Vortex layers of small thickness

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Abstract

We consider a 2D vorticity configuration where vorticity is highly concentrated around a curve and exponentially decaying away from it: the intensity of the vorticity is $O(1/\varepsilon)$ on the curve while it decays on an $O(\varepsilon)$ distance from the curve itself. We prove that, if the initial datum is of vortex-layer type, Euler solutions preserve this structure for a time which does not depend on ε. Moreover the motion of the center of the layer is well approximated by the Birkhoff-Rott equation.

Contents

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1 Introduction and main ideas

The main feature of an incompressible flow is vorticity, i.e. the possibility, for particles flow, to rotate. In fact (and this is the main difference between a fluid and a solid) a fluid, when subject to a tangential stress, for example due to interaction with a physical boundary, it displays a sliding motion that ultimately leads to rotational flow and to the appearance of vortices. Vorticity is therefore a powerful tool to understand and characterize fluid motion, both mathematically as well as from the physical point of view.

In many instances, particularly for high Reynolds number flows, vorticity tends to occupy small portions of the space often assuming very high values. Point vortices, vortex filaments and vortex tubes, vortex sheets and shear layers, boundary layers are examples where vorticity is able to strongly influence and organize the flow in large portions of the space, even if being essentially supported on sets of small, or even zero, measure. From the mathematical point of view, these are challenging problems, for which one cannot usually invoke the classical Yudovich theory [52]. The significant progresses that have been accomplished in the late eighties and early nineties [18, 19, 16, 37, 22, 33, 48, 43, 38] have left unanswered fundamental questions, like uniqueness of the solution, even in the 2D case where the absence of vortex stretching makes the dynamics of the vorticity as simple as possible; for vorticity distributions with non distinguished sign, even existence of the solution is currently unknown. Another important question, which is still open, is whether, imposing that the initial vorticity is a δ-function supported on a curve, the solution keeps this structure.

In this paper we shall consider, for the 2D Euler equations, an initial datum with a vortex layer structure: this means that, initially, a high intensity $O(\varepsilon^{-1})$ vorticity ω^{in} is distributed around a planar curve φ^{in}; and that the vorticity decays to zero away from the curve on an $O(\varepsilon)$ scale. We shall assume that the decay is exponential and that, initially, both the vorticity distribution and the curve have analytic regularity. Our hypotheses imply no restriction on the sign of the vorticity.

The data we shall consider can be handled using the general theory of Yudovich, so that existence of the solution is not an issue. However, Yudovich theory does not give any information on the structure of the solution. What we shall prove is that the solution mantains the vortex layer structure for a time which is small, but independent from the thickness of the layer. Moreover we shall prove that the dynamics of the vortex layer is ruled, to the leading order in ε, by the Birkhoff-Rott (BR) equation; our results can therefore be considered as a rigorous justification of the BR equation as a valid model for the description of the Euler dynamics of vortex layers.

The BR equation [44, 7] is an integrodifferential equation that governs the motion of a curve $y=\varphi(x,t)$ on which the vorticity is concentrated as a Dirac δ-function. Assuming the regularity of the curve, as well as of the vorticity intensity, one can prove the equivalence of the solution of the BR equation with the solution of the Euler equations in their weak formulation, see [39]; in [36], after introducing the notion of weak solution of the BR equation, the same consistency result was proved under weaker, likely optimal [35], regularity hypotheses.

Proving regularity results for the BR equation is not a simple matter. The problem was known to be subject to the Kelvin-Helmholtz instability [21], being higher modes of a small perturbation of the flat profile exponentially amplified. The asymptotic analysis of Moore [41] suggested that the problem was indeed ill-posed, and that the ill-posedness mechanism revealed itself in the appearance of a curvature singularity in the shape of the curve; a phenomenon leading, at later times, to an infinite roll-up of the curve [8]. Subsequent careful computations [28] confirmed these results. The nonlinear ill-posedness of the BR equation in H^m with $m>3/2$ was finally proved in [13], see also [20]. The analytic setting seems therefore to be a natural framework where to look for solutions of the BR equation. The exponential decay of the spectrum is in fact able to tame, at least for a short time, the exponential amplification of the modes, a situation that is typical for many flows with highly concentrated vorticity, see e.g. [46, 47, 14, 30, 29]. In fact, in [49], the well-posedness of the BR equation for analytic data was achieved. Later, in [12], it was proved that if one has a small analytic perturbation of a flat profile, then long time well-posedness of the BR equation follows, with an existence time that goes like $|\log\varepsilon|$, being ε the size of the perturbation.

Many attempts have been taken to regularize the BR equation to go beyond the singularity time: see [5] where the dynamics of a vortex sheet as ruled by the regularized Euler-α equations was proved to be well posed globally in time; [27, 1, 2] where the effects of surface tension were taken into account; or [51], where the notion of chord-arc curves was introduced to follow the solution of the BR equation also beyond the singularity time. In [51], among other things, the author proved that if the chord-arc curve "does not roll up too fast" then the BR solution is indeed analytic, significantly improving the result of Lebeau [32].

Among the attempts of regularization of the vortex-sheets solutions, those concerning the study of small thickness layers, are the ones more closely related to the present paper. In [40, 17], through a formal asymptotic analysis, corrections to the BR equations that take into account the thickness of the layer were derived. However these corrections prevent neither singularity formation, nor ill-posedness for non analytic data. Numerical computations performed considering small thickness layers [3, 15, 11, 10] show that the BR singularity, and the subsequent infinite roll-up of the sheet, are not just a mathematical curiosity, but are the signature of the formation of vortex cores. This means that the vorticity is not any more confined close to the center of the layer and has the tendency to concentrate forming vortical structures, a phenomenon that, besides its intrinsic importance because it signals a new stage in the layer dynamics where a strong interaction between the outer and the inner flow sets in, is also reminiscent of separation of the boundary layer and transition to turbulence for wall bounded flows [23]. The only rigorous analysis concerning vortex layers is in [6] where, still in the context of interface dynamics, and for vortex layers of uniform vorticity, it was established the convergence to the BR dynamics in the small thickness limit. In fact uniform vorticity across the layer allowed the authors to define the two interfaces bounding the layer and to study their motion through a system of two coupled BR-like equations. The analytic norm used in [6] allowed to prevent zero-time loss of regularity of the two interfaces and to show that the zero-thickness limit is governed by the BR equation. In the present paper, on the other hand, we make no assumption on the vorticity distribution (except the exponential decay away from a curve); this means that the present setting could be appropriate to tackle the important problem of the justification of the BR equation as zero viscosity limit of the Navier-Stokes solution, where the presence of diffusion does not allow uniform vorticity distribution.

1.1 Main ideas and plan of the paper

The main ideas which our results are based on are the following.

1.1.1 The use of a comoving reference frame

The physical systems has a fast $O(\varepsilon^{-1})$ scale, across the layer, and a slow $O(1)$ scale, along the layer. The cartesian coordinates (x,y) mix these scales; it is therefore natural to analyze the dynamics in a frame adapted to the curve. In fact we write Euler equations, in the vorticity formulation, using a frame that moves with the layer and that, as one of the coordinate lines, has a curve centered in the layer. The other coordinate line is parallel to the cartesian y-line, and the coordinate is rescaled with ε to blow-up the layer. The resulting comoving curvilinear coordinates (ξ,Y) are not orthogonal, and therefore it follows a more complicated form of the Euler equations; however, in the new coordinates, fast $O(\varepsilon^{-1})$ variations, across the layer, and $O(1)$ variations, along the direction tangent to the layer, are separated. We also point out that this reference frame, that can be used in the whole plane as long as the curve is a graph, has the additional advantage of avoiding the singularity one encounters in using the tangent-normal reference frame, as in [40, 17].

1.1.2 Analyticity, Hölder norms and potential estimates

We use Hölder norms in an analytic setting. The analytic setting seems to us necessary to avoid instabilities and ill-posedness, that are present in the BR equation. Hölder function spaces are a natural environment where to establish potential estimates that, in our case, are necessary to give bounds of the velocity in terms of the vorticity. In fact we shall prove potential estimates that bound Hölder norms of the velocity in terms of the Hölder norms of the rescaled vorticity, see Proposition 8.1. This means that we do not have the gain of one order of regularity (typical of the classical potential theory) but we get that our bounds are $O(1)$, being expressed in terms of the rescaled vorticity $\tilde{\omega} \equiv \varepsilon\omega$. Our potential estimates are therefore uniform in ε. We also stress that these potential estimates do not depend on the analytic setting (as it is clear from the proof where it is evident that analyticity is only a complication, see Appendix E and Appendix E.3) and we believe that are of independent classical interest.

1.1.3 Decomposition of the velocity generated by a vortex layer

Another key ingredient in our procedure is a general decomposition formula for the velocity generated by a vortex layer through the Biot-Savart law. We shall see that the velocity can be decomposed as the velocity predicted by the BR kernel (which, assuming that all the vorticity in concentrated on the curve, gives the overall motion of the curve) plus a term that depends, in a very simple way, from the local distribution of the vorticity, plus a remainder term that, inside the layer, is $O(\varepsilon)$. Although this remainder, at the formal level, is easily seen to be $O(\varepsilon)$, to give a rigorous proof of this fact, requires higher order regularity of the vorticity. Having decomposed the rescaled vorticity as $\tilde{\omega} = \omega_0 + \varepsilon\omega_1$, and applying the decomposition formula to the velocity generated by ω_0, we push the remainder to the equation ruling ω_1.

1.1.4 The *a-priori* analyticity of the skeleton of the layer

The resulting equation for ω_0, which is now convected by the BR term plus the local term, is particularly simple: integration across the layer gives a set of four equations that do not involve the finer structure of the layer (i.e. the ω_0 in its dependence on the *normal* coordinate), but only the *skeleton* of the layer, i.e. the curve $\varphi_0(x,t)$, the vorticity intensity $\gamma_0(x,t)$, the quantity

$$\int_0^\infty \omega_0(x,Y,t)dY' - \int_{-\infty}^0 \omega_0(x,Y,t)dY'$$

which is the only information needed on how the vorticity is distributed inside the layer, and the Lagrangian factor X_0 that keeps track of the tangential motion along the curve, see (2.9) and (3.17). One can therefore solve for this system and establish the *a-priori* analyticity of the skeleton of the layer.

With this *a-priori* analyticity, one can then solve the equation for the leading order vorticity ω_0, and for the corrections ω_1 and φ_1. The fact that both ω_0 and ω_1

remain exponentially decaying away from φ gives the persistence of the vortex-layer structure, for a time that does not depend on the size of the layer.

The convergence of the motion of the layer to the dynamics predicted by the BR equation (the *justification* of the Birkhoff-Rott equation) is our final result, and it is a rather easy consequence of our analysis.

We shall also establish a *far-field* estimate proving, rigorously, that the velocity field generated by a vorticity layer distributed around a curve φ, away from the layer and to the leading order, can be computed assuming the vorticity to be concentrated on the curve φ; the rate of decay of the velocity field to the far-field approximation, in terms of the distance from the layer, is also given.

The plan of the paper is as follows. In the next Section we shall write the Euler equations adapted to the curve. In Section 3 we shall derive the decomposition formula for the velocity generated by a vortex layer, and use this formula to write the equations ruling the leading order vorticity and the correction. In Section 4 we define the function spaces, while in Section 5 we state formally our main results. In Section 6 we give some preliminary mathematical results, like the Cauchy estimate for an analytic function, the Abstract Cauchy-Kowaleski Theorem, and the estimates for the BR operators. In Section 7 we construct, through a short time existence Theorem, the *skeleton* of the layer. In Section 8 we give the potential estimates that will be a crucial ingredient in the analysis of the equations governing the correction ω_1. In Section 9 we construct the vorticities ω_0, ω_1 and the correction to the curve φ_1. In Section 10 we prove that, to the leading order, the motion of the layer, is governed by the BR equation. In Section 11 we draw some Conclusions. The paper has several Appendices where we have postponed most of the technical details.

2 Euler equations adapted to the layer

Euler equations, for an incompressible 2D flow, are written as:

$$\partial_t \omega + u \cdot \nabla \omega = 0\,,$$

where the velocity u can be recovered from the vorticity as $u = -\nabla^{\perp}\Delta^{-1}\omega$. We shall consider these equation in the strip $(x,y) \in [-\pi/2, \pi/2] \times \mathbb{R}$, with data periodic in x. One can see that the velocity field has, in terms of vorticity, the following expression:

$$(2.1)\quad u(x,y,t) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{x-\pi/2}^{x+\pi/2} \frac{\sinh\left[2\left(y-y'\right)\right]}{\sin^2\left(x-x'\right)+\sinh^2\left(y-y'\right)}\,\omega\left(x',y',t\right)dx'dy'\,,$$

$$(2.2)\quad v(x,y,t) = -\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{x-\pi/2}^{x+\pi/2} \frac{\sin\left[2\left(x-x'\right)\right]}{\sin^2\left(x-x'\right)+\sinh^2\left(y-y'\right)}\,\omega\left(x',y',t\right)dx'dy'\,.$$

A precise statement on the class of initial data where we shall solve Euler equations will be given later. Roughly speaking we shall consider that there exist a curve

$y=\varphi^{in}(x)$ such that the initial vorticity decays exponentially fast away from the curve; i.e.

$$\omega^{in}(x,y)\sim\frac{1}{\varepsilon}e^{-\mu|y-\varphi^{in}(x)|/\varepsilon}\,, \tag{2.3}$$

where $\varepsilon>0$ gives the thickness of the layer and $\mu>0$ is the rate of exponential decay. The position of the curve $y=\varphi(x,t)$ will be determined imposing that it is transported by the flow generated by the vorticity. One has therefore to solve:

$$\partial_t\omega+u\partial_x\omega+v\partial_y\omega\;=\;0\,, \tag{2.4}$$

$$\partial_t\varphi+u^{\varphi}\partial_x\varphi\;=\;v^{\varphi}\,, \tag{2.5}$$

$$\omega(x,y,t=0)\;=\;\omega^{in}(x,y)\,, \tag{2.6}$$

$$\varphi(x,t=0)\;=\;\varphi^{in}(x)\,, \tag{2.7}$$

where the u, v, are calculated using (2.1)-(2.2), while u^{φ}, v^{φ} are the fluid velocity components on the curve φ.

