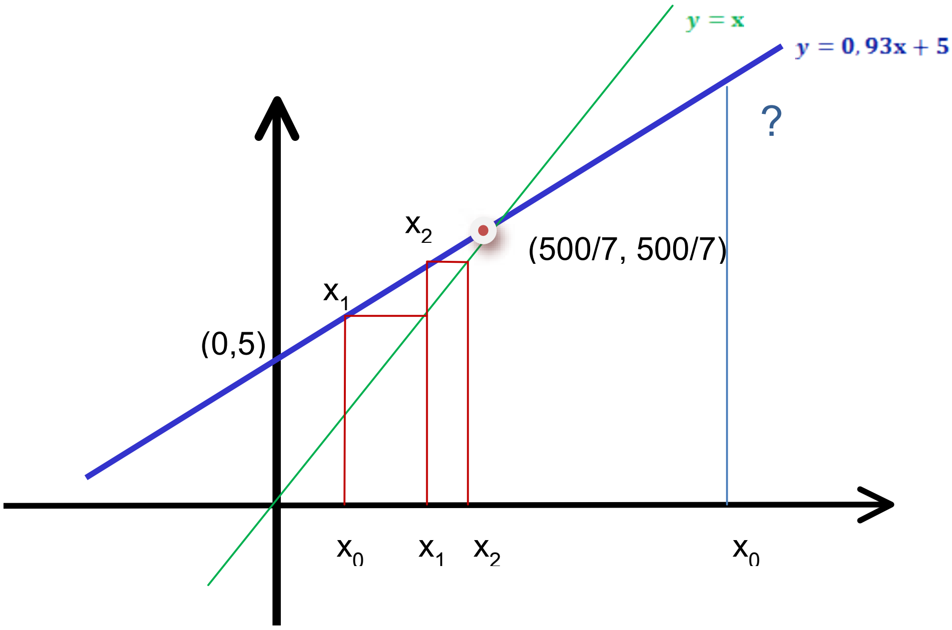


Figure 5. Graphic visualization

In general, how does the dynamical system behave over time (regardless of the starting point)?
The system has an attractive equilibrium point ***Pe (500/ 7,500 / 7)*** at the intersection of the two graphs ***y = 0.93x + 5*** and ***y = x.***

4. Modeling and formalization: Dynamical Systems of one-dimensional linear applications

Let ***L: R → R*** be a linear application, that is, ***L (x) = ax*** with ***a*** ∈ ***R***

4.1. Case 1: *a <1.*

p=1. The first option to investigate the ***dynamical system generated*** by this application would be to take a starting point, for example, 1, and calculate its orbit (projections on the axis OX of the points obtained).
$1, a^2, a^3, a^4, a^5, \ldots$
The orbit converges to point 0.
p≠1. For any other starting point the result stays the same because the orbit of the generic point ***p*** is

$p, a\,p, a^2\,p, a^3\,p, a^4\,p, a^5\,p, \ldots\ldots$

The Dynamical Analysis that follows this finding is:

a) If |a| < 1, all orbits converge to point 0

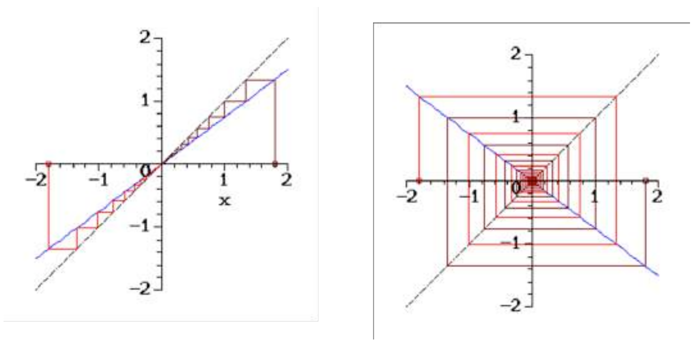


Figure 6. Dynamical Analysis-1

b) If |a| > 1, any orbit other than point 0 diverges in modulo to infinity

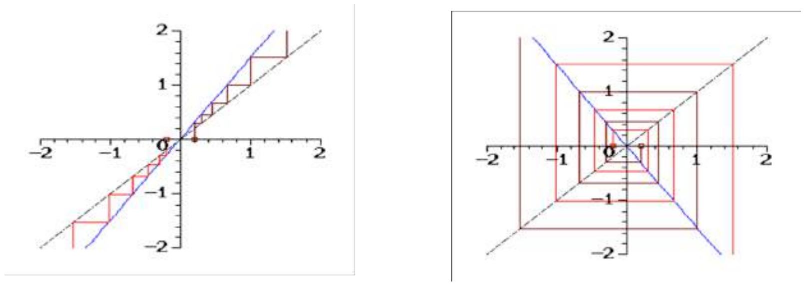


Figure 7. Dynamical Analysis-2

c) If a = 1, all points are invariant

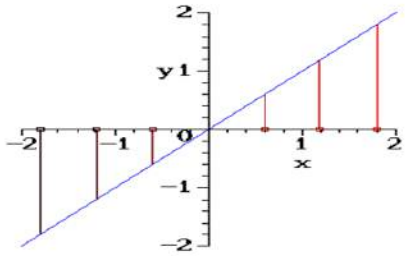


Figure 8. Dynamical Analysis-3

d) If a = -1, all points except 0 are of the period 2

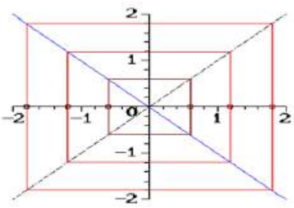


Figure 9. Dynamical Analysis-4

5. Ideas concerning other Dynamic Systems

Studying linear dynamics, analogous studies can be done with multiple other dynamic systems.

5.1 The quadratic family

The quadratic family is constructed by the applications

$f(x) = x^2 + p, p \in R$

5.2. The logistic family

The logistic family generates from the applications: $\boldsymbol{f_c: [0,1] \rightarrow [0,1]}$ of the form

$f(x) = cx\,(1-x), c \in R$

5.2.1. Examples: Logistics Family Graphics

Given the family

$\boldsymbol{f_c: [0,1] \rightarrow [0,1]}\ / f_c\,(x)=cx(1-x), c \epsilon R$

we can see the behavior of f in differing iterations.

Example 1: c=0.75

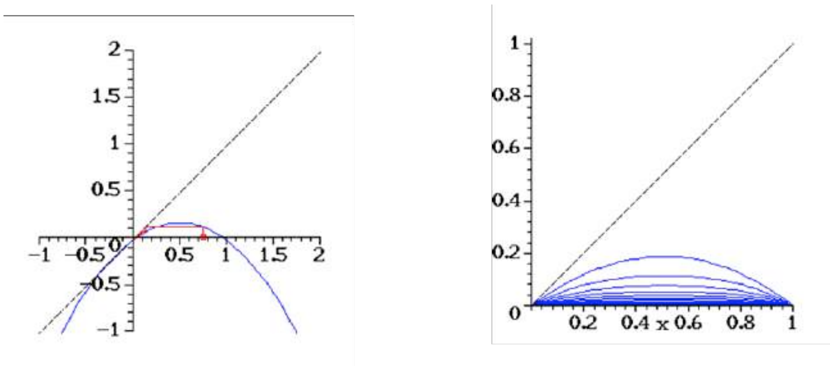


Figure 10. The interval [0,1] is invariant

Example 2: c=1

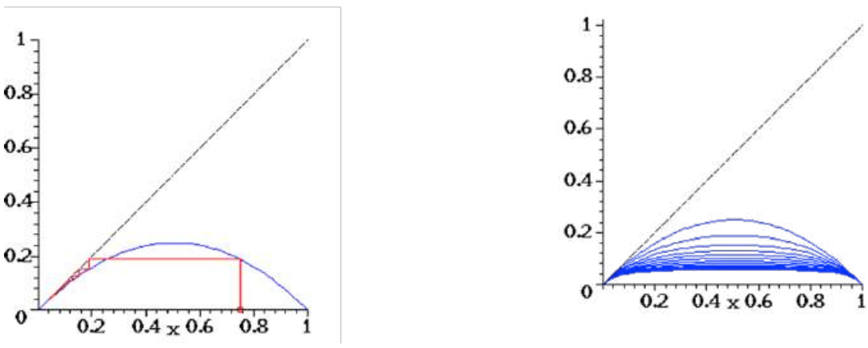


Figure 11. The point **0** has become indifferent although it still attracts the orbits of all points of (0,1). Point 1 is still fixed

Example 3: c=1.5

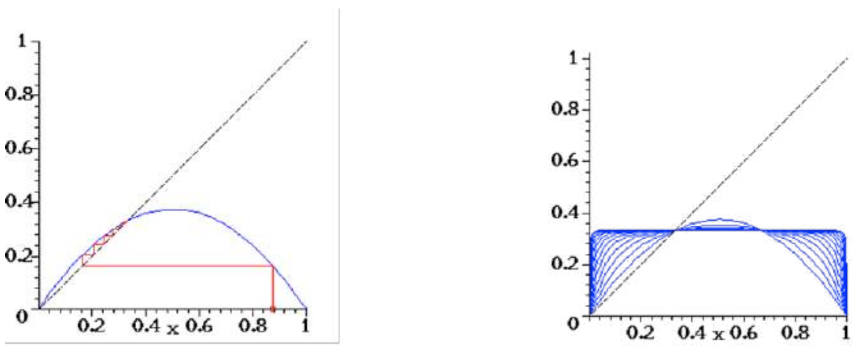


Figure 12. The orbit of any point of (0,1) tends to 1/3.

Example 4: c=2

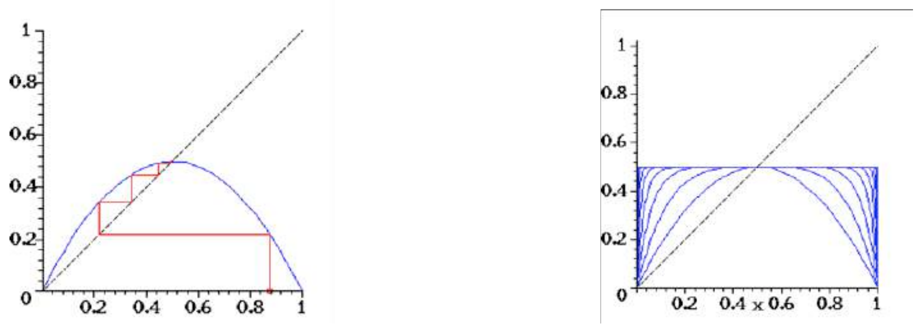


Figure 13. The orbit of any point of (0,1) tends to 0.5 which is now super attractive.

Example 5: c=2.5

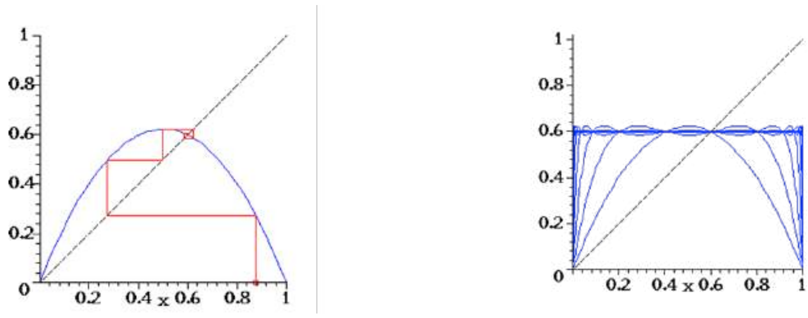


Figure 14. The orbit of any point of (0,1) tends to 0.6 which is an attractive point.

NOTE. The fundamental idea of this experience with secondary school students is to introduce dynamic systems. This is done by encouraging research on new concepts through mathematical models that are used to solve real-life problems.
In addition, in order to fulfil the task sufficiently, our students need to deepen their concepts of:

A) Fixed point
B) Eventually fixed point
C) Attractive point
D) Repulsive spots
E) Indefinite points

6. Conclusion

In this paper, I drew attention to certain mathematical models (as well as other abstract systems) that are defined as dynamic systems**.** They are applied to explain various phenomena of real-life situations. Furthermore, it showed that they are, possibly contrary to common beliefs, cognitively available to our secondary school students.

With the help of the presented case studies, I aim to provide answers to questions that are raised by the teachers of Mathematics. In general, teaching and learning science should include activities that enable the individual to construct (Blanch et al., 2004; Izquierdo et al., 1999) their way of feeling, thinking, speaking, and acting about the world around us by choosing those scientific models as one of the possible referential points. This is especially true regarding the teaching of mathematics. By applying these models, the path of the student should be guided by the path of research (Bonil et al., 2010). Furthermore, these models should strengthen the students' ability to contrast their self-generated information to the pre-existing information. Working on DSs showed to be one way to exercise the students' imagination or to understand, for example, certain approaches in ecology. A critical question that could be raised by students is *Do mathematical models provide answers to certain influences on the fundamental conditions of certain species?*

As a final conclusion, I would like to end with the following argument, quoted from the Real Decreto 1513/2006 (Ministerio de Educación y Cincia):
"... *The real possibility of using mathematical activity in contexts as varied as possible.*

Therefore, their development in compulsory education will be achieved to the extent that mathematical knowledge is applied spontaneously to a wide variety of situations, coming from other fields of knowledge and everyday life ... ".
"*... It is necessary to apply those skills and attitudes that allow mathematically reasoning, understanding mathematical argumentation and expressing and communicating in mathematical language, using the appropriate support tools, and integrating mathematical knowledge with other types of knowledge to give a better answer to the situations of life of different levels of complexity ...* "

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Mathematisation as didactic principle seen through teachers' descriptions of mathematical modelling

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Abstract. Considering the 'popularity' of mathematical modelling as an arena for using mathematisation as didactic principle, teachers' conceptions of mathematical modelling in relation to how they view their main task as mathematics teachers, as well what use the mathematics students learn at school might be, was investigated drawing on interview data. While the teachers generally separated 'reality' and 'mathematics', the potential diversity of mathematical descriptions and the problematic nature of any "translation" were generally not discussed. Overall goals of teaching mathematics such as understanding, interest and usefulness were emphasised rather than relating modelling to mathematisation as didactic principle.

Résumé. Compte tenu de la «popularité» que la modélisation mathématique a gagnée en tant que domaine d'utilisation de la mathématisation comme principe didactique, les conceptions des enseignants de la modélisation mathématique par rapport à la façon dont ils considèrent leur tâche principale en tant que professeurs de mathématiques, ainsi que ce que les étudiants en mathématiques apprennent à l'école pourrait être, a été étudié en tirant parti des données d'entrevue. Bien que les enseignants se séparent généralement de la «réalité» et des «mathématiques», la diversité potentielle des descriptions mathématiques et la nature problématique de toute «traduction» n'ont généralement pas été discutées. Les objectifs généraux d'enseignement des mathématiques tels que la compréhension, l'intérêt et l'utilité ont été soulignés plutôt que de rapprocher la modélisation de la mathématisation en tant que principe didactique.

a) Introduction

In mathematics education, mathematical modelling has become 'popular' as an arena for using mathematisation as a didactic principle making it possible to draw on real world contexts while not using standard word problems. As Jablonka and Gellert (2007), elaborating on mathematisation as a didactic principle, observe:

> Since problems given in textbooks generally do not claim to mirror problematic situations authentically, the mathematisation required from the students is essentially an artificial activity (Jablonka & Gellert, 2007, pp. 2-3)

Mathematical modelling is often promoted with the claim to provide potentially more authentic and hence less "artificial" situations as well as the idea of potential adjustment of a model. Mathematisation as didactic principle in this view is contrasted with modelling, as the latter is driven by extra-mathematical interests (Jablonka, 1996; Skovsmose, 1990). Referring to Basil Bernstein's notions of horizontal and vertical discourse and Treffer's (1987) descriptions of horizontal and vertical mathematisation, Jablonka and Gellert (2007) further write:

> The fiction is, that abstraction from extra-mathematical contexts to mathematical concepts and structures is possible and straightforward, but, actually, this process is a step from the contradictory world to a coherently organised esoteric sphere that has long since cut off its everyday roots. (p. 3)

Based on these observations, this paper looks at a group of teachers' conceptions of mathematical modelling in relation to their ideas of their main task as mathematics teachers and of what use the mathematics students learn at school might be.

b) Method

This paper draws on interview data collected by Frejd (2011), who investigated upper secondary mathematics teachers' conceptions of mathematical modelling, using a grounded theory approach[1]. From the re-analysis of the data presented here, linking the answers to the interview question A: *What is your interpretation of the notion of mathematical modelling?*, provided by the 18 teachers to their answers to the interview questions B: *How would you describe your main task as a teacher of mathematics?*, and C: *What use do you think your students will have of the mathematics they learn?*, this paper intends to illuminate these teachers' views of mathematisation, its relation to what they see as modelling, and/ or its importance as a didactic principle.

A thematic analysis of the available interview data seemed adequate for the rather open research interest of this paper. According to Braun and Clarke (2006), a thematic analysis is "a method for identifying, analysing and reporting patterns (themes) within data" (p. 78), giving it both theoretical freedom and flexibility. While assuming that what the teachers expressed during the interviews reflected their experiences and opinions on what was discussed, one also needs to point out that formulations of key notions and questions, as well as the interview settings themselves, are constituent elements of the interview discussions. As the overall contexts and social relationships that might explain the opinions put forward by the participants (latent themes) were not considered, this paper has tried to identify semantic themes that reflect the main patterns found in the data (Braun & Clarke, 2006, p. 86).

c) Results of the thematic analysis

In tables 1, 2, and 3, the results of the thematic analysis are summarised, including the themes and some examples of codes constituting these themes. In Table 4, the coding of the answers of each individual teacher to questions A, B and C are shown.

On question A, five main themes were defined, as illustrated in Table 1.

Table 1. Themes/ examples of codes, question A (interpretation of 'mathematical modelling')

A	Themes	Codes
A1	Describing or explaining something (in 'reality') in mathematical terms	Finding a mathematical function to describe an event Investigating some kind of association Using variables to simplify a relationship Translating an everyday problem, or any problem, to a symbolic language Making students formulate themselves mathematically Describing reality in a mathematical way
A2	Constructing a theory for some situation that you describe with math	Creating a model for how reality works Describing a system you do not know
A3	Making simulations	One can make mathematical simulations
A4	Applied tasks	Practical tasks A kind of formula to be able to calculate in a situation
A5	Open tasks and problem solving	Open tasks where students need to make own assumptions

The most common conception of mathematical modelling among the teachers has been identified as theme A1: *Describing or explaining something (in reality) in mathematical terms* (13 out of 18; cf. Table 4 below).

[1] See Frejd (2011) for a description of the selection of interviewees and the interviewing methods.

The reference to algebra, symbolic language, and mathematical function, suggest that this theme can be interpreted as referring to a mathematisation process involving some transition from a real-world context to the "esoteric sphere" of school mathematics. Notably, the teachers generally formulate this process as being unproblematic, as a kind of direct "translation" of a real-world situation into "symbolic language" (cf. Jablonka & Gellert, 2007, p. 5). The following excerpt, though, suggests that this process might not be seen as that 'simple' by all of these teachers:

> Einstein writes in his equations that he cannot solve them but he is superb in expressing himself in the symbolic language that is mathematics. That's what I interpret as mathematical modelling. (T7)

Some teachers, however, do not explicitly link modelling to specific (school) mathematics, but refer to theory construction or simulation techniques or use terms like "open problems" or "practical tasks". The themes A2 to A4 suggest an array of functions of modelling, from theoretical understanding, simulation to use of standard techniques. A5 points to making assumptions and explicitly refers to the classroom setting.

On question B, *How would you describe your main task as a teacher of mathematics?*, there appeared to be more variation in the answers than on question A, coded in 8 themes (see Table 2). A main concern for several teachers seemed to be the promotion of students' understanding of the mathematics they are learning (theme B1, 7 out of 18):

> The goal is to try to make everybody understand (T1)

Underlying this theme seems to be some teachers' concern about students finding mathematics "difficult", possibly related also to themes B2 and B3, as well as B6 (each 3 out of 18). While one teacher pointed to the more general "thinking models" that come with learning mathematics (B7), the three remaining themes are more utility oriented. None of the teachers discussed engaging in modelling and teaching related strategies as a goal.

Table 2. Themes and examples of codes, question B (main task as math teacher)

B	Themes	Codes
B1	Promoting students' understanding	Understanding different contexts Killing the myth that math is difficult Making it easy for the students
B2	Teaching the students as much mathematics as possible	Understanding their thinking to be able to teach what they are supposed to learn
B3	Stimulating the students	Making students interested Making students like mathematics
B4	Firm basis for further studies	That they get a good ground to build on
B5	Managing life in society	Mathematics useful outside school
B6	Fostering self confidence	What makes students not learn mathematics is the lack of self confidence
B7	Developing thinking models	A good way to train the brain thinking models
B8	Mathematics is necessary	We simply need mathematics

On question C, though, obviously with some overlaps with question B, several teachers pointed to mathematics as a subject promoting a way of thinking involving symbols, patterns, and logic (theme C5, 8 out of 18; see Table 3), a theme partly linked to theme C3 (only 1 out of 18). The most common answers, however, were coded within theme C2 (9 out of 18), emphasising mathematics as a tool in science and "everywhere", possibly linked to themes C1 and C6 (3 and 2, respectively, out of 18). Self confidence in mathematics was seen as useful in terms of what could perhaps be interpreted as a type of mathematical literacy (theme C4, 3 out of 18).

Table 3. Themes and examples of codes, question C (usefulness of math for students)

C	Themes	Codes
C1	Practical use in society	Mathematics is necessary Everybody has use of math in society
C2	A tool for use in other (school) subjects	Used in all sciences For use everywhere
C3	A general way of formulating and solving problems	This way of formulating problems one always needs in life
C4	Having mathematical self confidence	Check calculations they encounter in society Self confidence to use the tools when needed
C5	A way of thinking	Logical thinking; using a symbolic language; training thinking; thinking patterns
C6	Generally useful	Usefulness generally, in mathematics, science, societal
C7	Much of it is not useful	Most of it one would not need to manage one's life

From Table 4 it becomes clear that the answers to question C by some teachers were more comprehensive than the answers to A and B, with C2 and C5 often coming together. One can also observe that some teachers are coded differently on all questions (T5 and T6, T7 and T8, and others), suggesting very different conceptions on teaching goals and modelling.

Table 4. Distribution of teacher (T1, etc.) answers over the themes (A, B, C); for example, the answers by teacher T5 were coded as A1 and A3, B1, and C2.

Theme	T1	T2	T3	T4	T5	T6	T7	T8	T9
A	1	1	1	1	1, 3	5	1	4	2
B	1	4, 5	2	2	1	3	6	1	1
C	1	-	1, 2	4	2	5	5	4	2

Theme	T10	T11	T12	T13	T14	T15	T16	T17	T18
A	1	1	1	2	1	1	1	1	5
B	1	3	3	6	6	1, 5	2, 7	1	8
C	2, 5	2, 5, 7	2	2, 6	5	1, 3, 5, 7	2, 5	2, 4, 5	6

d) Discussion

While it is certainly the case that many teachers in this study did not elaborate a conception of mathematical modelling (Frejd, 2011), there is some considerable variety in their views on the function of mathematical descriptions or techniques with respect to 'reality'. What they have in common, however, is a sharp distinction between this 'reality' on the one hand, and mathematics on the other. Based on this, one could allege that they hold an epistemological view that is based on this duality. Mathematical modelling is seen as a providing some mirror of reality or, for some, mathematical techniques are directly useful for solving practical problems. The potential diversity of mathematical descriptions and the problematic nature of any "translation" were generally not discussed. Motivation of students might be one major reason for engaging with some forms of applied tasks. Mathematisation as didactic principle, with the goal of developing mathematical structures and methods, or mathematical modelling, as driven by particular interests and so producing a range of models for the same problem, were not explicitly separated. As a didactic principle, however, mathematisation was also interpreted in a formal sense, for developing generic thinking tools. Describing 'reality' (A1) was mostly linked to the theme promoting understanding (of mathematics) (B1). That also themes B2, B3, B5, as well as C2, were associated with A1 (see Table 4) indicate that for these teachers, drawing on real world contexts in the teaching of mathematics is seen to promote their overall goals of teaching mathematics such as understanding, interest and usefulness. In his study, from which these data were drawn, Frejd (2011) also asked the question *Why do you use modelling in your teaching?* to those (9) of the teachers reporting they do use it. Of the answers, one was coded as *To practice the transfer between different discourses*. To problematize this "transfer" is absent in the teachers' discourse.

