Comparison of different 2nd order formulations for the solution of the 2D groundwater flow problem over irregular triangular meshes

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Abstract: - Mixed and Mixed Hybrid Finite Elements (MHFE) methods have been widely used in the last decade for simulation of groundwater flow problem, petroleum reservoir problems, potential flow problems, etc. The main advantage of these methods is that, unlike the classical Galerkin approach, they guarantee local and global mass balance, as well the flux continuity between inter-element sides. The simple shape of the control volume, where the mass conservation is satisfied, makes also easier to couple this technique with a Finite Volume technique in the time splitting approach for the solution of advection-dispersion problems. In the present paper, a new MHFE formulation is proposed for the solution of the 2D linear groundwater flow problem over domain discretized by means of triangular irregular meshes. The numerical results of the modified MHFE procedure are compared with the results of a modified 2nd spatial approximation order Finite Volume (FV2) formulation [2], as well as with the results given by the standard MHFE method. The FV2 approach is equivalent to the standard MHFE approach in the case of isotropic medium and regular or mildly irregular mesh, but has a smaller number of unknowns and better matrix properties. In the case of irregular mesh, an approximation is proposed to maintain the superior matrix properties of the FV2 approach, with the consequent introduction of a small error in the computed solution. The modified MHFE formulation is equivalent to the standard MHFE approach in both isotropic and heterogeneous medium cases, using regular or irregular computational meshes, but has a smaller number of unknowns for given mesh geometry.

Key-Words: - groundwater, finite elements method, mixed hybrid finite elements method, finite volumes method, positive-definite matrix, M-property, Raviart-Thomas basis function

1 Introduction

The numerical solution of a groundwater flow problem, according to the Eulerian approach, yields a set of discrete values of potential head or pressure and velocity, referred to different points of a mesh dividing the computational domain in elements.

The Partial Differential Equations (PDEs) governing system is given by the mass conservation equation of the fluid phase and by the Darcy formula for the velocity.

In the past decades, Finite Differences (FD), Finite Volumes (FV) and Finite Elements (FE) methods have been widely applied for the solution of the linear groundwater flow problem, as well as of the pollutant and/or heat transport in porous media. See for example [12, 11, 13, 16].

The FD method is the oldest one, is based on a homogeneous and regular domain decomposition and is easy to implement. The FV and FE methods are more flexible in discretizing complex geometry domains as well as complex boundary conditions.

For a correct solution of the transport problem, the velocity field needs to preserve mass conservation, both locally and globally. Unlike FD and FV methods, in FE approach the velocity field is calculated by differentiation of the potentials inside the elements, the discrete normal fluxes are discontinuous across inter-element boundaries and the local mass conservation is not warranted.

One way to circumvent this problem is to proper exploit the local subdomains where the mass conservation property is satisfied. For example, in the Galerkin technique applied on 2D triangulation, the subdomains can be defined as the Voronoi (or Thiessen) polygons. Application of this approach in 3D is very complicated and provides a strong increment of the computational time. On the other hand, the Finite Volume (FV) approach is mass conservative because the subdomain where the mass balance is applied is the same mesh element.

The Mixed Finite Element (MFE) methods provide and attractive framework for these types of problems: by simultaneously approximating the

potential head and normal fluxes, the computed normal fluxes are continuous across inter-element edges and the local and global mass balance is automatically achieved. MFE methods have been extensively used for the solution of parabolic problem in many application fields (groundwater flow problems, petroleum reservoir problems, potential flow problems, …), but in elliptic problems (i.e. steady state problems) the matrix of the system becomes ill-conditioned [9, 17].

An improved technique is represented by the Mixed Hybrid Finite Element (MHFE) method. The MHFE discretizes spatially the flow equations in a set of continuity equations across all the edges of the mesh, using the average potentials along these edges as unknowns. In time splitting techniques for the simulation of advection-dispersion problems, the solution of the dispersive component by means of MHFE methods can be easily coupled with the solution of the convective components obtained by means of other formulations (for example [1-4, 6, 7]).

If the mesh is triangular and the element parameters are isotropic, it can be proved [17] that the MHFE is algebraically equivalent to a FV approach, where the potential gradients are computed between the circumcentres of the triangles. In the FV approach the unknowns are the potentials in the circumcentres, that are less than the mesh edges and, if the mesh is regular or mildly irregular, the final system matrix is symmetric positive-definite with good conditioning properties.

A 2nd spatial approximation order FV formulation (FV2), recently proposed [5] and a new modified formulation of the standard MHFE method (MMHFE) are presented in the present paper for the solution of the 2D linear groundwater flow problem discretized over generally unstructured triangular meshes.

Numerical results of the FV2 scheme are shown to be equivalent to the results obtained by the standard MHFE method in the case of regular and mildly regular meshes. In the case of irregular meshes, a simple approximation is proposed; the approximation is aimed to maintain the good properties of the final matrix, even if this simplification is paid with a reduction of the accuracy provided by the second order spatial approximation.

In the second proposed methodology, the standard MHFE formulation is mixed with the FV2 formulation. The first one is restricted only to some of the obtuse triangles, called "degenerate", the second one to all the other triangles. The matrix of the resulting linear system is symmetric and positive-definite, even though the "M-property" could not be guaranteed in the cases of degenerate obtuse triangles. Numerical results are the same obtained by a standard MHFE formulation, in both cases of isotropic and heterogeneous medium, using regular or irregular computational meshes.

A comparison of the computational costs of the proposed methodologies is also carried out.

2 The Physical Problem

Consider the following evolutionary problem on the domain $\Omega \in R^2$ of a saturated porous medium (see for example [8]):

$$S_0 \frac{\partial H}{\partial t} + \nabla \cdot \mathbf{G} = Q \qquad (1,a),$$

$$\mathbf{G} = -K\nabla H \qquad (1,b),$$

with the following boundary conditions:

$$H = H_d \quad \text{on } \Gamma_D \qquad (2,a),$$

$$q_N = -K \frac{\partial H}{\partial n_\Omega} \quad \text{on } \Gamma_N \qquad (2,b),$$

where n_Ω is the outward unit normal to Ω. The initial condition are:

$$H(\mathbf{x},0)=H_0(\mathbf{x}) \qquad (2,c).$$

Eq. (1,a) represents the mass balance equation of the fluid in the porous medium, where the velocity field $\mathbf{v}$ is expressed by means of the Darcy law:

$$\mathbf{v} = -K\nabla H \qquad (3),$$

In Eqs. (1)-(3) $\mathbf{G}$ is the diffusive flux, equivalent to the Darcy velocity $\mathbf{v}$ provided by Eq. (3), H is the potential head, K the hydraulic conductivity, assumed constant in time, S_0 is the storativity coefficient, assumed constant in space and time, Q is the source or sink term. Γ, Γ_D and Γ_N are respectively the boundary of Ω, the part of Γ with given piezometric head H_d (Dirichlet condition) and the part of Γ with given flux q_N (Neumann condition).