To prove that the vorticity $\omega(x,y,t)$ will satisfy, for $t>0$, a behavior like (2.3), is one of the main goals of this paper.

2.1 The comoving frame

To separate fast and slow variations, it is natural to introduce a coordinate system adapted to the curve. In the plane (x,y) let $y=\psi(x,t)$ be a generic curve, periodic in x of period π. Moreover let $(u(x,y,t),v(x,y,t))$ a velocity field π-periodic in x. We make the change of coordinates $(x,y,t)\to(\xi,Y,\tau)$ defined as:

$$x=\xi+X(\xi,\tau)\,,\qquad y=\varepsilon Y+\psi(\xi,\tau)\,,\qquad \tau=t\,, \tag{2.8}$$

where the lagrangian factor X is given by

$$X(\xi,\tau)\equiv\int_0^{\tau}u^{\psi}(\xi,\tau')d\tau'\,. \tag{2.9}$$

In equation (2.8) and in the rest of the paper we have used the same symbol to denote the curve as a function of the eulerian variable x or the Lagrangian variable ξ: when we write $\psi(\xi,\tau)$ we shall in fact mean $\psi(x(\xi,\tau),\tau)$. Moreover notice that the functions $\psi(\xi,\tau)$ and $X(\xi,\tau)$ are π-periodic in ξ.

One can see that the Euler equations and the equation ruling the motion of a curve φ, written in the above defined coordinate system, are the following:

$$\partial_\tau\tilde\omega+\frac{(u-u^{\psi})}{1+X_\xi}\partial_\xi\tilde\omega+\frac{1}{\varepsilon}\left[-\partial_\xi\psi\frac{(u-u^{\psi})}{1+X_\xi}+(v-v^{\psi})\right]\partial_Y\tilde\omega\;=\;0\,, \tag{2.10}$$

$$\partial_\tau\varphi+\frac{u^{\varphi}-u^{\psi}}{1+X_\xi}\partial_\xi\varphi\;=\;v^{\varphi}\,, \tag{2.11}$$

$$\tilde\omega(\xi,Y,t=0)\;=\;\tilde\omega^{in}(\xi,Y)\,, \tag{2.12}$$

$$\varphi(\xi,t=0)\;=\;\varphi^{in}(\xi)\,, \tag{2.13}$$

where we have introduced the rescaled vorticity $\tilde{\omega} \equiv \varepsilon\omega$. The incompressibility condition in the new variable reads as

$$\frac{\partial_\xi u}{1+X_\xi} + \frac{1}{\varepsilon}\left[-\frac{\partial_\xi \psi}{1+X_\xi}\partial_Y u + \partial_Y v\right] = 0 \,. \tag{2.14}$$

The details of the derivation of the above equations are given in Appendix A. Here we anticipate that in our study of the Euler equations for vortex layer data, we shall set $\psi = \varphi$; i.e. we shall use, as base curve of the reference frame, the curve $\varphi = \varphi_0 + \varepsilon\varphi_1$.

We make the following formal Remarks concerning the above Eq.(2.10). First, given that the time derivative is comoving, then $\partial_\tau\tilde{\omega}$ is $O(1)$. The operator ∂_ξ is the derivative along the curve (the ξ line-coordinates are $Y = const$, therefore they are parallel to the curve). It follows that $\partial_\xi\tilde{\omega} = O(1)$. In (2.10) the $O(\varepsilon^{-1})$ terms are therefore only in the last term, involving the derivative across the curve. On the other hand, using the incompressibility condition, one can see that:

$$\left[-\frac{\partial_\xi \psi}{1+X_\xi}(u-u^\Psi) + (v-v^\Psi)\right] = -\varepsilon\int_0^Y \frac{\partial_\xi u}{1+X_\xi}dY' \,, \tag{2.15}$$

so that (2.10) can be written as

$$\partial_\tau\tilde{\omega} + \frac{(u-u^\Psi)}{1+X_\xi}\partial_\xi\tilde{\omega} - \int_0^Y \frac{\partial_\xi u}{1+X_\xi}dY'\partial_Y\tilde{\omega} = 0 \,, \tag{2.16}$$

and therefore also the last term appearing in (2.10) is again, formally, $O(1)$. However the use of the incompressibility condition leads to a not less severe difficulty, i.e. the appearance of a linearly growing in Y term (given by the integration in Y of $\partial_\xi u$, a quantity that has no decay properties in Y), combined with a loss of a derivative given by the appearance of $\partial_\xi u$; notice that the potential estimate given in Proposition 8.1, allowing to bound the velocity in terms of the vorticity, does not give the usual gain of one derivative; this is the price one has to pay to make the potential estimate uniform in ε. The simultaneous presence of loss of exponential decay and of loss of regularity does not allow to get enough contractiveness.

Therefore, to make rigorous the above formal analysis, a deeper understanding of the structure of the velocity generated by the vortex layer is required.

3 The asymptotic expansion

We solve the above problem (2.10)-(2.13) in two steps: the first at $O(1)$ and the second at $O(\varepsilon)$, writing:

$$\tilde{\omega} = \omega_0 + \varepsilon\omega_1 \,, \qquad \varphi = \varphi_0 + \varepsilon\varphi_1 \,. \tag{3.1}$$

To understand what are the equations at different order we have to inquire on the structure of the velocity.

3.1 The structure of the velocity

Here we find it convenient to use the variables (x,Y) and use the following expression for the velocity generated by a rescaled vorticity Ω (the real vorticity being $\varepsilon^{-1}\Omega$) concentrated around a curve ψ:

$$u - iv = \sum_{n=\infty}^{\infty} \int_{-\infty}^{\infty} \int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} \frac{\Omega(x',Y')}{K_{\psi}^{\varepsilon}(x-x',Y-Y')} dx'dY' , \tag{3.2}$$

where (with an abuse of notation) we have defined:

$$K_{\psi}^{\varepsilon}(x-x',Y-Y') = x - x' + i\left[\psi(x) - \psi(x') + \varepsilon(Y-Y')\right] ,$$
$$K_{\psi}^{0}(x-x') = x - x' + i\left[\psi(x) - \psi(x')\right] ,$$

The definition of K_{ψ}^{0}, that will be present in the definition of the Birkhoff-Rott operator, will be useful later.

Using (3.2) one can derive, see Appendix B, the following expression for the velocity written in complex variable notation:

$$u - iv = M(\Omega,\psi) + R(\Omega,\psi) \tag{3.3}$$

where the operators M and R are defined as follows:

$$\begin{aligned} M(\Omega,\psi) &= \\ BR[\gamma,\psi] + \frac{1}{2}\left[\int_{-\infty}^{Y} \Omega(x,Y')dY' - \int_{Y}^{\infty} \Omega(x,Y')dY'\right] \frac{(-1+i\partial_x\psi(x))}{(1+\partial_x\psi^2(x))} &= \\ BR[\gamma,\psi] + \frac{1}{2}\left[\int_{Y}^{\infty} \Omega(x,Y')dY' - \int_{-\infty}^{Y} \Omega(x,Y')dY'\right] \tilde{t}^* & \end{aligned} \tag{3.4}$$

being γ the vorticity intensity, i.e. the total vorticity across the layer:

$$\gamma(x) = \int_{-\infty}^{\infty} \Omega(x,Y')dY' ,$$

$BR[\gamma,\psi]$ the Birkhoff-Rott operator:

$$BR[\gamma,\psi] = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} \frac{\gamma(x')}{K_{\psi}^{0}(x-x')} dx' , \tag{3.5}$$

and the vector $\tilde{t}^*$ is the complex conjugate of $\tilde{t} = \tilde{t}^u + i\tilde{t}^v$, which is a vector parallel to the tangent to the curve ψ with components:

$$\tilde{t}^u = \frac{1}{1+\partial_x\psi^2} , \qquad \tilde{t}^v = \frac{\partial_x\psi}{1+\partial_x\psi^2} . \tag{3.6}$$

The remainder R is defined as follows:

$$R(\Omega,\psi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} T(x,x',Y') \left[\frac{1}{K_{\psi}^{\varepsilon}} - \frac{1}{K_{\psi}^{0}}\right] dx'dY' ,$$

where for notational simplicity we have suppressed the dependence of the K's from x, x' and $Y-Y'$, and where we have defined:

$$T(x,x',Y') = \Omega(x',Y') - \Omega(x,Y') + \Omega(x,Y')\left(\tilde{t}^*(x') - \tilde{t}^*(x)\right)\left(1 + i\partial_x \psi(x')\right). \tag{3.7}$$

A crucial property of the above quantity is the fact that $T(x,x,Y') = 0$, that will be used in the proof of Proposition 8.3.

If one uses the variable ξ instead of x, one can write the following expression for the velocity:

$$u - iv = \sum_n \int_{-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\Omega(\xi',Y')}{\mathscr{K}_\psi^\varepsilon(\xi-\xi',Y-Y')} J(\xi')d\xi'dY', \tag{3.8}$$

where we have defined:

$$\mathscr{K}_\psi^\varepsilon(\xi-\xi',Y-Y') = \xi - \xi' + X(\xi) - X(\xi') + i\left[\psi(\xi) - \psi(\xi') + \varepsilon(Y-Y')\right]$$

with $J = 1 + \partial_\xi X(\xi,\tau)$. The above expression can be recovered from (3.2) with the use of the change of variable (2.8).

One can define the corresponding operators $\mathscr{BR}$, $\mathscr{M}$ and $\mathscr{R}$ that now depend on the Lagrangian factor X also:

$$\mathscr{BR}[\gamma,\psi,X] = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\gamma(\xi')}{\mathscr{K}_\psi^0(\xi-\xi')} J(\xi')d\xi', \tag{3.9}$$

$$\mathscr{M}(\Omega,\psi,X) = \mathscr{BR}[\gamma,\psi,X] + \frac{1}{2}\left[\int_Y^\infty \Omega(Y')dY' - \int_{-\infty}^Y \Omega(Y')dY'\right]\tilde{t}^* \tag{3.10}$$

$$\mathscr{R}(\Omega,\psi,X) = \sum_n \int_{-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \mathscr{T}(\xi,\xi',Y')\left[\frac{1}{\mathscr{K}_\psi^\varepsilon} - \frac{1}{\mathscr{K}_\psi^0}\right] J(\xi')d\xi'dY', \tag{3.11}$$

where $\tilde{t}$ has the expression given in (3.6) with $\partial_x\psi = [1+X_\xi]^{-1}\partial_\xi\psi$, and $\mathscr{T}$ is the same as T in (3.7) expressed in the variable ξ. One can therefore write the decomposition formula for a velocity field generated by a vorticity Ω concentrated on a layer close to a curve $\psi(\xi)$ in the coordinates (ξ,Y) as

$$u - iv = \mathscr{M}(\Omega,\psi,X) + \mathscr{R}(\Omega,\psi,X). \tag{3.12}$$

The decomposition given in (3.12) is the main result of this Subsection. In fact we shall see that, under suitable hypotheses, the contribution given by $\mathscr{R}$ is small inside the layer, see Proposition 8.3. This will allow us to write an equation for the leading order vorticity ω_0 where the vorticity will be convected only by the velocity $\mathscr{M}_0 \equiv \mathscr{M}(\gamma_0,\varphi_0,X_0)$, and delaying the contribution of $\mathscr{R}$ in the equation for the correction ω_1. The physical meaning of the contribution of $\mathscr{M}$ is apparent: the velocity of the curve, as given by the Birkhoff-Rott operator, plus a local–in the sense that depends on the vorticity at the points (ξ,Y')– contribution of the vorticity integrated across the layer; this contribution has direction along the tangent to the curve. Given that the contribution of $\mathscr{M}$ ignores the details of how the vorticity is distributed in the layer, as well as for the fact that the velocity at a point x is influenced by the vorticity at other points x' only through the $\mathscr{BR}$ operator which governs the global motion of the curve, we shall name $\mathscr{M}$ as the macroscopic velocity.