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Investigative tasks: possibilities to develop teachers' technological pedagogical content knowledge

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Abstract. The aim of this paper is to discuss the research undertaken in a continued mathematics teacher education focusing on investigative tasks to teach spatial position geometry, using Cabri 3D software. The theoretical basis about the formative process came from Imbernón's ideas about teacher education in and to change, collective reflection from Zeichner ideas and from development of TPACK defined by Mishra and Koehler. The qualitative research, in the sense of Bogdan & Biklen, was part of a broader project of the Education Observatory Brazilian Program. The subjects were nine teachers and the investigative tasks focused here occurred in six four-hour meeting with the intention of providing reconstruction of teaching practice. Data collection was done by direct observation, audio and video recordings of the meetings and the materials produced by the subjects. Data analysis was interpretative. In this paper, the episode related to "Skew quadrilateral" task was analysed and established conclusions. We found evidence of reflection and construction of new references to the practice by teaching strategies with investigative tasks and use of digital technology.

Keywords: Continuing Education, TPACK, Reconstruction of Practice, Spatial Geometry, *Cabri 3D*.

Résumé. Le but de cet article est de discuter les résultats d'une enquête menée dans un processus de formation continuée des enseignants de mathématiques, axés sur les tâches d'investigation pour l'enseignement de la géométrie de position à l'aide du logiciel Cabri 3D. L'intention a été de fournir aux enseignants l'expérience avec des situations didactiques qui faisaient appel aux technologies, afin de les proposer une réflexion sur les processus d'enseignement et d'apprentissage de la géométrie. La base théorique du processus de formation se concentre sur les idées d'Imbernon en vue de former les enseignants « dans » et « pour » le changement (c'est pour instruire ces enseignants en ces temps d'incertitude et de changement social rapide); sur la réflexion collective, centrée dans les idées de Zeichner; et sur le développement de TPACK (technological pedagogical content knowledge – connaissance du contenu pédagogique et technologique), définit par Mishra et Khoeler. La recherche est qualitative, au sens donné par Bogdan & Biklen, et représente une partie d'un projet plus vaste appelé "Programa Educacional Brasileiro Observatório da Educação". Neuf enseignants ont participé de la recherche et les tâches d'enquête ont été développées pendant six réunions de quatre heures chacune, avec l'intention de rendre favorable la (re)construction de la pratique d'enseignement. La récolte des données a été réalisée par: l'observation directe, par des enregistrements audio et vidéo des réunions et par la collecte des documents produits par les neuf enseignants. Pour l'analyse de ces donnés, nous avons utilisé la méthode interprétative. Dans cet article, l'épisode qui nous avons choisi pour présenter c'est la tâche « Quadrilatère Reverse », lequel a été décrit, analysé et nous a permis d'établir des conclusions. Nous avons rencontré des évidences de réflexion et de construction de nouvelles références pour la pratique d'enseignement à partir de l'utilisation en tant que stratégies des tâches d'investigation et des technologies numériques.

Mots-clés: Formation Continue, TPACK, Reconstruction de la Pratique, La Géométrie Spatiale, *Cabri 3D*.

1. Introduction

The vertiginous social and technological changes of the late twentieth century and the early years of the twenty-first century have introduced a new paradigm of living in society. The different ways for people to communicate and relate, as well as the speed and ease of access to information have set up a new *modus operandi*. It is in this reality that young people are immersed and must deal with the technological artefacts in their future professional and social life. These changes have affected students' expectations regarding school and teachers' role.

The mathematics teaching practice needs to be reviewed considering this current scenario and the various technological possibilities that are available in the students' daily lives. In our schools, the use of technologies for teaching often serves for curriculum content illustration and appearance "modernization" without, however, modifying the teaching essence. We understand that the pedagogical use of technology involves the integration of different semiotic representations of the concepts, in a way that the manipulation and the exploration of objects help the establishment of relations, as well as the modelling and investigation of problem situations. (LOBO DA COSTA & PRADO, 2015). Thus, technology can be an ally to *think-with*, in the sense given by Papert (1994), favouring the construction of meaning.

A relevant pedagogical strategy in teaching is to use investigative tasks, which can be developed integrating technological resources. The experimentations with these resources are favoured by the manipulation/application of mathematical concepts, rising hypothesis and creating representations on the software interface. For example, the students as they investigate the conjectures posed, explore several mathematical possibilities, may find possible solutions, in addition, these investigations could also subsidize the validation of results found.

> [...] to bring to the classroom the spirit of genuine mathematical activity, thus constituting a powerful educational metaphor. The student is called to act as a mathematician, not only in the formulation of questions and conjectures and in the performance of tests and refutations, but also in the presentation of results and in the discussion and argumentation with his colleagues and the teacher (PONTE, BROCADO & OLIVEIRA, 2003, p.23).

In this sense, it is necessary that the formative actions, supported by Imbernón's (2006) and Zeichner's (1993) ideas, give the teachers the opportunity of *learning-by-doing*. That is, to experience investigative tasks using the technological resources and reflect on this learning experience, seeking to relate it to the possibilities of reconstructing their own teaching practice, from the construction of pedagogical, technological and mathematical content knowledge.

From this problematic, the continued education addressed in this study was designed to promote the discussion of investigative tasks involving technology. In this paper, we report on one of these tasks for the Space Geometry teaching.

2. The Research

The qualitative research, according to Bogdan & Biklen (1994), aimed to understand how investigative tasks can boost the construction of technological pedagogical knowledge of content - TPACK (Mishra & Khoeler, 2006). The tasks, developed in a process of continuing mathematics teacher education, focused on the teaching of Spatial Position Geometry, using *Cabri 3D software*. This education process was part of a broader Project of the Education Observatory Brazilian Program, financed by CAPES. The subjects were nine teachers and the investigative tasks focused here occurred in six four-hour meeting with the intention of providing construction of teaching practice. The instruments of data collection were: direct observation, protocols of the activities and audio and video recordings of the formative meetings. The analysis was interpretive.

In this paper, we discuss the development of an investigative task called "Skew Quadrilateral". This term is used is used here to indicate a quadrilateral with non-coplanar vertices and no-coplanar sides. We also use "skew lines" meaning nonparallel lines in space that do not intersect.

Initially we proposed that the teachers investigated with the *Cabri 3D software* the conditions for four points in the space being vertices of a quadrilateral. The group constructed several quadrilaterals and in the collective discussion the established conclusion was that the four vertices should be non-collinear points three-to-three. Teachers of the group stated that this may be a useful activity to discuss with their students the conditions of the vertices for the determination of a quadrilateral. Similarly, an investigative task could also be proposed to discuss the positioning conditions of points for the existence of a triangle with vertices at these points. However, all the quadrilaterals constructed by the group of teachers had the four vertices in the same plan. Using the software, the participants moved the vertices of the constructed quadrilaterals and investigated the different possibilities regarding the types of quadrilaterals and erroneously concluded, at that moment, that a quadrilateral is always a 2D figure. To allow the group in confronting the conclusion to a situation in which there is no plan to which all the vertices of the constructed quadrilateral belong. After that, we proposed a task aimed to lead them in investigating the existence of quadrilaterals that are not 2D figure. The task and an example of skew quadrilateral construction (AEGF) are in figure1.

Skew Quadrilateral

1) Construct a rectangular parallelepiped ABCDEFGH, using the *Cabri 3D*.
2) Consider 4 distinct and non-coplanar vertices of the parallelepiped and construct a quadrilateral.
3) Investigate the relative positions of the straight-line support on the sides of this quadrilateral, which is called the skew quadrilateral.
List the positions found.
4) Investigate and answer:
• Are the diagonals of a skew quadrilateral always in skew lines?
• Are two opposite sides of a skew quadrilateral always in skew lines?

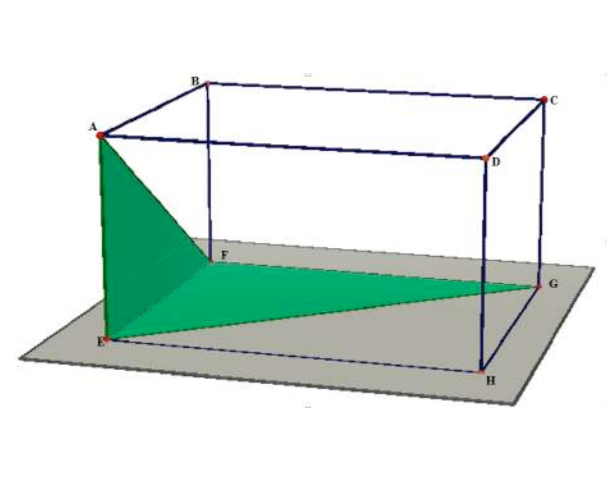

Figure 1. Skew Quadrilateral Investigative Task

Source: Adapted from Muraca (2011).

We observed that by the instructions of the task, the four vertices of the quadrilateral are non-coplanar points, which made it impossible to construct a 2D quadrilateral. At the time of the collective discussion, the participating teachers stated that they did not know this type of quadrilateral and some of them asked if this figure could still be considered as such a quadrilateral. It was necessary to return to the definition of quadrilateral for the establishment of consensus. Therefore, this investigative task helped to develop teacher's content knowledge. With regards to the requested investigation about the relative positions between the diagonals as well as opposite sides of the skew quadrilateral, the representation with the software and the movement that allowed for modifications of positions, both were essentials to promote changes in participating teachers' ideas and for the establishment of new conclusions. That is, the diagonals are always skew lines, the same occurring with two opposite sides. The following figure shows the diagonals representation of the skew quadrilateral and the diagonals support lines, which are skew lines.

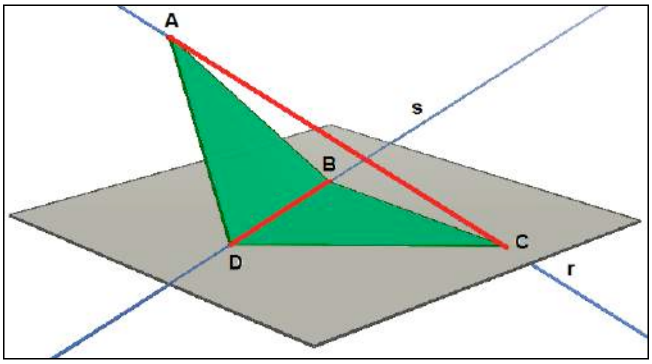

Figure 2. Diagonals' supports lines of a skew quadrilateral representation

Source: Muraca (2011 p. 129).

In the collective discussion, one of the teachers emphasized the possibilities of using this task as a didactic strategy to help his students to assign meaning to the skew lines. As for the sides of the skew quadrilateral, when asked about the positions found they were unanimous in saying that any two sides of the skew quadrilateral are either concurrent or are skew lines. Hence, the discussion focused on how to prove this statement. We reached the conclusion that this investigative task aided the process of concept construction, the representation on the software was able to provide a new vision related to the figure.

Continuing the investigations, we proposed the task described in Table 1.

Table 1. Task of analysis and rewriting of statements about the Skew Quadrilateral

Skew Quadrilateral Task
Read the sentences below; rewrite them to express true statements - that is, valid for all cases. Affirmation 1: Given the distinct coplanar points A, B, C, D, it is possible to construct a skew quadrilateral whose vertices are those points. Affirmation 2: Given the non-coplanar distinct points A, B, C and D, it is possible to construct a skew quadrilateral whose vertices are those points. Statement 3: It is possible to construct a skew triangle.

Source: Adapted from Muraca (2011).

The teachers were asked to use the software to investigate and analyse the didactic possibilities of this task before rewriting the sentences, listing the possible difficulties of their students in a similar task.

We notice that Affirmation 1 is false, since if the points are coplanar the quadrilateral will be contained in a single plane, so it is not skew. As for Affirmation 2, it is true, because if the four vertices are not in the same plane, there is no plane containing the quadrilateral, so it is a skew one. Finally, in relation to Affirmation 3, it is false, because the vertices of a triangle are always three non-collinear points, that is, they are always coplanar points, so the triangle is always a 2D figure, from which it is possible to conclude that there is no skew triangle.

The last of the three statements caused the most controversy. Seven of the nine teachers conjectured that their students would claim that if there is a skew quadrilateral should exist a skew triangle. Some of the teachers' observations were as follows:

> **Teacher A and Teacher B:** *It is possible to construct a skew triangle, if the three points are not coplanar.*
> **Teacher C:** *If you have the distinct points A, B and C not coplanar, you can construct a skew triangle whose vertices are those points.*

The analysis of the rewriting proposals of the participating teachers and the previous analysis before the application of this type of task to the students, allows us to conclude that, in the perception of these teachers, the explorations and investigations with the software are not enough to lead to the correct conclusion. Since in this case, the postulate of determining the plan needs to be accessed. However, they consider that the investigations and explorations allowed to pose and test conjectures, but it is necessary that such conjectures be validated in some way and the software alone does not play that role. At that moment they understand that it is the moment of pedagogical mediation.

The participating teachers, when analysing the potential of the tasks experienced, were unanimous in recognizing that they could potentiate the gathering of conjectures, exploration and research on the validity of statements made about relative positions, both between lines and between planes, and they can students to be better prepared to face demonstrations. Particularly, participant teachers considered the activity on the skew quadrilateral as great potential for the discussion with the students about positions between the lines in the space as well as plans. We analyse that the participant teachers mobilized technological, content and pedagogical knowledge during the meetings.

3. Final Considerations

This study evidenced possibilities for continued education in order to boost the construction / reconstruction of knowledge by the participants. The educational process provided experience of learning situations by investigative tasks, integrating the technological resources, helped not always to deepen the conceptual understanding but also to rethink teaching. Discussions throughout the tasks showed that as teachers gave meaning to their own learning using the tools of the software, they came to recognize that this could be a new way of learning and that this kind of learning may occur with their students. We found that during this formative period there were some indications of the construction of technological, pedagogical and mathematical knowledge by the participants as well as reflection on the practice, but in a limited way. This can be understood by the fact that the teacher's experience in the investigative task was centred on the learning itself (software, type of activity, content exploration, etc.). However, we consider that this experience and the sharing of the learning process itself with its peers and the trainer were fundamental to lead them to reflect on the possibilities for the teaching of Spatial Position Geometry envisaged by the investigative approach using technological resources.

This situation showed us that the process of continued education in the TPACK perspective, aimed at facilitating the reconstruction of the practice is not a simple process, nor does it occur immediately. It is necessary to contemplate continuity actions in the formative process giving the teacher an opportunity to construct new references for their practice and also to concretize them and to discuss the actions undertaken and the strategies used in practical situations, with their peers. However, the concretization requires new constructions of knowledge that the continued education should provide so that an investigative approach with the use of technology was put into practice with the students and the application discussed and shared in the group, so that, there is reflection on the practice towards its reconstruction.

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Problem-based learning: an investigative approach to teach optimization problems

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Abstract. This paper presents a research that aimed to identify possibilities of an investigative approach to teach functions from geometric problems. The proposal was carried out as part of a broader Brazilian project of the Programa Observatório da Educação and developed with a group of nine high school teachers from public schools of São Paulo city, throughout two four-hour meetings. The theoretical support related to problem-based learning (PBL) came from Savery's summary, investigative tasks from Ponte's studies and jeux des cadres from Douady definition. The activities were posed in the geometric framework and implemented using concrete materials and technological resources. The research methodology was qualitative; the data was comprised of teachers' productions (digital files and paper and pencil registers). The interpretative analysis highlighted that, in the participant teachers' view, these investigative activities may help the students to understand better the aspects related with the study of functions, to integrate several mathematics field contents and to develop research capacity and perseverance in the search for results, particularly regarding to the use of different strategies, validation techniques and results control.

Resumé. Cet article présente une recherche visant à identifier les possibilités d'une approche d'investigation pour enseigner les fonctions à partir de problèmes géométriques. La proposition a été réalisée dans le cadre d'un projet plus large au Brésil, le Programme Observatoire de l'éducation et a été développé avec un groupe de neuf enseignants du secondaire des écoles publiques de São Paulo, lors de deux réunions de quatre heures. Le soutien théorique lié à l'apprentissage par problème (PBL) provient du résumé de Savery, des tâches d'enquête des études de Ponte et des jeux de cadres de la définition de Douady. Les activités ont été posées dans le cadre géométrique et mises en œuvre avec des matériaux concrets et des ressources technologiques. La méthodologie de la recherche était qualitative, les registres composées de productions d'enseignants (fichiers numériques et registres en papier et crayon). L'analyse a souligné que, dans l'opinion des enseignants participants, ces activités d'investigation peuvent aider les élèves à mieux comprendre les aspects liés à l'étude des fonctions, à intégrer plusieurs contenus de mathématiques et à développer la capacité de recherche et la persévérance Dans la recherche de résultats, en particulier en ce qui concerne l'utilisation de différentes stratégies, techniques de validation et contrôle des résultats.

1. Introduction

A challenge for teachers while teaching mathematics is to adopt an approach that allows the student to assume an active and conductive attitude towards learning. In this sense, leading the student to adopt an investigative posture in relation to mathematics, to raise conjectures, test them, explore proposed situations, act autonomously, discover ways and validate found solutions provide a better understanding of the mathematization processes.

Problem-Based Learning (PBL) uses problems related to everyday life to stimulate research by and among students. Specialists elaborated these problems with the aim of developing skills foreseen in the school curriculum. This approach to teaching is particularly challenging, but it depends on the problems selected and the activities in which students engage, that should encourage them and allow independent and autonomous actions. Thus, we present a PBL proposal for the study of the variations in functions coming from geometric problems, which we developed in the framework of a broader project of the Observatory Program of Education of CAPES.

~~2.~~ Theoretical foundations

The theoretical foundation was built upon ideas summarized by Savery (2006) on Problem-Based Learning (PBL), Ponte (2003) on investigative tasks and Douady (1992) and Douady & Perrin-Glorian (1989) on "*jeux des cadres*".

For Savery (2006), PBL is an approach focused on student work and driven from real-world situations. For that reason, choose mathematical resources suitable for resolution providing opportunities like those that will be faced in life in society. Students must participate and collaborate with the group, which seeks to develop skills such as critical thinking. The activities should be finalized with a discussion of the concepts and principles used, to highlight the relevant general aspects.

Investigative tasks are open-ended and designed to be applied for teaching and learning purposes. We follow Ponte (2003, p. 94), for whom "investigating means working from issues that interest us and are not quite clear initially, but which we can clarify and study in an organized way." When developing investigative tasks, students should have moments of exploration, conjecture, hypothesis testing, justification and validation, with the accompaniment, help and guidance of the teacher. The implementation of such tasks can boost the exploration of mathematical ideas related to it, as well as mobilize diverse strategies for its solution and validation. Serrazina, Vale, Fonseca & Pimentel (2002) consider an investigative task as close to that of problem solving, as both seek to engage students in complex processes of thought, provide participants with an opportunity to develop both cognitive abilities and attitudes. Unlike the problem-solving process that is driven by a specific question to answer, the approach in an investigative process takes in account some other questions related to the situation.

The activities we propose for research involve problems dealt with in the scope of Geometry, more especially, area of plane figures, a context that leads to paths in which consider the use of Algebra, promoting a change of frames (*jeux des cadres*). For Douady and Perrin-Glorian (1989), the construction of mathematical knowledge is associated with the explicit use and adaptability of tools in a process of formulating and solving problems. In this process, the mathematical concepts intervene interactively, as object or as tool, characterizing the tool-object dialect. The changes of frames are the different formulations of a problem that may allow new access and different ways of coping with the difficulties identified in the resolution process through alternative tools or techniques not available in the previous frame. Given a problem formulated in a frame, the learner's experiences and knowledge may lead him to translate it into another frame and thus reinterpret the initial questions in this new context, establishing possible correspondences. For Douady (1992) an important characteristic of Mathematics is the ability to translate a problem into several frames: algebraic, numerical, geometric, analytical, etc ..., causing us to have several resolution tools. At the initiative of the teacher, a problem must be suitably chosen to allow the approach from different perspectives, thus increasing the possibilities of establishing relations and strategies of resolution. It is noteworthy that these are the main aspects explored in the activity here reported and analysed, which was extracted from the sequence of problems discussed in the continued education.

3. The Research

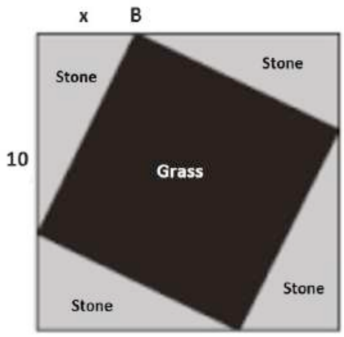
The research of qualitative methodology according to Bogdan & Biklen (1994), is characterized by being focused on the understanding of processes and meanings, with a detailed record of the object of study, being essentially descriptive. The research participants were comprised of nine mathematics teachers active at public high schools in São Paulo City. The data were collected through video recordings of the meetings and collection of the records produced by the teachers. The analysis was interpretive. Specifically, for the accomplishment of the investigative tasks, focus of this paper, two meetings of four hours each took place, in the scope of the other activities of the project. In the meetings, research tasks involving knowledge about functions and the calculation of areas were proposed, starting from problems stated in the geometric framework, which were first explored, discussed and analysed by teachers using both concrete material and *Geogebra* software. Then the teachers were invited to apply them in the classroom. The methodological procedures used by the researchers at the meetings were: (1) to propose problems that allowed the exploration and investigation of the situation posed, with the use of concrete materials and the use of *Geogebra*; (2) developing investigative activities that would lead to change of frames; (3) identify possibilities of an investigative approach to teach optimization geometric problems.

In this article, we discuss and analyse one of the research activities.

3.1 Description and Analysis

The optimization problem that subsidized the investigative activity is presented below.

A mayor wants to build a squared square of 10 m on the side, which will have four triangular stone garden beds and a square grass garden bed, as in figure.



The mayor has not yet decided what the grass area will be, so the length of the AB segment is indicated by x in the figure

A) Calculate the area of the grass garden bed to x = 2.

B) Write the expression from the grass area as a function of x and sketch its graph.

C) It is known that the grass garden bed costs R $ 4.00 per square meter and the stone garden beds cost R $ 3.00 per square meter. Use this information to answer the following two items.

What is the smallest amount of money the mayor must have to build the five garden beds?

If the mayor has only $ 358.00 to spend on the five garden beds, what is the area of the largest grass garden bed that the square can have?

Figure 1. Garden beds' problem

Source: OBMEP 2005 Collection - N3 - Phase 2

In order to guide the research activity, we proposed that teachers made a model of the square on paper and used this model to explore the problem.

Teachers have doubled the triangles to "inside" the square. In this way they could observe that, with the folds, a new square appears in the interior, as in the models of 'figure 2'.

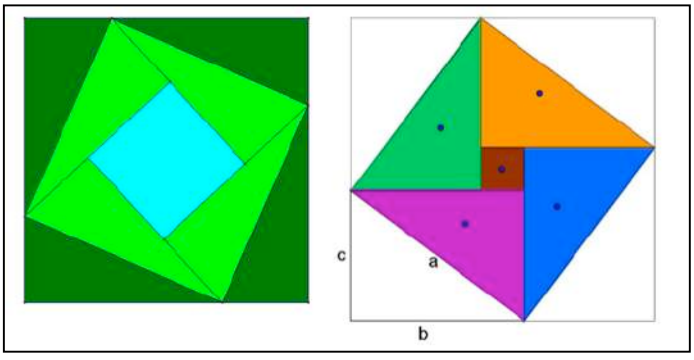
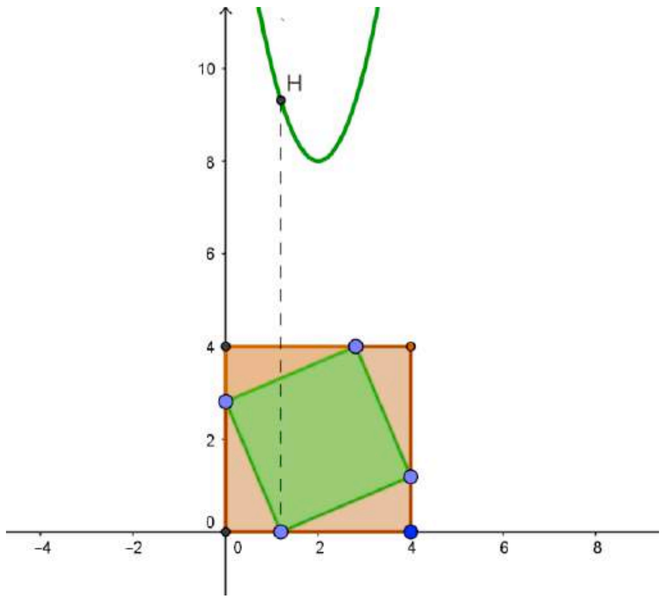


Figure 2. Paper foldings built for research
Source: Project "OBEDUC Practices"collection

Next, a model was elaborated using Geogebra; exploring this model the participants observed that the area of the inner square varied from an initial value to a minimum value and grew again. The teachers found the function that describes the area and prepared, in pairs, the resolutions to collectively discuss their conclusions.