3 Formulation of the Problem

3.1 The Mixed Hybrid Finite Element method formulation

Let T be a triangulation of Ω with m triangles T_j ($j = 1,\ldots, m$), N nodes and L edges e_j, ($j = 1,\ldots, L$). The piezometric head is approximated as:

$$H \simeq \sum_{l=1,m} H_l \psi_l \qquad (4),$$

where ψ_l are $P_0(T_l)$ scalar basis function, taking on the value one on the triangle T_l and zero elsewhere.

The velocity $\mathbf{v}$ in element l is approximated as:

$$\mathbf{v}_l = \sum_{j=1,3} FL_{jl}\mathbf{w}_{jl} \qquad l=1,...,m \qquad (5),$$

where FL_{jl} is the flux leaving element l from side j and $\mathbf{w}_{jl}$ are basis of the lowest-order Raviart-Thomas space $\mathbf{X}_T$, defined as:

$$\mathbf{w}_{jl} = \frac{1}{2|T_l|}\begin{pmatrix} x - x_{jl} \\ y - y_{jl} \end{pmatrix}, \quad j=1,2,3 \qquad (6),$$

with x_{jl} and y_{jl} the coordinate vertices (see figure 1).

Important properties of the $\mathbf{X}_T$ space are [10, 15, 17]:

$$\nabla \cdot \mathbf{v}_l \quad \text{is constant on } T_l \qquad (7,a),$$

$$\mathbf{v}_l \cdot \mathbf{n}_i \quad \text{is constant over each edge } e_i \qquad (7,b).$$

Multiplying both members of Eq. (1,b) for the weight function $\mathbf{w}$, integrating over the element T_l and applying the Green integration rule, you get:

$$\int_{T_l} K_l^{-1}\mathbf{v}_l \cdot \mathbf{w}_{il}d\Omega - \int_{T_l} H\nabla \cdot \mathbf{w}_{il}d\Omega + \\ + \int_{\partial T_l} TP\,\mathbf{w}_{il}\cdot\mathbf{n}_l d\Gamma = 0 \qquad (8).$$

Integrating Eq. (1,a) and merging Eq. (1,b) you get:

$$\frac{S_0^{t_k+1}|T|H^{t_k+1}}{\Delta t} + \int_{T_l}\nabla \cdot \mathbf{v}_l d\Omega = \frac{S_0^{t_k}|T|H^{t_k}}{\Delta t} + \int_{T_l}Q_l d\Omega \quad (9).$$

The equivalence of the element fluxes across the inter-element edges provides:

$$\int_{\partial T_l} \mathbf{v}_l \cdot \mathbf{n}_l d\Gamma + \int_{\partial T_l} \mathbf{v}_r \cdot \mathbf{n}_r d\Gamma = 0 \qquad e_j \in T_l \cap T_r \qquad (10).$$

The Neumann and Dirichlet boundary condition become:

$$\int_{\partial T_l} \mathbf{v}_l \cdot \mathbf{n}_l d\Gamma = b_N \qquad e_j \in \Gamma_N \cap T_l \qquad (11),$$

$$TP_j = H_d \qquad e_j \in \Gamma_D \qquad (12),$$

where $i = 1, 2, 3$, $l = 1,\ldots, m$, $j = 1,\ldots, L$ and TP are the unknown Lagrange multipliers further defined.

The Lagrange multipliers in Eq. (8) are expressed as:

$$TP = \sum_{j=1,L} TP_j \mu_j \qquad (13),$$

where μ_j is a piecewise constant basis function with value equal to one on the edge e_j and zero elsewhere and TP_j represents the average value of the potential head on e_j. According to the properties expressed in Eqs. (7), the flux law (1,b) can be written in variational form as:

$$\int_{T_l} \mathbf{v}_l \cdot \mathbf{w}_{il} = \sum_{j=1,3} FL_{jl}\int_{T_l} \mathbf{w}_{il} \cdot \mathbf{w}_{jl} = \\ = K_l \int_{T_l} \nabla H \cdot \mathbf{w}_{il} = K_l\left(H_{T_l} - TP_{il}\right) \qquad (14).$$

Eq. (14) can be written as:

$$FL_{il} = K_l \sum_{j=1,3} A_{ij}^{-1}\left(H_{T_l} - TP_{jl}\right) \qquad (15),$$

where $\mathbf{A} = \left[A_{ij}\right]$, $A_{ij} = \int_{T_l} \mathbf{w}_{il} \cdot \mathbf{w}_{jl}$. Using Eq. (15) and a fully-implicit time discretization, the mass balance Eq. (9) becomes:

$$H_{T_l}^{t_k+1} = \frac{K_l}{\beta}\sum_{i=1,3}\alpha_i TP_{il}^{t_k+1} + \frac{\delta}{\beta}H_{T_l}^{t_k} + \frac{Q_{sl}}{\beta} \qquad (16),$$

where $\alpha_i = \sum_{j=1,3} A_{ij}^{-1}$, $\delta = S_0|T_l|/\Delta t$, $\beta = \delta + \alpha K$, $\alpha = \sum_{i=1,3}\alpha_i$ and $Q_{sl} = \int_{T_l} Q_l d\Omega$; replacing Eq. (16) in Eq. (15), one gets:

$$FL_{il}^{t_k+1} = K_l \left[\begin{array}{l} \dfrac{K_l \alpha_i}{\beta} \sum_{j=1,3} \alpha_j TP_{jl}^{t_k+1} + \\ -\sum_{j=1,3} A_{ij}^{-1} TP_{jl}^{t_k+1} + \dfrac{\delta\alpha_i}{\beta} H_{T_l}^{t_k} + \dfrac{\alpha_i Q_{sl}}{\beta} \end{array} \right] \quad (17).$$