3.2 The equations at different orders

In this Subsection we shall define the equations satisfied by ω_0 and ω_1. We shall write these equations using the coordinates (ξ, Y, τ) with $\psi = \varphi$. To specify these equations first we have to establish how, correspondingly to the expansion (3.1), we decompose the velocity. We shall write $u = u_0 + \varepsilon u_1$, (u_0, v_0) being the velocity generated by ω_0 and (u_1, v_1) the velocity generated by ω_1. However, in the previous Subsection, we have seen that a vortex layer vorticity generates a velocity field containing a term, the remainder $\mathscr{R}$, which formally is $O(\varepsilon)$. Therefore the term u_0 contains the term $\mathscr{M}$ which is $O(1)$, and the term $\mathscr{R}$ which is $O(\varepsilon)$. Using the complex notation we can write:

(3.13)
$$u - iv = u_0 - iv_0 + \varepsilon(u_1 - iv_1) = \mathscr{M}(\omega_0, \varphi, X) + \mathscr{R}(\omega_0, \varphi, X) + \varepsilon\,(u_1 - iv_1)\,.$$

It is natural to have the leading order vortcity ω_0 convected by $O(1)$ quantities, i.e. by $\mathscr{M}$. However the term $\mathscr{M}(\omega_0, \varphi, X)$ depends also by the correction terms through $\varepsilon\varphi_1$ and the Lagrangian factor X, see (3.16) below; we therefore define:

(3.14)
$$\mathscr{M}_0(\omega_0, \varphi_0, X_0) = \mathscr{BR}[\gamma_0, \varphi_0, X_0] + \frac{1}{2}\left[\int_Y^{\infty} \omega_0(Y')dY' - \int_{-\infty}^{Y} \omega_0(Y')dY'\right]\tilde{t}_0^*$$

where the vector $\tilde{t}_0^*$ is the complex conjugate of $\tilde{t}_0 = \tilde{t}_0^u + i\tilde{t}_0^v$, having defined:

$$\tilde{t}_0^u = \frac{1}{1 + \partial_x\varphi_0{}^2}\,, \qquad \tilde{t}_0^v = \frac{\partial_x\varphi_0}{1 + \partial_x\varphi_0{}^2}\,. \tag{3.15}$$

The difference with respect to $\mathscr{M}(\omega_0, \varphi, X)$ is in the fact that the BR operator is computed using the curve φ_0 instead of φ and the leading order approximation (see (3.17) below) X_0 instead of X; analogously, the vector $\tilde{t}_0$ is tangent to the curve φ_0 rather than φ. In the rest of this paper, to make the notation less cumbersome, we shall adopt the shorthands:

$$\mathscr{BR} = \mathscr{BR}(\omega_0, \varphi, X), \quad \mathscr{M} = \mathscr{M}(\omega_0, \varphi, X), \quad \mathscr{R} = \mathscr{R}(\omega_0, \varphi, X),$$
$$\mathscr{BR}_0 = \mathscr{BR}(\omega_0, \varphi_0, X_0), \quad \mathscr{M}_0 = \mathscr{M}_0(\omega_0, \varphi_0, X_0)$$

unless the dependence of the operators $\mathscr{BR}$, $\mathscr{M}$ and $\mathscr{R}$ from their arguments needs to be made explicit.

The Lagrangian factor X is implicitly defined using the exact expression of the velocity, i.e. as:

$$X(\xi, \tau) = \int_0^{\tau} [\mathscr{M}^u(\omega_0, \varphi, X) + \mathscr{R}^u(\omega_0, \varphi, X) + \varepsilon u_1]_{Y=0}\, d\tau'\,. \tag{3.16}$$

On the other hand, the leading order approximation of the Lagrangian factor, that we shall denote X_0, is implicitly defined solely in terms of $\mathscr{M}_0^u$, i.e. as:

$$X_0(\xi,\tau) = \int_0^\tau \mathscr{M}_0^u(\omega_0,\varphi_0,X_0)|_{Y=0}\, d\tau' . \tag{3.17}$$

We write the equations satisfied by $\tilde{\omega}$ using the reference frame adapted to the curve φ, so that in Eq.(2.16) one has to take $\psi = \varphi$, while the Lagrangian factor has the expression given in (3.16). Moreover, (u^Ψ, v^Ψ) means $(u,v)_{Y=0}$.

We now write the equations for the leading order vorticity ω_0 and for the approximation of the curve φ_0 as well for the $O(\varepsilon)$ corrections ω_1 and φ_1. The details of the derivation can be found in the Appendixes C and D. The equation for ω_0 is:

$$\partial_\tau \omega_0 + \left[\mathscr{M}_0^u - \mathscr{M}_{0|Y=0}^u\right] \frac{\partial_\xi \omega_0}{1+\partial_\xi X_0} - \int_0^Y \partial_\xi \mathscr{M}_0^u dY' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} = 0 . \tag{3.18}$$

The rationale behind (3.18) is in the decomposition of the velocity given in (3.13) and in the fact that, inside the layer, the term $\mathscr{R}$ (and, obviously, $\varepsilon\,(u_1 - iv_1)$ also) is an $O(\varepsilon)$ term, see Proposition 8.3. It is therefore natural to have the leading order vorticity to be convected by the $O(1)$ part of the velocity. The choice of using $\mathscr{M}_0$ instead of $\mathscr{M}$ allows to have the dynamics of ω_0 to depend explicitly only on φ_0 instead of the correct curve $\varphi = \varphi_0 + \varepsilon\varphi_1$.

The effects of the remainder term $\mathscr{R}$, of $\mathscr{M} - \mathscr{M}_0$, and of $\varepsilon\,(u_1 - iv_1)$ are therefore postponed to the next order equation for ω_1.

We couple the above equation (3.18) with the following equation for the motion of the curve φ_0:

$$\partial_\tau \varphi_0 = \mathscr{M}_{0|Y=0}^v , \tag{3.19}$$

and with the following equation for the approximated Lagrangian factor X_0

$$\partial_\tau X_0 = \mathscr{M}_{0|Y=0}^u , \tag{3.20}$$

that derives from the definition (3.17).

The equation for ω_1 can be derived from (2.10) with $\psi = \varphi$, and from (3.18):

$$\partial_\tau \omega_1 + \frac{1}{\varepsilon}\left[\frac{\mathscr{M}^u - \mathscr{M}^u_{Y=0}}{1+\partial_\xi X} - \frac{\mathscr{M}^u_0 - \mathscr{M}^u_{0|Y=0}}{1+\partial_\xi X_0}\right]\partial_\xi \omega_0 +$$
$$\frac{1}{\varepsilon}\frac{\mathscr{R}^u + \varepsilon u_1 - (\mathscr{R}^u + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_0 -$$
$$\frac{1}{\varepsilon}\left[\frac{1}{1+\partial_\xi X}\int_0^Y \partial_\xi \mathscr{M}^u dY' - \frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}^u_0 dY'\right]\partial_Y \omega_0 -$$
$$\frac{1}{\varepsilon}\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi(\mathscr{R}^u + \varepsilon u_1)dY'\right]\partial_Y \omega_0 +$$
$$\frac{u_0 + \varepsilon u_1 - (u_0+\varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_1 -$$
$$\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi u_0 dY'\right]\partial_Y \omega_1 +$$
$$\left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0}\right]\partial_Y \omega_1 = 0 \tag{3.21}$$

Notice that the terms in the first and third lines above are, formally, $O(1)$ because inside the layer:

$$\mathscr{M} - \mathscr{M}_0 = O(\varepsilon) \qquad X - X_0 = O(\varepsilon);$$

the above two formal statements will be made rigorous later: the first one in Lemmas I.1 and I.2 of Appendix I; the second is an obvios consequence of the estimates on $\mathscr{M} - \mathscr{M}_0$ and on $\mathscr{R}$. The terms in the second and the fourth lines of (3.21) are $O(1)$ because, inside the layer $\mathscr{R} = O(\varepsilon)$.

Equation (3.21) has to be coupled with the following equation for φ_1

$$\partial_\tau \varphi_1 = \frac{1}{\varepsilon}\left[\mathscr{M}^v - \mathscr{M}^v_0 + \mathscr{R}^v + \varepsilon v_1\right], \tag{3.22}$$

whose derivation can be found in Appendix D, and with an equation ruling the evolution of the Lagrangian factor X. From (3.16) we know that

$$\partial_\tau X = \left[\mathscr{M}^u + \mathscr{R}^u + \varepsilon u_1\right]_{Y=0}; \tag{3.23}$$

Introducing the correction X_1 defined as:

$$X = X_0 + \varepsilon X_1$$

and because of (3.20), one writes the evolution equation for X_1:

$$\partial_\tau X_1 = \frac{1}{\varepsilon}\left[\mathscr{M}^u - \mathscr{M}^u_0 + \mathscr{R}^u + \varepsilon u_1\right]_{Y=0} \tag{3.24}$$

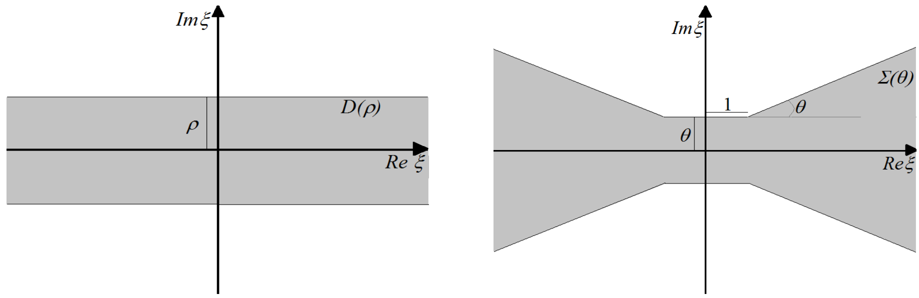

FIGURE 4.1. The domain of analyticity $D(\rho)$ and $\Sigma(\theta)$

In (3.21) and (3.22) the velocities u_1 and v_1 are generated by the vorticity ω_1 and are therefore given by the Biot-Savart law

$$u_1(\xi,Y,\tau) = \int_{\xi-\frac{\pi}{2}}^{\xi+\frac{\pi}{2}} \int_{-\infty}^{\infty} \mathscr{H}_u(\xi-\xi',Y-Y')\omega_1\left(\xi',Y',\tau\right) J(\xi,\tau)\,d\xi' dY' \tag{3.25}$$

$$v_1(\xi,Y,\tau) = \int_{\xi-\frac{\pi}{2}}^{\xi+\frac{\pi}{2}} \int_{-\infty}^{\infty} \mathscr{H}_v(\xi-\xi',Y-Y')\omega_1\left(\xi',Y',\tau\right) J(\xi,\tau)\,d\xi' dY' \tag{3.26}$$

where the expression for the kernels $(\mathscr{H}_u,\mathscr{H}_v)$ is given in (8.1) and (8.2), and the Jacobian J is written explicitly in (8.5).

The main results of this Subsection are: the derivation of the evolution equations for ω_0, φ_0 and X_0, (3.18), (3.19) and (3.20) respectively; and the derivation of the equations for ω_1, φ_1 and X_1, (3.21), (3.22) and (3.24) respectively. The $O(\varepsilon)$ velocity (u_1,v_1), appearing in the equation for ω_1 is given by (3.25)-(3.26).

In Section 7 we shall integrate in Y equation (3.18) and derive the equations of the skeleton of the layer, therefore obtaining φ_0, X_0 and the vorticity intensity γ_0. In Section 9 we shall solve for ω_0, ω_1, φ_1 and X_1. The equations are coupled by the fact that the frame where the equation for ω_0 is written, has base curve the exact $\varphi = \varphi_0 + \varepsilon\varphi_1$.

4 Functional setting

In this Section we shall describe the function spaces we shall use throughout the paper. In what follows we shall denote by $D(\rho)$ the periodic strip of the complex plane of width $\rho > 0$:

$$D(\rho) \equiv \{\xi + i\eta :\ \xi \in \mathbb{R}/\pi\mathbb{Z}, |\eta| < \rho\}\,,$$

and with $\Sigma(\theta)$, where $0 < \theta < \pi/4$, the conoid in the complex plane:

$$\begin{aligned}\Sigma(\theta) \equiv\ & \{Y + i\lambda : |Y| \le 1,\ |\lambda| < \theta\} \bigcup \\ & \{Y + i\lambda : |Y| \ge 1,\ |\lambda| < \theta + (|Y|-1)\tan\theta\}\end{aligned} \tag{4.1}$$

In the rest of the paper α with $0 < \alpha < 1$ will express the Hölder modulus of continuity. The positive constants μ and β have the following meaning: μ will give the rate of exponential decay of the vorticity away from the layer, while β will give the rate at which the domains of analyticity shrink in time.