The model (as in figure 3) allowed to investigate what happens with the area of the new square when we vary the segment chosen as the leg of the right triangle (cathetus) in the original square.



The research allowed us to investigate some aspects of the problem and to answer the question: "What is the function that expresses the area of the new square (the grass garden bed)? and also relate the variation of the garden bed area to the variation of the function found.

Figure 3. Imagem no Geogebra (Source: Project "OBEDUC Practices"collection)

It is noteworthy that the properties observed in the construction of the paper model supported the construction of the model in *Geogebra*, which in turn allowed new explorations and consequently added new conclusions. These manipulations guided the change of the geometric frame to the algebraic, to obtain the solution of the problem. There were three separate referrals for the calculations of the areas and the costs that were discussed, compared and validated in the whole group.

The analysis of the statements of the participating teachers evidenced the conviction about this approach in helping the students understand aspects related to the functions, in the establishment of connections between them and the areas of the geometric figures studied. In addition, it facilitated the integration of content in different fields of mathematics and the development of research capacity and perseverance in the search for results, especially regarding the use of different strategies and results control.

Here are some of the statements made by teachers:

> The student today has great difficulty to abstract and this proposal that he realize that by decomposing it is possible to transform one figure into another is interesting. (Prof. Al)
> The most difficult for them is to associate and analyse the mathematics involved. (Prof.Cel)

However, participating teachers warned that:

> For these investigations, the student must have constructed the concept of area, perimeter .... Knowledge that needs to be used in this situation. (Prof. Cel)
> It takes a job before to teach Geogebra, because in it you can move the point and hence maintain the properties of the figure [rectangle] ... (Prof. Al)

4. Final considerations

The analysis of the data allowed us to conclude that in this Problem-Based Learning approach we identify as possibilities for teaching and learning: the use of experiments and investigations with passages through concrete and *Geogebra*, which can attract Students' interest in learning, especially with the pre-construction work. The intradisciplinary work, since the activities enable to deal with several associated contents, such as ~~area,~~ functions, which helps to establish connections between such contents and between the geometric and algebraic frames.

Finally, it is worth mentioning that the participants in the teacher continuous education process emphasized that they realized the need for the teacher to prepare, to master the subject, to be open to the students' suggestions and to be aware that in solving the problems many unexpected strategies can emerge, as in this case, to calculate the area and estimate the costs. In this sense, the preparation, planning and performance of the teacher are fundamental for conducting a class from the perspective of the PBL and, on the other hand, it is expected that the student adopts an investigative stance, compatible with society increasingly mathematically which is inserted.

Acknowledgment

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La modélisation mathématique et processus de mathématisation dans la formation des enseignants

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Résumé. Le curriculum de Québec a une approche par compétences où la première à développer chez les élèves du secondaire dans la discipline des mathématiques est *la résolution de situations- problème*. Les enseignants et les auteurs des manuels scolaires étaient habitués à suivre un programme d'études axées sur la résolution de problèmes (en général problèmes fermés) et cela a impliqué une résistance au changement pour des raisons naturelles. Une situation problème, en principe, est un problème en contexte et de type ouvert, impliquant les processus de modélisation et de mathématisation pour sa résolution. Dans ce document, nous proposons un modèle qui impliquerait que les enseignants de mathématiques en formation puissent construire une structure structurante dans le sens de Bourdieu (dans un apprentissage socioculturel), qui permettrait la résolution de situations-problème et la construction du concept de covariation entre variables et les fonctions.

Abstract. In Quebec, the curriculum is based on competences. Firstly, the secondary-school-curriculum of mathematics entails situation-based problems. In the past, teachers and authors of school textbooks followed a programme of studies that focused on the resolution of problems (in general closed problems). This habit is hard to change. A situation-based problem, in principle, is an open-type-problembased on a specific context. In order to resolve situation-based problems, students need to apply processes of modelling and of mathématisation. In this paper, we offer a model which implicates that the teacher-trainersof mathematics construct a founding structure in the sense of Bourdieu (in a socio-cultural training) which would advance the resolution of situation-based problems and the conceptualizing of co-variation between variables and functions.

1. Introduction

Au Québec nous suivons un programme par compétences qui stipule que la première compétence est la résolution de situations-problème, dont les caractéristiques, sont : 1^er^ compétence : La situation n'a pas été présentée antérieurement en cours d'apprentissage; L'obtention d'une solution satisfaisante exige le recours à une combinaison non apprise de règles ou de principes dont l'élève a fait ou non l'apprentissage; Le produit ou sa forme attendu n'a pas été présenté antérieurement.. La 2^e^ compétence est de déployer un raisonnement mathématique et la 3^e^ doit promouvoir la communication avec un langage mathématique (MELS 2007, p. 22). Aussi, le ministère de l'Éducation stipule que l'enseignement doit suivre des hypothèses d'apprentissage socioconstructivistes.

Cette nouvelle approche pour les enseignants, les auteurs de manuels scolaires et les étudiants a donné lieu à une longue période d'adaptation (13 ans) dans le milieu éducatif. Le ministère de l'Éducation a décidé de suivre un cadre théorique de Perrenoud (1999/2000):

> Une compétence est une capacité d'action efficace face à une famille de situations, qu'on arrive à maitriser parce qu'on dispose à la fois des connaissances nécessaires et de la capacité de les mobiliser à bon escient, en temps opportun, pour identifier et résoudre de vrais problèmes. (MELS, 2007 p. 16)

Le ministère de l'Éducation a pris en compte le cadre théorique de Duval (1995) avec celui de Perrenoud.

L'approche théorique de Duval porte sur le renforcement des connaissances à partir d'une perspective constructiviste (centrée sur l'individu) ; un objet ou concept mathématique est construit chez l'individu à travers l'articulation entre différentes représentations de l'objet mathématique ou concept. Toutefois, en vertu de la promulgation par le ministère utilisant l'approche théorique de Duval, cette construction est liée à des représentations institutionnelles, et donc, est loin de faire référence à une compétence liée à la résolution de situations problème. Cette dernière affirmation est basée sur le fait que le cadre théorique ci-dessus, n'a pas pris en compte dans sa structure la construction des concepts via des représentations non institutionnelles qui peuvent jouer un rôle important dans les concepts de construction, de modélisation mathématique et processus liés à la mathématisation.

Nous nous demandons pourquoi le ministère de l'Éducation n'a pas privilégié la modélisation mathématique et les processus de mathématisation correspondants comme objet d'étude pour le développement de la 1re compétence dans la discipline des mathématiques.

2. La modélisation mathématique et les processus de mathématisation

Le paradigme de la modélisation mathématique a été étudié pendant des décennies, mais c'est dans ce siècle qu'il est considéré comme le plus important. Selon Lerman (2014) :

> Research results indicated that the identification of problem-solving strategies and the process of modeling their use in instruction was not sufficient for students to foster their comprehension of mathematical knowledge and problem-solving approaches. (p. 498)

Ainsi, les chercheurs se sont intéressés de plus en plus à ce problème (Blum et al. 2007, Quiroz et al. 2015; Quiroz 2016). La modélisation mathématique considère les processus de mathématisation lors de l'exécution d'une tâche mathématique. Par exemple Lerman (2014) mentionne que:

> Mathematization provides a particular challenge for mathematics education as it becomes important to develop a critical position to mathematical rationality as well as new approaches to the construction of meaning. (p. 442).

La recherche montre que les élèves en difficulté d'apprentissage ne parviennent pas à développer des éléments de contrôle dans la résolution qui leur permet d'atteindre un résultat cohérent exempt de contradictions (Saboya et al. 2015).

Notre modèle cherche à promouvoir la construction d'une structure structurante dans le sens de Bourdieu (1980) qui se traduirait par la construction des concepts et des structures de contrôle dans la résolution de situations problème en contexte ; la structure développée sous une approche socioculturelle de l'apprentissage. Pour ce faire, nous devons suivre une méthodologie dans la construction d'activités (voir Hitt, Saboya & Cortés 2017).

Notre modèle (Hitt, 2013) est immergé dans une perspective de la théorie de l'activité de Leontiev (1978). Compte tenu de la méthode d'enseignement ACODESA (Hitt-Gonzalez Martin, 2015, Hitt-Savoie-Cortés 2016, Hitt-Saboya-Cortés 2017) des activités *ad hoc* on été élaborées, permettant de guider l'enseignant(e) en formation et des étudiant(e)s dans les processus de mathématisation.

En 2007 (voir Hitt-Passaro), nous avons présenté les résultats des élèves de secondaire 2 et un groupe d'enseignant(e)s en formation en utilisant la même activité d'un randonneur autour d'une piste. Dans le cas des enseignants, en fonction de la piste proposée par eux-mêmes (situation du type ouvert), la difficulté algébrique peut être plus dans un cas que l'autre. Si les représentations spontanées ont été produites par une équipe faible, cette équipe n'avancera pas dans le processus de mathématisation. Voici les questions proposées aux futurs enseignants, questionnaire réduit (avec les élèves nous avons demandé aussi les processus inverses que consiste du passage d'une représentation graphique à un dessin qui détermine l'emplacement du mât, voir Hitt et González-Martín 2015).

Page 1 Renseignements personnels et de l'équipe	

Page 2

*Un randonneur entreprend une longue randonnée en forêt. Il suit une **piste** qui lui permet de revenir à son point de départ à la fin de la randonnée. Durant sa promenade, il ne repasse **jamais au même endroit** et il ne fait qu'un seul tour de piste.*

*Un **poste de secours** est situé **à l'intérieur de la région délimitée par la piste**. Un grand mât avec un drapeau permet au randonneur de repérer l'emplacement du poste de secours quel que soit l'endroit où il se trouve sur la piste.*

Trace une piste de ton choix et place le poste de secours à l'intérieur en respectant l'énoncé.

Ma piste de randonnée :

Page 3
La distance entre le randonneur et le poste de secours varie selon l'endroit où se trouve le randonneur sur la piste. Décrit cette variation

Page 4
Trouve une nouvelle manière de présenter le phénomène décrit à la page 3 sans que le dessin de la piste y apparaisse.

Page 4 continuation
En utilisant la réponse à la question précédente, décrit en mots de quelle façon la distance entre le randonneur et le poste de secours varie selon les endroits où se trouve le randonneur sur la piste.

Cependant, un exemple d'une proposition de covariation entre variables impliquant une relation non fonctionnelle avait été donné par une équipe qui est tombée dans des contradictions sans fournir une approche cohérente. Une équipe de futur enseignant(e)s a proposé la troisième figure (voir plus bas), demandant des processus algébriques plus complexes, la construction d'une fonction en parties (Variables : distance parcourue et distance de la personne au mât).

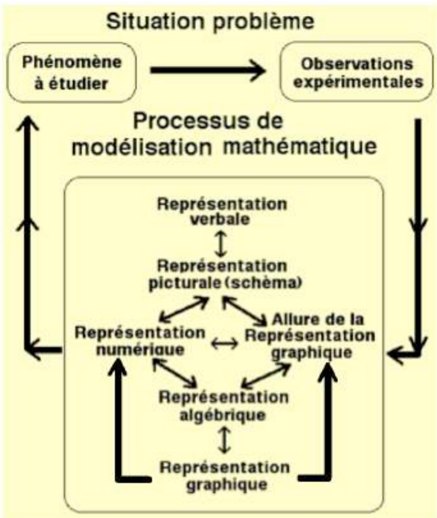


Figure 1. Modèle suivi dans l'expérimentation

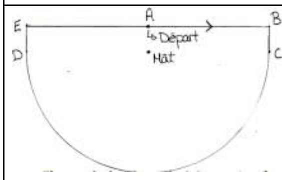
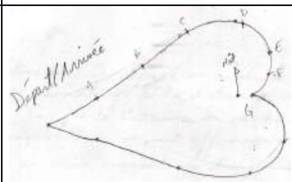
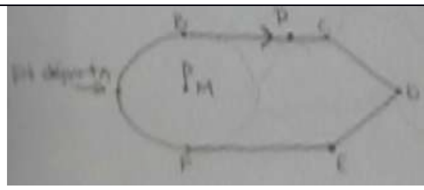
Niveau élémentaire	Niveau intermédiaire relation non fonctionnelle	Niveau avancé
		

Figure 2. Différents productions des étudiants

3. Discussion

Notre proposition est immergée dans une perspective socioculturelle, que favorise l'utilisation d'objets physiques (par exemple la corde dans la covariation entre les variables : distance parcourue et distance de la personne au mât, avec des pistes irrégulières), favorise l'émergence de représentations spontanées dans un processus de communication dans la salle de classe, et cette communication permettra son évolution vers les représentations institutionnelles.

Compte tenu d'un programme de mathématiques axé sur le développement des compétences mathématiques, et de résolution de situations-problème, nous proposons la construction d'une structure structurante qui permettrait le développement des compétences dans les processus de modélisation mathématique et processus de mathématisation.

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Student-teachers' re-inventing mathematisation as a didactic principle

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Abstract. As part of their field experiences in a mathematics education masters program, student-teachers conduct mini teaching experiments with individual students. In this paper, we reflect on the experiences we had with three student-teachers who worked with a sixth grade student on the multiplication of fractions. Their goal was to support the student's development of understanding the multiplication algorithm that she had already learned. Overall, initial results indicate that these student-teachers came to be involved in practices that supported the development of a new way of designing their instruction as they strove to assist the student in mathematising her activity. On the other hand, the analysis of this case study brings to the fore some of the challenges involved in fostering mathematisation as a didactic principle.

Résumé. Les étudiants-enseignants d' un programme de maîtrise en mathématiques effectuent de petites expériences d'enseignement avec des étudiants individuels, dans le cadre de leurs expériences sur le terrain. Dans cet article, nous réfléchissons aux expériences que nous avons eues avec trois étudiants-enseignants qui ont travaillé avec une élève de la sixième année de l'école primaire sur la multiplication des fractions. Leur objectif était de soutenir le développement de l'élève pour comprendre l'algorithme de multiplication qu'elle avait déjà appris. Dans l'ensemble, les résultats initiaux indiquent que ces étudiants-enseignants sont devenus impliqués dans des pratiques qui ont soutenu le développement d'une nouvelle façon de concevoir leurs instructions alors qu'ils s'efforçaient d'aider l'élève à mathématiser son activité. D'autre part, l'analyse de cette étude de cas met en évidence certains des défis impliqués dans la promotion de la mathématisation en tant que principe didactique.

1. Introduction

Two years ago (starting in 2015) we embarked on a research project that seeks to understand how to support teachers in developing an inquiry-based and student-centered approach in teaching mathematics. This project is conducted with a group of teachers (two in-service and 12 pre-service) in the context of a masters program. In the previous year, our primary goal was to support them in starting to develop a vision of instruction in which building on students' reasoning would be necessary for supporting progressive mathematisation of their ideas. Activities in which teachers were engaged gave them opportunities: (1) to enhance their understanding of fractions, (2) to learn how to interpret students' ideas, (3) to rehearse aspects of the practices they were supposed to develop, and (4) to sensitize themselves to the constraints of the institutional setting of schooling in our country. A preliminary analysis of our data shows that teachers had already started to re-organize their knowledge and beliefs. This year (2016 – present) we organized a design experiment concerning their field experiences in public schools of Athens. Our aim was to investigate how to support teachers in starting to reorganize their practices, as they are involved in a cycle of creating, implementing, and revising their plans for teaching.

One of our primary considerations in organizing our student teachers' field experiences was the institutional setting of the schools (Cobb, McClain, Lamberg, & Dean, 2003). In the centralized educational system of our country, the materials (textbooks and teacher guides) that teachers use in their classrooms are guiding them towards instruction that is mainly focused on procedural fluency, rather than on conceptual understanding. We judged that student teachers should interact with students over an extended period of time. Otherwise, they would only come across students' typically procedural reasoning. By engaging the student-teachers in conducting mini teaching experiments (8-10 half-hour meetings) with individual students, we hoped that they could familiarize themselves with the challenges of supporting the students' mathematical growth. Importantly, their experiences might be extremely useful, as most of their future students would be coming from classrooms with a procedural orientation.

In this paper, we focus on a group of three student-teachers that worked with a sixth-grade student on the multiplication of fractions. Though the child knew how to multiply two fractions, her conceptual understanding of the algorithm was absent. Thus, the group decided to focus their work on supporting her development of an increasingly sophisticated understanding of the procedure. The fact that the end product of mathematisation was already known makes the analysis of this case particularly interesting. Our goal of analysis was to identify some of the key challenges involved in student-teachers' coming to use progressive mathematisation in teaching, as well as in our attempts to support them.

2. Conceptual framework

Student-teachers' field experiences are typically organized on the basis of a particular assumption concerning the relation between theory and instructional practice (cf. Cobb & Bowers, 1999). Student teachers are called to apply the theoretical principles they already know from their university courses to their instructional practice. Within this perspective, the principles of a theory, such as Realistic Mathematics Education instructional theory (Gravemeijer, 1994; Van den Heuvel-Panhuizen, & Drijvers, 2014), could be directly transferred to practice. In contrast to this assumption, we viewed student teachers' activity in conducting their teaching experiments as a source for re-inventing these principles. Thus, based on our ongoing assessments of their activity, we identified some of these principles as goals of their learning. On the other hand, research in practice-based teacher education was a basis for designing supports for student teachers' development of their practices (Ball, Sleep, Boerst, & Bass, 2009; Borko, 2004; Grosssman, Hammerness, & McDonald, 2009; Lampert et al., 2013). In particular, this research literature indicates that teacher education should include opportunities: (1) to enact the targeted practices in practical settings of graduated difficulty with the support of skilled others, (2) to analyze and critique representations of the targeted practices, (3) to implement instructional materials that can be conducive to the desired practices, and (4) to work as a community of learners with the guidance of skilled others.

3. Methodology

In approaching our work as a professional design study (Cobb, Jackson, & Dunlap, 2014), we set three interrelated goals for student-teachers' learning[2]. These goals were oriented towards the progressive mathematisation of students' activity and included: (1) setting clear goals for students' learning, (2) selecting and sequencing tasks in relation to each other, and (3) facilitating students' learning by encouraging them to express their reasoning and building on it.

To support the student-teachers' realizing these goals, we asked each group of them to complete specially designed forms concerning their plans for each of their sessions with the children as well as their evaluation of each meeting. Apart from giving them written feedback on these forms, we engaged them in discussing their plans and evaluations during our weekly meetings at the university. In these meetings, student-teachers were also called to analyze selected short video-clips of their prior sessions with the children. In short, it was their involvement in pedagogies of investigation and enactment (Grossman et al., 2009) that constituted the design of supporting the development of their practices.

The collected data to analyze and to document student-teachers' progress consisted of video-recordings of their sessions with the children and of our weekly meetings, as well as the student-teachers' planning and evaluation forms.

In analyzing student-teachers' learning, we tried to find evidence regarding how and to what extent their practices were changing in terms of the set goals. In order to identify the challenges they faced in their attempts to support students' learning, we also focused on analyzing the role of our supports in terms of setting clear goals for student-teachers' learning, continuously assessing their progress, and asking them to justify their pedagogical decisions.

4. Expected conclusions

Initial results of the ongoing analysis suggest that the three student-teachers on whom we focused our analysis showed considerable progress across the goals we had identified for their practices. Notably, it appears that the student-teachers would need further support, with respect to setting clear goals for their student's learning, and the sequencing of tasks based on assessments of her progress.

[2] The identification of these goals drew upon the research literature on the development of teaching practices, as well as upon difficulties detected as they were rehearsing activities among themselves in their prior semester.

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Pre-service elementary school teachers' ideas about fractions

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Abstract. This paper focuses on a study carried out to analyse pre-service Primary school teachers' knowledge about fractions. It addresses three questions: 1) What ideas do Pre-service Primary School teachers have about fractions? 2) How do these teachers understand ordering and equivalence of fractions? And 3) How these teachers understand the representation of fractions? A survey was carried out with 86 pre-service Primary school teachers from several parts of Portugal. The survey comprised 52 questions related to the concept of fraction, the invariants of rational numbers (ordering and equivalence), fraction representations, and the interpretations of fractions. Results revealed that Pre-service Primary school teachers have several difficulties concerning fractions. Several difficulties were identified in these pre-service teachers: some do not recognise the need to divide the whole into equal parts in fractions representations; they find difficult to relate parts of unit to 2 or more units; the property of the density of rational number set is not recognised by students, many believe they can count the number of fractions between 0 and 1; many cannot find the double of a given fraction; and many of them revealed difficulties with the interpretations of fractions, in spite of having to teach them in a close future.

Résumé. Cet article se concentre sur une étude réalisée pour analyser les connaissances préalables des enseignants du primaire sur les fractions. L'étude aborde trois questions: 1) Quelles idées les enseignants des écoles primaires ont-ils des fractions? 2) Comment ces enseignants comprennent l'ordre et l'équivalence des fractions? Et 3) Comment ces enseignants comprennent la représentation des fractions? Un sondage a été réalisé avec 86 enseignants du primaire de plusieurs régions du Portugal. L'enquête comprenait 52 questions liées à la notion de fraction, les invariants des nombres rationnels (ordre et équivalence), les représentations de fraction et les interprétations des fractions. Les résultats ont révélé que les enseignants de l'école maternelle ont plusieurs difficultés en ce qui concerne les fractions. Plusieurs difficultés ont été identifiées dans ces enseignants de pré-service: certains ne reconnaissent pas la nécessité de diviser l'ensemble en parties égales dans les représentations de fractions; Ils trouvent difficile de relier des parties de l'unité à 2 unités ou plus; La propriété de la densité de l'ensemble des nombres rationnels n'est pas reconnue par les étudiants, d'autres croient pouvoir compter le nombre de fractions entre 0 et 1; Beaucoup ne peuvent pas trouver le double d'une fraction donnée; Et beaucoup d'entre eux ont révélé des difficultés d'interprétation de fractions, en dépit d'avoir à leur enseigner dans un proche avenir.

4. Framework

This study aims to understand pre-service teachers' knowledge of fractions. For that, it addresses three questions: 1) What ideas do future Primary School teachers have about fractions? 2) How do these teachers understand ordering and equivalence of fractions in different interpretations of fractions? And 3) How these teachers understand the representation of fractions in those interpretations?

The concept of fraction is considered fundamental for a successful and proper development of children's mathematical thought. Nevertheless, is also one of the most complex concepts that children learn during the elementary grades. Knowing the concept of fraction demands the understanding of the logical aspects of fractions (ordering and equivalence) and the ability to use distinct modes of representation, in different interpretations of this concept (Behr et al., 1983; Nunes, Bryant, Pretzlik, Wade, Evans & Bell, 2004; Mamede & Nunes, 2008).

Pre-service teachers must be competent in the domain of rational numbers in order to be able to develop fruitful practices with their primary school students. But are future teachers adequately prepared to teach rational numbers to their students? Little research has been developed in order to explore this issue regarding the Portuguese reality.

In the domain of rational numbers, the Portuguese mathematics curriculum for primary school demands the use of mathematical and pedagogical knowledge quite challenging for teachers. They are supposed to be fully acquainted with the representation, ordering and equivalence of fractions, as well as with different interpretations of fractions. Teachers are also supposed to help students to establish the link between different representations of rational numbers (fractions, decimals, percentage), and compute with fractions and decimals.

2. Teachers' difficulties with fractions

Literature suggests that very often teachers have the same difficulties of their students and have the same misconceptions (see Lamon, 2003). Previous research conducted with Portuguese elementary school teachers has revealed that teachers have several difficulties concerning the teaching of rational numbers. These difficulties comprise conceptual and didactical features. Pinto and Ribeiro (2013) analysed 27 pre-service teachers' mathematical knowledge about fractions – interpretations of fractions, representation of fractions, the concept of unit, ordering and equivalence of fractions, and the density of the rational number set. Results indicate that most of the teachers felt comfortable only with part-whole interpretation of fractions; only 50% of them could identify the unit when a part of the whole was given; 43% and 97% of teachers revealed problems with equivalence and ordering of fractions, respectively; and 73% of them possess erroneous ideas concerning the density of the rational number set.

Concerning the didactical issues, Cardoso and Mamede (2013) interviewed two primary school teachers to understand how they explored the concept of fractions with their 3rd graders, when quotient interpretation is involved, as recommended by the curriculum guidance. Their results showed that teachers did not possess any lesson plan to teach fractions, because usually they do not teach them. Teachers believe that their students were not capable of learning all the content included in the curricula, nevertheless they had never tried to teach them fractions before. Perhaps the devaluation of students' abilities could be hiding some of the teacher's difficulties with fractions. Indeed, one of the teachers could not identify a pictorial representation of a fraction in a quotient interpretation context. These authors pointed out that the fragility of teachers' knowledge regarding these issues compromises their teaching of rational numbers (see Cardoso & Mamede, 2013; Pinto & Ribeiro, 2013), and consequently, the development of students' rational number sense.