The final system to solve is obtained by writing the continuity of the fluxes between two adjacent elements A and B:

$$FL_{iA}^{t_k+1} + FL_{iB}^{t_k+1} = 0 \quad (18).$$

For the edge i, the corresponding diagonal term is:

$$m_{ii} = m_{ii}^A + m_{ii}^B, \quad m_{ii}^A = K_A \left(A_{ii}^{-1} - \frac{K_A \alpha_i^2}{\alpha} \right) \quad (19),$$

while the off-diagonal term is:

$$m_{ij}^A = K_A \left(A_{ij}^{-1} - \frac{K_A \alpha_i \alpha_j}{\beta} \right) \quad (20).$$

The matrix of the system is symmetric and positive-definite and for the solution of the system a preconditioned conjugate gradient can be used.
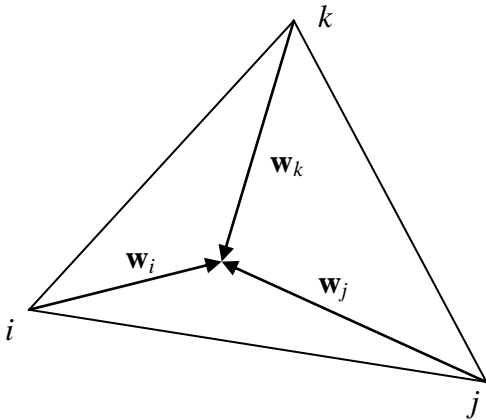
Fig. 1. The Raviart-Thomas basis functions

3.2 A 2nd order Finite Volume formulation (FV2)

It can be proven [17] that the mixed hybrid finite element method is algebraically equivalent to the following second order finite volume approach.

Call H_A the potential in the circumcentrum C^A of the element A and TP_{Aj} the potential head in the centre P_{jm}^A of the jth edge of the same element. Assume a linear variation of the potential inside the element and a potential continuity along the edges; according to these hypotheses the change of the potential between the C^A and the point P_{jm}^A is given by (see figure 2):

$$H_A - TP_{Aj} = FL_{Aj} \frac{h_j^A}{K_A b_j^A} \quad (21),$$

where FL_{Aj} is the edge flux, b_j^A in the length of edge j and h_j^A is the minimum distance of C^A from the edge j. If the distance h_j^A is taken positive or negative according to the position of C^A with respect to edge j, oriented in counterclokwise direction. Relationship expressed by Eq. (21) still holds if the angle in vertex j is greater than 90 degrees and the point C^A falls outside the triangle. Assuming m to be the local index of the same edge in the element B next to A, it is possible to cancel the unknown TP_{Aj} by summing Eq. (21) with the similar equation holding for the edge m in element B, that is:

$$TP_{Bm} - H_B = -FL_{Bm} \frac{h_m^B}{K_B \, b_m^B} \quad (22).$$

Because $TP_{Bm} = TP_{Aj}$ and $FL_{Bm} = -FL_{Aj}$, you get:

$$H_A - H_B = \frac{FL_{Aj}}{b_j^A} \left(\frac{K_B h_j^A + K_A h_m^B}{K_A K_B} \right) \quad (23).$$

Summing the fluxes through the three edges of element A, you get the finite volume equation (assuming zero source term):

$$|T_A| S_{0A} \frac{\partial H_A}{\partial t} = \sum_{j=1,3} \xi_{jm}^A \left(H_E - H_A \right) \quad (24),$$

where S_{0A} and $|T_A|$ are respectively the storage coefficient and the area of element A, E is the index of the element next to the jth edge of element A and:

$$\xi_{mj}^A = \frac{b_j^A K_A K_B}{Kh_{AB}} \quad (25,a),$$

$$Kh_{AB} = K_A h^B + K_B h^A \quad (25,b).$$

After time discretization, Eqs. (24)-(25) form a linear system of order NE, if NE is the element

number. If the quantity Kh_{AB} always satisfies the condition:

$$Kh_{AB} > 0 \qquad (26),$$

the linear system, coupled with the proper boundary conditions, has always the so called M-property (a non-singular matrix where the diagonal terms are always positive and the off-diagonal terms are negative or null); the M-property guarantees that local maxima or minima not appear in the solution in a domain without sinks or sources [17]), as well as the positive definite condition. In this hypothesis, the FV formulation is easier than the MHFE formulation, because it requires the solution of a system smaller than in the MHFE formulation, with a matrix that always satisfy the M-property.

Unfortunately, if the maximum angle is greater than 90 degrees in one or more elements the positive sign of Kh_{AB} cannot be guaranteed. To overcome this difficulty, the following procedure is proposed:

Call A and B two elements sharing the same edge, numbered with index j in element A and index m in element B (see figure 2). If condition expressed in Eq. (26) is originally not satisfied, it can be met by changing the location of points C^A and C^B. Compute first the quantity

$$Kh_{AB}^{max} = K_A h_0^B + K_B h_0^A \qquad (27),$$

that is the value of Kh_{AB} attained when the points C^A and C^B have a distance from the edge j of element A equal to the heights h_0^A and h_0^B of the elements A and B with respect to the common edge. This distance from edge j is the maximum possible for points C^A and C^B to avoid negative distances from the other two sides of each element. In a second step compute the distances of the new points C'^A and C'^B. A possible choice is to find the root β of the following equation:

$$\left(h_0^A \beta + h^A (1-\beta) \right) K_B + \left(h_0^B \beta + h^B (1-\beta) \right) K_A = \\ = \varepsilon Kh_{AB}^{max} \qquad (28),$$

where ε is a small quantity (say 0.001). The new distances are obtained by setting:

$$h'^A = h_0^A \beta + h^A (1-\beta) \qquad (29,a),$$

$$h'^B = h_0^B \beta + h^B (1-\beta) \qquad (29,b).$$

The increment of the distances h^A and h^B from the common edge can be obtained by reducing the distance of C^A and C^B from the nodes P_j and P_m, opposite (respectively in element A and B) to the common edge. To guarantee that distances from the other edges of the same elements A and B remain positive, for any possible angle values, C'^A and C'^B must be kept respectively on the lines $\overline{P_j C^A}$ and $\overline{P_m C^B}$.

The proposed strategy is equivalent to assume the equality $TP_{Bm} = TP_{Aj}$ even if the projections of C'^A and C'^B on the same edge are not the same, as happens for the circumcentres. For this reason, the use of a very small ε in Eq. (8) implies very small extradiagonal coefficient and a slower convergence, but the use of a much larger value could affect the quality of the solution.