For a function $f : D(\rho) \to \mathbb{C}$ we introduce the notation:

$$|f|_\rho \equiv \sup_{(\xi,\eta)\in D(\rho)} |f(\xi+i\eta)| \,,$$

$$|f|_\rho^{(\alpha)} \equiv \sup_{(\xi,\eta),(\bar\xi,\eta)\in D(\rho)} \frac{\left|f(\xi+i\eta)-f(\bar\xi+i\eta)\right|}{\left|\xi-\bar\xi\right|^\alpha}$$

For a function $g : D(\rho)\times\Sigma(\theta) \to \mathbb{C}$ we introduce the following notations:

$$|g|_{\rho,\theta} \equiv \sup_{(\xi,\eta)\in D(\rho),\,(Y,\lambda)\in\Sigma(\theta)} |g(\xi+i\eta,Y+i\lambda)| \,,$$

$$|g|_{\rho,\theta}^{(\alpha)} \equiv \sup_{\substack{(\xi,\eta),\,(\bar\xi,\eta)\,\in\, D(\rho)\\ (Y,\lambda),\,(\bar Y,\lambda)\,\in\,\Sigma(\theta)}} \frac{\left|g(\xi+i\eta,Y+i\lambda)-g(\bar\xi+i\eta,\bar Y+i\lambda)\right|}{\left[(\xi-\bar\xi)^2+(Y-\bar Y)^2\right]^{\alpha/2}}$$

and

$$|g|_{\rho,\theta,\mu} \equiv \sup_{(\xi,\eta)\in D(\rho),\,(Y,\lambda)\in\Sigma(\theta)} |e^{\mu|Y|}g(\xi+i\eta,Y+i\lambda)| \,,$$

$$|g|_{\rho,\theta,\mu}^{(\alpha)} \equiv \sup_{\substack{(\xi,\eta),\,(\bar\xi,\eta)\,\in\, D(\rho)\\ (Y,\lambda),\,(\bar Y,\lambda)\,\in\,\Sigma(\theta)}} \frac{\left|e^{\mu|Y|}g(\xi+i\eta,Y+i\lambda)-e^{\mu|\bar Y|}g(\bar\xi+i\eta,\bar Y+i\lambda)\right|}{\left[(\xi-\bar\xi)^2+(Y-\bar Y)^2\right]^{\alpha/2}}$$

Definition 4.1. Let $f : D(\rho)\to\mathbb{C}$ analytic. Then we say $f \in B_\rho^\alpha$ when:

$$\|f\|_\rho^{(\alpha)} \equiv |f|_\rho + |f|_\rho^{(\alpha)} < \infty$$

Definition 4.2. Let $g : D(\rho)\times\Sigma(\theta)\to\mathbb{C}$ analytic. Then we say $g\in B_{\rho,\theta}^\alpha$ when:

$$\|g\|_{\rho,\theta}^{(\alpha)} \equiv |g|_{\rho,\theta} + |g|_{\rho,\theta}^{(\alpha)} < \infty$$

Definition 4.3. Let $g : D(\rho)\times\Sigma(\theta)\to\mathbb{C}$ analytic. Then we say $g\in B_{\rho,\theta,\mu}^\alpha$ when:

$$\|g\|_{\rho,\theta,\mu}^{(\alpha)} \equiv |g|_{\rho,\theta,\mu} + |g|_{\rho,\theta,\mu}^{(\alpha)} < \infty$$

Definition 4.4. Let $f : D(\rho)\to\mathbb{C}$ analytic. then we say $f\in B_\rho^{m,\alpha}$ when:

$$\|f\|_{m,\rho}^{(\alpha)} \equiv \sum_{j\le m} |\partial_\xi^j f|_\rho + |\partial_\xi^m f|_\rho^{(\alpha)} < \infty$$

Definition 4.5. Let $g : D(\rho)\times\Sigma(\theta)\to\mathbb{C}$ analytic. Then we say $g\in B_{\rho,\theta}^{m,\alpha}$ when:

$$\|g\|_{m,\rho,\theta}^{(\alpha)} = \sum_{i+j\le m} |\partial_\xi^i\partial_Y^j g|_{\rho,\theta} + \sum_{i+j=m} |\partial_\xi^i\partial_Y^j g|_{\rho,\theta}^{(\alpha)} < \infty$$

Definition 4.6. Let $g : D(\rho)\times\Sigma(\theta)\to\mathbb{C}$ analytic. Then we say $g\in B^{m,\alpha}_{\rho,\theta,\mu}$ when:

$$\|g\|^{(\alpha)}_{m,\rho,\theta,\mu}=\sum_{i+j\le m}|\partial^i_\xi\partial^j_Y g|_{\rho,\theta,\mu}+\sum_{i+j=m}|\partial^i_\xi\partial^j_Y g|^{(\alpha)}_{\rho,\theta,\mu}<\infty$$

Definition 4.7. Let $t\in[0,T]$ and let β and ρ be such that $\beta T<\rho$. A function $f(\cdot,\cdot)$ will be said to be in $B^{m,\alpha}_{\rho,\beta,T}$ when $f(\cdot,t)\in B^{m,\alpha}_{\rho-\beta t}$ $\forall t\in[0,T]$ and when:

$$\|f\|^{(\alpha)}_{m,\rho,\beta,T}\equiv\sum_{j\le m}\sup_{0\le t\le T}|\partial^j_\xi f(\cdot,t)|_{\rho-\beta t}+\sup_{0\le t\le T}|\partial^m_\xi f(\cdot,t)|^{(\alpha)}_{\rho-\beta t}<\infty$$

Definition 4.8. Let $t\in[0,T]$ and let β, ρ and θ be such that $\beta T<\rho$, $\beta T<\theta$. A function $g(\cdot,\cdot,\cdot)$ will be said to be in $B^{m,\alpha}_{\rho,\theta,\beta,T}$ when $g(\cdot,\cdot,t)\in B^{m,\alpha}_{\rho-\beta t,\theta-\beta t}$ $\forall t\in[0,T]$ and when:

$$\begin{aligned}\|g\|^{(\alpha)}_{m,\rho,\theta,\beta,T}\equiv&\sum_{i+j\le m}\sup_{0\le t\le T}|\partial^i_\xi\partial^j_Y g(\cdot,\cdot,t)|_{\rho-\beta t,\theta-\beta t}+\\&\sum_{i+j=m}\sup_{0\le t\le T}|\partial^i_\xi\partial^j_Y f|^{(\alpha)}_{\rho-\beta t,\theta-\beta t}<\infty\end{aligned}$$

Definition 4.9. Let $t\in[0,T]$ and let β, ρ and θ be such that $\beta T<\rho$, $\beta T<\theta$. A function $g(\cdot,\cdot,\cdot)$ will be said to be in $B^{m,\alpha}_{\rho,\theta,\mu,\beta,T}$ when $g(\cdot,\cdot,t)\in B^{m,\alpha}_{\rho-\beta t,\theta-\beta t,\mu}$ $\forall t\in[0,T]$ and when:

$$\begin{aligned}\|g\|^{(\alpha)}_{m,\rho,\theta,\beta,\mu,T}\equiv&\sum_{i+j\le m}\sup_{0\le t\le T}|\partial_\xi\partial^j_Y g(\cdot,\cdot,t)|_{\rho-\beta t,\theta-\beta t,\mu}+\\&\sum_{i+j=m}\sup_{0\le t\le T}|\partial^i_\xi\partial^j_Y g|^{(\alpha)}_{\rho-\beta t,\theta-\beta t,\mu}<\infty\end{aligned}$$

Remark 4.10. For a function $g(\xi,Y)$ defined on $D(\rho)\times\Sigma(\theta)$ one can define the Hölder modulus of continuity with respect to ξ and Y separately:

$$|g|^{(\alpha,\xi)}_{\rho,\theta,\mu}\equiv\sup_{\substack{(\xi,\eta),\,(\bar\xi,\eta)\in D(\rho)\\(Y,\lambda)\in\Sigma(\theta)}}\frac{\left|e^{\mu|Y|}g(\xi+i\eta,Y+i\lambda)-e^{\mu|Y|}g(\bar\xi+i\eta,Y+i\lambda)\right|}{|\xi-\bar\xi|^\alpha}$$

$$|g|^{(\alpha,Y)}_{\rho,\theta,\mu}\equiv\sup_{\substack{(\xi,\eta)\in D(\rho)\\(Y,\lambda),\,(\bar Y,\lambda)\in\Sigma(\theta)}}\frac{\left|e^{\mu|Y|}g(\xi+i\eta,Y+i\lambda)-e^{\mu|\bar Y|}g(\xi+i\eta,\bar Y+i\lambda)\right|}{|Y-\bar Y|^\alpha}$$

One can define, analogously to Definition 4.5, the quantities $\|g\|^{(\alpha,\xi)}_{m,\rho,\theta}$ and $\|g\|^{(\alpha,Y)}_{m,\rho,\theta}$, and, analogously to Definition 4.6, the quantities $\|g\|^{(\alpha,\xi)}_{m,\rho,\theta,\mu}$ and $\|g\|^{(\alpha,Y)}_{m,\rho,\theta,\mu}$. Clearly one has the norms equivalence:

$$c\left(\|g\|^{(\alpha,\xi)}_{m,\rho,\theta}+\|g\|^{(\alpha,Y)}_{m,\rho,\theta}\right)\le\|g\|^{(\alpha)}_{m,\rho,\theta}\le C\left(\|g\|^{(\alpha,\xi)}_{m\rho,\theta}+\|g\|^{(\alpha,Y)}_{m,\rho,\theta}\right)$$

$$c\left(\|g\|^{(\alpha,\xi)}_{m,\rho,\theta,\mu}+\|g\|^{(\alpha,Y)}_{m,\rho,\theta,\mu}\right)\le\|g\|^{(\alpha)}_{m,\rho,\theta,\mu}\le C\left(\|g\|^{(\alpha,\xi)}_{m\rho,\theta,\mu}+\|g\|^{(\alpha,Y)}_{m,\rho,\theta,\mu}\right)$$

The above Remark will be useful when we shall prove the potential estimates bounding the velocity in terms of the vorticity; in fact, in the Biot-Savart kernel the tangential variable ξ and the transversal variable Y appear with different scales, and it will be easier to give separate estimates for the Hölder modulus of continuity taken with respect to ξ and with respect to Y.

Remark 4.11. In what follows, to make the notation simpler, when the index denoting the number of derivatives is equal to 0, we shall omit it. For example $B^{\alpha}_{\rho,\theta}$, $B^{\alpha}_{\rho,\theta,\mu}$, $B^{\alpha}_{\rho,\theta}$, $B^{\alpha}_{\rho,\theta,\mu,\beta,T}$ will be used instead of $B^{0,\alpha}_{\rho,\theta}$, $B^{0,\alpha}_{\rho,\theta,\mu}$, $B^{0,\alpha}_{\rho,\theta}$, $B^{0,\alpha}_{\rho,\theta,\mu,\beta,T}$ respectively. The same notation will be used for norms.

5 Main results

Consider the Euler equations written in the comoving frame adapted to the curve, Eqs.(2.10)-(2.13).

Remark 5.1. To solve Euler equations we need to introduce the correction term ω_1. This necessity can be understood in the following way. In the Euler equations appears an $O(\varepsilon^{-1})$ term, see (2.10). Using the incompressibility condition this $O(\varepsilon^{-1})$ is shown to be, formally, $O(1)$, see (2.16). The linearly growing term deriving from the integration in Y of a non decaying quantity, together with the presence of the ∂_ξ derivative of the velocity and of the ∂_Y derivative of the vorticity, cannot be handled directly. The solution is in the decomposition formula (3.3) (or, in the Lagrangian coordinates we are using, formula (3.12)). The term $\mathscr{M}$ has the special structure being given by the BR term (that can be shown to be *a-priori* analytic, see Theorem 5.2 and Section 7 below) plus a term (involving the integration of the vorticity) whose behavior at $|Y|\to\infty$ can be shown to be *a-priori* analytic. The remainder term $\mathscr{R}$ is formally $O(\varepsilon)$ and this fact will be proven to be true inside the layer. However the proof, to avoid logarithmic singularities, involves the use of delicate cancellation properties that require higher regularity of the vorticity ω_0, still to be proved at $O(1)$. The solution to this impasse is to introduce the correction ω_1 and to put the effects of the term $\mathscr{R}(\omega_0,\varphi_0)$ in the equation for ω_1.

Therefore we look for a solution of the Euler equations in the form

$$\tilde{\omega}=\omega_0+\varepsilon\omega_1\,,\quad \varphi=\varphi_0+\varepsilon\varphi_1\,,\quad X=X_0+\varepsilon X_1\,;$$

the necessity to include the lagrangian factors as unknown derives from the fact that we are solving Euler equation in a lagrangian frame. The unknowns (ω_0,φ_0,X_0) solve equations (3.18)-(3.20) with initial data

$$\omega_0(\xi,Y,t=0)=\tilde{\omega}^{in}(\xi,Y)\,,\quad \varphi_0(\xi,t=0)=\varphi^{in}(\xi)\,,\quad X_0(\xi,t=0)=0\,, \tag{5.1}$$

while (ω_1,φ_1,X_1) solve equations .(3.21), (3.22) and (3.24) with initial data

$$\omega_1(\xi,Y,t=0)=0\,,\quad \varphi_1(\xi,t=0)=0\,,\quad X_1(\xi,t=0)=0\,. \tag{5.2}$$

Please notice that at time $t = 0$ the variable x and ξ coincide so that, in the specification of the initial data, one can use indifferently x or ξ.

The reason why the lagrangian factors X_0 and X_1 enter dynamically in the equations to be solved can be understood in the following way: looking at the definition of the zero-th order approximation X_0 (3.17), one sees that to compute X_0 one has to know the first order approximation of the velocity at $Y = 0$ (i.e. $\mathscr{M}_0$) which, in turn, to be computed, needs the knowledge of the lagrangian factor, see (3.14). Analagously, to compute X_1, one need to know the first order correction u_1 which can be recovered from the Biot-Savart law (3.25), where the lagrangian factors appear throuh the kernel $\mathscr{H}_u$, see (8.1).

To solve these equations we first construct the skeleton of the layer, namely $(\varphi_0, \gamma_0, X_0)$, being γ_0 the leading order vorticity intensity, i.e. the integral in Y of ω_0, see (7.1).