In this scenario, it becomes of utmost importance to have an insight into pre-service primary school teachers' preparation to teach fractions. The identification of pre-service teachers' difficulties is relevant to improve the future teaching and learning of rational numbers, as it interferes with the mathematisation process of the students.

3. Methods

A survey was conducted with 86 pre-service elementary school teachers (mean age: 22 years, 5 months), from several parts of Portugal. The survey comprised 52 questions related to fractions (18 related to the concept and properties of fractions; 13 concerning the invariants of rational numbers - ordering and equivalence; 13 about the representation of fractions and concept of unit; and 8 about the interpretations of fractions).

4. Results

The mean of correct responses was 31,5 (standard deviation of 6,56). Table 1 resumes the means and standard deviation of the proportions of pre-service teachers' correct responses.

Table 1. Mean (standard deviation) of proportions of correct responses by each Type of questions.

Type of questions	Mean (standard deviation)
Concept of fraction	.68 (.17)

Invariants of fractions	.60 (.22)
Representation of fractions and the concept of unit	.58 (.21)
Interpretations of fractions	.62 (.15)

There were 18 questions concerning the concept of fractions, but only 2 teachers (2,3%) could succeed in all of them; only 31,4% of the teachers could succeed in at least 75% of the questions.

In one of the questions regarding the concept of fractions, most of the participants presented an incomplete answer when asked to identify pictures that could represent 1/3, as those of the Figure 1.

Figure 1. Pre-service teachers were asked to circle the pictures that represent 1/3.

Only 12 teachers (14%) presented a total correct answer; 70 (81,4%) presented an incomplete answer considering only the last picture; and 4 presented responses indicating that the size of the part into which the whole is divided is irrelevant to represent fractions. In another question related to the concept of fraction in which it was asked "How many ninth are there in 3 units?", 31 participants (36%) answered correctly, 18 teachers (20,9%) presented no answer, 12 participants (14%) referred '27/9', and almost 29% gave other incorrect answers. When asked "How many fractions are there between 0 and 1? Justify your answer.", only 36 teachers (41,9%) answered correctly, 37 presented no answer (43%), and 13 presented an incorrect response (15,1%). Pre-service teachers' justifications reveal that some of those who gave a correct answer could not present a written justification, only 13 gave an explanation based on the property of the density of rational number set (15,1%); 47 presented no justifications at all (54,7%), and the remaining justifications were incorrect.

When asked whether "4/8 is 2 times bigger than 2/4", 34 participants (39,5%) believed that that was true, but 52 considered the sentence false; when they were asked whether "4/8 results from multiplying 2/4 by 2", 40 pre-service teachers (46,5%) could not recognise it as a false sentence. This reveals some of the difficulties of prospective teachers with fractions. It is not expected that future elementary school teachers could accept 4/8 as the double of 2/4.

Regarding the invariants of fractions, 13 questions were presented to the pre-service teachers concerning ordering and equivalence of fractions, but only 1 teacher (1,2%) could succeed in all of them, and 10 teachers (11,6%) succeeded in 12 questions. Only 9,75% of the teachers could succeed in at least 75% of the questions.

Concerning the invariants of fractions, in the problem of the Figure 2, teachers had to compare fractions in each situation. In the case A) 56 teachers (65,1%) identified the correct fraction, 12 gave a wrong answer (14%) and 18 students (20,9%) could not answer; in the case B) 47 students (54,7%) identified the correct fraction, 14 gave a wrong answer (16,3%) and 25 students (29,1%) could not answer; in the case C) 52 students (60,5%) identified the correct fraction, 9 gave a wrong answer (10,5%), and 25 students (29,1%) could not answer; in the case D) 49 students (57%) identified the correct fraction, 14 gave a wrong answer (16,3%) and 23 students (26,7%) could not answer.

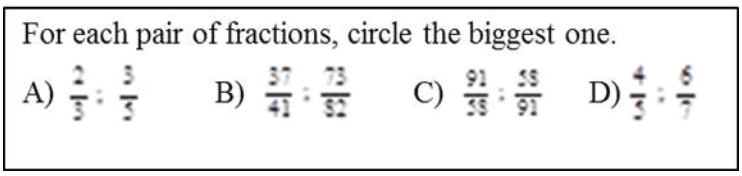

Figure 2. A problem of ordering of fractions presented to students.

In the case A) 30 pre-service teachers were not able to compare correctly 2/3 and 3/5, which are fractions that children use in the 3rd grade; probably, for the same reason 37 teachers failed in the case D), comparing 4/5 and 6/7. In some of the correct answers of these questions, students had to reduce fractions to the same denominator to compare them in order to produce a correct answer. Nevertheless, 4th graders cannot use this procedure to compare fractions. In case B), 47 pre-service teachers (54,7%) presented a correct answer, but 39 failed to compare these fractions (45,4%), being unable to recognise, for instance, that one denominator is the double of the other, but the correspondent numerators are not. In the case C), 52 teachers (60,5%) succeeded, but 34 (39,6%) could not realise that one of the fractions was smaller than 1 and the other was bigger. This lack of knowledge when dealing with fractions compromises the implementation of meaningful classes about rational numbers conducted by these pre-service teachers. More results will be presented in the conference regarding pre-service knowledge about fractions.

5. Final remarks

Pre-service elementary school teachers need to improve their ideas about fractions. Several difficulties were identified in these pre-service teachers: some do not recognise the need to divide the whole into equal parts in fractions representations; they find difficult to relate parts of unit to 2 or more units; the property of the density of rational number set is not recognised by students, many believe they can count the number of fractions between 0 and 1; many cannot find the double of a given fraction; and many of them revealed difficulties with the interpretations of fractions, in spite of having to teach them in a close future. Discussion and the educational implications of our findings will be presented in the conference and in the version of the paper for the proceedings.

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Comment interpréter le cycle de modélisation avec l'Espace de Travail Mathématique ? Etude de la trajectoire d'un problème.

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Résumé. À travers la trajectoire d'un problème, le lièvre et la tortue, nous étudions comment le cycle de modélisation peut être interprété à l'aide du modèle des Espaces de Travail Mathématiques. Il s'agit de préciser le rôle de l'enseignant dans la circulation du travail mathématique lors de l'activation d'un cycle de modélisation.

Abstract. Through the problem's trajectory, the hare and the turtle, we study how the cycle of modelling can be interpreted in the Mathematical Working Space model. It's a question of specifying teacher's role in the circulation of the mathematical work during the activation of a modelling cycle.

1. Introduction

Notre recherche porte sur l'intégration de la simulation informatique par des enseignants au niveau grade 9 après une formation sur les probabilités et statistiques. Elle prend appui sur la situation « du lièvre et de la tortue ». On lance un dé : s'il tombe sur 6 alors le lièvre gagne sinon la tortue avance d'une case. Kiet (2015) et Gaydier (2013) ont analysé le travail de l'élève dans cette situation et notre étude s'intéresse à l'enseignant, s'appuyant sur la trajectoire de problème (Kuzniak & al., 2013). Elle recouvre l'ensemble des processus de transformation d'un problème à travers diverses institutions allant d'une formation pour des enseignants à sa mise en place dans les classes. Ce problème peut être résolu avec différents modèles mathématiques. Nous examinerons la circulation du travail mathématique provoquée par l'enseignant dans cette activité de modélisation ; notre analyse s'appuiera sur le cadre des Espaces de Travail Mathématique (ETM) (Kuzniak, 2011) et du cycle de modélisation (Blum & Leiss, 2007).

2. Cadre théorique

L'analyse a priori des modèles utilisés par l'enseignant et de l'implémentation de la situation en classe s'appuient sur les Espaces de Travail Mathématique (ETM). Ce modèle permet de repérer la spécificité du travail mathématique. Un ETM est une structure d'accueil de ce dernier et repose sur l'articulation de deux plans : celui épistémologique et celui cognitif. Le premier s'articule autour de trois pôles : le référentiel théorique (propriétés, théorèmes et définitions), le représentamen (signes) et les artefacts matériels ou symboliques (logiciel, techniques de calculs). Le plan cognitif contient trois pôles : les processus de visualisation, de construction et de preuve.

Cet ensemble est représenté par un diagramme formé d'un prisme à bases triangulaires dont les arêtes verticales font le lien entre ces deux plans et définissent :

- la dimension sémiotique, reliant représentamen et visualisation
- la dimension instrumentale, reliant artefact et construction
- la dimension discursive, lien entre référentiel et preuve.

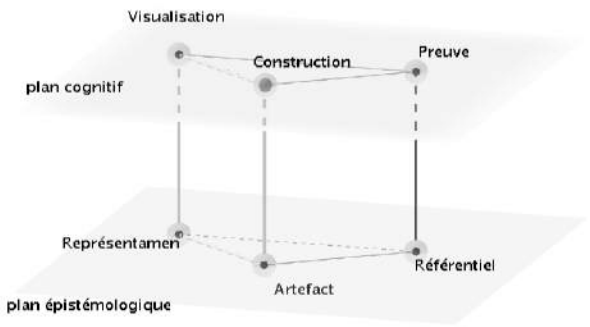


Figure 1. Modèle des Espaces de Travail Mathématiques (Kuzniak, 2011)

Le modèle des ETM permet d'identifier les genèses activées par l'enseignant au regard de celles favorisées par les élèves. Nous analyserons les plans activés dans le déroulement de la résolution de la tâche pour décrire la circulation à travers les plans [Sem-Dis], [Sem-Ins] et [Ins-Disc]. Notre but est de repérer en quoi le travail organisé par l'enseignant pourrait ne pas être complet (Kuzniak et al., 2016). L'étude des ETM idoines prévus pour une classe (potentiels) et de ceux mis en place par des enseignants (après formation) nous permettra également de préciser certaines connaissances spécifiques nécessaires à l'enseignant pour mener à bien cette situation.

L'implémentation de cette situation en classe sera aussi décrite avec le cycle de modélisation (Blum & Leiss, 2007). Quels ajustements sont possibles au sein du cycle et en quoi la responsabilité de l'enseignant est-elle engagée ?

3. Méthodologie de recherche

Nous avons donc choisi d'analyser différentes mises en œuvre de cette situation à travers la notion de trajectoire d'un problème (Kuzniak et al., 2013). Un avatar est une réalisation particulière envisagée par un professeur. Nous avons choisi un premier avatar, qui nous permettra de décrire les choix d'un professeur pour sa classe. Nous préciserons le modèle mathématique retenu par celui-ci, et décrirons la circulation prévue a priori dans l'ETM. Grâce à l'observation de sa séance, nous mettrons en évidence certains blocages et rechercherons les facteurs qui empêchent éventuellement un travail complet. Pour ce faire, nous nous appuierons sur les dimensions et plans du modèle des ETM.

Nos données émanent d'un questionnaire, d'entretiens pré et post séance de classe. Cette dernière a été filmée et couplée à des enregistrements sonores.

4. Description du travail prévu dans l'ETM idoine potentiel

L'énoncé présentait un parcours avec cinq cases et une d'arrivée. La séance était imaginée sur 2h.

Phase 1 exploration de la situation :(10 min) recherche individuelle où elle attendait une amorce de résolution au brouillon. Des dés étaient disponibles sur son bureau, permettant d'éventuels lancers.

Cette phase dans l'ETM idoine de classe, devait prendre appui sur la dimension sémiotique, voire se situer sur la dimension instrumentale pour certains, le dé étant un artefact pouvant favoriser l'expérimentation et l'émergence d'échanges autour de la situation ensuite.

Phase 2 recherche en travail de groupes : (3 élèves) (amplitude variable selon l'avancée, entre 50min et 1h50) avec restitution de production écrite et fichier numérique éventuel attendus.

L'enseignante avait imaginé que certains élèves (phases 1 ou 2) feraient des expériences aléatoires avec un dé jeté manuellement préalablement à une simulation. L'accès à un ordinateur a été prévu, en ce qu'il permettrait de créer une simulation numérique, si certains élèves le jugeaient utile. Elle pensait que les groupes s'achemineraient vers un travail dans le plan [sem-ins], avec l'usage du tableur, et espérait la production de simulations à partir de feuilles de calcul vierges, sans autre consigne que celle de la phase 1.

Ainsi, des ETM idoines de groupes devraient alors se façonner, grâce aux échanges d'idées individuelles sur la situation au sein des groupes. L'enseignante prévoyait de prendre des indices et d'interagir si nécessaire. Ces divers ETM idoines, organisés autour des travaux de 3 élèves, constitueraient des entités au service de celui de la classe, en particulier en phase 3. Elle avait préparé la veille trois simulations tableur présentant toutes 6 lancers indépendants pour chaque course.

Phase 3 d'institutionnalisation: Ce bilan, s'appuyant sur des productions de groupes sélectionnées par l'enseignante, était évoquée dans la préparation. Flou dans son contenu potentiel, il dépendrait du temps de consacré à la phase 2. Des représentants de groupes viendraient exposer à la classe leur travail, sur un temps limité et ce dans un ordre choisi par l'enseignante.

5. Description du travail effectif dans l'ETM idoine de classe

Voici une description partielle de l'ETM idoine de classe, repris phase après phase suite à nos observations de la mise en œuvre dans la classe.

Phase 1 d'exploration de la situation : (10 mn) Dans cette phase de recherche individuelle, peu d'élèves ont utilisé un dé pour effectuer des expériences. Des incompréhensions des règles du jeu furent fréquentes et multiples avant même que les élèves mathématisent la situation (asymétrie de la règle d'avancement sur le parcours). Le travail a débuté ou dans le plan [sem-ins] ou dans le plan [sem-dis]. Pour passer de la situation modèle au modèle réel, certains ont lancé des dés, notant pour chaque course les gagnants, tandis que d'autres, dans le plan [sem-dis] donnaient spontanément des probabilités erronées (pour que le lièvre gagne 1/6 voir 6/36, pour que la tortue gagne 5/6), mobilisant le modèle d'équiprobabilité sur le dé.

Phase 2 recherche en travail de groupes : (1h50) Elle est ici partiellement relatée.
La structuration de la feuille de calcul, a dévoilé des difficultés d'interprétation des règles du jeu auxquelles l'enseignante a dû faire face.

Un premier ETM idoine de groupe présentait une simulation tableur avec 7 lancers de dés par course alors qu'au maximum 6 sont nécessaires. L'enseignante a alors imposé un recours aux dés physiques pour jouer des courses à la main, afin de modifier un confinement dans la dimension instrumentale. La condition d'arrêt a alors été réajustée à 6 lancers dans le tableur. Un travail a eu lieu dans le plan [sem-dis], en appui sur des artefacts (dés et objets matérialisant les animaux), non prévu dans la circulation initiale.

Un autre groupe, souhaitait, au tableur, faire des relances de dé uniquement si le 6 n'était pas sorti, et a demandé de l'aide face à des difficultés. L'enseignante a alors imposé 6 lancers systématiques par course, privilégiant un modèle (celui de la loi binomiale) jugé plus facile sur le tableur que celui initialement choisi (lié à la loi géométrique). La circulation initiée a été détournée par l'enseignante, privilégiant le plan [ins-dis] au plan [sem-ins]. Les élèves ont exécuté les 6 lancers, verbalisant une résistance au modèle imposé.

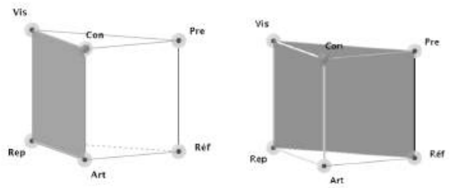
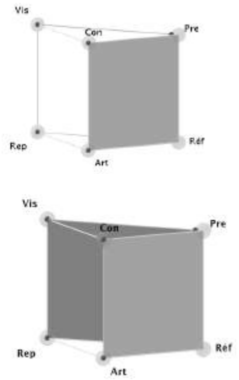
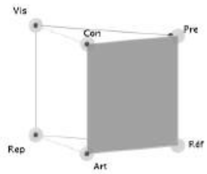
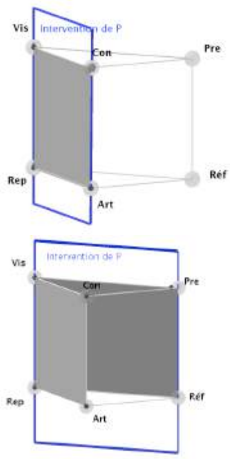
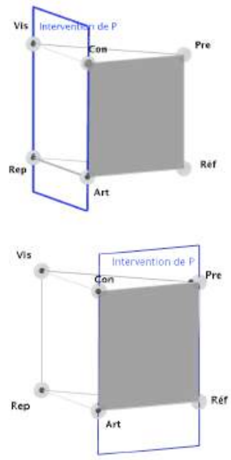
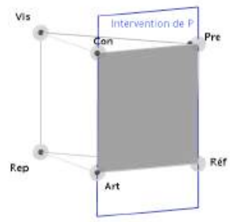
Phase 3 d'institutionnalisation : Dans chaque ETM idoine de groupe, l'enseignante a pris appui sur le fichier tableur présent et la loi faible des grands nombres, faisant relancer les simulations, observer et verbaliser la stabilisation des fréquences. Aucune réponse au problème n'a été partagée en classe entière.
Le tableau 1 complète cette étude.

6. En conclusion

Le logiciel choisi n'est pas neutre et influe sur la circulation du travail mathématique à travers un cycle de modélisation non linéaire. Dans les ETM idoines de groupes, les circulations, a priori différentes ont été rendues homogènes par les interventions de l'enseignante. Si elle a semblé favoriser l'émergence de diverses procédures de résolution, elle n'a retenu que l'approche fréquentiste. Les simulations ont été ajustées, avec un modèle imposé (l'enseignante le considérant plus aisé dans le tableur). La phase 3 aurait pu permettre de confronter les valeurs observées par approche fréquentiste et des calculs exacts de probabilités d'un ETM idoines de groupe, mais l'enseignante n'a pas exploité ces derniers, privilégiant un travail dans les plans [sem-ins] et [ins-dis].

Tableau 1. Circulation dans les ETM idoines de groupe/classe, interventions du professeur (P)

	Situation modèle Modèle réel	**Modèle réel Modèle mathématique**	**Modèle mathématiques Résultats mathématiques**

	Phase 1, début **phase 2**	**Phase 2** usage du tableur	**Phase 3**
Circulation			
Interventions deP			

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Continuing education for teachers and the teaching of statistics at elementary levels

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Abstract. This article aims to analyze a teacher development experiment that explores statistical literacy oriented to develop the professional perspective in a group of twenty-three mathematics teachers working at the early levels of elementary education in public schools in the state of Sao Paulo, Brazil. As data collection instruments this qualititative-methodology approach used a questionnaire and recorded activities proposed by the teachers. Data analysis used the studies of Zeichner & Linhares on Reflective Practice, D'Ambrosio & Skovsmose's Critical Mathematics Education, and Statistical Literacy as proposed by Batanero, Godino & Gal. Results showed that there is a gap between the statistical literacy tenets presented at curricular orientation discussions and teacher practices, which is developed on the basis of activity templates available through external evaluation models and textbooks that emphasize the use of procedural statistical concepts in mathematics classes at Elementary Education levels. The need to review the curriculum of teacher development courses became evident: teachers should be able to not only repeat what is prescribed by official documents, but also and mainly, to be able to put into practice - with critical awareness and autonomy - activities that enable students to develop a critical reading of reality, a key factor to fully exercise their civil rights.

Résumé. Dans cet article on vise à analyser une expérience de développement de l'enseignant qui explore l'alphabétisation statistique axée sur le développement de la perspective professionnelle dans un groupe de vingt-trois enseignants en mathématiques travaillant au début de l'enseignement primaire dans les écoles publiques de l'État de São Paulo au Brésil. Guidé par une méthodologie quantitative, en tant qu'instruments de collecte de données on a utilisé un questionnaire et des activités enregistrées proposées par les enseignants. Pour l'analyse des données on a utilisé les études de Zeichner & Linhares sur la pratique réflexive, l'éducation mathématique critique d'D'Ambrosio & Skovsmose et l'alphabétisation statistique proposée par Batanero, Godino & Gal. Les résultats ont montré qu'il existe un écart entre les principes de l'alphabétisation statistique présentés lors des discussions sur l'orientation curriculaire et les pratiques des enseignants, qui sont élaborés sur la base de modèles d'activités disponibles à l'aide de modèles d'évaluation externe et de manuels qui mettent l'accent sur l'utilisation de concepts statistiques de procédure dans les classes de mathématiques aux niveaux d'enseignement primaire. La nécessité d'examiner le curriculum des cours de développement des enseignants est devenue évidente: les enseignants devraient pouvoir non seulement répéter ce qui est prescrit par les documents officiels, mais aussi et surtout pouvoir mettre en pratique - avec une conscience critique et une autonomie - des activités qui permettent aux étudiants développer une lecture critique de la réalité, un facteur clé pour exercer pleinement la citoyenneté.

1. Introduction

The evidence that statistics is increasingly present in the everyday life of persons has made the teaching of this subject at Elementary Levels more prominent in debates within the field of Mathematics Education. Presently, the easy access to digital technology and the use of specific software that generates calculations and graphs have contributed to the use of statistical methods for information and knowledge sharing. The use of concepts and statistical procedures focuses on the analysis of several situations, both for simple information and data involved in reality and for more complex kinds. Many times, information is transmitted in various media formats through graphs representing situations in different knowledge fields. These graphs must be read and interpreted so people can understand information in a critical manner when making decisions and to exercise their citizenship. Gal (2005) emphasizes the importance of statistical literacy for the development of citizenship as " [...] needed if adults are to be fully aware of trends and phenomena of social and personal importance" (Gal, 2005, p.49). According to that author, statistical literacy is to know how to critically interpret and evaluate information by comparing it against the arguments related to the data presented in various contexts so we can adequately discuss the meaning of statistical information. Hence, to be statistically literate is not a simple task: it requires knowledge construction. In Brazil, even though Statistical Education is included in the official curricular documents of elementary school, this content has not been effectively treated during mathematics classes in the perspective proposed by Batanero, Díaz, Contreras & Roa (2013) or Batanero & Godino (2005) and Gal (2005), who emphasize practices that allow students to experiment with investigative processes to learn statistical concepts. Statistical Literacy is a social demand based, for instance, on Freire's ideas (1994) about the importance of having education to develop individuals capable of reading the world in a critical manner. These ideas are also sponsored by researchers of Mathematical Education such as D'Ambrosio (2014) and Skovsmose (2008).

In the present Brazilian scenario in of research has shown that statistics concepts developed at school by mathematics teachers have the kind of approach that, in general, only prioritizes procedural use based on the application of formulas and calculations. This practice has its root in the initial development of teachers' education and in an entrenched practice based in educational concepts, which focus on teaching reproduction of models and procedures. Such a fact reinforces the existence of a gap between the need of the school to offer statistical literacy to students and the actual teaching practices. The gap, in its turn, constitutes a set of issues that has instigated researchers to re-think the initial and continued education of mathematics teachers.

Having these issues in mind, this article presents a study that has been developed within the scope of a teachers' continuing education course with participants of projects promoted by the Brazilian government known as Education Observatory. One of its projects takes place in the Postgraduate Studies in Mathematical Education at Anhanguera University of Sao Paulo, in partnership with the Department of Education of the State of Sao Paulo, and encompasses the development of actions both in research-related fields and in teacher education in the reality of elementary school. The goal of this research is to analyze an educational experiment on Statistical Literacy oriented to the professional development of a group of 23 teachers, presently teaching mathematics at the early levels of elementary school (6 to 10 years of age). Before the analysis is presented, however, we will show the basis of the curricular guidelines related to the teaching of Statistics, since the development experiment was planned and performed based on their tenets.

2. Curricular guidelines

Since 1990 we have used the National Curricular Guidelines (PCN, in Portuguese), as a guidance for teachers in their educational tasks. According to the official document, mathematics has an important role in the basic development of Brazilian citizens when it states that, "*in order to exercise citizenship it is necessary to know how to calculate, measure, argue with, and treat information statistically"* (BRASIL, 1997, p.25). It also underscores that "*students should not only learn how to read and interpret graphic representation, but also be able to describe and interpret their reality using mathematical knowledge"* (BRASIL, 1997, p.49). For that to happen, the teaching of mathematics should enable the development of an investigative spirit, which means, in the case of statistical collection and organization of information, to create records to transmit information, and the elaboration of simple and two-entry tables and graphs, text production based on the interpretation of graphs and tables (BRASIL, 1997). The new national curricular guidelines are currently under revision and a document entitled Common Core National-Based Curriculum

(BCNN, in Portuguese) is in its final phase of development. As far as statistics is concerned, the BNCC (Brasil, 2016) suggests the development of statistical work through the collection and organization of data from a survey based on students' interest with emphasis on those contents from the first school year (age 6). The fact is that curricular guidelines produced in a clear and well-argued basis by experts do not guarantee that teachers will incorporate them to the point they will be put into practice in the classroom. Also, it is worth pointing out that textbooks insist on favoring procedural characteristics on the use of statistics. Besides this, when an activity is presented as fulfilling the investigate features to be developed with students, teachers end up restricting the activity and focusing only on its procedural features, strongly influenced by their background education. Hence the need to invest both in their initial development years to review this issue and in their continuing education to enable in-service teachers to construct new references related to content, and their teaching and autonomy in the re-creation of their own practice.