The adopted approximation is similar to the one adopted in the first order finite volume formulation to relate the fluxes with the variation of the average potentials from one element to the next. Its effect on the final solution is to shift locally the accuracy of the solution from the second order to the first order size.
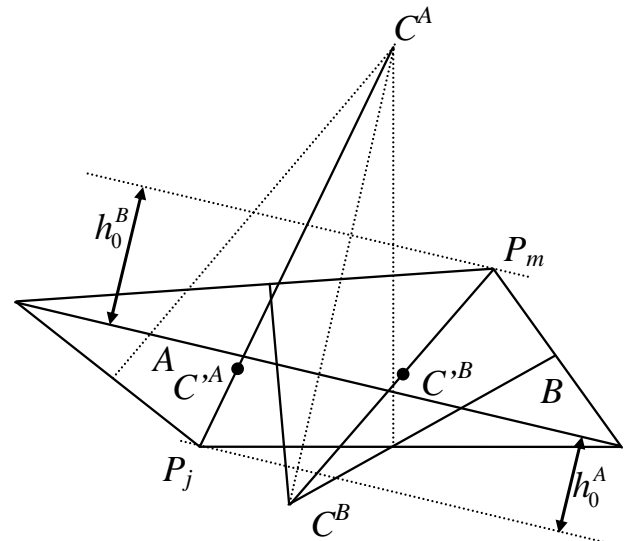

Fig. 2. Modification of the circumcentrum position for the proposed FV procedure

In the non-linear case, with storativity and hydraulic conductivity parameters changing in time as function of the potential heads, the procedure expressed by Eqs. (27)-(29) has to be repeated at each time step.

3.3 A modified Mixed Hybrid Finite Element method formulation (MMHFE)

Define a set of obtuse triangles, that we call "degenerate" (DG), such that:

$$\left(Kh_{DB} < 0 \text{ and } B \notin DG \right) \text{ or}$$
$$\left(h_j^A < 0 \text{ and } B \in DG \right) \tag{30},$$

where B is the index of the jth triangle next to D and condition (30) holds for any value of $j = 1, 2, 3$.

Observe that the partition of the all element set in DG and not DG triangles is not unique. A simple partition is given by putting in DG all the obtuse triangles, but a strong reduction of the size of DG can be obtained with a simple coding.

Observe that Eq. (23) can be written in the form:

$$FL_{Aj} = \frac{K_A K_B}{Kh_{AB}} b_j^A \left(H_A - H_B \right) \tag{31},$$

where A is the index of any element and B is the index of the jth element close to A.

The same left hand side, according to Eq. (21), can be written in the form:

$$FL_{Aj} = \frac{K_A b_j^A}{h_j^A} \left(H_A - TP_{Aj} \right) \tag{32}.$$

This implies that the mass balance equation, in the finite volume formulation, can be written for the not DG elements in the form (assuming zero source term):

$$\sum_{j=1,2,3} FL_{Aj} = S_{0A} \left| T_A \right| \frac{\partial H_A}{\partial t} \tag{33},$$

where FL_{Aj} is computed according to Eq. (31) if element B is not DG, according to Eq. (32) if element B is DG.

The flux continuity equation (Eq. (18)), in each side of the DG elements, can also be written using the same flux formulation (31) or (32).

The final system will be given by a number of continuity equations (33) equal to the size of the not DG set, plus a number of equations equal to the number of sides of the elements in the DG set.

After time discretization, the unknowns of the final system are the average potential in each side of the DG elements, plus the potentials in the circumcentrum of the not DG elements. The results obtained by the solution of Eqs. (33) and (18) are equal to the results obtained solving the all system

of Eqs. (18) in the original MHFE formulation, and the matrix properties are the same, but the number of equations can be much smaller.

In the non linear case the partition of all the elements in DG and not DG elements has to be updated after each time step, because the optimal partition depends on the parameter values.

To reduce the total computational time, it is convenient to renumber the elements, in order to leave all the obtuse ones in the same index range.

Observe for example in figure 3,a an acute triangle with two adjacent obtuse triangles. Triangles initially numbered as 1,3,2 (triangle 1), 3,5,4 (triangle 2) and 1,5,3 (triangle 3) become 1,5,3 (triangle 1), 1,3,2 (triangle 2) and 3,5,4 (triangle 3) (see figure 3,b).
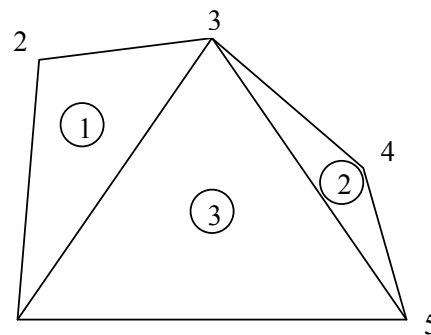

Fig. 3,a. An example of irregular mesh with acute and obtuse triangles
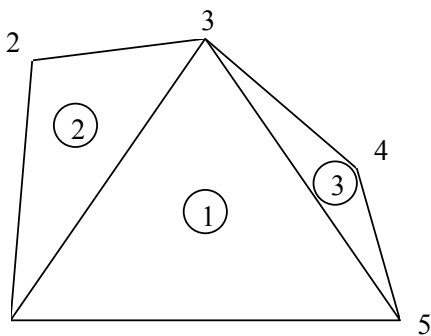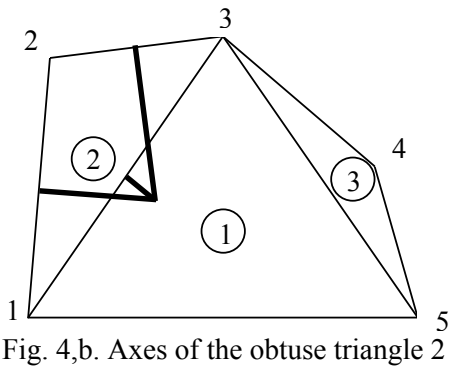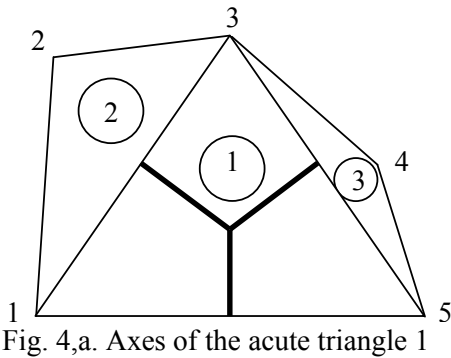

Fig. 3,b. The mesh of figure 3,a after the reordering of triangles

In figures 4,a and 4,b the axes of triangles 1 and 2 with distances h_j^1 and h_j^2 are shown. Since the element sides are counterclockwise oriented, distances are positive if circumcentres are on the left of the side, negative otherwise.