This is accomplished in the following Theorem that is one of the main results of this paper.

Theorem 5.2 (Construction of the skeleton of the layer)**.** *Suppose* $\tilde{\omega}^{in} \in B^{1,\alpha}_{\rho_0,\theta_0,\mu_0}$ *and* $\varphi^{in} \in B^{2,\alpha}_{\rho_0}$ *with* $\|\varphi^{in}\|^{(\alpha)}_{2,\rho_0} < 1/4$*. Then there exist* $\bar{\rho}_0 < \rho_0$*,* $\bar{\theta}_0 < \theta_0$ $\bar{\mu}_0 < \mu_0$*,* T_0 *and* $\bar{\beta} > 0$ *such that* $\gamma_0 \in B^{1,\alpha}_{\bar{\rho}_0,\bar{\beta},T_0}$ *and* $\varphi_0, X_0 \in B^{2,\alpha}_{\bar{\rho}_0,\bar{\beta},T_0}$*.*

Remark 5.3. The equations ruling the dynamics of γ_0 can be derived from the equation for ω_0, i.e. Eq.(3.18), through integration in Y, see (7.2) below. However, in this equations, it appears the quantity $\gamma_0^+ - \gamma_0^-$, which expresses how the curve is off the barycenter of the layer. This new variable, therefore, will have to be considered in the dynamics. This will be accomplished in Section 7, where we shall solve (7.2), (7.4), (7.5) and (7.6) for $(\gamma_0, \gamma_0^+ - \gamma_0^-, \varphi_0, X_0)$.

Having proven that the layer, macroscopically, has analytic regularity, one has to pass to determine how the vorticity is distributed, namely ω_0 and ω_1, the correction to the position of the curve φ_1, as well the correction to the lagrangian factor X_1.

Theorem 5.4 (Construction of the vorticity distribution)**.** *Suppose* $\tilde{\omega}^{in} \in B^{1,\alpha}_{\rho_0,\theta_0,\mu_0}$ *and* $\varphi^{in} \in B^{2,\alpha}_{\rho_0}$*, with* $\|\varphi^{in}\|^{(\alpha)}_{2,\rho_0} < 1/4$*. Then there exist* $\rho_1, \theta_1, \mu_1, \beta_1, T_1$*, with* $\rho_1 < \bar{\rho}_0 < \rho_0$*,* $\theta_1 < \bar{\theta}_0 < \theta_0$ $\mu_1 < \bar{\mu}_0 < \mu_0$*,* $T_1 < T_0$ *and* $\beta_1 > \bar{\beta} > 0$ *such that Eqs.*(3.18)*,* (3.21)*,* (3.22)*,* (3.24) *admit a unique solution* $(\omega_0, \omega_1, \varphi_1, X_1)$ *in* $[0, T_1]$ *with* $\omega_0 \in B^{1,\alpha}_{\rho_1,\theta_1,\mu_1,\beta_1,T_1}$*,* $\omega_1 \in B^{\alpha}_{\rho_1,\theta_1,\frac{\mu_1}{2},\beta_1,T_1}$ *and* $\varphi_1, X_1 \in B^{1,\alpha}_{\rho_1,\beta_1,T_1}$*.*

Notice how the vorticity correction ω_1, as compared to the leading order ω_0, has lower regularity in the spatial variables (x, Y), and a slower exponential decay when $|Y| \to \infty$

We now state the convergence of the dynamics of the vortex layer (as ruled by the Euler equations) to the dynamics of the Birkhoff-Rott equation. We find more convenient to state the convergence result using the eulerian variable x rather than the lagrangian variable ξ. We remark that the vorticities ω_i and the curves φ_i, that in the above Theorem have been constructed using the Lagrangian variable ξ, given the analytic invertibility of the change of variable $x \to \xi$, they can be expressed as analytic functions of the eulerian variable x.

Let a vortex layer initial datum $(\tilde{\omega}^{in}(x,Y), \varphi^{in}(x))$ be given. The above Theorem 5.2 and Theorem 5.4 have constructed the solution to the Euler equations in the form of a vortex layer solution $(\tilde{\omega}, \varphi)$ with $\tilde{\omega} = \omega_0(x,Y,t) + \varepsilon\omega_1(x,Y,t)$ and $\varphi(x,t) = \varphi_0(x,t) + \varepsilon\varphi_1(x,t)$. Moreover, taking $\tilde{\rho}_1 < \rho_1$, $\tilde{\theta}_1 < \theta_1$, $\tilde{T}_1 < T_1$, one can assume $(\tilde{\omega}, \varphi) \in B^{1,\alpha}_{\tilde{\rho}_1,\tilde{\theta}_1,\mu_1,\beta_1,\tilde{T}_1} \times B^{2,\alpha}_{\tilde{\rho}_1,\beta_1,\tilde{T}_1}$. To simplify the notation we rename $\tilde{\rho}_1$, $\tilde{\theta}_1$ and $\tilde{T}_1$, and assume $(\tilde{\omega}, \varphi) \in B^{1,\alpha}_{\rho_1,\theta_1,\mu_1,\beta_1,T_1} \times B^{2,\alpha}_{\rho_1,\beta_1,T_1}$.

We can therefore compute the vorticity intensity of the vortex layer:

$$\gamma(x,t) = \int_{-\infty}^{\infty} \tilde{\omega}(x,Y',t)dY'$$

and say that $\gamma \in B^{1,\alpha}_{\rho_1,\beta_1,T_1}$

Consider the Birkhoff-Rott equations:

$$\partial_t \gamma_s + \partial_x (U\gamma_s) = 0 \tag{5.3}$$

$$\partial_t \varphi_s + U\partial_x \varphi_s = V \tag{5.4}$$

where

$$(U,V) = BR[\gamma_s, \varphi_s]$$

being BR the Birkhoff-Rott operator defined in (3.5). Initialize the BR equations with the initial vorticity intensity of the vortex layer and with the same curve which the layer is based on:

$$\gamma_s(x,t=0) = \int_{-\infty}^{\infty} \tilde{\omega}^{in}(x,Y')dY' \qquad \varphi_s(x,t=0) = \varphi^{in}(x) \tag{5.5}$$

From the results of [49, 12] we know that there exist $\rho_s > 0$, $\beta_s > 0$, $T_s > 0$ with $\rho_s - \beta_s T_s > 0$ such that the BR equations admit a unique solution $(\gamma_s, \varphi_s) \in B^{1,\alpha}_{\rho_s,\beta_s,T_s} \times B^{2,\alpha}_{\rho_s,\beta_s,T_s}$.

Define $\rho = \min(\rho_1, \rho_s)$, $\beta = \max(\beta_1, \beta_s)$ and $T = \min(T_1, T_s)$.

Next Theorem states that the BR solution is a good approximation of the vortex layer solution.

Theorem 5.5 (Convergence to the BR solution)**.** *Let $(\tilde{\omega}, \varphi) \in B^{1,\alpha}_{\rho,\theta,\mu,\beta,T} \times B^{2,\alpha}_{\rho,T}$ the VL-solution of the Euler equation with VL initial datum $(\tilde{\omega}^{in}, \varphi^{in})$. Let $(\gamma_s, \varphi_s) \in B^{1,\alpha}_{\rho,T} \times B^{2,\alpha}_{\rho,T}$ the solution of the BR equations with initial datum given by* (5.5)*. Then:*

$$\|\gamma_s - \gamma\|^{(\alpha)}_{1,\rho,\beta,T} \leq c\varepsilon \qquad \|\varphi_s - \varphi\|^{(\alpha)}_{2,\rho,\beta,T} \leq c\varepsilon$$

6 Mathematical preliminaries

In this Section we present some miscellaneous results that will be used throughout the paper.

6.1 The Abstract Cauchy-Kowalewski Theorem

We first state a fixed-point Theorem, the so called Abstract Cauchy-Kowalewski Theorem. Consider the equation

$$u + F(u,t) = 0\,. \tag{6.1}$$

Let $\{X_\rho : \ 0 < \rho \le \rho_0\}$ be a scale of Banach spaces with norms $|\cdot|_\rho$, such that $X_{\rho'} \subset X_{\rho''}$ and $|\cdot|_{\rho''} \le |\cdot|_{\rho'}$ when $\rho'' \le \rho' \le \rho_0$.

Theorem 6.1 (ACK). *Suppose that there exist $R > 0$, $\rho_0 > 0$, and $\beta_0 > 0$ such that for $0 < \tau \le T \le \rho_0/\beta_0$ the following statements hold:*

(1) *if ρ is such that $0 < \rho \le \rho_0 - \beta_0\tau$, then the function $F(0,t)\colon [0,\tau] \mapsto \{u \in X_\rho : \sup_{0\le t\le\tau} |u(t)|_\rho < \infty\}$ is continuous and*

$$|F(0,t)|_{\rho_0-\beta_0 t} \le R_0 < R;$$

(2) *if ρ', ρ are such that $0 < \rho' < \rho \le \rho_0 - \beta_0\tau$, then the function $F(u,t)\colon [0,\tau] \mapsto X_{\rho'}$ is continuous for all u such that $\{u \in X_\rho : \sup_{0\le t\le T} |u(t)|_\rho \le R\}$;*

(3) *if ρ' and $\rho(s)$ are such that $\rho' < \rho(s) \le \rho_0 - \beta_0 s$ and if u^1 and $u^2 \in \{u : u(t) \in X_{\rho_0-\beta_0 t} : \sup_{0\le t\le\tau} |u(t)|_{\rho_0-\beta_0 t} \le R\}$, then*

$$|F(u^1,t) - F(u^2,t)|_{\rho'} \le C \int_0^t ds \frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'}$$

where C is a constant independent of t, τ, u^1, u^2, ρ, ρ', $\rho(s)$.

Then there exists $\beta > \beta_0$ such that for all $0 < \rho < \rho_0$ Eq. (6.1) has a unique solution $u(t) \in X_{\rho_0-\beta t}$ with $t \in [0, \rho_0/\beta]$. Moreover, $\sup_{\rho<\rho_0-\beta t} |u(t)|_\rho \le R$.

For a proof of the above Theorem see e.g. [45, 9] and [34, 14] where a version that allows mild singularities in time is given.

6.2 Cauchy estimates

We now present some Lemmas, that will be used in the analysis of the various nonlinear terms present in Eqs.(3.18) and (3.21); the proof of these Lemmas are based on the use of the Cauchy estimate of a derivative of an analytic function. Throughout this section $l \ge 0$ is an integer.

Lemma 6.2. *Let $f \in B^{l,\alpha}_{\rho''}$. If $\rho' < \rho''$ then*

$$\|\partial_\xi f\|^{(\alpha)}_{l,\rho'} \le \frac{\|f\|^{(\alpha)}_{l,\rho''}}{\rho'' - \rho'} \,. \tag{6.2}$$

If the derivative is with respect to the Y variable, because of the angular shape of the region of analyticity, shrinking the strip of analyticity, one can also bound linear growth in Y.

Lemma 6.3. *Let $f \in B^{l,\alpha}_{\rho'',\theta'',\mu'}$. If $\theta' < \theta''$ and If $\mu' < \mu''$*

$$\|\partial_Y f\|^{(\alpha)}_{l,\rho',\theta',\mu'} \le \frac{\|f\|^{(\alpha)}_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \mu' \|f\|^{(\alpha)}_{l,\rho',\theta',\mu'} \tag{6.3}$$

$$\|Y\partial_Y f\|^{(\alpha)}_{l,\rho',\theta',\mu'} \le \frac{\|f\|^{(\alpha)}_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \mu' \frac{\|f\|^{(\alpha)}_{l,\rho',\theta',\mu''}}{\mu'' - \mu'} + \|f\|^{(\alpha)}_{l,\rho',\theta',\mu'} \,. \tag{6.4}$$

We finally state the following Lemma that can be easily proven using the above Cauchy estimate.

Lemma 6.4. *Let $f \in B^{l,\alpha}_{\rho',\theta'}$ and $g \in B^{l,\alpha}_{\rho'',\theta'',\mu'}$. If $\rho' < \rho''$, then*

$$\|f\partial_\xi g\|^{(\alpha)}_{l,\rho',\theta',\mu'} \le c\|f\|^{(\alpha)}_{l,\rho',\theta'} \frac{\|g\|^{(\alpha)}_{l,\rho'',\theta',\mu'}}{\rho'' - \rho'}$$

and, if $\theta' < \theta''$

$$\|f\partial_Y g\|^{(\alpha)}_{l,\rho',\theta',\mu'} \le c\|f\|^{(\alpha)}_{l,\rho',\theta'} \frac{\|g\|^{(\alpha)}_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'}$$

6.3 Estimates on the Birkhoff-Rott operator and on the macroscopic velocity

We introduce the notation:

$$\delta\gamma = \gamma^{(1)} - \gamma^{(2)}, \quad \delta\psi = \psi^{(1)} - \psi^{(2)}, \quad \delta X = X^{(1)} - X^{(2)}, \quad \delta\Omega = \Omega^{(1)} - \Omega^{(2)}.$$

We state the following results:

Proposition 6.5. *Let $\gamma^{(i)} \in B^{1,\alpha}_\rho$, $\psi^{(i)} \in B^{2,\alpha}_\rho$ and $X^{(i)} \in B^{2,\alpha}_\rho$ for $i = 1,2$.*
Then $\mathscr{BR}[\gamma^{(i)}, \psi^{(i)}, X^{(i)}] \in B^{1,\alpha}_\rho$ and

$$\|\mathscr{BR}[\gamma^{(1)}, \psi^{(1)}, X^{(1)}] - \mathscr{BR}[\gamma^{(2)}, \psi^{(2)}, X^{(2)}]\|^{(\alpha)}_{1,\rho} \le c\left(\|\delta\gamma\|^{(\alpha)}_{1,\rho} + \|\delta\psi\|^{(\alpha)}_{2,\rho} + \|\delta X\|^{(\alpha)}_{2,\rho}\right)$$

For the proof see [49] or [12, 13].