3. Research and development

This qualitative research was carried out in the context of a continuing education course, based on Zeichner (1993) and Linhares (2011) regarding the reflective practice related to Statistical Literacy and the sharing of experience among the participants. A total of 23 teachers who teach mathematics in the early years of public schools in the city of Sao Paulo took part in the experiment. Data collection was made through the following: profile-questionnaire, protocols of teachers' activities and recordings (videotape and texts) of on-site meetings. The diagnostic activities clearly showed that the teachers in the group had many doubts regarding the reading and construction of graphs. This finding made us redirect the development proposal in two goals: to provide teachers with the construction of statistical literacy and to analyze and reflect upon their practices in the teaching of statistics.

4. Analysis and results

Information collected throughout the course allowed us to analyze the teachers' knowledge regarding the themes and literacy process of descriptive statistics, such as different forms of graphical representation, the measures of central tendency and some initial aspects of measures of dispersion. The chart below brings an example of one activity proposed in the course.

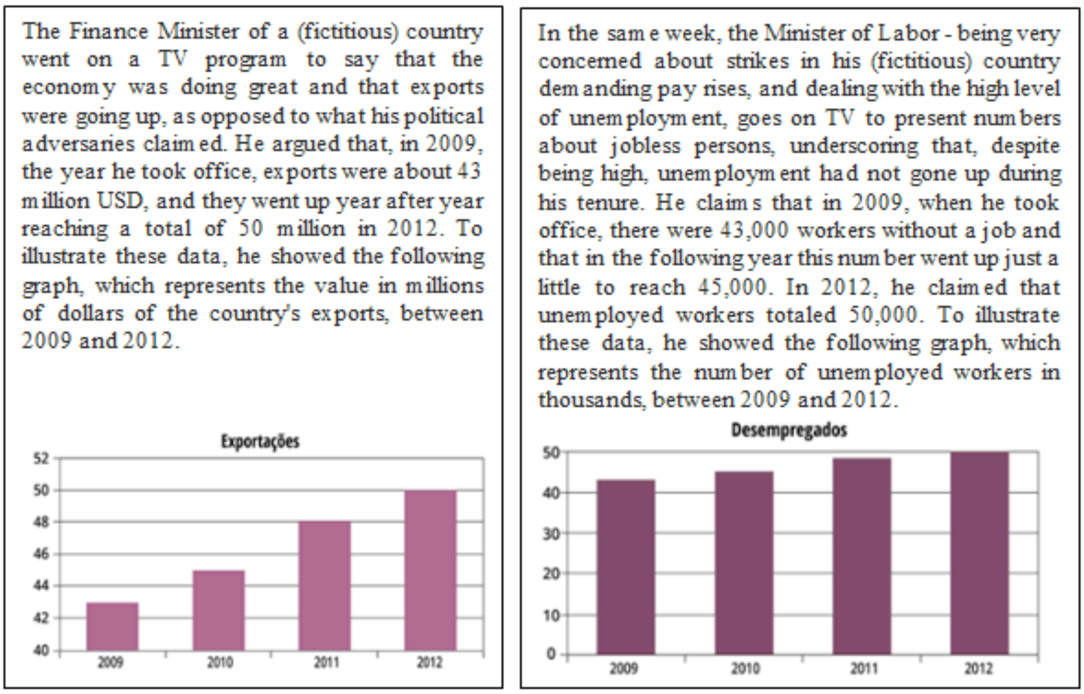

Figure 1: Activity Scenario

The reading and interpretation of the two graphs in Figure 1 generated lots of discussion in the group, mainly regarding the labelling of data in the graph and the scale used in its construction. This reflection enabled participants to become aware of the need to be statistically literate to be able to read information in a critical manner. In further proposed activities, we noticed that teachers emphasize the building of bar graphs, including data used for continuous variables or for frequency distribution grouped in class intervals, instead of using line graphs or histograms.

We also noticed that the argument used by the majority of participants to analyze and make decisions to solve situations that involved central tendency values was based on the analysis of the values of mean, mode and median used in isolation. It was necessary to retrieve problem-situations that favored the need to relate the values of these measures to accurately understand the data presented.

As for the focus of the development to provide the sharing of practices by statistics teachers with their students, we found out that there is an effort to work with the children's daily life issues such as birthday dates, favorite games, and others. However, when the teachers propose a data survey to the students, the activity is reduced to organizing a list and building a graph, usually in columns, and reading information, such as locating the most and least voted. This fact was discussed during the development course. This reflection allowed teachers to acknowledge that the proposed activities in their classes are directly affected by activities used in external evaluations and in textbooks, whose focus is in the reading of information in tables or column-graphs and problem-solving, simply from reading. This finding shows that there is a gap between the tenets of statistics teaching understood as a kind of literacy presented in the curricular guidelines and the teacher's practice that is developed based on models of activities available in external evaluations and in textbooks that tend to reinforce the procedural use of statistical concepts in mathematics classes at elementary levels.

5. Final remarks

The statistical literacy process provided the participants with the contact with everyday situations that demanded decision-making for which they needed to read critically the information presented. This approach in the formative context stimulated the reflection on the teaching practice and the re-reading of the curricular guidelines. The participating teachers had the opportunity to experience a process of statistical literacy development. This study showed that curricular guidelines should be analyzed and understood by in-service teachers and in initial development courses. For that, the need to review the curriculum of teachers' initial development courses is key to ensure that future teachers will be able not only to repeat what is prescribed by official documents, but also and mainly, to put it into practice with critical awareness and autonomy and to propose activities that enable students, from the very early school years, to build statistical literacy so that children can develop a critical reading of reality and exercise their citizenship.

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Collaborative inquiry: A professional learning approach for middle school mathematics teachers

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Abstract. This research studied the use of a collaborative inquiry approach as a tool in the professional learning of middle school mathematics teachers to challenge their pedagogical practice in the middle schooling context. It involved the building of a professional learning community among middle school mathematics teachers to challenge and share their pedagogical practices. This grounded theory study adopted the collaborative inquiry approach via the use of site-based programmed planning time to enable the teachers to discuss, deliberate and share pedagogical practices so as to grow their professional practice. The research was undertaken over a three-year period, and included the use of team planning recordings, student focus groups, teacher survey responses and individual teacher interviews as data to triangulate the teachers' collective professional journey and trajectory of growth and development in their pedagogical understanding and practices. The findings of this research confirm the importance of such an on-site built-in professional learning model which harnesses the benefits of the site-based context in establishing a professional learning community that enables a group of teachers to consolidate professional learning needs for their collective growth. This approach takes into account the teacher differentiated learning needs and has the added benefits of creating a positive team and school culture that connects a critical mass among the teachers to reflect their practice. Such professional learning approach was found to be effective in developing a shared positive disposition and potentially initiated pedagogical practice change. It adheres to adult learning principles such as focussing on relevant topics to their work and encouraging dialogues to share personal experience. The impact of the teachers' pedagogical change on the students' engagement was found to be largely dependent on the wider context of the culture and attitude of the school, the teachers' ability to link mathematics to other disciplines, and the mental models and beliefs of the teachers.

Résumé. Cette recherche a étudié l'utilisation d'une approche d'enquête collaborative comme outil d'apprentissage professionnel des enseignants des mathématiques du collège pour remettre en question leur pratique pédagogique dans le contexte de l'école intermédiaire. Il s'agissait d'établir une communauté d'apprentissage professionnel parmi les enseignants des mathématiques du collège pour contester et partager leurs pratiques pédagogiques. Cette étude basée sur la théorie ancrée a adopté l'approche de l'enquête collaborative par l'utilisation du temps de planification programmé sur le site pour permettre aux enseignants de discuter, de délibérer et de partager des pratiques pédagogiques afin de développer leur pratique professionnelle. La recherche a été entreprise sur une période de trois ans et a inclus comme donnés l'utilisation d'enregistrements de planification d'équipe, des groupes de discussion d'étudiants, des réponses aux enquêtes auprès des enseignants et des entrevues individuelles avec des enseignants, pour trianguler le parcours professionnel collectif des enseignants et la trajectoire de croissance et de développement dans leur compréhension pédagogique et pratique. Les résultats de cette recherche confirment l'importance d'un tel modèle d'apprentissage professionnel sur site intégré qui exploite les avantages du contexte basé sur le site en établissant une communauté d'apprentissage professionnel qui permet à un groupe d'enseignants de consolider les besoins d'apprentissage professionnel pour leur croissance collective. Cette approche prend en compte les besoins d'apprentissage différenciés des enseignants et offre des avantages supplémentaires en créant une équipe positive et une culture scolaire qui relie une masse critique parmi les enseignants pour refléter leur pratique. Une telle approche d'apprentissage professionnel s'est révélée efficace pour développer une disposition positive partagée et un changement de pratique pédagogique potentiellement lancé. Il adhère aux principes d'apprentissage des adultes, notamment en mettant l'accent sur des sujets pertinents à leur travail et en encourageant les dialogues pour partager leurs expériences personnelles. L'impact du changement pédagogique des enseignants sur l'engagement des étudiants a largement dépendu du contexte plus large de la culture et de l'attitude de l'école, de la capacité des enseignants à associer les mathématiques aux autres disciplines et des modèles et croyances mentaux des professeurs.

1. Aim of study

The objective of this research was to evaluate the effectiveness of collaborative inquiry approach in developing middle school mathematics teachers' understanding and implementation of effective pedagogy.

The following research sub-questions were addressed:

a. What challenges and benefits would the implementation of the collaborative inquiry as a means of professional learning approach present for the school and the teachers?
b. What would be the essential elements in implementing effective collaborative inquiry as a professional learning approach for teachers in the school?
c. How would these essential elements be incorporated in a theoretical model to inform the effectiveness of collaborative inquiry as an professional learning approach for middle school mathematics teachers in schools?

This study used the grounded theory approach in conjunction with qualitative research methodology in the collection and analysis of data (Creswell, 2008). This qualitative design determined the impacts of collaborative inquiry approach of professional learning on site regularly for a group of mathematics teachers' development of understanding and implementation of effective pedagogies (Nelson, 2003; Nelson, 2008; Nelson, Slavit, Perkins, & Hathorn, 2008; Shadish & Luellen, 2006).

2. Methodology: Nature and Appropriateness of Grounded Theory Approach

I adopted the constructivist approach to grounded theory research, first used by Glaser and Strauss in 1967, in that all the reality was constructed and both constructed perspectives of the respondents and I were equally valued (Oktay, 2012). Grounded theory was chosen for this research because it was designed to study the interactions between individuals and their practice setting employing the symbolic interaction theory (Oktay, 2012). I focussed on the interactions of the teachers within a collaborative inquiry setting as a team of year level teachers focusing on the interrogation of their own pedagogical practice and its effectiveness.

This multi-stage process with the gathering of data after the selection of topic using the researcher's theoretical sensitivity. My literature review focused on the current understanding of various concepts relating to the research problem and the emerging thinking as each phase of the data was being analysed and new understandings emerged. Consequently, it is essential that a heightened state of theoretical sensitivity by me eradicated the theoretical bias. Teppo (2015) suggested that as the research progressed, literature review would provide additional source of data for locating pattern for the emerging ideas and concepts from the data analysis. During the final stages of the research, further literature was reviewed to place the emerged theory in the existing theoretical framework of the area of study and extending the current framework as advocated by Teppo (2015).

Theoretical sampling in grounded theory was used as the study progressed when the core concepts of the theory emerged through a range of sampling strategies such as surveys, observations of teaching and dialogues and focus groups (Oktay, 2012; Teppo, 2015). Theoretical saturation was reached when the theory and data fitted together (Oktay, 2012). At this stage, the theory emerged and was consolidated.

This research design was divided into three phases; and in each phase, inductive logic was used to hypothesise the theory based on literature review and/or data analysis; then applying deductive logic based on the data collected and analysed to generate and/or refine the theory. After that, the research design was reviewed and refined to further test and refine the theory.

3. Data Sources

This longitudinal study undertaken at a Kindergarten to Year 12 co-educational school on the Gold Coast, Australia. The school has been founded 35 years ago and it has four distinct phases of learning, namely early years, junior years, middle years and senior years. The research participants were teachers and students from the middle years. These teachers taught middle years' mathematics, i.e. year six to year nine.

The twelve middle years' mathematics teachers were invited to the information session about this research after the Head of College had granted approval for this research to take place at the site. The teachers were provided with the approach of the study, i.e. the researcher would utilise the timetabled year level weekly team planning sessions as collaborative inquiry sessions so that the teachers would not need to invest extra time to be involved in this research project. The types of data source and frequency of data collection at each stage of the research is summarised in table 1 below.

Table 1. Summary of data sources.

Stage of research	Types of data source	Frequency
Year 1	Collaborative inquiry sessions	Weekly over approximately 37 weeks in an academic year.
	Student focus group sessions	In two groups at the end of the academic year.
Year 2	Collaborative inquiry sessions	Similar to year 1.
	Teacher online survey	Once in the middle of the academic year.
Year 3	Teacher individual interviews	Twice, six months apart, in the year.

There were eight teacher participants over the life of the study. However, as in most longitudinal study, teachers' relocation and changing schools, resulted in the reduction of original number down to four in the final year.

The collaborative inquiry sessions were recorded and analysed using the grounded theory approach to analyse and synthesise as outlined above. Furthermore, student focus groups, teacher online survey and teacher participants' individual interviews were the other data sources to ensure internal validity and reliability of the theory emerging from the data analysis. Triangulation of data from the various sources was applied at each phase of data analysis to ensure that the emerging theory was validated at each state to eliminate bias, error and anomalies. During each phase of the data analysis and interpretation, I always maintained a heightened state of critical self-reflection and reminded myself to stick to the appropriate modes of representation to honour the ethics of research (Taylor & Wallace, 2007).

4. Findings

My first key finding is that the benefits of collaborative inquiry as a professional learning tool for teachers at a school site outweigh its challenges which can be managed through careful planning (Cox, 2010; Small, 2011; Wilkins & Shin, 2011). Consequently, schools should endeavour to implement such a professional learning approach for all the teachers. The school leadership's role of supporting the teachers' professional learning and growth is to provide a conducive and relevant learning platform and structure to make learning relevant while meeting adult learning principles (Rickley, 2008; Terhoff, 2002). A built-in approach for teachers' learning is deemed more relevant when it is structured into the teachers' work routine than is the bolt-on professional learning approach of external presenters at conferences, workshops and seminars, as these presenters do not have the same understanding of the contextual constraints and preferences (Kruse, Louis & Bryk, 1994; T. H. Nelson, et al., 2008).

My second finding is that for collaborative inquiry to be effective at a school, the teacher professional learning must incorporate one or more of these four elements depending on the context and setting of the school within its internal and external environment. These elements are positive teacher mental model and belief, effective pedagogy, optimal environment and authentic assessment. All teachers are pre-disposed by their background to hold a certain mental model; they can either have growth or fixed mindsets (Brahier, 2005; Loughran, 2005; Strauss, 2001). Collaborative inquiry will be an effective professional learning tool for teachers if the teachers' mental models and beliefs are aligned with the school culture and values as well as its preferred pedagogical practice (Bessoondyal, 2005; Bishop, 2006; Darling-Hammond & McLaughlin, 2011; Louis, Anderson & Riedel, 2003; Provest, 2003). To create an optimal environment, the teachers need to have a tool kit of effective pedagogy (Bessondyal, 2005; Mowlaid & Rahimi, 2010; Nickson, 2000; Stacey, 2008; Utlay, 2004). The pedagogical practice of teachers dictates how they run their classrooms and how the tone of the classrooms is set. As a team of teachers sharing the same understandings about the students at the same site with the same expectations from legislative requirements, the teachers in this study shared collaborative insights about teaching and learning challenges unique to the site. It can be concluded that it would be helpful for the teaching community to share ways of developing assessments, assessment practice and teaching approaches that would address the site-based needs of students. The teachers can work together to develop strategies to address students' weaknesses by using appropriate differentiation strategies incorporating higher order thinking elements to assess the students. Similarly, the teachers through their collaborative insights can link teaching practice to examination skills required to assist the students to achieve (Brahier, 2005; Moroco & Solomon, 1999; B. S. Nelson, 1999; Wong, 2009).

The last finding was the development of a theoretical model shown figures 1, 2 and 3. The model was developed based on the findings of the grounded theory study supported by a progressive literature review to verify my thinking and progress of theory sampling and saturation. However, the model has not been tested in context for refinement so as to apply this emerging theory in practice. It is grounded in the theory emerging from this research, and articulates that, for collaborative inquiry to be an effective professional learning tool for teachers at school, the students must be at the core of the inquiry represented by the sphere in the centre of the model. It has not fully explored the various ways collaborative inquiry could be implemented in the specific school context and the logistics involved in making this tool come to life. Hence, a model of collaborative inquiry akin to the one developed in this study can be developed and shared with school leaders to support Australian Institute of Teaching and School Leaders' advocacy of teacher professional standard.

5. Conclusions

Collaborative inquiry provides the teachers with an effective professional learning approach which meets their learning needs in context and embedded in their job schedule. To teach, the teachers need to first engage the students in learning. To engage students, teachers need to create optimal environment conducive for learning which require effective pedagogy to make learning relevant, practical and useful. The effective pedagogy contains both generic and school relevant pedagogical strategies consistent with the school's mission, vision and values. Students' learning must be continually assessed authentically to ascertain their progress in the mastery of the learning. However, teachers bring with them mental models and beliefs which require alignment with the team or school's values and approaches so that consistency of practice can be attained. Collaborative inquiry is well placed to challenge these mental models and beliefs while offering the teachers regular forums to close the gaps between their perceptions and practice. All these elements about teaching and learning are captured in the model in figures 1, 2 and 3.

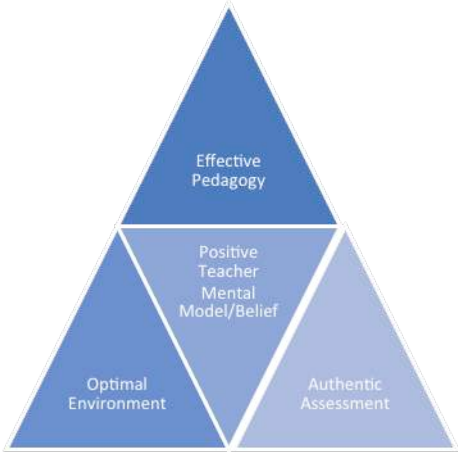

Figure 1. Net (two-dimensional representation of a three-dimensional model) of the tetrahedron containing inter-connecting evolutionary elements.

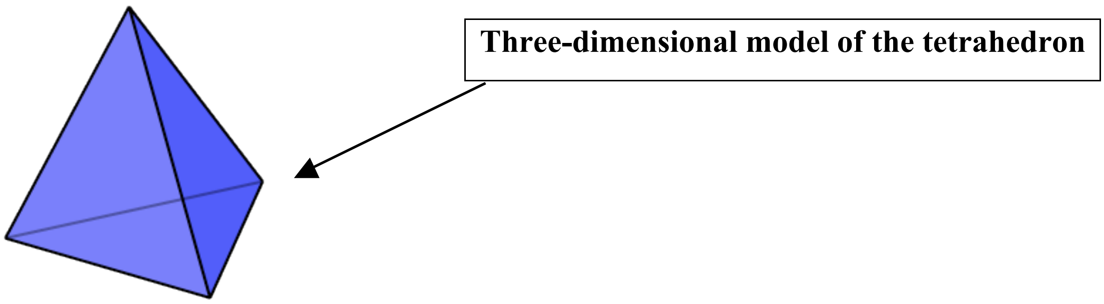

Figure 2. Three-dimensional model of the tetrahedron containing equally important inter-connecting evolutionary element

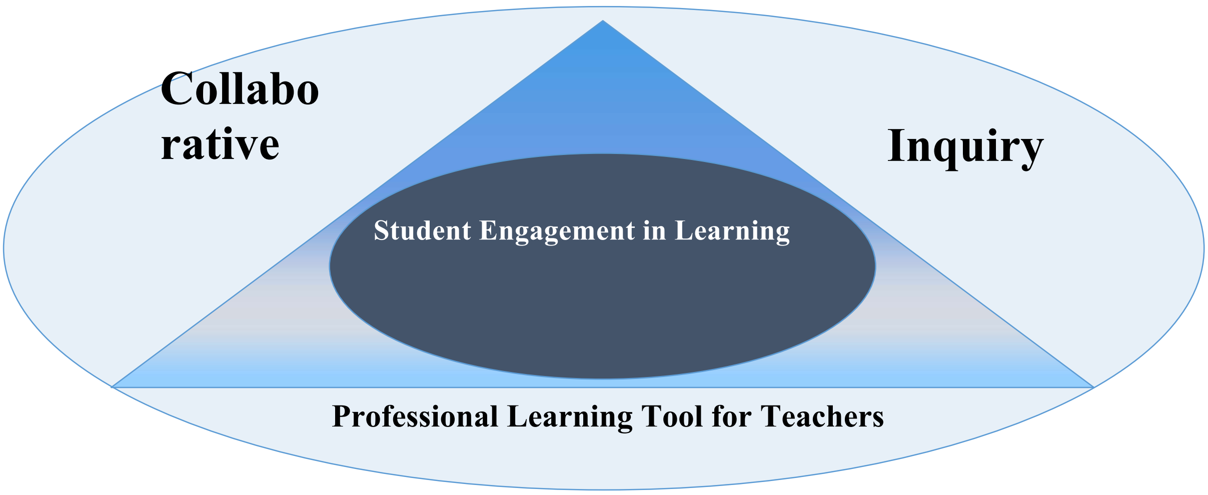

Figure 3. Collaborative inquiry: A professional learning tool for teachers.

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WORKSHOPS / ATELIERS

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MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

"Quaderni di Ricerca in Didattica (Mathematics)", n. 27, Supplemento n.2, 2017
G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Activities for an accessible and inclusive Maths learning

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Abstract. According to institutional national requests, Italian schools have to be as much inclusive as possible, respect diversity and apply individualized education. This article reports on the project "Methodologies, technologies, materials and activities for an accessible and inclusive learning of Mathematics", based on the design of inclusive tasks for students, their experimentation in the classes and the analysis of the teaching experiments. Inclusion is intended as the involvement of students with different abilities and special needs in the class activities in laboratorial ways with the use of materials and artefacts. Inclusion is introduced in relation to the tasks and the teaching practices to be used. Students' perceptions of inclusion are examined through a questionnaire referring to cognitive and emotional aspects.

Résumé. Selon les demandes nationales institutionnelles, les écoles italiennes doivent être aussi inclusives que possible, respecter la diversité et appliquer une éducation individualisée. Cet article traite du projet "Méthodologies, technologies, matériaux et activités pour un apprentissage accessible et inclusif des mathématiques", basé sur la conception de tâches intégrées pour les étudiants, leur expérimentation dans les cours et l'analyse des expériences pédagogiques. L'inclusion est l'implication d'étudiants ayant des capacités différentes et des besoins spéciaux dans les activités de classe en laboratoire avec l'utilisation de matériaux et d'artefacts. L'inclusion est introduite en relation avec les tâches et les pratiques d'enseignement à utiliser. Les perceptions des élèves sur l'inclusion sont examinées à l'aide d'un questionnaire se référant aux aspects cognitifs et émotionnels.

1. Aims, frame and development of the project

It is perceived as essential by institutions and teachers that learning scenarios are shaped on the students and on their different capabilities, with a change in perspective: from the focus on "what students do not do" to "what students do" (Healy & Powell, 2012) to achieve the goal of a didactic that could be a common denominator for all the students and that does not leave back anybody (MIUR, 2013).

Our project[1], developed by the Mathematics Department of the University of Turin (Italy) under the direction of Professor Ornella Robutti and the supervision of Professor Ferdinando Arzarello, is based on this idea: that inclusive activities are useful not only to students with particular and special needs, but to every student, and the whole class itself as a community of students can be advantaged by an inclusive educational approach. For this reason, the first aim of our project is to plan inclusive activities for students with different abilities, different learning approaches, different (special) needs, but also for all the students. A second aim deals with teachers and their involvement not only in experiencing these activities, but also in designing them along with the academic team (professors, master degree students). The project has been followed by

[1] The project is funded by Fondazione Cassa di Risparmio di Torino.