Fig. 4,a. Axes of the acute triangle 1

Fig. 4,b. Axes of the obtuse triangle 2

Assume condition $Kh_{12} = K_1h_2^1 + K_2h_1^2 > 0$ holds. In this case a possible DG set is given by the only element number 3. The unknowns of the problem are the three average potential of the sides of element 3, plus the potentials in the circumcentres of elements 1 and 2 (see figure 5).
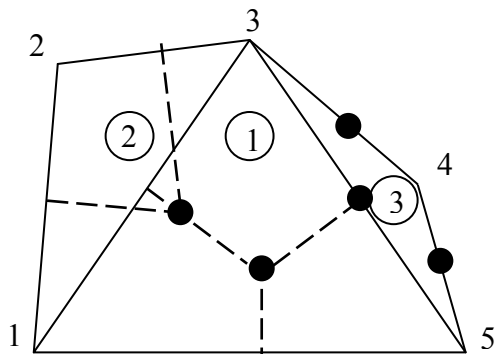

Fig. 5. The unknown locations of the problem

4 Numerical Tests

A computational domain 1 m x 1 m has been assumed for the numerical simulations. Two meshes have been used. The first is structured with 153 equilateral or rectangular triangular elements (see figure 6,a), the other one is unstructured with 128 elements (see figure 6,b). The unstructured mesh has been found in [14]. On the right vertical boundary of the domain the potential head is assumed equal to 1 m (Dirichlet condition); the bottom boundary is assumed impervious to fluxes,

while on the other two boundaries an incoming flux is assumed (Neumann condition), equal respectively to 0.12 m^2/s and 0.125 m^2/s for the two meshes.

Two series of simulations have been carried out, assuming homogeneous and heterogeneous hydraulic conductivity. The storage coefficient S_0 is assumed equal to 0.1 s^{-1}; in the homogeneous case the conductivity K is equal to 0.1 m/s, while in the second heterogeneous case the value of K inside element i is given by:

$$K_i = e^{\left(sin\left(\frac{\pi x_{gi}}{2\Delta x}\right) + sin\left(\frac{\pi y_{gi}}{2\Delta y}\right)\right)/10} \quad (34),$$

where x_{gi} and y_{gi} are the coordinates of the centers of mass of the element and parameters $\Delta x = \Delta y$ are assumed equal to 1/8; in figure 7 the contour lines of the points with the same value of K are shown for the unstructured mesh.

Numerical results of the standard and modified MHFE schemes, as well as of the previously proposed FV2, for the two meshes and both homogeneous and heterogeneous conductivity cases have been compared. The time step Δt used for the simulations is 1.25 s and 2.5 s, respectively for the structured and unstructured mesh. See also [5].

In figures 8-9 the equipotential contours are shown for the cases of structured and unstructured meshes and heterogeneous K at the simulation time $T = 500$ s: observe that for both meshes, the contour lines obtained with all the three methods are undistinguishable. A similar behavior occurs in the test-cases for homogeneous medium, but for brevity results are not reported here.

The effect of the mesh size has been also investigated. Starting from the meshes used before, at each refinement level $i + 1$, each triangle of the previous i[th] mesh is subdivided in 4 equal triangles connecting the midpoints of the three sides (see figure 10) and Δt is halved.

In figures 11-14 results of the proposed numerical procedures are shown for the case of homogeneous and heterogeneous conductivity for the 2nd mesh refinement level (2448 elements for the structured mesh and 2048 elements for the unstructured one). Numerical results of both MHFE formulations and those ones of the FV2 formulation are the same, in the case of structured meshes; for the unstructured meshes results of the FV2 formulation differ from the MHFE and the MMHFE formulation near the boundaries and in the zones with higher mesh irregularity (see for example the contour lines values 2.6 or 2.4).

In tables 1,a and 1,b the RMS errors with respect to the solutions of the MHFE method are shown for the proposed FV2 procedure and the standard Galerkin FE formulation. Of course, RMS errors of the proposed MMHFE formulation are zero. In the Galerkin approach, the potential head in the element has been computed as the mean value of the potentials at the three nodes..

The RMS errors are computed as:

$$ \text{RMS} = \frac{1}{NE} \sum_{i=1,NE} \sqrt{\left(H^i_{FV(GA)} - H^i_{MHFE} \right)^2} \qquad (35). $$

Generally, the error of the Galerkin approach is greater than the one of the FV2 scheme. This is due a) to some difference in the discretization of the boundary conditions, that are assigned to the edges in both the FV2 and MHFE schemes and to the nodes in the Galerkin scheme, and b) to the smaller degree of freedom (the number of nodes) of this last one with respect to the others (respectively the number of elements for FV2, the number of edges for MHFE and an intermediate value for MMHFE). FV2 scheme provides a much smaller error, with respect to the Galerkin scheme, in the case of heterogeneous domain, due to the strong effect of the flux continuity at the line where the parameter change occurs. We can also observe that the error increases in the FV2 method for the more refined unstructured meshes. This is because the adopted refinement provides a proportional increment of the irregular elements, that could be avoided using more sophisticated mesh generation techniques.

The mean values per element of the CPU times are reported in tables 2,a-2,d for the four numerical procedures. An Intel Core 2 E 6850 3.00 GHz processor has been used for the numerical simulations. In 15,a-15,d the mean CPU times are reported for the four procedures, as well as for a standard 1st spatial approximation order FV formulation.

The mean required CPU time is mainly related to the number of unknowns. The CPU time of the MMHFE scheme is lower than the one of the standard MHFE for all the investigated tests, and very close to the FV2 CPU time for the structured meshes. In this last case, because the number of unknowns is the same for the two methods, the CPU difference is due to the time required in MMHFE for the partition of the elements in DG and not DG elements (even if the DG set comes out to be empty). The computational cost required by the FV2 method is approximately twice the one required by the FE Galerkin scheme, but it is always smaller

than the ones of both the MHFE methods for the case of unstructured mesh.