Proposition 6.6. *Let $\Omega^{(i)} \in B^{1,\alpha}_{\rho,\theta,\mu}$, $\psi^{(i)} \in B^{2,\alpha}_\rho$ and $X^{(i)} \in B^{2,\alpha}_\rho$ for $i = 1,2$.*
Then $\mathscr{M}[\Omega^{(i)}, \psi^{(i)}, X^{(i)}] \in B^{1,\alpha}_{\rho,\theta}$ and

$$\|\mathscr{M}[\Omega^{(1)}, \psi^{(1)}, X^{(1)}] - \mathscr{M}[\Omega^{(2)}, \psi^{(2)}, X^{(2)}]\|^{(\alpha)}_{1,\rho,\theta} \le c\left(\|\delta\Omega\|^{(\alpha)}_{1,\rho,\theta,\mu} + \|\delta\psi\|^{(\alpha)}_{2,\rho} + \|\delta X\|^{(\alpha)}_{2,\rho}\right)$$

Given the expression (3.10) for $\mathscr{M}$, the proof of Proposition 6.6 is an immediate consequence of Proposition 6.5.

7 Construction of the skeleton of the layer: proof of Theorem 5.2

A preliminary crucial step in the construction of the vortex layer is to show that $\gamma_0(\xi,\tau)$ and $\varphi_0(\xi,\tau)$, remain analytic up to a time T_0.

We define the following quantities:

(7.1)
$$\gamma_0(\xi) = \int_{-\infty}^{\infty} \omega_0(\xi,Y')dY', \quad \gamma_0^+(\xi) = \int_0^{\infty} \omega_0(\xi,Y')dY', \quad \gamma_0^-(\xi) = \int_{-\infty}^{0} \omega_0(\xi,Y')dY'$$

We integrate in Y, from $-\infty$ to ∞, Eq.(3.18) and find the following evolution equation for γ_0:

$$\partial_\tau \gamma_0 - \mathscr{M}_0^u|_{Y=0} \frac{\partial_\xi \gamma_0}{1+\partial_\xi X_0} + \frac{\partial_\xi \left(\gamma_0 \mathscr{B}\mathscr{R}_0^u\right)}{1+\partial_\xi X_0} = 0; \tag{7.2}$$

to derive the above equation we have used

$$\int_{-\infty}^{\infty} dY\,\omega_0(\xi,Y) \int_Y^{\infty} dY'\,\omega_0(\xi,Y') = \int_{-\infty}^{\infty} dY\,\omega_0(\xi,Y) \int_{-\infty}^{Y} dY'\,\omega_0(\xi,Y')\,.$$

From (3.14) and (3.6) one derives

$$\mathscr{M}_0|_{Y=0} = \mathscr{B}\mathscr{R}_0 + \frac{1}{2}\left[\gamma_0^+ - \gamma_0^-\right]\tilde{t}_0^* \tag{7.3}$$

where

$$\tilde{t}_0 = \frac{1 + i[1+\partial_\xi X_0]^{-1}\partial_\xi \varphi_0}{1+[1+\partial_\xi X_0]^{-2}\partial_\xi {\varphi_0}^2}$$

and recognizes that in the equation for γ_0, equation (7.2), it appears also the quantity $\left[\gamma_0^+ - \gamma_0^-\right]$. To get a closed system one has therefore to find an equation for $\left[\gamma_0^+ - \gamma_0^-\right]$.

Integrating in Y Eq.(3.18), first from 0 to ∞ and then from $-\infty$ to 0 one gets the following equations:

$$\partial_\tau \gamma_0^+ + \frac{1}{1+\partial_\xi X_0}\left[-\frac{1}{2}\gamma_0^+ \partial_\xi \gamma_0\, \tilde{t}_0^u - \frac{1}{2}\gamma_0^+ \gamma_0^- \partial_\xi \tilde{t}_0^u + \gamma_0^+ \partial_\xi \mathscr{B}\mathscr{R}_0^u\right] = 0$$
$$\partial_\tau \gamma_0^- + \frac{1}{1+\partial_\xi X_0}\left[\frac{1}{2}\gamma_0^- \partial_\xi \gamma_0\, \tilde{t}_0^u + \frac{1}{2}\gamma_0^+ \gamma_0^- \partial_\xi \tilde{t}_0^u + \gamma_0^- \partial_\xi \mathscr{B}\mathscr{R}_0^u\right] = 0$$

Subtracting the two equations one derives:

$$\partial_\tau\left(\gamma_0^+ - \gamma_0^-\right) + \frac{1}{1+\partial_\xi X_0}\left\{-\frac{1}{2}\gamma_0 \partial_\xi \gamma_0 \tilde{t}_0^u - \frac{1}{4}\left[\gamma_0^2 - \left(\gamma_0^+ - \gamma_0^-\right)^2\right]\partial_\xi \tilde{t}_0^u + \left(\gamma_0^+ - \gamma_0^-\right)\partial_\xi \mathscr{B}\mathscr{R}_0^u\right\} = 0 \tag{7.4}$$

The two above equations (7.2) and (7.4) are two equations for γ_0 and $\left(\gamma_0^+ - \gamma_0^-\right)$. Given that the operators $\mathscr{BR}_0$ and $\mathscr{M}_0$ depend, beside γ_0, also on φ_0 and X_0, (7.2) and (7.4) have to be coupled with the equation ruling the motion of the curve φ_0:

$$\partial_\tau \varphi_0 = \mathscr{M}_0^v|_{Y=0} \tag{7.5}$$

and the equation ruling the dynamics of X_0, that can be derived directly from the definition of X_0 given in (3.17)

$$\partial_\tau X_0 = \mathscr{M}_0^u|_{Y=0} \tag{7.6}$$

The equations are initialized with

$$\gamma_0(\xi,\tau=0) = \gamma_0^{in}(\xi), \quad \left(\gamma_0^+ - \gamma_0^-\right)(\xi,\tau=0) = (\gamma_0^+ - \gamma_0^-)^{in}$$
$$\varphi_0(\xi,\tau=0) = \varphi^{in}(\xi), \quad X_0(\xi,\tau=0) = 0$$

where, obviously, $\gamma^{in}(\xi)$ and $(\gamma_0^+ - \gamma_0^-)^{in}$ can be computed from ω^{in}.

An immediate consequence of the estimate on the $\mathscr{BR}$ and $\mathscr{M}$ operators, and of the expression of $\mathscr{M}_0|_{Y=0}$ given in (7.3), is the following estimate on $\mathscr{M}_0|_{Y=0}$

Proposition 7.1. *Let $\gamma_0^{(i)} \in B_\rho^{1,\alpha}$, $\varphi_0^{(i)} \in B_\rho^{2,\alpha}$ and $X_0^{(i)} \in B_\rho^{2,\alpha}$ for $i=1,2$. Then $\mathscr{M}[\gamma_0^{(i)},\varphi_0^{(i)},X_0^{(i)}]|_{Y=0} \in B_\rho^{1,\alpha}$, and*

$$\|(\mathscr{M}[\gamma_0^{(1)},\varphi_0^{(1)},X_0^{(1)}] - \mathscr{M}[\gamma_0^{(2)},\varphi_0^{(2)},X_0^{(2)}])_{|Y=0}\|_{1,\rho}^{(\alpha)} \le$$
$$c\left(\|\delta\gamma_0\|_{1,\rho}^{(\alpha)} + \|\delta(\gamma_0^+ - \gamma_0^-)\|_{1,\rho}^{(\alpha)} + \|\delta\varphi_0\|_{2,\rho}^{(\alpha)} + \|\delta X_0\|_{2,\rho}^{(\alpha)}\right)$$

The system (7.2), (7.4), (7.5) and (7.6), with the use of the ACK Theorem, and with the help of Propositions 6.5 and 6.6, can be easily proven to admit a unique solution $\left[\gamma_0, \gamma_0^+ - \gamma_0^-, \varphi_0, X_0\right]$ in the appropriate analytic function space.

The following Proposition therefore holds:

Proposition 7.2. *Suppose $\omega^{in} \in B^{1,\alpha}_{\rho_0,\theta_0,\mu_0}$ and $\varphi^{in} \in B^{2,\alpha}_{\rho_0}$, with $\|\varphi^{in}\|_{2,\rho}^{(\alpha)} < 1/4$. Then there exist $\beta_0 > 0$ and $T_0 > 0$ such that the system* (7.2), (7.4), (7.5) *and* (7.6) *has a unique solution $\left[\gamma_0, \gamma_0^+ - \gamma_0^-, \varphi_0, X_0\right]$ with $\gamma_0 \in B^{1,\alpha}_{\rho_0,\beta_0,T_0}$, $\gamma_0^+ - \gamma_0^- \in B^{1,\alpha}_{\rho_0,\beta_0,T_0}$, $\varphi_0 \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}$ and $X_0 \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}$.*

The above Proposition states the *a priori* analyticity of the zero-th order total vorticity strength γ_0 and of the zero-th order approximation of the base curve φ_0.

This ends the proof of Theorem 5.2

Remark 7.3. In the equation for ω_0, that we shall analyze in Section 9, the functions that we have just constructed, γ_0, $\gamma_0^+ - \gamma_0^-$, φ_0 and X_0 appear through their first derivatives. One can see this, for example considering the convective term along the Y direction where is present the term $\partial_\xi \mathscr{M}_0^u$; this operator involves the BR operator (and therefore γ_0 and X_0) as well $\gamma_0^+ - \gamma_0^-$ and φ_0. Therefore we shall need higher regularity. This can be accomplished simply by shrinking the analyticity strip and by making shorter the time of the existence of the solution; therefore we can say that $\gamma_0 \in B^{2,\alpha}_{\tilde{\rho}_0,\tilde{\beta}_0,\tilde{T}_0}$ where $\tilde{\rho}_0 < \rho_0$, $\tilde{\beta}_0 > \beta_0$ and $\tilde{T}_0 < T_0$. To keep the notation simpler we rename the new widths of analyticity and the new time of existence to say that:

$$\gamma_0 \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}, \quad \gamma_0^+ - \gamma_0^- \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}, \quad X_0 \in B^{3,\alpha}_{\rho_0,\beta_0,T_0}, \quad \text{and} \quad \varphi_0 \in B^{3,\alpha}_{\rho_0,\beta_0,T_0}.$$

8 Potential estimates

In this Section we shall state some basic estimates that will be crucial in our development.

Define:

$$\begin{aligned} H_u(x,y) &= \frac{1}{8\pi^2}\frac{\sinh(2y)}{\sin^2(x)+\sinh^2(y)} \\ H_v(x,y) &= -\frac{1}{8\pi^2}\frac{\sin(2x)}{\sin^2(x)+\sinh^2(y)}. \end{aligned}$$

The velocity field (U,V) generated by a vorticity Ω has the following expression:

$$\begin{aligned} U(x,y,t) &= \int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}}\int_{-\infty}^{\infty} H_u(x-x',y-y')\Omega\left(x',y',t\right)dx'dy' \\ V(x,y,t) &= \int_{x-\pi/2}^{x+\pi/2}\int_{-\infty}^{\infty} H_v(x-x',y-y')\Omega\left(x',y',t\right)dx'dy' \end{aligned}$$

We want now to write the velocity in a frame adapted to the curve $\varphi(\xi)$. Recall that the comoving variables (ξ,Y) adapted to the curve φ_0 are defined as:

$$x = \xi + X(\xi,\tau) \qquad y = \varepsilon Y + \varphi(\xi)$$

where the Lagrangian factor X is given by (3.16).