21 teachers[2] from Middle School (grades 6-8) and High School (grades 9-13), who designed the activities, carried out them in their classes, and observed students and analysed results. Aims of the project are:

- design deeply inclusive didactic activities, planned for all the students with a particular attention to special educational needs ones;
- create a didactic environment that promotes collaboration and peer instruction;
- encourage students' interest through meaningful activities;
- foster the "to be well at school" paying attention to emotional aspects;
- take care of the accessibility of the materials (forms and files) given to the students, to avoid that the way problems are posed could become itself a problem (Zan, 2016).

As institutional reference we used the National Curriculum with its recommendation on inclusion: *"The Italian school develops its educational action in line with the principles of inclusion of people"* and put the student *"at the centre of educational action in all its aspects: cognitive, emotional, relational [...]"* with special care for *"students with disabilities or with special educational needs, through appropriate organizational and teaching strategies, to be considered in the normal training supply design."* (MIUR, 2012, p. 14). Hence "*school's aims are to be defined as of the learner and the definition and implementation of educational and teaching strategies should always take into account the uniqueness and complexity of each person, his articulated identity, his aspirations, capabilities and his fragility"* (MIUR, 2012, p. 5).

Recent studies on special needs (Lewis & Fischer, 2016) point out that a consensual operational definition of mathematical learning disability (MLD) is yet to be developed and there are raising questions about the reliability of the measures across studies and the validity of the identification approaches used to identify specific students with an MLD. As a matter of fact, there is a big unevenness in the classification methods used that rarely take into consideration non-cognitive elements. As a consequence, the same diagnosis could be assigned to students with a different cognitive profile (Karagiannakis & Baccaglini-Frank, 2014), making even more difficult for teachers to identify an effective didactic action. For that reason, our first interest is not the characterisation of the students with special needs present in the class, but the attention to their involvement in activities, in order to have them engaged in the tasks and interactive with their mates.

To do this, we refer to two pedagogues of the last century: Célestin Freinet and John Dewey. Although with different socio-political implications, both put the students at the centre of the educational process: "*from their basic needs [...]the techniques - manual and intellectual - to master, the subject to be taught, the system of learning, the education mode will derive, in a renovated school that emphasizes the individual's enthusiasm, his creative and active faculties, his ability to always move forward, up to a full realization of his capabilities"* (Freinet, 1973). Freinet's teaching approach comes from the observation of how children learn to walk, run, ... spontaneously, driven by curiosity and proceeding through trials and errors, enriching the experience with a natural creativity that helps them to overcome obstacles. Therefore, school should cultivate this natural predisposition to learning that cannot be separated from the survey through direct experience. Also, Dewey, theorist of the "active school", sees in the experience, as interaction between human being and environment, the education process' basis: starting from real and significant problems, which involve and encourage the curiosity, the student makes hypothesis and tests them, by trials and errors, to arrive with the teacher's help to the logical rigour. In the educational process, the student plays an active leading role: "learning by doing" in a rich and stimulating environment, avoiding the transmission of a pre-packaged knowledge from the teacher. The teacher has the responsibility of encourage students' involvement and the development of real and concrete interests and creativity, strengthening the motivational factor

[2] The teachers involved: S. Abbati, B. Baldi, S. Beltramino, A. Berra, E. Calemma, A. Cena, P. Curletti, M. Dalè, A. Drivet, S. Fratti, L. Genoni, A. Ghersi, P. Gulino, C. Idrofano, D. Pavarino, F. Raina, A. Rongoni, D. Sasso, C. Soldera, G. Trinchero, R. Valentini.

(Dewey, 1987).

The use of "Mathematical laboratory" (UMI, 2003) as teaching practice involves students in social interaction because they work in groups and interact together, discuss and argue on the task. Moreover, using materials and artefacts, students learn in a perceptive-motor way, not only in a symbolic way, feeling them deeply engaged in the task.

Last but not least, teachers have been aware of including every student in the activities taking into account cognitive and emotional aspects.

During teaching experiments, in order to enhance the inclusion, special attention was paid to:

- class setting: desks were organized in small blocks to promote interaction;
- work in small groups (3-4 people): using the scaffolding technique when perceiving the possibility of a fruitful Zone of Proximal Development (Vygotskij, 1980);
- discussion (in the working group and plenary): the students have the opportunity to share their ideas with classmates and, guided by the teacher, to reach a shared solution.
- The teaching experiments were carried out following a methodology made of different tools:
- initial questionnaire (series of exercises, sometimes taken from INVALSI[3] tests, useful to understand the previous knowledge of the following activity's topic);
- final questionnaire (which coincides with the initial one, proposed after the activity to check whether the understanding of the concept was changed significantly);
- "satisfaction questionnaire" (common to all the activities, with questions about the emotional-motivational sphere, to understand what had been the emotions of the students involved);
- questionnaire INVALSI model (containing 3-4 questions like INVALSI ones, proposed a few weeks after the activity to see if the students had embraced the concepts introduced).

2. Activities

During the first two years of the project several tasks for junior and senior high school students were designed, regarding Shapes, Numbers and Calculus. The activities realized were tested in 31 classes[4], involving approximately 600 students, 15% of whom with special needs.

We will briefly take into consideration one of them, "The mainmast"[5], addressed to 6 grade students and dealing with the concepts of perpendicular line and distance between a point and a straight line, focusing on the choices teachers made to enhance inclusion.

The activity was centred on a laboratorial approach: students divided into small heterogeneous groups worked with everyday materials (figure 1) promoting a perceptive-motor approach (figure 2), useful to enhance their sense of achievement.

Figure 1. Students drawing the mainmast and then exploring the difference between perpendicularity and verticality with the help of everyday materials.

[3] National Institute for the evaluation of the educational system, subject to the supervision of the Ministry of education

[4] Teachers that have tested the activities in their classes are: B. Baldi, P. Curletti, S. Fratti, L. Genoni, P. Gulino, C. Idrofano, D. Pavarino, F. Raina, A. Rongoni, D. Sasso, C. Soldera, G. Trinchero, R. Valentini.

[5] This activity is inspired by M@t.abel activities. M@t.abel is a national project, started in autumn 2005, addressed to teachers that proposes a new approach to the teaching and learning of mathematics. http://www.scuolavalore.indire.it/superguida/matabel/

Figure 2. A group of students is exploring the concept of distance between a point an a straight line.

Forms to be filled were avoided, to prevent decoding and writing difficulties and, where necessary, their accessibility was promoted using capital letters, double spacing and short sentences with familiar words (figure 3).

Figure 3. An example of accessible form.

As far as learning is concerned, students achieved good results. The analysis of INVALSI style test proposed in 5 classes where the experimentation has taken place and in 3 classes where the topic has been taught in a traditional way (for a total of 150 students involved, 34 of whom with special needs) is encouraging: in the first group, students have achieved, on average, better results than in the second one. If we have a look at special need students results (figure 4), they are even more satisfying: while in the classes without experimentation none of them (with the exception of one student) have achieved more than 2/4 (pass mark), in the other classes 18 students over 24 have achieved more than 2/4 and 8 of them have achieved full marks.

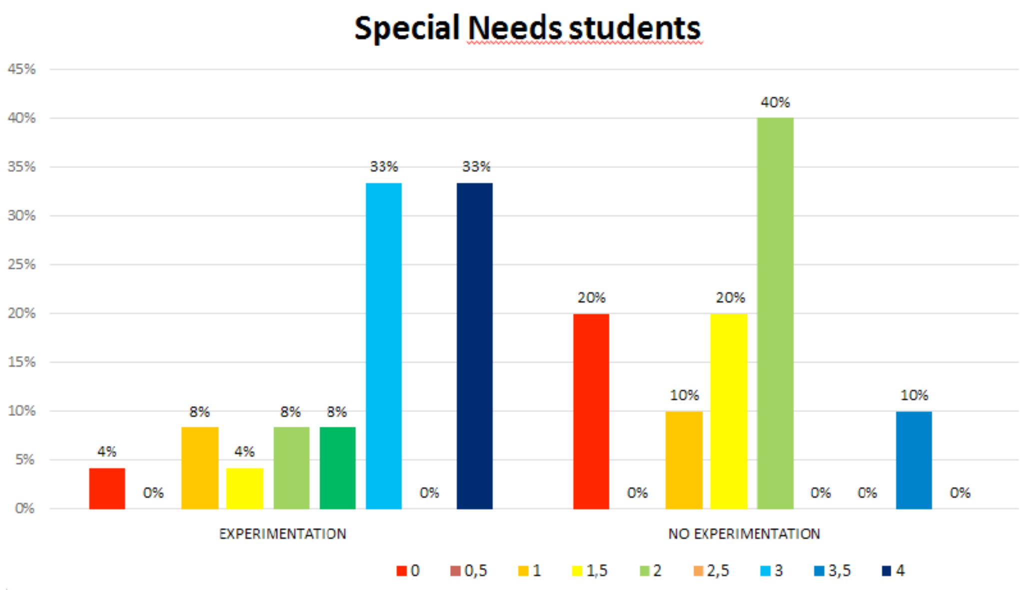

Figure 4. The INVALSI style test analysis.

Finally, the analysis of the "satisfaction questionnaire" has showed a positive feedback concerning inclusion and involvement during the activity: a high percentage of pupils (more than 75%) have written to have done a good job, sign of a positive perception of themselves and of their work. More than 95% of the pupils have understood the instructions and perceived the topic clearer than before, sign of the effectiveness of the strategies taken in the design of the task and of the didactic methodologies used. Furthermore, during the activities students felt at their ease, having experienced peacefulness (33%), happiness (25%) and freedom of express their ideas (24%).

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Mathematical Working Spaces: a construct to make sense of modelling based teaching/learning situations

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Abstract. The goal of the proposed workshop is to present and discuss the idea of Mathematical Working Spaces as a construct useful for conceptualizing several aspects of the modelling process with regard both to epistemological and cognitive dimensions of teaching/learning. After presenting the general framework, we propose two phases of work: starting from a situation of modelling and reflecting on specific questions, and a concluding discussion in relationship with questions offered by the discussion document.

Résumé. Le but de cet atelier proposé est de présenter et discuter l'idée des Espaces de Travail Mathématique comme un modèle utile pour conceptualiser plusieurs aspects du processus de modélisation des dimensions épistémologiques et cognitives de l'enseignement et de l'apprentissage. Après avoir présenté le cadre général, nous proposons deux phases de travail: une première situation de modélisation et de réflexion sur des questions spécifiques, et une discussion de conclusion en relation avec les questions proposées par le document de discussion.

Our proposition deals with the following concerns as expressed in the discussion document. We emphasize particular keywords in bold font.

*The mathematisation of social, economic and technological relations in the form of formal structures is a **double-edged sword**. On the one hand, it has proven effective and efficient in terms of developing more and more complex structures (...) On the other hand, once established as the standard (or only) way of describing, predicting and prescribing social, economic, ecologic, etc. processes, **it severely reduces the possibilities** of finding non-formal, non-quantifiable, nonmathematical solutions.*

***Mathematical Modelling** (...) is another **orientation for curriculum construction** that attracts worldwide attention. Within Mathematical Modelling, the **authenticity of everyday situations** is of relevance. From these everyday situations a '**real world model'** is generated and, further, the 'real world model' is translated into a '**mathematical model'**, which can be used for calculation or other mathematical procedures.*

The goal of the proposed workshop is to present and discuss the idea of Mathematical Working Spaces (Kuzniak, Tanguay & Elia, 2016) as a construct useful for conceptualizing several aspects of the modelling process with regard both to epistemological and cognitive dimensions of teaching/learning. In the introduction, we present a general framework and an example. Then we propose two phases of work: starting from a situation of modelling and reflecting on specific questions, and a concluding discussion in relationship with questions offered by the discussion document.

Introduction

1.1 General Framework

We are interested in mathematics taught at upper secondary level. Our objective is to give meaning to this mathematics as a tool for understanding the sensible world. That is why we are interested in modeling processes involving models of a varied nature. On the one hand, it is important that students access mathematical models involving formalized and computationable objects, related for example to algebra or calculus, rather than purely numerical descriptions. The goal is to give meaning to formalism and calculation as a tool for understanding the sensory world and thus to remedy a weakness in teaching: the transmission of a meaningless formalism. On the other hand, it is also important that students encounter and use more informal models, often closer to the common experience and learn to compare and control the solutions obtained in both types of models in view of the context to be modeled.

The context to be modeled is itself a space. It is the space where questions can be asked that motivate a representation in other spaces, the modeling spaces. As the announcement says, it is important that the questions at stake are 'authentic', that is to say belong to a field of concern whose relevance students can grasp. However, there is no reason why these concerns should be limited to 'everyday life'. Actually, in most cases, the context motivating mathematical modelling at upper secondary level is not directly in students' everyday experience and special strategies are required to make them question this context. Think for instance of relativity, a model of the real world very far from everyday experience and without practical application, and which nevertheless can be a relevant topic for secondary students. Think also of vocational fields where mathematical models are often proposed, that are too simplistic to be relevant[6].

We see then modeling as the appropriation of spaces of different natures. Mathematical models are inscribed in modeling spaces based on specific theories, sign systems and artefacts. There are also modeling spaces where the theory remains more implicit, and where the rules relating to sign systems and artefacts are mainly understood in usage. These are the spaces where the more informal models that we mentioned earlier are inscribed. We propose, as in the example below, to consider these different spaces as 'Mathematical Working Spaces' (MWS)

1.2 An introductory example

A problem for a surveyor is the calculation of the surface area of a farmer's field. We consider the common case of flat quadrilateral fields. Assuming that the lengths of the sides and of one diagonal are known, it is possible to consider different working spaces where models can answer the question. In the first one, we consider a sheet of paper, the scaling procedures, the instruments for drawing and measuring, and a formula for calculating the area of a triangle. In this first working space, splitting the field into two triangles and measuring altitudes makes it possible to answer the question in a practical way. A second working space is that of the elementary geometry where the so-called Heron formula makes it possible to calculate the area of a triangle knowing the length of its sides without drawing or measurement. The solutions in the two working spaces share a common general strategy: splitting into two triangles, but do not share the other means of action, the justifications of these actions and the resulting conceptualizations.

Kuzniak (2013) shows how the framework of "Mathematical Working Spaces" allows considering the specificities of each working space and their complementary. In the example, the first two MWS spaces do not necessarily organize themselves in a hierarchy where the mathematical model would have the pre-eminence. The "measuring" MWS allows the problem to be satisfactorily solved with limited theoretical apparatus. The "elementary geometry" MWS avoids drawing and measuring and therefore the accuracy is not limited by the measurement on a reduced scale and the imprecision of drawing. These two different MWS are guided by two different geometrical paradigms named GI and GII according Kuzniak (2013). The procedure in this space allows automatization, for example by way of a program on a calculator. The implementation on a calculator for "black box" use offers surveyors a "demathematisated" working space. The "measuring" MWS favors the use of instruments and therefore the associated genesis, while the "elementary geometry" MWS fosters the use of signs (semiotic genesis). In both spaces, discursive genesis may be called upon to justify the procedure used: We have noted that splitting into triangles is a procedure shared by the two working spaces and can thus connects the two models.

Working theme 1: Mathematisation and demathematisation in understanding modelling processes

While we acknowledge the 'modelling cycle' (Blum, Galbraith, & Niss, 2007) as an overarching modelling trajectory, we also stress recursive processes inside this trajectory, designing, amending and fine-tuning diverse models inscribed in particular working spaces possibly related to different mathematical domains. These processes establish a continuous balance between mathematisation and demathematisation. This is

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6 For instance, the Lighthouse example (Blum, Ferri 2009) assumes that the observer is at sea level and neglects the influence of the tide and other factors upon the height of the lighthouse over the sea. This would disqualify a real seafarer!

illustrated by the following situation.

2.1 Situation: Sea floor areas

We consider here three-dimensional models of sea floor areas. These models are very important for human activities and provide impressive pictures and videos. However, they are most often unquestioned in relation to the processes of data acquisition and treatment they receive, and then in relation to their accuracy. As a part of their project in an in-service master class, two student-teachers aimed at questioning both processes and at evaluating the model by confronting it with reality. It means that they wanted to acquire and process data for a variety of configurations on the sea floor in order to compare it to the original configuration. Configurations were chosen simple: flat floor, sloped floor, half sphere… The students had to recreate the data acquisition, of course not at sea!

Since "real life" data acquisition is accomplished from a boat navigating an area and carrying a multi-beam echo sounder, they decided to build a *mock-up*, traveling on two horizontal axes above a floor, using 5 sticks to represent beams. Moving the boat with 3 positions on one axis and 6 on the other, they obtained a 3 x 6 x 5 table of data. Then they put into operation various treatments inspired by relevant scientific documentation.

2.2 Activity

The participants in the workshop will first look at the treatments that have to be applied in order to get a 3D model. Then they will consider the interest of designing a *mock-up* and of acquiring data from this *mock-up* as compared to using data acquired in 'real life'. Finally, they will discuss how data acquisition in the *mock-up* and mathematical treatments imply two different working spaces whose interrelation can help understanding.

Working theme 2: Working spaces for organizing students' modelling work

In this second theme, we propose to consider the classroom implementation of a modelling process involving models in several mathematical working spaces.

3.1 Situation

We consider here the shape of a main cable of a suspension bridge. Tasks for students at secondary level most often assume that the shape is a portion of a parabola, but do not question the reasons why it is quadratic, whereas for instance cable-stayed bridges have straight cables. It seemed to us that these reasons are accessible to 12th grade students, with the condition that they understand the process of modelling from which this result is derived: the tension along the cable progresses linearly, and so does the slope. Moreover, this process of modelling offers an opportunity for students to mobilize their knowledge in calculus at a synthetic level while investigating a real-world situation.

We consider four models and the corresponding Mathematical Working Spaces.

1. A discrete model is derived from the finite number of suspensors: a main cable is represented by a broken line and the deck is represented by a collection of weights hung on the cable by way of equally distant suspensors. The "static working space" organized around the first Newton Law is suitable for a study of the tension along the cable.
2. Knowing the tension at each suspension point, it is possible to establish recurrence formulas for the coordinates of these points. Students work here in a "coordinate geometry" working space.
3. It is then interesting to program the recurrence formulas in order to graph the broken line. Using a dynamic programming environment, students can appreciate the role of parameters, like the constant value of the horizontal component of the tension. This is the algorithmic working space.
4. Given the large number of suspensors, one can look for a curve, limit of the broken line modeling the cable when this number tends to infinity. The work on this "continuous model" is done inside a "mathematical functions working space" governed by classical symbolism and rules in calculus.

3.2 Activity

The participants in the workshop will first characterize the four working spaces and the potential geneses inside each working space. Then, taking advantage of reports on classroom implementation (Lagrange et al., 2015), the participants will discuss how this characterization helps designing and evaluating classroom situations.

Conclusion

We will conclude by discussing with the participants how the work done helps to reflect on starting questions offered for subtheme 1 by the discussion document:

- *What qualifies a real-world context as a point of departure and/or point of arrival of a didactic arrangement that builds on mathematisation?*
- *What are specific cognitive, social or discursive processes that occur in learning environments that have mathematisation as a pivot?*
- *Which material arrangements support students' learning of mathematics by mathematisation (e.g. artefacts, physical experiences, learning spaces, etc.)?*
- *Which epistemologies of mathematics are built?*

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Faire entrer les élèves dans la mathématisation horizontale. Des « fictions réalistes » et un dispositif de « résolution collaborative »

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Résumé. Cet atelier présente un travail développé au sein du groupe ResCo de l'IREM de Montpellier autour de l'enseignement et l'apprentissage de la modélisation mathématique. Notre objectif est de faire travailler les élèves sur la mathématisation horizontale et de leur faire prendre conscience que cette phase nécessite de faire des choix. Pour atteindre cet objectif, nous proposons des situations spécifiques appelées « fictions réalistes » et un dispositif adapté de résolution collaborative de problèmes basé sur des échanges entre pairs. L'atelier proposera aux participants de vivre notre dispositif en accéléré et d'étudier des productions d'élèves pour illustrer notre propos.

Abstract. This workshop introduces a work developed within the group ResCo of IREM of Montpellier around education and training of mathematical modelling. Our objective is to make work the pupils on the horizontal mathématisation and to make them become aware that this stage requires to make choices. To attain this objective, we offer specific situations called " realistic inventions " and a device adapted by collaborative resolution of problems based on exchanges between peers. The workshop will offer our device to the participants in a fast version and to study pupils' productions to illustrate our purpose

1. Faire entrer les élèves dans la mathématisation horizontale

La modélisation est un enjeu majeur en mathématiques et la modélisation mathématique est de plus en plus mise en avant dans les curriculums. En France par exemple, les nouveaux programmes mentionnent six compétences mathématiques présentes du primaire au lycée (de 6 à 18 ans) : Chercher, Raisonner, Calculer, Modéliser, Représenter, Communiquer. Dans cet atelier, nous nous intéressons à la possibilité de faire entrer les élèves dans une activité de modélisation qui produisent des apprentissages autour de la compétence Modéliser.

Pour préciser notre objectif, nous nous appuyons sur les travaux de Treffers (1978) qui distingue mathématisations horizontale et verticale. « La mathématisation horizontale qui part du monde de la vie pour arriver au monde des symboles et la mathématisation verticale qui se déplace à l'intérieur de ce monde des symboles. » (IREM Paris 7, 2011).

Le but principal du travail présenté ici est d'amener les élèves à prendre conscience de la nécessité de faire des choix dans la phase de mathématisation horizontale en les confrontant à des situations non mathématisées (que nous appelons fictions réalistes) et en abordant ces situations au travers d'un dispositif de résolution collaborative de problèmes entre classes dans lequel la confrontation des choix possibles, entre pairs, contribue à ce que les élèves cernent et comprennent mieux les enjeux de l'activité de modélisation. Nous entendons par modélisation mathématique la démarche de construction d'un modèle mathématique permettant de mettre en relation les éléments choisis d'un fragment de réalité en lien avec la question à étudier (Yvain, 2016).

Le travail présenté ici est issu d'une collaboration entre enseignants et chercheurs développée au sein de l'IREM de Montpellier, dans le groupe de travail ResCo[7] (ResCo, 2014) et le LéA CheRPAM[8]. L'atelier

7 Résolution Collaborative de problèmes (http://www.irem.univ-montp2.fr/).
8 Chercher et Résoudre des Problèmes pour Apprendre des Mathématiques, Lieu d'Éducation Associé à l'Institut

propose de présenter des fictions réalistes, de vivre le dispositif en accéléré et d'étudier des productions d'élèves. Les données proposées porteront sur le dispositif mis en place en 2017 en cours d'expérimentation. Cet atelier est associé à une communication de S. Yvain qui apportera un éclairage complémentaire, du point de vue de la recherche en didactique des mathématiques.

2. Des situations spécifiques : les fictions réalistes

Selon Israël (1996), la modélisation mathématique est un « fragment de mathématique appliqué à un fragment de réalité ». Il ajoute que « non seulement un seul modèle peut décrire différentes situations réelles, mais le même fragment de réalité peut être représenté à l'aide de modèles différents » (p.11).

Ainsi, notre choix s'est porté sur la construction de situations qui décrivent des fragments de réalité et posent une question sur ce fragment de réalité. Les figures 1 et 2 présentent deux exemples, sur lesquels nous reviendrons plus loin (les énoncés sont tels qu'ils ont été donnés aux élèves).

L'arbre
Des botanistes du Jardin des Plantes ont rapporté un arbre exotique inconnu, dont on vient de découvrir l'espèce. Pour étudier cette nouvelle espèce, les botanistes ont réalisé les croquis de l'arbre chaque année depuis 2013.

Schémas de l'arbre en novembre 2013, novembre 2014 et novembre 2015.

Les botanistes veulent faire construire une serre pour protéger l'arbre. Ils estiment qu'il aura atteint sa maturité en 2023. Pour les aider dans ce projet, prévoyez comment sera l'arbre en 2023 ?

Figure 1. Énoncé de la fiction réaliste « l'arbre » (échelle modifiée)

En voie d'extinction ?
Sur une petite île du Pacifique vivent deux espèces animales étudiées par les scientifiques depuis le 1er septembre 2012. Il s'agit d'un rongeur et d'un petit carnivore, prédateur du rongeur. Les scientifiques ont effectué des relevés de populations sur l'île qui leur permettent de faire les estimations suivantes :

Date	Rongeurs	Carnivores
1er septembre 2012	2000	30
1er septembre 2013	25000	25
1er septembre 2014	101000	50
1er septembre 2015	3500	70
1er septembre 2016	950	18

Populations de rongeurs et de carnivores sur l'île.