The partition cost depends on the complexity of the adopted procedure. If the chosen DG set is given by all the obtuse triangles, the cost is basically zero; in our case, a simple partitioning procedure with a cost almost proportional to the number of elements has been adopted. Finally observe that, increasing the order of the refined mesh, the growth of the computational costs is less than linear for all the numerical procedures.
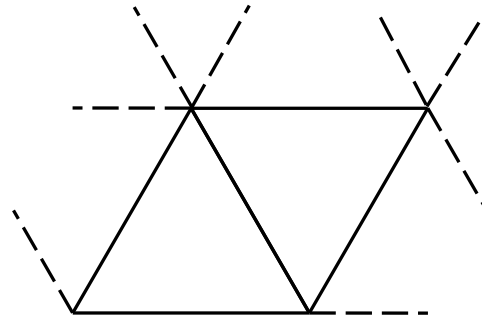

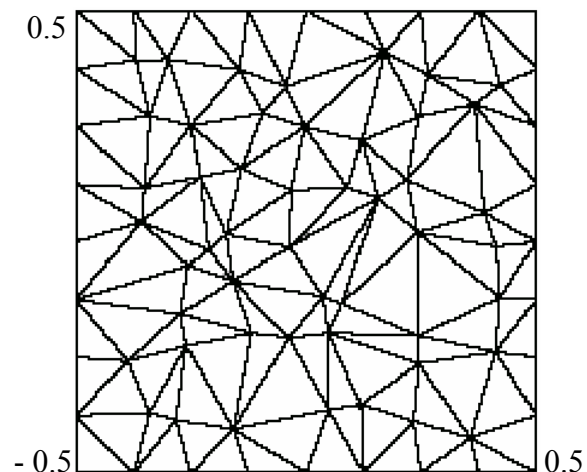
Fig. 6,a. The equilateral elements of the structured mesh

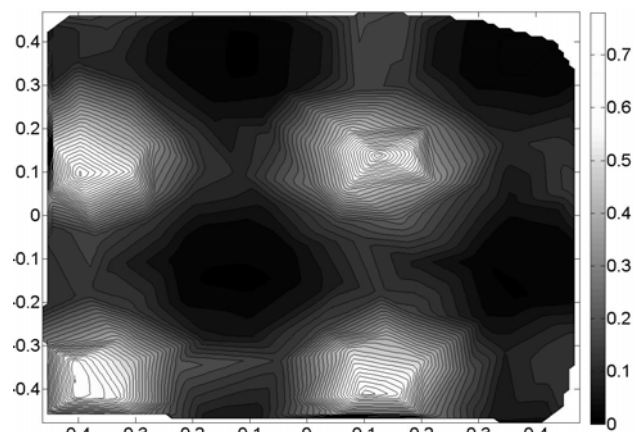

Fig. 6,b. The unstructured mesh

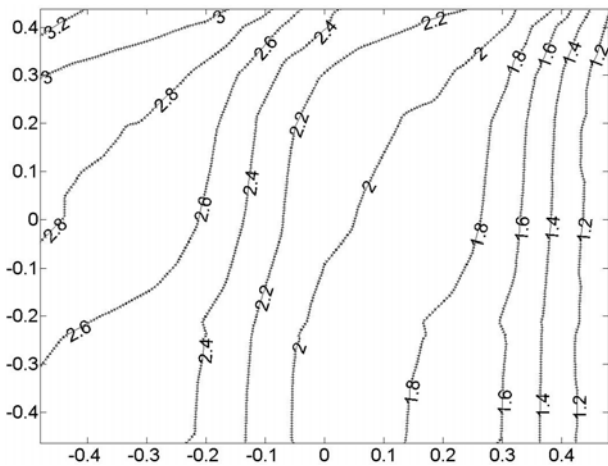Fig. 7. Spatial distribution of K in the heterogeneous case

Fig. 8. Equipotential contours − heterogeneous medium, structured mesh (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)
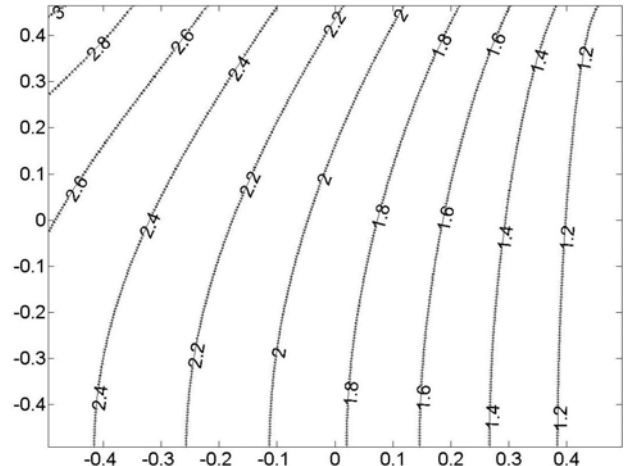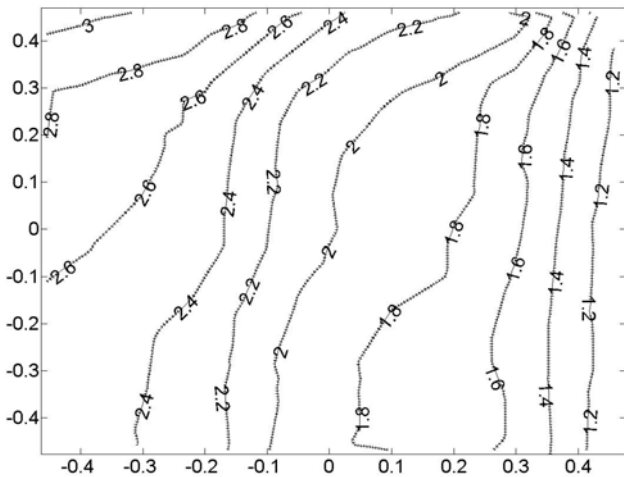

Fig. 11. Equipotential contours − homogeneous medium, structured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)

Fig. 9. Equipotential contours − heterogeneous medium, unstructured mesh (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)
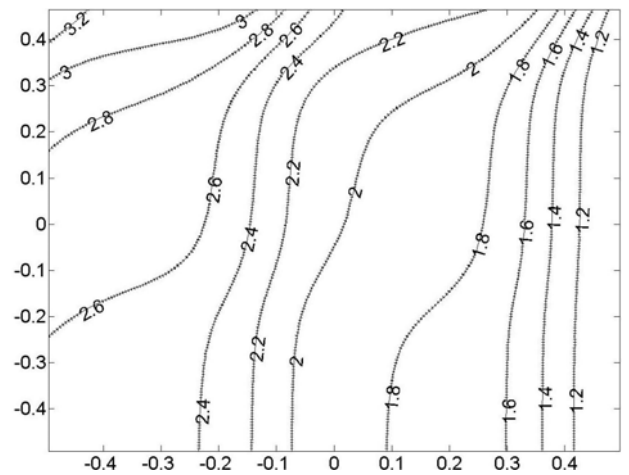