In terms of the comoving variables ξ and Y the Biot-Savart kernel takes the form:

$$\mathscr{H}_u(\xi,\xi',Y-Y') = \frac{1}{8\pi^2}\frac{\sinh 2\left[\varepsilon(Y-Y')+\varphi(\xi+X)-\varphi(\xi'+X')\right]}{\sin^2(\xi-\xi'+X-X')+\sinh^2\left[\varepsilon(Y-Y')+\varphi(\xi+X)-\varphi(\xi'+X')\right]} \tag{8.1}$$

$$\mathscr{H}_v(\xi,\xi',Y-Y') = -\frac{1}{8\pi^2}\frac{\sin 2(\xi-\xi'+X-X')}{\sin^2(\xi-\xi'+X-X')+\sinh^2\left[\varepsilon(Y-Y')+\varphi(\xi+X)-\varphi(\xi'+X')\right]}, \tag{8.2}$$

so that the velocity field, in its dependence on the real variables ξ and Y is written as:

$$U(\xi,Y,\tau) = \int_{\xi-\frac{\pi}{2}}^{\xi+\frac{\pi}{2}}\int_{-\infty}^{\infty}\mathscr{H}_u(\xi,\xi',Y-Y')\Omega\left(\xi',Y',\tau\right)J(\xi',\tau)\,d\xi'dY' \tag{8.3}$$

$$V(\xi,Y,\tau) = \int_{\xi-\frac{\pi}{2}}^{\xi+\frac{\pi}{2}}\int_{-\infty}^{\infty}\mathscr{H}_v(\xi,\xi',Y-Y')\Omega\left(\xi',Y',\tau\right)J(\xi',\tau)\,d\xi'dY' \tag{8.4}$$

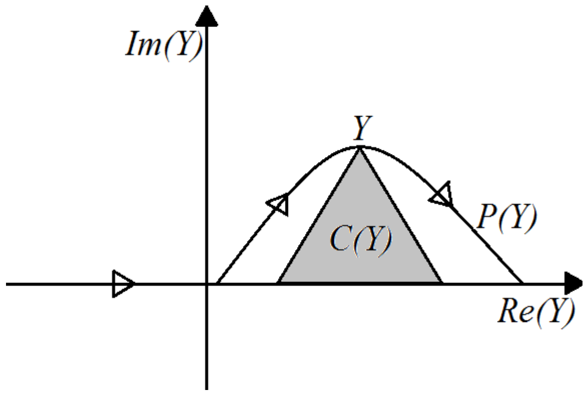

FIGURE 8.1. The path $P(Y)$

where:

$$J(\xi,\tau) = 1 + \partial_\xi X(\xi,\tau). \tag{8.5}$$

The above expressions assume ξ and Y real. Following [4] we now show how the above expressions can be extended analytically for $(\xi,Y) \in D(\rho)\times\Sigma(\theta)$. First we notice that, concerning the variable $\xi = \xi_R + i\xi_I$, by a contour deformation, one can take the path of integration parallel to the real axis and passing through ξ, i.e. $\xi' = \xi'_R + i\xi_I$; the expressions (8.3) and (8.4) for U and V can be written as:

$$U(\xi,Y,\tau) = \int_{\xi_R-\frac{\pi}{2}}^{\xi_R+\frac{\pi}{2}} \int_{P(Y)} \mathscr{H}_u(\xi,\xi',Y-Y')\Omega\left(\xi'_R + i\xi_I, Y', \tau\right) J\, d\xi'_R dY' \tag{8.6}$$

$$V(\xi,Y,\tau) = \int_{\xi_R-\frac{\pi}{2}}^{\xi_R+\frac{\pi}{2}} \int_{P(Y)} \mathscr{H}_v(\xi,\xi',Y-Y')\Omega\left(\xi'_R + i\xi_I, Y', \tau\right) J\, d\xi'_R dY' \tag{8.7}$$

where $P(Y)$ is a path of integration in $\Sigma(\theta)$. By contour deformation we can choose $P(Y)$ such that $P(Y) \subset (\Sigma(\theta) - C(Y)) \cup \{Y\}$ where the cone $C(Y)$ is defined in the following way:

$$C(Y) \equiv \{a + ib \in \Sigma(\theta) : |Y_R - a| < |Y_I - b|,\, |Y_I| > |b|,\, Y_I \cdot b > 0\}.$$

With the same procedure adopted in Proposition I.1 of [4] one can see that (8.6) and (8.7) define two analytic functions in $D(\rho)\times\Sigma(\theta)$. We now show how to conveniently parametrize $P(Y)$ choosing, as parameter, the variable Y'_R, see Fig.8.1.

In fact, take

$$P(Y) = Y'_R + ip(Y,Y'_R) \tag{8.8}$$

where $p(Y,Y'_R)$ is chosen to satisfy: 1) $p(Y,Y'_R) = 0$ when $|Y_R - Y'_R| > |Y_I| + \Delta$, with $\Delta > 0$; and 2) $p(Y,Y'_R) = Y'_R + iY_I$ in a sufficiently small neighborhood of Y. This choice of the path of integration will allow, when evaluating the difference between velocities calculated at two points (ξ,Y) and $(\bar{\xi},\bar{Y})$ (which is needed to bound the Hölder modulus of continuity), to choose the same path of integration when $Y_I=\bar{Y}_I$, or two paths of integration differing by a small circuit, of size $|Y_I-\bar{Y}_I|$ otherwise, see Appendix E.3.

The function $p(Y,Y_R')$ can be chosen to be C^2, with bounded norm, and with bounded derivatives. The bound can be chosen to be independent from Y. Using the path $P(Y)$ the velocity can therefore be written

$$U(\xi,Y,\tau) = \int_{\xi_R-\frac{\pi}{2}}^{\xi_R+\frac{\pi}{2}} \int_{P(Y)} \mathscr{H}_u(\xi,\xi',Y-Y_R'-ip(Y,Y_R'))\cdot \tag{8.9}$$
$$\Omega\left(\xi_R'+i\xi_I,Y_R'+ip(Y,Y_R'),\tau\right) J\left(1+i\partial_{Y_R'}p\right)d\xi_R'dY_R'$$

$$V(\xi,Y,\tau) = \int_{\xi_R-\frac{\pi}{2}}^{\xi_R+\frac{\pi}{2}} \int_{P(Y)} \mathscr{H}_v(\xi,\xi',Y-Y_R'-ip(Y,Y_R'))\cdot \tag{8.10}$$
$$\Omega\left(\xi_R'+i\xi_I,Y_R'+ip(Y,Y_R'),\tau\right) J\left(1+i\partial_{Y_R'}p\right)d\xi_R'dY_R'$$

8.1 The basic potential estimates

The following Proposition gives an estimate of the $\|\cdot\|^{(\alpha)}$-norm of the velocity (U,V) in terms of the $\|\cdot\|^{(\alpha)}$-norm of the rescaled vorticity.

Proposition 8.1. *Let (U,V) be expressed by* (8.9) *and* (8.10)*, where $\Omega\in B^{\alpha}_{\rho,\theta,\mu}$, $\varphi\in B^{1,\alpha}_{\rho}$ and $X\in B^{1,\alpha}_{\rho}$, with $\|\varphi\|^{(\alpha)}_{1,\rho}\le C_\varphi$, and with $\|X\|^{(\alpha)}_{1,\rho}\le C_X$, being C_φ,C_X two constants sufficiently small (for example smaller than $1/4$). Then $U,V\in B^{\alpha}_{\rho,\theta}$, and the following estimates hold:*

$$\|U\|^{(\alpha)}_{\rho,\theta}\le c\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu} \tag{8.11}$$

$$\|V\|^{(\alpha)}_{\rho,\theta}\le c\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu} \tag{8.12}$$

The proof of the above Proposition has been postponed to Appendix E.

We remark that the above Proposition would be valid also in the usual Hölder spaces, as it is apparent form the proof where it is clear that analyticity is only a complication.

The velocity (U,V), as given by (8.3) and (8.4) depends also on the curve φ and on the Lagrangian factor X, as it is apparent from the expressions (8.1) and (8.2) of the Biot-Savart kernel. Denoting with $(U[\Omega,\varphi,X],V[\Omega,\varphi,X])$ the velocity field in its dependence from $[\Omega,\varphi,X]$ we now give the Lipschitz estimate.

Proposition 8.2. *Let $[\Omega^{(i)},\varphi^{(i)},X^{(i)}]\in B^{\alpha}_{\rho,\theta,\mu}\times B^{1,\alpha}_{\rho}\times B^{1,\alpha}_{\rho}$ for $i=1,2$. Then the following estimate holds:*

$$\|U[\Omega^{(1)},\varphi^{(1)},X^{(1)}]-U[\Omega^{(2)},\varphi^{(2)},X^{(2)}]\|^{(\alpha)}_{\rho,\theta}\le c\left[\|\delta\Omega\|^{(\alpha)}_{\rho,\theta,\mu}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}+\|\delta X\|^{(\alpha)}_{1,\rho}\right]$$

$$\|V[\Omega^{(1)},\varphi^{(1)},X^{(1)}]-V[\Omega^{(2)},\varphi^{(2)},X^{(2)}]\|^{(\alpha)}_{\rho,\theta}\le c\left[\|\delta\Omega\|^{(\alpha)}_{\rho,\theta,\mu}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}+\|\delta X\|^{(\alpha)}_{1,\rho}\right]$$

The proof of the above Proposition is postponed to Appendix F.

The two above Propositions will be applied in Section 9 to the velocity field (u_0, v_0), (u_1, v_1) generated by the vorticity ω_0 and ω_1.

8.2 Approximation of the velocity field inside the layer.

An important ingredient in the construction of the solution ω_1 are the next results saying that the term $\mathscr{R}$ is $O(\varepsilon)$ inside the layer and that outside the layer the growth in Y is linear in Y so that it will be easily tamed by the exponential decay of the vorticity. We stress that for these results to hold one must have sufficient regularity for the vorticity, the curve and the lagrangian factor.

Proposition 8.3. *Let $\Omega \in B^2_{\rho,\theta,\mu}$, $\psi \in B^3_\rho$ and $X \in B^3_\rho$. Then the following estimate holds:*

$$\|\mathscr{R}(\Omega, \psi, X)(\cdot, Y)\|^{(\alpha)}_\rho \le c\varepsilon(1+Y)$$

where the value of the constant c depends only on $\|\Omega\|_{2,\rho,\theta,\mu}$, $\|\psi\|_{3,\rho}$ and $\|X\|_{3,\rho}$.

If one looks at the expression (3.11) one can see how the size $O(\varepsilon)$ derives from the difference between $1/\mathscr{K}^\varepsilon_\psi$ and $1/\mathscr{K}^0_\psi$: however the difference of the two kernels gives rise to a stronger singularity that must be compensated by the fact that $\mathscr{T}(\xi, \xi', Y') = 0$ at $\xi' = \xi$. To exploit this fact, however, one must have higher regularity. The details can be found in Appendix G.1.

We now consider the more specific case when $\Omega = \omega_0$, $\psi = \varphi = \varphi_0 + \varepsilon\varphi_1$ and $X = X_0 + \varepsilon X_1$.

Proposition 8.4. *Let $\omega_0 \in B^{1,\alpha}_{\rho,\theta,\mu}$, $\varphi_0, X_0 \in B^{2,\alpha}_\rho$, while $\varphi_1, X_1 \in B^{1,\alpha}_\rho$. Then the following estimate holds:*

$$\|\mathscr{R}(\omega_0, \varphi_0 + \varepsilon\varphi_1, X_0 + \varepsilon X_1)(\cdot, Y)\|^{(\alpha)}_\rho \le c\varepsilon(1+Y)$$

where the constant c depends on the norms of ω_0 in $B^{1,\alpha}_{\rho,\theta,\mu}$, of φ_0, X_0 in $B^{2,\alpha}_\rho$, and of φ_1, X_1 in $B^{1,\alpha}_\rho$.

The idea behind the proof of the above Proposition is that, shrinking the strip of analyticity of ω_0, φ_0 and X_0, one can get the regularity required by Proposition 8.3. On the other hand, the higher order correction deriving from φ_1 and X_1, it is already $O(\varepsilon)$, and one does not need to use the smallness of the difference between $1/\mathscr{K}^\varepsilon_\psi$ and $1/\mathscr{K}^0_\psi$, so that higher regularity for φ_1 and X_1 is not necessary. Some more details is given in Appendix G.2.

We now pass to proving the Lipschitz property of the operator $\mathscr{R}$. We denote:

$$\delta\mathscr{R} = \mathscr{R}(\Omega^{(1)}, \varphi_0 + \varepsilon{\varphi_1}^{(1)}, X_0 + \varepsilon X_1^{(1)}) - \mathscr{R}(\Omega^{(2)}, \varphi_0 + \varepsilon{\varphi_1}^{(2)}, X_0 + \varepsilon X_1^{(2)})$$

$$\delta\omega_0 = \omega_0^{(1)} - \omega_0^{(2)}, \quad \delta\varphi_1 = {\varphi_1}^{(1)} - {\varphi_1}^{(2)}, \quad \delta X_1 = X_1^{(1)} - X_1^{(2)}$$

Notice how, in computing the variation of $\mathscr{R}$, we are holding fixed the leading order approximation of the base curve φ_0 and of the lagrangian factor X_0. This is appropriate because the Proposition below will be used in constructing (beside ω_0 and

ω_1) the corrections φ_1 and X_1, given that the leading order φ_0 and X_0 have already been constructed in Section 7, Proposition 7.2, and therefore will be considered as given.

Proposition 8.5. *Suppose* $\omega_0^{(i)} \in B^{1,\alpha}_{\rho,\theta,\mu}$, φ_0 *and* $X_0 \in B^{2,\alpha}_{\rho}$, *while* ${\varphi_1}^{(i)}$ *and* $X^{(i)} \in B^{1,\alpha}_{\rho}$. *Then the following estimate holds:*

$$\|\delta\mathscr{R}\|_{\rho}^{(\alpha)} \leq c\varepsilon\left[(1+Y)\|\delta\omega_0\|_{1,\rho,\theta,\mu} + \|\delta\varphi_1\|_{1,\rho}^{(\alpha)} + \|\delta X_1\|_{1,\rho}^{(\alpha)}\right]$$

where the constant c depends on the norms of φ_0 *and* X_0 *in* $B^{2,\alpha}_{\rho}$.