Les scientifiques craignent la disparition de ces deux espèces.Pour savoir si leur crainte est fondée, prévoyez l'évolution des effectifs des deux espèces pour les 10 années à venir.

Figure 2. Énoncé de la fiction réaliste « En voie d'extinction ? »

Ces situations sont des fictions réalistes au sens de Ray (2013). Selon lui, une fiction réaliste est « une situation a priori on mathématique, le contexte de cette situation est fictif mais réaliste, une prise ne charge efficace de cette situation demande une phase de modélisation, la phase de modélisation peut renvoyer à plusieurs problèmes mathématiques selon les choix qui sont faits » (p. 29). Nous ajoutons à cela deux autres caractéristiques :

Français de l'Éducation.

5. la fiction réaliste est conçue comme une transposition d'une problématique de modélisation issue des pratiques scientifiques professionnelles (travail du modélisateur) ;

6. les variables didactiques (Brousseau, 1998) de la fiction réaliste sont choisies de manière à favoriser l'entrée dans la mathématisation horizontale.

Pour permettre un travail de la mathématisation horizontale, ces situations sont placées dans un contexte (un fragment de réalité) qui soulève une question nécessitant la construction d'un modèle. Dans les deux cas présentés, l'enjeu de modélisation est porté par une question de prévision : c'est l'inaccessibilité (dans le temps) qui motive la construction d'un modèle.

3. Un dispositif de résolution collaborative entre classes

Pour faire vivre l'enjeu de modélisation, nous proposons ces situations au sein d'un dispositif de résolution collaborative de problèmes. Ce dispositif, proposé de la Sixième (grade 6) à la Terminale (grade 12), repose sur des échanges entre des classes (via une plate-forme), regroupées par trois, qui travaillent sur la même fiction réaliste, pendant cinq semaines, à raison d'une séance d'une heure par semaine.
La résolution collaborative s'organise en différentes phases :

- *Phase 1* (semaine 1) : Chaque classe découvre le sujet. Les élèves entrent dans la résolution du problème en rédigeant les questions qu'ils se posent, qui sont ensuite envoyées aux deux autres classes.

- *Phase 2* (semaine 2) : Chaque classe poursuit ses recherches en construisant des réponses aux questions reçues, qui sont communiquées aux autres classes.

- *Phase 3* (semaine 3) : Chaque classe prend connaissance des réponses des deux autres classes à ses questions.

Toutes les classes reçoivent la fiction réaliste relancée (Yvain & Gardes, 2014) : un énoncé dans lequel des choix de mathématisation horizontale ont été effectués par le groupe ResCo, en tenant compte des échanges entre les classes.

- *Phase 4* (semaines 3 et 4) : En s'appuyant sur les choix fixés par la fiction réaliste relancée, chaque classe poursuit la résolution du problème initial et confrontent leur démarches et leurs résultats sur la plate-forme (mathématisation verticale).

- *Phase 5* (semaine 5 ou plus) : Le groupe ResCo envoie aux enseignants une « clôture » : une résolution possible du problème ainsi modélisé en précisant des éléments de mathématiques travaillés et des informations sur la modélisation mathématique dans le contexte concerné afin que chaque enseignant puisse faire une synthèse et une institutionnalisation dans sa classe.

La phase de questions-réponses vise à déclencher le processus de mathématisation horizontale. Le fait de laisser vivre les questions permet de rendre visible les interrogations des élèves, et de les traiter collectivement (Aldon et al., 2014). C'est dans la phase des réponses que, d'une part, les questions pertinentes pour la mathématisation horizontale vont se dégager et que, d'autre part, vont apparaître différents choix pour modéliser le fragment de réalité considéré. Les élèves expriment d'autant plus librement leur questionnement qu'il est à destination d'autres élèves. À leur tour, ils vont recevoir des questions émanant d'autres classes, parfois communes aux leurs. Formuler des réponses à ces questions leur permet souvent de mieux identifier les grandeurs pertinentes.

Dans la partie suivante nous illustrons nos propos en montrant comment les élèves entrent dans l'activité de mathématisation horizontale lors du travail sur la fiction réaliste « L'arbre », dans laquelle était engagée une soixantaine de classes en 2016.

4. Retour sur les expérimentations de l'IREM de Montpellier : l'arbre

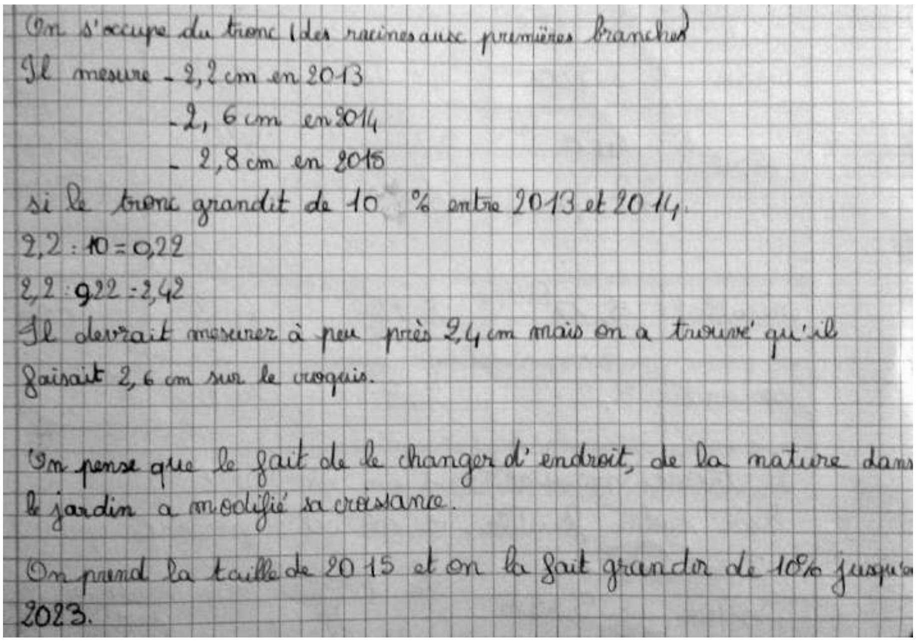
La fiction réaliste « l'arbre » décrite en figure 1, a été conçue comme une transposition d'un problème de modélisation historiquement et épistémologiquement important en sciences de la vie qui est la modélisation de la croissance des végétaux (Varenne, 2007).

Dans l'atelier, nous revenons sur certaines phases de la résolution collaborative mise en œuvre en 2016 pour illustrer l'activité des élèves relativement à la mathématisation horizontale.

Lors de la première phase, les élèves se posent des questions que l'on peut classer selon différentes catégories : question pour préciser l'énoncé (« De quelle espèce est l'arbre ? »), questions autour des choix mathématiques (« Est-ce que l'arbre grandit proportionnellement ? »), questions autour de données (« Peut-on faire des mesures sur le dessin pour faire nos calculs ? »). L'analyse des questions posées montre que les élèves entrent déjà dans la mathématisation horizontale. Les réponses produites par les élèves montrent qu'ils s'approprient l'enjeu de mathématisation horizontale en faisant des choix et en émettant des hypothèses :

- « Comment et de quelle manière pousse l'arbre chaque année ? », « Nous avons mesuré la plus grande longueur et la plus grande largeur en mettant l'arbre dans un rectangle. Ce rectangle ne grandit pas régulièrement. On pense que l'arbre pousse de manière irrégulière. » (classe de 6e).
- « Est-ce que le tronc et les branches poussent toujours à la même vitesse. Si oui, de combien chaque année ? », « Nous avons mesuré la largeur du tronc [...] et on a vu que ça grandit de 2 mm la première année, puis 3 mm la seconde. On prévoit que ce sera de 4 puis 5 puis 6 mm de plus chaque année » (classe de 6e).

La fiction réaliste relancée apporte elle aussi des témoignages dans ce sens : Dans la figure 3, des élèves requestionnent les choix de la fiction réaliste relancée. Ils ont besoin de trouver une justification à ces choix avant d'entrer dans la mathématisation verticale.


On s'occupe du tronc (des racines aux premières branches)
Il mesure - 2,2 cm en 2013
- 2,6 cm en 2014
- 2,8 cm en 2015
si le tronc grandit de 10 % entre 2013 et 2014.
2,2 : 10 = 0,22
2,2 0,22 = 2,42
Il devrait mesurer à peu près 2,4 cm mais on a trouvé qu'il faisait 2,6 cm sur le croquis.

On pense que le fait de le changer d'endroit, de la nature dans le jardin a modifié sa croissance.

On prend la taille de 2015 et on la fait grandir de 10% jusqu'e
2023.

Figure 3. Extrait d'un compte-rendu d'élèves, phase 4 (classe de 6e).

5. Conclusions et perspectives

Des fictions réalistes inspirées de problèmes de modélisation issus des pratiques scientifiques professionnelles peuvent permettre de faire vivre en classe la mathématisation horizontale, à condition de les mettre en œuvre dans un dispositif adapté. La fiction réaliste « En voie d'extinction ? » est celle qui a été proposée en 2017 par le groupe ResCo. Les données et leur analyse seront présentées dans l'atelier.

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Functions of operations and operands in school mathematics and physics: a complex interdisciplinary (de)mathematised phenomenology

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Abstract. In this laboratory of interdisciplinary reflection, we consider the school unit as an open learning organisation. This systemic view allows us to observe the implicit processes of (de)mathematisation that take place during the teaching-learning of school mathematics and physics through common to both mathematical formulae. The participants are invited to reflect upon specific examples with operations and operands in mathematics and physics, to realise the diverse implicit intentionalities that give meaning to cognitive reflexes and conventions that remain hidden for the learners and for the teachers of different disciplines. The present school reality promotes the compartmentalisation of knowledge, hindering the identification of both the specific to each discipline and the invariant across disciplines meanings. Such invisible obstacles may become visible, thus engineerable, within a reformed model of school that facilitates the emergence of the complex interdisciplinary teaching-learning space amongst and within school mathematics and physics.

Résumé. Dans ce laboratoire de réflexion interdisciplinaire, nous considérons l'unité scolaire comme une organisation d'apprentissage ouverte. Cette vision systémique nous permet d'observer les processus implicites de la (de)mathématisation qui se déroulent pendant l'enseignement et l'apprentissage des mathématiques et de la physique à l'école par des formules mathématiques communs. Les participants sont invités à réfléchir à des exemples spécifiques avec des opérations et des opérandes en mathématiques et en physique, pour réaliser les diverses intentionnalités implicites qui donnent un sens aux réflexes cognitifs et aux conventions qui restent cachés pour les étudiants et les enseignants des différentes disciplines. La réalité actuelle de l'école favorise la compartimentation des connaissances, ce qui entrave l'identification à la fois de la discipline spécifique et des invariants á travers des disciplines. De tels obstacles invisibles peuvent devenir visibles, donc gérable, dans un modèle d'école réformé qui facilite l'émergence de l'espace interdisciplinaire d'enseignement et d'apprentissage entre les mathématiques scolaires et la physique scolaire.

1. The school unit learning system: emerging interdisciplinary realities

These Mathematics and physics lie at the heart of the contemporary curricula, reflecting their crucial role in the contemporary societies. In particular, the ability to identify, link and apply scientific and mathematical models within the broader virtual-sociocultural practices appear to be an internationally acknowledged critical issue of education. Within this complex reality, in this laboratory, we view the school unit as a system to investigate the inter-acting and inter-related constructions that affect both mathematics and physics education within the complex and multi-levelled labour of the school unit (Bertalanfy, 1968; Davis & Simmt, 2003). The school unit functions as an intelligent, learning organisation with members of diverse roles, relationships and intentionalities transforming and being transformed by the organisation itself. Learning is conceptualised as linking links (Moutsios-Rentzos & Kalavasis, 2016). In the school unit, learning involves the transformation of the system, as the links transform the produced meaning, thus resulting to a new state of equilibrium. The systemic reflections upon experiences and/or actions identify change, whilst through their communication the new learning emerges at the individual and collective foreground. Such processes and states may be scarcely explicit within the same school course, whilst they seem to be completely missing across courses of different disciplines. It is argued that a school unit needs to explicitly address these phenomena to facilitate the students appropriately linking links within and across courses.

In this Reflective Interdisciplinary Laboratory, we focus on the (de)mathematisation processes (Gellert & Jablonka, 2007) that co-exist within and across school mathematics and physics, positing that appropriate *interdisciplinary reflective practices* may facilitate the construction of both intra-/inter-disciplinary meanings and learning. Through interdisciplinary individual and collective, descriptive, comparative and critical reflections (Jay & Johnson, 2002; Nissilä, 2005), common practices may emerge that would allow linking mathematics and physics, whilst clearly differentiating each discipline, thus obtaining internally-referenced (for the learners) meaning, rather than remaining an externally imposed differentiation. The co-development of the new qualities of the teaching-learning interdisciplinary experiences is expected to appropriately transform the learning abilities of the school unit, in line with the contemporary educational and social requirements and expectations.

2. Operations and operands in school mathematics and physics

Number operations lie at the heart of mathematics; from the early years, till university mathematics. In school mathematics, number operations work with three main types of operands (numerical, symbolic, geometrical), whilst magnitudes may be employed as a context (for example, word problems; Verschaffel, Greer & De Corte, 2000). Though the context may seem to be epistemologically irrelevant to mathematics, it lies at the heart of physics, giving physical meaning to the mathematical formulae, notably operations. Moreover, context seems to be crucial for mathematics education as, for example, through horizontal and vertical mathematisations, the students may be guided to meaningfully re-invent a mathematical idea (Gravemeijer, 1994).

The workshop simulates an interdisciplinary re-visit of the meaning of operations and operands in school mathematics and physics, organised in three parts. At the crux of the workshop lies the collective, comparative reflections upon signs and expressions of operations and operands that are used both in mathematics and physics, in order for the participants to experience that this common use in both sciences use indicates the historical and epistemological dialectic within each discipline and, importantly, between the two disciplines. First, we shall reflect upon operations and operands within mathematics. Drawing upon the ontological differences amongst algebraic and geometrical object (Duval, 2006), the algebraic operations in the expression of geometrical relationships may conceal the complexity of the geometrical meanings (Moutsios-Rentzos, Spyrou & Peteinara, 2014). For example, in an indicative activity of the first phase, we shall consider the algebraic expression of the Pythagorean Theorem. The algebraic addition of "a^2+b^2" is broader than its restriction to the set of positive real numbers; a restriction enforced by the geometrical nature of the operands. The participants will reflect upon the geometrical meaning of "a^2+b^2". For example: Does it refer to the addition of areas and, in that case, in which way does the relationship of areas characterise the angle of a triangle and what is the geometrical meaning of addition? Does it refer to the addition of two divisions the sum of which is constant ("$a^2/c^2+b^2/c^2=1$")? In that case, in which way does the ratio relationship characterise the angle of a triangle? And, what is the geometrical meaning of division?

In the second part, we shall consider the transformations of meanings that occur in the operations when the operand is a physical notion. For example, in an indicative activity of the first phase, we consider the definition of electric capacitance, as found in the textbook of the second grade of the Greek Lyceum (age 16 years old), is: "Capacitance C of a capacitor is the scalar physical quantity, which is equal to the quotient of the electric charge Q of the capacitor over the electric potential V of the capacitor. $C=Q/V$" (Alexakis et al., 2013, p. 32). The participants will be asked to calculate the capacitance of a given capacitor when the electric potential is doubled. Given the fact that the definition uses the word "quotient" and drawing upon the operation of division as a procedure, the participants may infer that the capacitance is halved. Nevertheless, it is argued that the nature of the operand, the physical meaning of capacitance (that is, the fact that the capacitance of a given object is a characteristic of the capacitor and as such remains constant), gives meaning to the performed operations, clarifying the nature of the relationship and in specific that 'C' is a constant, whilst Q, V are variables and, thus, the relationship is of the form "$y=ax$" i.e. an analogy. Following these, a more mathematically and physically compatible definition may be chosen; for example, the *physical notion* of capacitance may be defined as *the constant ratio* of the held charge over the applied electric potential. Drawing upon the *mathematical* notion of ratio, the students may appropriately infer that the *physical* capacitance remains the same, implying that the charge held is also doubled, thus *mathematically facilitating* their gaining *deeper physical understanding*.

The workshop will conclude with a reflective discussion upon the answers that in-service physicists gave to the same questions to further elucidate the (de)mathematisation phenomenology within the school unit. Such a process may facilitate the construction of meaningful teaching-learning bridges across the two courses, which importantly will help in gaining deeper understanding of both the taught content of the two distinct courses *and* their noematic convergences/divergences. We posit that through such interdisciplinary linkings, a novel quality of *interdisciplinary meta-learning* emerges, within which each discipline is viewed as a way of experiencing a phenomenon, thus highlighting both the distinctness of the disciplines and their convergences. Hence, the interdisciplinary linkings allow the conatruction of a meaningful communication space between mathematics and physics, thus facilitating appropriate "didactical planification towards to a meaningful learning as linking links" (Moutsios-Rentzos & Kalavasis, 2016, p. 97). We argue that such a quality of learning seems to be appropriate for the contemporary integrated, virtual society.

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Designing mathematics walks

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Abstract. There is a lot of mathematics to be found in the streets. The world around us provides us with many opportunities to come up with 'mathematical' tasks based on everyday situations. Realistic problems lead to authentic presentations of questions and problems to be tackled by pupils of various levels. In maths classes in school, pupils are confronted with a descriptions of everyday situations. When they read and look to the pictures in the book, pupils have to picture the situation in order to use their mathematical knowledge to solve the questions about everyday situations. These (extended) descriptions of realistic situations are not needed in mathematics walks, because you meet the situation in the street and that makes the questions more obvious. Moreover, pupils are encouraged by the discovery that maths knowledge and skills actually help them to cope with many real-life problems. In mathematics walks we aim to bring pupils into daily-life situations in which they see that you can recognise, use and discover mathematics. Maths turns out to be versatile, interesting, enjoyable and sometimes surprising. Reasons enough to get out of the classroom and take a walk in the school's surroundings.

Résumé. Il y a beaucoup de mathématiques dans les rues. Le monde qui nous entoure nous offre de nombreuses occasions de poser des tâches 'mathématiques' basées sur des situations quotidiennes. Des problèmes réalistes conduisent à des présentations authentiques de questions et de problèmes à résoudre par des élèves de différents niveaux. Dans les cours de mathématiques à l'école, les élèves sont confrontés à une description des situations quotidiennes. Quand ils lisent et regardent les images dans le livre, les élèves doivent imaginer la situation afin d'utiliser leurs connaissances mathématiques pour résoudre les questions sur les situations quotidiennes. Ces descriptions (prolongées) de situations réalistes ne sont pas nécessaires dans les promenades mathématiques, car vous rencontrez la situation dans la rue et cela rend les questions plus évidentes. De plus, les élèves sont encouragés par la découverte que les connaissances et les compétences en mathématiques les aident à faire face aux nombreux problèmes de la vie réelle. Aux promenades mathématiques, nous visons à amener les élèves dans des situations de la vie quotidienne dans lesquelles ils voient que vous pouvez reconnaître, utiliser et découvrir les mathématiques. Les mathématiques se révèlent polyvalentes, intéressantes, agréables et parfois surprenantes. Raisons suffisantes pour sortir de la salle de classe et marcher dans les environs de l'école.

1. Using realistic contexts

During (realistic) mathematics education many times meaningful tasks are formulated in a realistic context. Students/pupils are invited to work on these problems sometimes with only common sense and without any algorithm or structure to find a solution. The teacher is guiding them in different ways to provide them with self confidence in combination with challenging tasks. By examining a number of different situations, the pupil discovers what is always 'the same' in these situations. In this way an abstract concept develops from the reality and is understood. (Gravemeijer, K.,1994).

Many mathematics teachers make use of realistic contexts to make mathematical concepts as comprehensible as possible. This is sometimes best done by placing the pupils in situations they can recognise. After all real-life situations appeal more to the imagination, so that abstract concepts gain meaning and are more easily absorbed. And this idea is not a very modern (Freudenthal, H. ,1991). Realistic contexts are then deployed as examples of how to use mathematics. A context with questions in which problems can be solved by mathematical means shows how mathematics can be applied. Contexts may also be used for their value in motivation, to show that mathematical skills are useful for making decisions in daily life. In short, in today's mathematics teaching, realistic contexts are incorporated for a variety of reasons.

2. Presentation of contexts in the classroom

The presentation in the classroom of a realistic context is generally provided by the authors of the math-book. They take a great deal of trouble to make the situation as clear as useful, with text, drawings, pictures or even video. After all, pupils must be able to envisage the situation to be able to decide 1) what information is relevant, 2) what information is needed to answer the question, or even 3) what question you could ask in such a situation which could be solved mathematically. Obviously, the teacher can use other means to make the reality appeal as vividly as possible to the imagination of the pupils in the classroom. An (imaginary) story, a photograph, drawing or video (i.e. see Mayer, 2010) could also lead to the development of mathematical problems. In addition, material can also form the context from which mathematical questions appear. For the purpose to bring a realistic situation into the classroom it has to be described and if possible supported by visual evidence. There is always the drawback that the description of a situation already provides a selection of the relevant information offered by the practical situation. There are, however, other possibilities to pose realistic contexts for the sake of mathematics education.

3. Presentation of contexts during a mathematics walk

Mathematics walks are designed to invite students (and their teachers) to walk down the streets in their own neighbourhood and look at the world around in a mathematical way. Students are not only invited to answer questions about what the teacher provide them with, but also ask themselves questions about the situation in focus. The advantage of a realistic situation in a mathematics walk is that the situation virtually presents itself. How is this working? What is related to what? What questions can be asked? What is relevant here? Can the questions about the situation be answered easily and quickly here, or is it important to work accurately and with precision? What are the possible ways of solving the problem? Such questions don't rise separately but pop up together and more or less they force themselves on you. Students have to find relevant measurements and data for calculations by themselves. That makes the application of mathematics more authentic. (Spijkerboer, 2000)

Figure 1. Does this slope meet the legal requirements for a wheelchair ramp?

Although the mathematics walks offer a great variety of contexts, it is hard to develop new concepts from it (in the street). However, after a mathematics walk it is possible to go back to a particular experience in the next classroom lesson. It is then quite possible to use the realistic contexts again to develop new mathematical concepts. For instance, counting the number of people going in and out of a shop can forecast the number of visitors of this shop during one week, month or year. The concepts of mean, variance and deviation is going to growth out of this task.

The most important point of contexts during mathematics walks is to learn to recognize mathematics in the world around us. As soon as something can be seen or calculated in realistic situations, it becomes clear how mathematics can be used. In mathematics walks there is plenty of scope for the use of knowledge and skills learnt in maths lessons. This knowledge and skills have to be applied in new situations. The pupil does not know beforehand what knowledge is useful in a particular situation; therefore, this can lead to a great diversity of approach. The development of individual ways of solving problems is an important objective of realistic mathematics education, and maths walks give an opportunity to practise this skill. Pupils are invited not to copy immediately the teacher's way of solving problems, but are invited to think themselves and discover their own ability to handle in new situations. In this way mathematics walks can help students (and teachers) to work on the higher levels of the OBIT-model (Spijkerboer & Santos, 2015).

4. Researchgroup

A number of ± 50 students at the 'Freie Universität Berlin' were invited to a summer course concerning didactics of mathematics education in primary school (2016). The students were observed during their task to design mathematics walks in the surroundings of their schools. The task was formulated like:

- Find in the neighbourhood of your school, or in the area around your home, 10 situations/contexts from which you can extract a mathematical question.
- Each situation is presented by a photograph with realistic questions about the situation. Please add the solutions too.
- Explain why this situation will be helpful/useful in mathematics education and for which grade it is suitable.

The contexts chosen by the students to fulfil the task formulated above, are interesting data to discover the ways aspiring primary teachers look at the world around with their mathematical eye. The students' work raises questions like; What is mathematics about for a young child?, What is useful in mathematics? and How is mathematics education designed in the head of the aspiring teachers?

During the workshop we will work on these questions that emerged from our own interaction with these data, to provide ourselves with personal advice to design a non-artificial mathematics walk in the neighbourhood of our own school for our own pupils. With the experiences of mathematics walks in not only Berlin, but also examples from mathematics walks in other cities it is possible to find criteria for suitable contexts in the street, to provide students during mathematics walks with challenging questions and interesting problems to solve. The experiences show that especially geometry is found in a great number of examples. Also counting and calculating is a favourite activity. It is hard and therefore more challenging to design tasks for pre-algebra or the use of statistics in the street.