Fig. 12. Equipotential contours − heterogeneous medium, structured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)
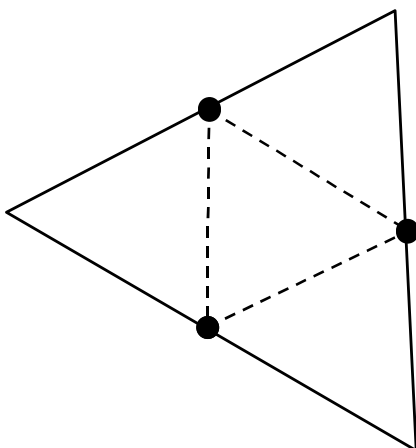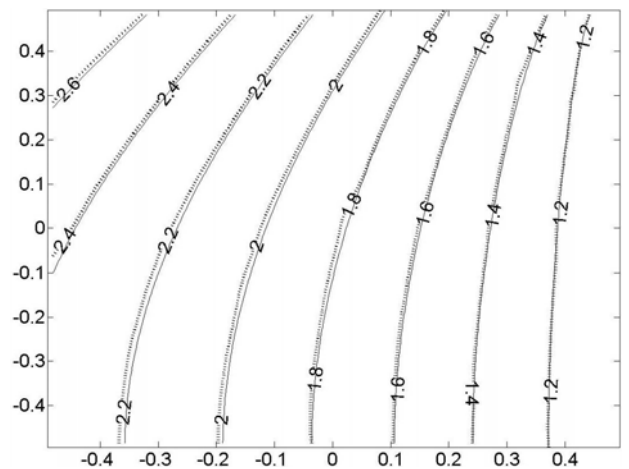

Fig. 10. Mesh refinement

Fig. 13. Equipotential contours − homogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)
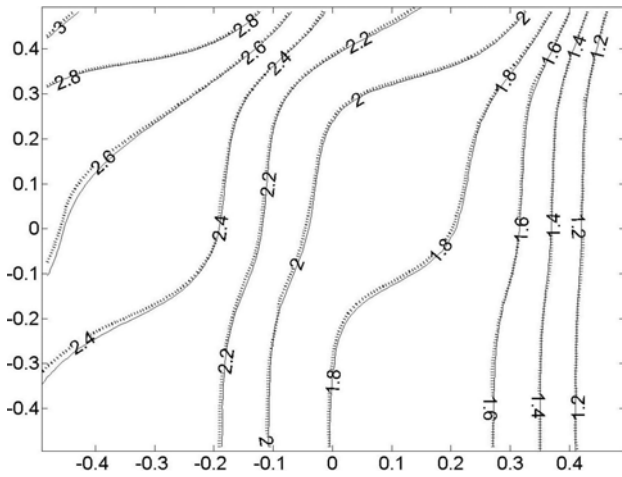
Fig. 14. Equipotential contours – heterogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, dotted lines FV2 method)

Table 1,a. RMS errors; homogeneous medium

NE	FV2	FE Galerkin
Unstructured mesh		
128	4.71E-04	2.02E-02
2048	7.15E-03	6.63E-03
Structured mesh		
153	0.00E+00	2.01E-02
2448	2.41E-05	6.37E-03

Table 1,b. RMS errors; heterogeneous medium

NE	FV2	FE Galerkin
Unstructured mesh		
128	4.97E-04	8.14E-02
2048	5.18E-03	1.28E-02
Structured mesh		
153	1-06E-12	6.62E-02
2448	2.39E-05	1.01E-02

Table 2,a. CPU mean values, homogeneous medium

NE	FV2	mod. MHFE	FE Ga	MHFE
Untructured mesh				
128	6.10E-06	7.32E-06	2.44E-06	1.34E-05
2048	3.01E-05	4.04E-05	9.06E-06	4.75E-05
Structured mesh				
153	5.11E-06	6.12E-06	2.04E-06	1.43E-05
2448	2.97E-05	3.22E-05	9.06E-06	4.78E-05

Table 2,b. CPU mean values, heterogeneous medium

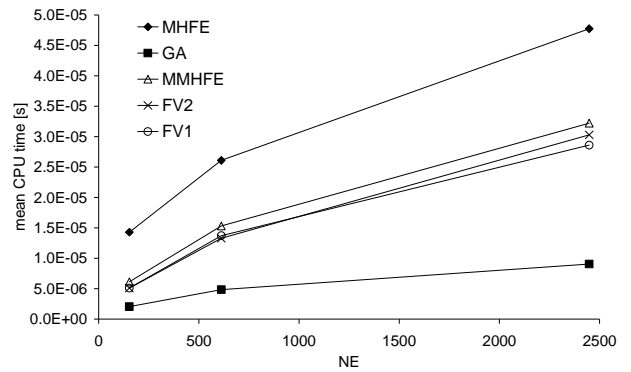| NE	FV2	mod. MHFE	FE Ga	MHFE
Unstructured mesh				
128	5.49E-06	8.54E-06	3.05E-06	1.40E-05
2048	3.14E-05	3.92E-05	8.74E-06	4.63E-05
Structured mesh				
153	5.11E-06	6.12E-06	2.04E-06	1.33E-05
2448	3.03E-05	3.05E-05	8.52E-06	4.67E-05

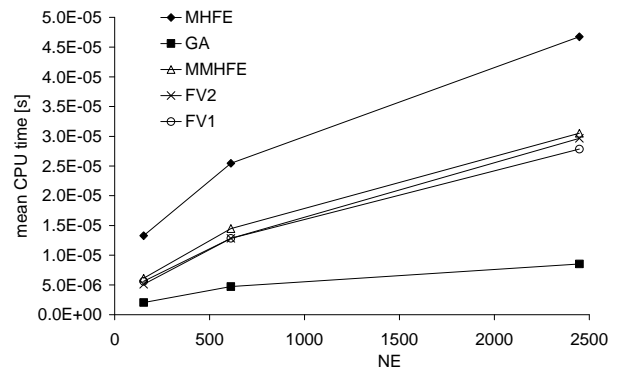Fig. 15,a. Mean CPU times – structured mesh, homogeneous medium

Fig. 15,b. Mean CPU times – structured mesh, heterogeneous medium
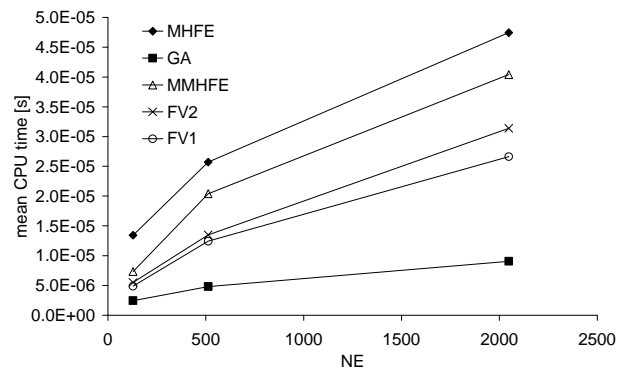

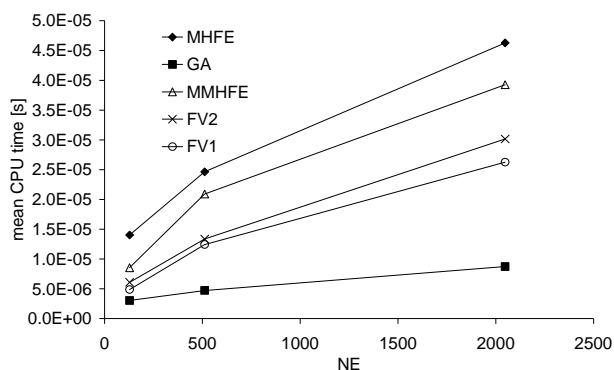Fig. 15,c. Mean CPU times – unstructured mesh, homogeneous medium

Fig. 15,d. Mean CPU times – unstructured mesh, heterogeneous medium

5 Conclusion

A modified formulation of the standard MHFE scheme (MMHFE) has been proposed in this paper for the solution of the linear groundwater flow problem over irregular triangular meshes, along with a modified second order finite volume approach. The formulation is based on the partition of all the elements in two sets, called "degenerate" (DG) and non DG.

The governing equations are: 1) the mass conservation equation for each non-degenerate obtuse triangle and 2) the continuity of the edge-fluxes through each side of the degenerate triangles.

A linear system, with a number of unknowns equal to the number of the non DG triangular elements plus a limited number of unknowns, one for each edge of the DG triangles, is obtained. The matrix system is symmetric and positive-definite; the "M-property" could not be satisfied for any of the DG triangles.

Numerical results of MMHFE are equivalent to the ones computed by the standard formulation, for all types of meshes and parameters distribution.

Results have also been compared with the results of a 2nd spatial approximation order Finite Volume formulation (FV2), recently proposed [5]. The classical FV2 formulation is changed by modifying the position of the circumcentrum for the obtuse triangles; this allows to maintain the "M-property" and the positive definite condition for all the obtuse triangles, but also provides an error with respect the MHFE and the MMHFE solutions.

The CPU time of the numerical methods have been investigated. Due to the lower number of unknowns, the mean CPU time required by the MMHFE scheme is lower than the one of the standard MHFE for all the investigated tests, specially for the structured meshes. The computational cost required by the FV2 method is

always smaller than the ones of both the MHFE methods for the unstructured mesh. For the structured mesh, the CPU mean times of FV2 and MMHFE schemes are very similar, and the difference is due mainly to the CPU time required in MMHFE for the partition of the elements in the DG and not DG set.

For structured meshes, results of the new FV2 procedure are very close to the ones provided by the MHFE scheme in both homogeneous and heterogeneous medium. For unstructured meshes, results of the new procedure differ from the ones of the MHFE methods, specially for very refined meshes, in the zone characterized by higher mesh irregularity.

It is important to underline that results provided by FV2 in cases of unstructured mesh, are much closer to the results of the MHFE methods than the ones computed by a standard Finite Volume 1st spatial approximation order, as shown in figures 16 and 17 (respectively for homogeneous and heterogeneous medium), in the case of the 2nd mesh refinement level.
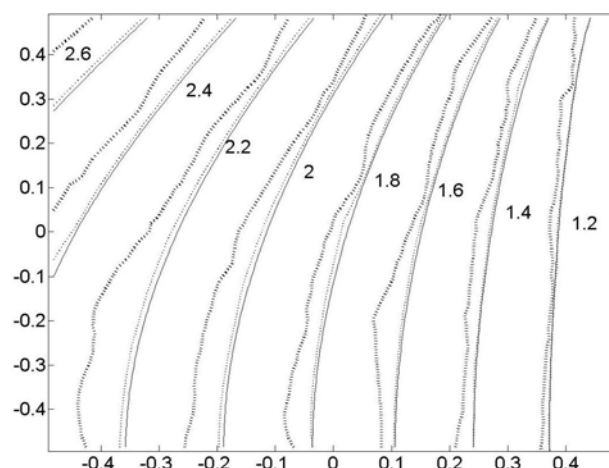

Fig. 16. Equipotential contours – homogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, small dotted lines FV2 method, large dotted lines 1st order FV method)
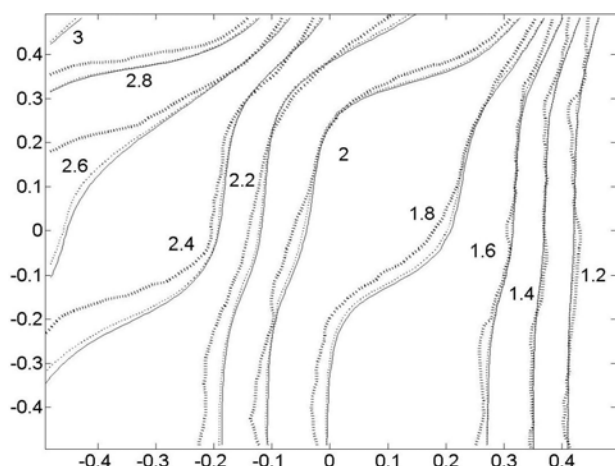
Fig. 17. Equipotential contours – heterogeneous medium, unstructured mesh 2nd refinement level (dashed lines: MHFE and MMHFE methods, small dotted lines FV2 method, large dotted lines 1st order FV method)

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