The exchange of those tasks will be a nice resource for the input given by classification of different contexts and problems for mathematical walks. You bring home many nice examples for use in mathematics lessons in different grades, with different purpose and with different results. Also tips how to organise this in daily education are given. The only question you have to answer for yourself is: Am I going to change my habits in mathematics education, designed by myself? (Spijkerboer, 2015)

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MATHEMATISATION: SOCIAL PROCESS & DIDACTIC PRINCIPLE

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MATHEMATISATION: PROCESSUS SOCIAL & PRINCIPE DIDACTIQUE

Emotional experiences of high school students in a mathematics class

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Cognitive psychology emerged in the 1970s, when psychologists began to analyze whether or not they could provide the necessary tools for the study of affect and emotion. Certainly, cognitive psychology, found no problem in explaining facts such as that the same question could be perceived from different perspectives. This was already encouraging, because the ability to look at a situation from different perspectives attracted attention as the fact, that different people often experience different emotions in response to the same event.

Ortony et al. (1988) state that there are three kinds of emotions, which are the result of focusing on one of the three prominent aspects of the world: events (eg joy and compassion) and their consequences, agents (eg, pride And reproach) and their actions, and objects (love and hate) pure and simple.

Reactions to agents differ in four emotions, which encompass the attribution group, and reactions to objects, which lead to the attraction group. Another aspect that is taken into account is the intensity of the emotions, since it varies according to the situation and the different people. For this, the authors elaborated a valuation structure, classifying it into three variables desirability, plausibility and attractiveness.

In this research, 30 students concentrated in groups of 8, 6 and 5 members were interviewed by adhering to an interview script. In group 1 there are 5 students and one interviewer. Below is one example of the analysis of one-hour interview that was performed in group 1. The presentation will show the complete analysis of two groups and preliminary results. One example:

Type of emotion: Boredom.
Trigger Situations: Teacher's Bad Explanation
Variables that affect the intensity: The continuous repetition of a single subject, The teacher does not dominate the content, The subject is not useful for life.

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Affective factors and mathematical thinking. emotional experiences of mathematics teachers

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Abstract. We interviewed mathematics teachers to find out which emotions they experience in their dayly activities. OCC theory was our base. We found emotions classified as Reactions, Group, Trigger Condition, Type of Emotion and Variables that affect the Intensity of Emotion.

Résumé. Nous avons interviewé des enseignants de mathématiques pour apprendre quelles émotions ils connaissent dans leurs activités quotidiennes. La théorie d'OCC était notre base. Nous avons trouvé des émotions classifiées comme les Réactions, le Groupe, la Condition de Gâchette, le Type d'Émotion et de Variables qui affectent l'Intensité d'Émotion.

File preparation and submission

The aim of this research was to identify the emotions experienced by active high school mathematics teachers during their activities.

Data were collected through extensive open-ended interview questions. The voluntary participation of 5 Mathematics professors in the "Víctor Mercante" Liceo pre-school of the National University of La Plata allowed extensive interviews with their personal history as students of Mathematics, his personal history as teachers of Mathematics, his self-concept as mathematicians and as teachers.

All the interviews were transcribed, which were video recorders in some cases and in others only recorded in audio, and analyzed according to the Theory of Cognitive Structure of Emotions (Ortony, Clore and Collins, 1988) mentioned here as OCC Theory. Identifying the triggering situation, words to signal the emotion, type of reaction, group of emotion belonging and variables that affect the intensity of the emotions.

The emotional experiences identified in the participants were classified according to Reactions, Group, Trigger Condition, Type of Emotion and Variables that affect the Intensity of Emotion. For the different types of Emotion identified, the respective transcription was analyzed, and we discussed the categories already mentioned by which they are recognized.

As for the evaluation of the emotional experiences in the class activities in front of the students, the reactions appear before the events and before the agents.

In the first case, the teachers have their predictions as to what is planned to carry out the mathematics class in order for the students to reach some level of achievement in the domain of the mathematical content at stake during the class, the mechanism to assess the goals, more specifically these events and according to the desirability as variable that intensifies the emotional reaction, were calculated with reference to the Goals of type I (interests). We analyze this as teachers refer to such reactions as "I am happy, satisfied with today's class," interpreting such a statement as a desirable event that has led to the emergence of a positively valued reaction.

In the second case, in contrast to the actions of the students, being in these cases the agents, it is the attribution variables that underlie the actions of the same and that the teachers value from the norms as a model of behavior. This analysis can be demonstrated, for example, in the statement: "Then, you see it as a frustration because, that is, I liked to live it, solve it, I liked solving things, finding solutions to problems. And now I see that it is a burden that is put on them: make them solve an exercise, right? ".

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Teaching practice as an object of reflection for the mathematics teacher: a proposal of intervention

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Abstract. In this article we present an ongoing research whose aim is to identify how the (high school) mathematics teacher transforms his or her own practice by reflecting upon it in benefit of a better-quality teaching in low socioeconomic contexts. This is an exploratory study and, for data collection, it involves video recording of classes of teachers (participants) as well as implementation of a course to discuss real teaching practice in the classroom through collaborative work. The theoretical framework is reflection-in-action. As results, we look forward to contributing to the improvement of teaching practice and seeking to close the gaps in mathematics learning.

Résumé. Dans cet article, nous présentons une recherche en cours dont l'objectif est d'identifier comment le professeur de mathématiques (du lycée) transforme sa propre pratique en la réfléchissant au bénéfice d'une amélioration qualitative de l'enseignement dans des contextes socio-économiques faibles. Il s'agit d'une étude exploratoire et, pour la collecte de données, il s'agit d'un enregistrement vidéo de classes d'enseignants (participants) ainsi que de la mise en œuvre d'un cours pour discuter de pratiques pédagogiques réelles en classe par un travail collaboratif. Le cadre théorique est la réflexion-dans-l'action. En tant que résultats, nous sommes impatients de contribuer à l'amélioration de la pratique pédagogique et cherchent à combler les lacunes dans l'apprentissage des mathématiques.

1. Introduction

Teacher training and refreshing, at any educational level, is a key factor that has an effect on the quality of education and the academic performance of the student (Adler, Ball, Krainer, Lin, & Novotna, 2005; Jaworski & Wood, 2008). In the last decades, education in Mexico has faced adversity partly because of a lack of teacher training. Sometimes, the way in which the high-school teacher teaches mathematics does not favor the curricular objectives, leading to the loss of interest among the students to continue learning. This is a severe problem in institutions located/placed in unfavorable contexts (e.g. poverty) since their teachers often have little to no experience. Reports indicate that 35% of the high-school teachers in Mexico have four years or less of experience; 80% of these teachers work at institutions whose students have vulnerable socioeconomic backgrounds (INEE, 2015). Recently, different programs to develop teaching skills have been implemented; however, a great deal of them are only training courses or short and sporadic workshops that often take place far from the reality in the classroom. Thus, there is no link between the mathematics curriculum and real teaching (Jaworski, 2006; Martínez-Rizo, 2013); in addition, they are specific refresher proposals, independent from one another. Interested on the problem described above, we have developed a research project regarding reflection on mathematics teaching practice (at high-school level) through collaborative work (Sfard, 2005). The aim of the work is to identify how that reflection helps teachers transform specific aspects to provide better quality teaching in vulnerable socioeconomic contexts. In this article we describe the theoretical and methodological proposal we will carry out to achieve the objective presented.

2. Reference framework

Mathematics teachers have the ability to reflect on the actions related to their practice and that reflection may arise in situations that for them could be problematic situations. Schön (1983) considers that "Stimulated by surprise, they [professionals] turn thought back on action and on the knowing which is implicit in action" (p. 50). A situation becomes a problem when teachers resort to the knowledge available to them in the moment, but that knowledge turns out to be poor or inadequate to face the situation. Reflection then demands teachers build the knowledge needed to confront the situation, thus improving their teaching practice. As Gilbert stated,

Reflection-in-action occurs when new situations arise in which a practitioner's existing stock of knowledge–their 'knowing-in-action'–is not appropriate for the situation. It involves reflecting on 'knowing-in-action'. 'Reflection-in-action' is a process through which hitherto taken for granted 'knowing-in-action' is critically examined, reformulated and tested through further action. It is a *process* of *research* through which the development of professional knowledge and the improvement of practice occur together (in much the same way as in action research). (1994, p. 516)

3. Methodology

This is an exploratory study in which mixed techniques to acquire information are used. It involves designing an intervention to promote reflection in (high-school) mathematics teachers and value the transformation of their teaching practice. The aim of the intervention work is to "the researchers stress that their studies are done *with* the teacher rather than about her, that they go to classrooms to listen to the teacher and to think with her rather than to tell her what to do, and that they [researchers] 'support teachers and learners to develop their own powers… rather than trying to make changes for them'" (Sfard, 2005, p. 401). To collect data, we will adapt a standardized observation guide focused on the pedagogical knowledge of the content, the use of resources and the teachers discourse and contents related to algebra and analytic geometry. The instrument will be applied to each participant, before, during, and at the end of the intervention. After each time the instrument is applied, we will observe and video record two classes per teacher. The data analysis will allow us to identify the dimensions of the practice on which the design of the intervention will be based. The intervention will consist of a course whose aim is to create reflection among the participants regarding their practice and for it. The course will promote collaborative work among researchers and teachers (Kieran, Tanguay & Solares, 2011; Sfard, 2005). To do so, activities will be implemented for the teacher to analyze and discuss his or her practice with other colleagues and the researchers involved. This kind of discussion will take place in terms of an action of reflective practice (Gilbert, 1994; Schön, 1983). Given the kind of study, the methodology agrees with that proposed by Guzmán, Marín e Inciarte (2004), where the participants analyze their practice, locate their real competences, determine the characteristics of teaching by competences and lay out the need of accompaniment during training. The participants will be ten high-school teachers, working at schools in low socioeconomic contexts.

4. Results and conclusions partial

The project here described is in its initial phase. We are currently designing the diploma course and reviewing the literature to determine the mathematics, algebra and analytic geometry contents that present a greater difficulty to the high-school teacher and whether this difficulty is linked to the way of teaching in the classroom. We have found that high-school teaching practice is poorly studied and of increasing importance due to the fact that high school is mandatory in Mexico. Particularly, little is known regarding how the teacher appropriates his or her practice and transforms it. We are interested on teachers of schools located in vulnerable socioeconomic contexts because they face greater challenges than their peers, who work in environments of less exclusion and poverty. We consider it is necessary to design and implement refresher actions that impact teaching practice in the short and long term and contribute to reduce the learning gap between high-school students. For example, we seek to implement intervention programs focused on providing students with spaces where the reality of the classroom can be subject of study. Besides collaborative work, we turn to reflection so that teachers analyze and transform certain areas to improve the quality of education and contribute to the students' learning. This research may provide valuable results to improve the current state of teaching practice. It particularly contributes to the need of knowing how teachers learn from their practice and how they can be given new opportunities to promote mathematics learning.

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Homo matematicus: the measure of all things

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Abstract: Where does it stem from that mathematics is not a subject beloved by all students? Probably because it is perceived as difficult, but also because its relationship to the real world often remains unclear. In this paper, I propose an approach that aims to illustrate the extend of how much mathematics lies within us, starting from the human body itself.

Résumé. D'où vient-il que ces mathématiques ne sont pas un sujet aimé par tous les étudiants? Probablement parce qu'il est perçu comme difficile, mais aussi parce que sa relation avec le monde réel reste souvent incertaine. Dans cet article, je propose une approche qui vise à illustrer l'étendue de la quantité de mathématiques qui se trouve en nous, à partir du corps humain lui-même.

1. Introduction

Thinking about the meaning of mathematics at school I was very impressed by the following consideration: "Everyone knows that something is wrong. The politicians say 'we need higher standards.' The schools say 'we need more money and equipment.' Educators say one thing and teachers say another. They are all wrong. The only people who understand what is going on are the ones most often blamed and least often heard: the students. They say 'math class is stupid and boring,' and they are right." (Lockhart, 2009).

For years, I conducted a highly significant investigation (although not statistically rigorous) about the relationship between students and mathematics. I set up a survey of middle and high school students that consisted of the following two questions:

1) How many of you love, or appreciate, math?

2) How many of you believe that math plays an important role in our society?

As a reaction to the first question, generally one or two hands arise. In response to the second question, I nearly get the consensus of all students.

Consequently, I am of the opinion that these outcomes should initiate a turn of action in regard to the teaching of mathematics, but all I can hear from the math teachers is the claim that "Unfortunately, there is nothing to do."

These questions were the basis from which I started my case study. In this study, I mostly applied common objects (Drivet, 2013) in order to explain how students approach mathematics. I have used many different artifacts, some with obvious disciplinary connotations (Abacus, Dice, Geoboard, Napier's Bones, Tangram, etc.), and others which are just a starting point to explore more or less usual mathematical themes (Bicycles, Glasses, Potatoes, Spaghetti Measures, T-Shirts, etc.). At the end of the implementation, which took two very intense hours, I observed that the students were still highly concentrated. What I realised further was that mathematics entails more than mere abstraction, memorization of formulas, procedures, and definitions to be repeated.

In general, this experience has strengthened my hypothesis that it is necessary to introduce students to a concrete approach to mathematics first, and then develop a more profound and systematic work.

One possible approach might be to connect mathematics with the formation of the human body and thereby adapting the teaching of the sophist philosopher Protagoras who claimed that "Man is the measure of all things".

How are we supposed to understand this sentence? Are we allowed to interpret it literally?

Is it possible to conclude that *homo matematicus* means that it is a "mathematical object"? It should be emphasized that the term "homo" does not have any gender significance.

Perhaps da Vinci was of this opinion when he drew the Vitruvian Man (Sinisgalli, 2006), one of the most famous symbolic unions of art and science.

2. Some examples

The ideal body

1,618… is the number which expresses the relationship of proportions called golden proportion, golden section, or divine proportion (Akhtaruzzaman & Shafie, 2011). The ideal body, according to ancient Greeks, was a body that had such a proportion.

For example, if you divide the height of the statue of Venus de Milo for the distance between the ground and her navel, you get this number.

The model and actress Laetitia Casta has been awarded the prize "woman of the new millennium". Interestingly, the proportions of her body are very close to the proportions of the golden section just like those of the Venus de Milo (Figure.1).

Figure 1. Venus de Milo and Laetitia Casta

The ideal face

Dr. Stephen R. Marquardt is a former maxillofacial surgeon in California. During the last decades, he has been working on a standard for judging the beauty of the human face.

He has determined the "Phi Mask" or "face mask" which is based upon segments of lines and forms that are related to each other through the golden proportion (Figure 2).

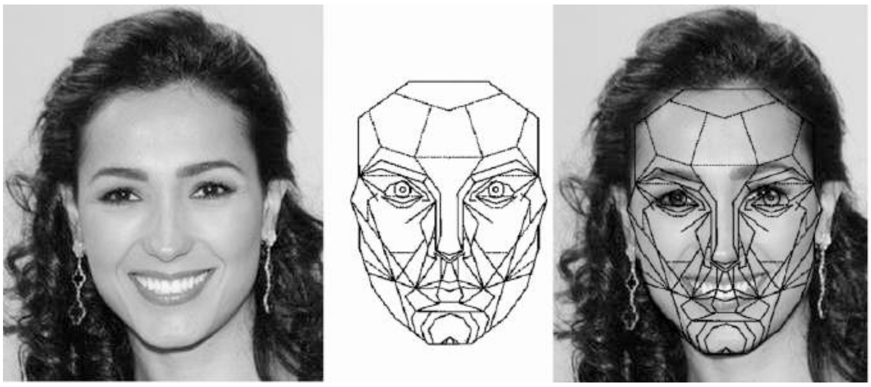

Figure 2. The mask is applied to the TV presenter Caterina Balivo

Hand and food

How can your hand be a suitable tool to determine the ideal amount of food? In order to do so, the human hand offers, for example, the following units of measure: the palm of the hand, the hand with stretched fingers, a clenched fist, the index finger, the tip of the index finger, the tip of the thumb, or cupped hands. (Figure 3) provides concrete examples:

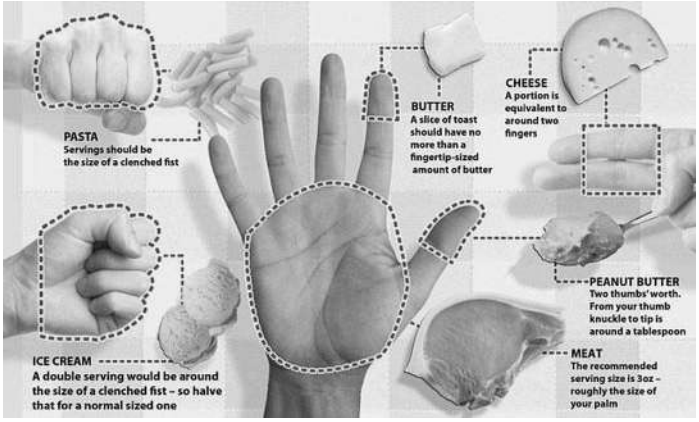

Figure 3. Examples of the relationship between hand and food

Hand and sky
If we stretch the arm and the fingers of the stretched arm, we are able to measure an angle of about 20° in the sky. If we spread our thumb but hold the other fingers tight, the estimated angle will be about 15°. If we hold all fingers tight together, the measured angle will be about 10°. The distance between the knuckles of the index and the little finger of the fist is equal to about 9°. The diameter of the thumb approximately equals about 2° and 30' while that of the the index finger corresponds to approximately 1° (Figure 4).

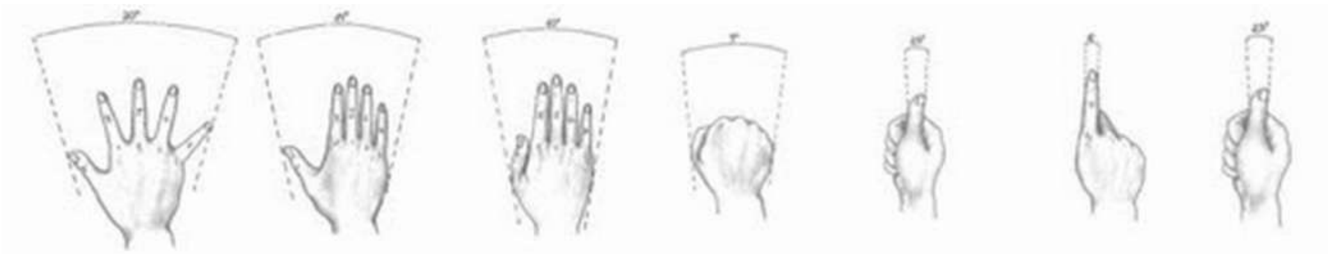

Figure 4. The estimated angles

Arm, foot and cathedrals
Originally, measurements of length took their names from parts of the human body and, in common usage, some of these measures still exist today.
In the Middle Ages, the builders of the beautiful French cathedrals applied a measuring instrument that consisted of five articulated rods of different lengths. These measures corresponded to (Figure 5):

- palm, equal to the width of a hand, excluding the thumb;
- 4 fingers, equal to the distance between the index and the little finger of an opened hand;
- span, equal to the distance between the thumb and the little finger of an opened hand;
- arm (or cubit), equal to the measured length of the forearm from the elbow to the end of the middle finger.
- foot, equal to the footprint length of the foot of a man.

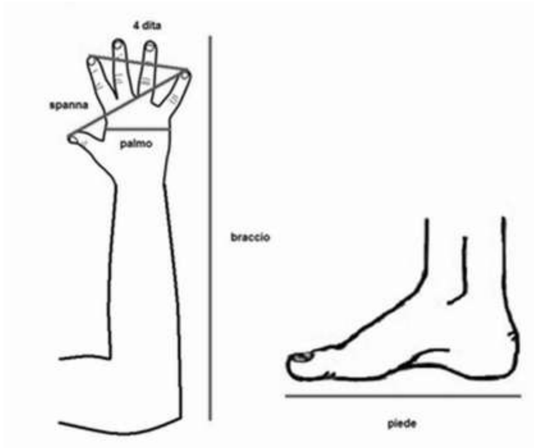

Figure 5. Arm and foot as measuring instruments

Skin and blood

The body surface area (BSA) is a very important anthropometric parameter. In medicine, for example, it is used to individualise nutritional or therapeutic programs.

The BSA is measured in m^2 and can be indirectly calculated on the ground of different equations, the simplest of which is Mosteller's that is based on the height (h) in cm and weight (p) in kg:

$$BSA = \sqrt{\frac{h \cdot p}{3600}}$$

The volume of blood of an individual can be calculated approximately by values of height (h) and weight (p) by using the Nadler's formula for males (V_m) and females (V_f):

$$V_m = 0{,}3669 \cdot h^3 + 0{,}03219 \cdot p + 0{,}6041$$
$$V_f = 0{,}3561 \cdot h^3 + 0{,}03308 \cdot p + 0{,}1833$$

Heart and love

The heart is a hollow muscular organ at the center of the chest cavity.

But is there an equation that shows in the shape of a heart in a diagram curve (which would at least be useful to impress your partner)?

The answer is positive (Figure 6):

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

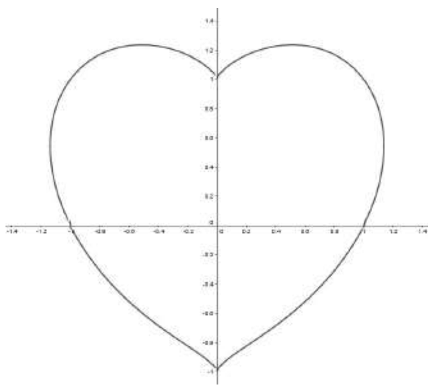

Figure 6. The heart curve

Could this be the secret formula of love, such as a chemical or a magic filter, that helps to classify one's feelings? Only mathematics can give us the tools for the best answer.

Is it possible to find a soul mate with a good approximation? We could think of the possibility of finding the right person relying on the "optimal stopping theory".

Underlying this theory is an algorithm that explains what you should do. You should reject 37% of the first

experiences and pick the next person that is better than everybody that you have seen before.
Of course, this does not guarantee to find a soul mate, but it optimises the chance of meeting a "good catch"

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Sitography

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https://www.ted.com/talks/hannah_fry_the_mathematics_of_love

Use of variety of models in teaching calculus as an effective means to enhance the interest to the subject, improve understanding and stimulate creative thinking of engineering students

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In this forum, we are going to demonstrate our long-term experience of using different models (mental, physical, handmade, graphical, or computer visualization of mathematical concepts or processes) in study of one or multivariable calculus in our college and we intend to hold a discussion on the matter too.

Despite of the growing tendency to use more and more computer models and mobile telephone application of mathematical concepts (which using also be discussed) we widely use the tangible models as well. We want to emphasize that tangible - it's not just visual, visual - it's not just what student can see on the picture or on the computer screen, but what he can see and touch physically and hold in the hands. We always try (and think that it is very important to learning process) to encourage our students to make an appropriate model by their own hands and by using of improvised materials. This kind of activity needs creative engineering thinking and can give positive impact to professional development of engineering students.

Our experience shows that such approach improves the understanding of important abstract notions and facts of Calculus. It also promotes creative engineering thinking of the students and stimulates their interest in learning of Calculus and its various applications.

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Some teaching motivations among Latin American teachers

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Abstract. We made a research which purpose was to identify factors that have motivated to some mathematics teachers to choose the teaching as a career. The study was made with mathematics teachers which wanted to get admission in the Mathematics Education Program of National Polytechnic Institute. The results found are similar to those reported in the literature. The love of mathematics and teaching stand out as the main factors.

Résumé. Nous avons fait une recherche quel but était d'identifier des facteurs qui ont motivé à certains enseignants de mathématiques pour choisir l'enseignement comme une carrière. L'étude a été faite avec les enseignants de mathématiques qui ont voulu recevoir l'admission dans le Programme d'Éducation de Mathématiques d'Institut Polytechnique national. Les résultats trouvés sont semblables aux annoncés dans la littérature. L'amour de mathématiques et d'enseignement ressort comme les facteurs principaux.

The aim of this research was to identify the emotions experienced by active high school mathematics teachers during their activities. The purpose of this project was to investigate the reasons or factors that led a group of mathematics teachers to choose their career. Using the virtual platform Moodle was possible to interview a group of mathematics teachers about their own motivations about this topic.

The characteristics of respondents in relation to their teaching practice were widely varied, such as the years of teaching service, the educational level and subjects taught, undergraduate studies, and even their place of residence, since the research involved teachers from Mexico and some Latin American countries. This allowed us to analyze if some of these characteristics had any repercussion in the reasons or factors that the teachers reported, for later contrast those motives with the findings that have been reported in previous research on the subject like Watt & Richardson (2012) or Bastick (2000).

The analysis of the information led us establish a categorization of the reasons that originated the entrance of this teachers to teaching. This factors were related to career "advantages" (job stability, holidays, working hours, etc.) or with an intrinsic desire to become teachers (appreciation for teaching, mathematics, working with children or young people, etc.). The love of mathematics and teaching stand out as the main factors.

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