The results in the above Propositions will be fundamental in compensating the ε^{-1} terms appearing in the equation for ω_1; in particular in the terms G_2 and G_4, defined in Section 9.2 below. The meaning of the above estimate is that the velocity inside the layer is well approximated by the operator $\mathscr{M}$ and, given that the $\mathscr{M} - \mathscr{M}_0 = O(\varepsilon)$, by $\mathscr{M}_0$ also. Looking at the expression (3.14) for $\mathscr{M}_0$ it means that:

$$u - iv =$$
$$\mathscr{BR}(\omega_0, \varphi_0, X_0) + \frac{1}{2}\left[\int_Y^{\infty} \omega_0(\xi, Y')dY' - \int_{-\infty}^{Y} \omega_0(\xi, Y')dY'\right]\tilde{t}_0^* + O(\varepsilon) \qquad \forall\, Y < C$$

C being a constant independent from ε. The above formula means that inside the layer the dominant contribution to the velocity is given by the Birkhoff-Rott motion of the curve plus a local (in the sense that involves only the vorticity at the point ξ) integral operator.

From Proposition 8.3, however, one immediately sees that the discrepancy grows with Y so that outside the layer the velocity field is not well approximated by $\mathscr{M}$: a different approximation is in fact valid far away from the layer.

8.3 The far field approximation

Outside the layer the velocity field converges to the velocity field generated by the vorticity concentrated on the curve; i.e. by a vortex sheet of total vorticity strength γ where $\gamma(\xi) = \int_{-\infty}^{\infty} \Omega(\xi, Y')dY'$. We therefore define the far field velocity as:

$$\begin{aligned} u^f + iv^f \equiv \mathscr{F}[\Omega, \psi] \equiv \\ \frac{1}{2\pi i}\sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\Omega(\xi', Y')}{\xi - \xi' + X - X' + i[\psi - \psi' + \varepsilon Y]} d\xi' dY' = \\ \frac{1}{2\pi i}\sum_n \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\gamma(\xi')}{\xi - \xi' + X - X' + i[\psi - \psi' + \varepsilon Y]} d\xi' \end{aligned}$$

(8.13)

where, as usual, $X = X(\xi)$, $X' = X(\xi')$, $\psi = \psi(\xi + X)$ and $\psi' = \psi(\xi' + X')$. We can now state the following Proposition:

Proposition 8.6. *Let* $\Omega \in B^2_{\rho,\theta,\mu}$, $\psi, X \in B^{2,\alpha}_\rho$ *with* $\|\psi\|^{(\alpha)}_{2,\rho} < 1/4$, $\|X\|^{(\alpha)}_{2,\rho} < 1/4$. *Moreover let* $|Y| > \varepsilon^{-1}$. *Then:*

$$\|u + iv - (u^f + iv^f)\|^{(\alpha)}_{1,\rho} \le c\left[\left(\frac{1}{|Y|} + f(Y)\right) + O(e^{-\mu/(3\varepsilon)})\right]$$

where $f(Y) \ge 0$ *has a rate of decay in* Y *rapid enough to make it integrable in* Y.

We recall that τ is present in the stretching factor $X = \int_0^\tau u(\xi, Y = 0, \tau')d\tau'$. From the proof one can see that the decay of $f(Y)$ is $O(Y^{-2})$. We also remark that the above estimate would also hold for $Y > \varepsilon^{-\kappa}$ with $0 < \kappa < 1$, the only difference being the fact that the exponentially small term would be $O(e^{-\mu/(2\varepsilon^\kappa)})$. The proofs of the above two Propositions are reported in Appendix G.

9 Construction of the vorticity distribution: proof of Theorem 5.4

This Section is devoted to the construction of the leading order vorticity ω_0, of the corrections ω_1 and φ_1 and X_1, therefore, to the proof of Theorem 5.4.

The equations are (3.18), (3.21), (3.22) and (3.24). We recast these equation in a form suitable for application of the ACK Theorem in the integrated in time form. We define the vector:

$$\Psi \equiv [\omega_0, \omega_1, \varphi_1, X_1]^T$$

so that equations (3.18), (3.21) and (3.22) can be written as

$$\Psi + K(\Psi, \tau) = 0 \tag{9.1}$$

where $K \equiv [K_{\omega_0}, K_{\omega_1}, K_{\varphi_1}, K_{X_1}]^T$. The operators K_{ω_0}, K_{ω_1}, K_{φ_1} and K_{X_1} express the righthandsides of equations (3.18), (3.21), (3.22) and (3.24), integrated in time; their explicit expressions is given below in (9.3), (9.4), (9.6) and (9.7). To simplify the notation we introduce the following definition

Definition 9.1. Let $\mathfrak{B}^\alpha_{\rho,\theta,\mu} \equiv B^{1,\alpha}_{\rho,\theta,\mu} \times B^\alpha_{\rho,\theta,\mu/2} \times B^{1,\alpha}_\rho \times B^{1,\alpha}_\rho$. Then for $\Psi \in \mathfrak{B}^\alpha_{\rho,\theta,\mu}$ we define the following norm:

$$|||\Psi|||^{(\alpha)}_{\rho,\theta,\mu} = \|\omega_0\|^{(\alpha)}_{1,\rho,\theta,\mu} + \|\omega_1\|^{(\alpha)}_{\rho,\theta,\mu} + \|\varphi_1\|^{(\alpha)}_{1,\frac{\rho}{2}} + \|X_1\|^{(\alpha)}_{1,\rho}.$$

Analogously, for $t \in [0,T]$ and β, ρ, θ, μ such that $2\beta T < \min(\rho, \theta, \mu)$ we define $\mathfrak{B}^\alpha_{\rho,\theta,\mu,\beta,T} \equiv B^{1,\alpha}_{\rho,\theta,\mu,\beta,T} \times B^\alpha_{\rho,\theta,\mu/2,\beta,T} \times B^\alpha_{1,\rho,\beta,T} \times B^\alpha_{1,\rho,\beta,T}$ and:

$$|||\Psi|||^{(\alpha)}_{\rho,\theta,\mu,\beta,T} = \|\omega_0\|^{(\alpha)}_{1,\rho,\theta,\mu,\beta,T} + \|\omega_1\|^{(\alpha)}_{\rho,\theta,\frac{\mu}{2},\beta,T} + \|\varphi_1\|^{(\alpha)}_{1,\rho,\beta,T} + \|X_1\|^{(\alpha)}_{1,\rho,\beta,T}.$$

Notice how, for the higher order vorticity ω_0, we are requiring more regularity, with respect to the variables x, Y, than for ω_1, and a faster exponential decay when $|Y| \to \infty$. Analogously for φ_1 and for X_1 we require less regularity than what we obtained for the leading order φ_0 and X_0.

The proof of Theorem 5.4 is based on the following Proposition:

Proposition 9.2. *Suppose that* $\Psi^{(1)}, \Psi^{(2)} \in \mathfrak{B}^{\alpha}_{\rho,\theta,\mu,\beta,T}$ *with:*

$$|||\Psi^{(i)}|||^{(\alpha)}_{\rho,\theta,\mu,\beta,T} \leq R\,.$$

Moreover let $\tau < T$ *and suppose that for all* $0 < s < \tau$ *one has that* $0 < \rho' < \rho(s) \leq \rho - \beta s$, $0 < \theta' < \theta(s) \leq \theta - \beta s$ *and* $0 < \mu' < \mu(s) \leq \mu - \beta s$*. Then the following estimate holds:*

$$|||K(\Psi^{(1)},\tau) - K(\Psi^{(2)},\tau)|||^{(\alpha)}_{\rho',\theta',\mu'} \leq$$

$$c\int_0^\tau \left[\frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho(s),\theta',\mu'}}{\rho(s)-\rho'} + \frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho',\theta(s),\mu'}}{\theta(s)-\theta'} + \frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho',\theta',\mu(s)}}{\mu(s)-\mu'} \right] ds\,. \tag{9.2}$$

To prove the above quasi-contractiveness property of the operator $K \equiv [K_{\omega_0}, K_{\omega_1}, K_{\varphi_1}, K_{X_1}]^T$, we pass to analyze each component.

9.1 Contractiveness of K_{ω_0}

The operator K_{ω_0} has the following expression

$$K_{\omega_0}(\Psi,\tau) \equiv \sum_{j=1}^{2} \int_0^\tau F_j(\omega_0, s)ds \tag{9.3}$$

with

$$F_1(\omega_0,\tau) = \left[\mathscr{M}^u_0 - \mathscr{M}^u_{0|Y=0}\right] \frac{\partial_\xi \omega_0}{1+\partial_\xi X_0}$$

$$F_2(\omega_0,\tau) = -\int_0^Y \partial_\xi \mathscr{M}^u_0 dY' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0}.$$

Notice how the operator K_{ω_0} depends explicitly only on ω_0. The fact that K_{ω_0} satisfies the estimate (9.2) is an obvious consequence of the following Proposition giving the quasi-contractiveness of the operators F_i.

Proposition 9.3. *Suppose* $\omega^{in} \in B^{1,\alpha}_{\rho_0,\theta_0,\mu_0}$ *and* $\varphi^{in} \in B^{2,\alpha}_{\rho_0}$*, with* $\|\varphi^{in}\|^{\alpha}_{2,\rho_0} < 1/4$*. Let* $\rho' < \rho < \rho_0$*,* $\theta' < \theta < \theta_0$*,* $\mu' < \mu < \mu_0$*; then the operators* F_i *satisfy the following estimate:*

$$\left\|F_i(\omega^{(1)}) - F_i(\omega^{(2)})\right\|^{(\alpha)}_{1,\rho',\theta',\mu'} \leq$$

$$c\left[\frac{\|\omega^{(1)} - \omega^{(2)}\|^{(\alpha)}_{1,\rho,\theta',\mu'}}{\rho-\rho'} + \frac{\|\omega^{(1)} - \omega^{(2)}\|^{(\alpha)}_{1,\rho',\theta,\mu'}}{\theta-\theta'} + \frac{\|\omega^{(1)} - \omega^{(2)}\|^{(\alpha)}_{1,\rho',\theta',\mu}}{\mu-\mu'} \right] \qquad i = 1,2$$

for any $\omega^{(1)}, \omega^{(2)} \in B^{1,\alpha}_{\rho,\theta,\mu}$.

The proof can be found in Appendix H.

9.2 Contractiveness of K_{ω_1}

The operator K_{ω_1} has the following expression:

$$K_{\omega_1}(\Psi,\tau) \equiv \sum_{j=1}^{7} G_j(\omega_0,\omega_1,\varphi_1,\tau) \tag{9.4}$$

with

$$G_1 \equiv \frac{1}{\varepsilon}\int_0^\tau \left\{ \left[\frac{\mathscr{M}^u - \mathscr{M}^u_{Y=0}}{1+\partial_\xi X} - \frac{\mathscr{M}^u_0 - \mathscr{M}^u_{0|Y=0}}{1+\partial_\xi X_0} \right] \partial_\xi \omega_0 \right\} ds$$

$$G_2 \equiv \frac{1}{\varepsilon}\int_0^\tau \left\{ \frac{\mathscr{R}^u + \varepsilon u_1 - (\mathscr{R}^u + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X} \partial_\xi \omega_0 \right\} ds$$

$$G_3 \equiv -\frac{1}{\varepsilon}\int_0^\tau \left\{ \left[\frac{1}{1+\partial_\xi X}\int_0^Y \partial_\xi \mathscr{M}^u dY' - \frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}^u_0 dY' \right] \partial_Y \omega_0 \right\} ds$$

$$G_4 \equiv -\frac{1}{\varepsilon}\int_0^\tau \left\{ \frac{1}{1+\partial_\xi X} \left[\int_0^Y \partial_\xi (\mathscr{R}^u + \varepsilon u_1) dY' \right] \partial_Y \omega_0 \right\} ds$$

$$G_5 \equiv \int_0^\tau \left\{ \frac{u_0 + \varepsilon u_1 - (u_0 + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X} \partial_\xi \omega_1 \right\} ds$$

$$G_6 \equiv -\int_0^\tau \left\{ \frac{1}{1+\partial_\xi X} \left[\int_0^Y \partial_\xi u_0 dY' \right] \partial_Y \omega_1 \right\} ds$$

$$G_7 \equiv \int_0^\tau \left\{ \left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0} \right] \partial_Y \omega_1 \right\} ds$$

The following Proposition gives the quasi-contractiveness of the operators G_i.

Proposition 9.4. *Suppose that $\Psi^{(i)} \in \mathfrak{B}^\alpha_{\rho,\theta,\mu,\beta,T}$ with $\|\Psi^{(i)}\|^{(\alpha)}_{\rho,\theta,\mu,\beta,T} < R$. Moreover let $\tau < T$ and suppose that for all $0 < s < \tau$ one has that $0 < \rho' < \rho(s) \le \rho - \beta s$, $0 < \theta' < \theta(s) \le \theta - \beta s$ and $0 < \mu' < \mu(s) \le \mu - \beta s$. Then, for $j = 1,\ldots,7$ the following estimates hold:*

$$\|G_j(\Psi^{(1)},\tau) - G_j(\Psi^{(2)},\tau)\|^{(\alpha)}_{\rho',\theta',\mu'/2} \le$$
$$c\int_0^\tau \left[\frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho(s),\theta',\mu'}}{\rho(s)-\rho'} + \frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho',\theta(s),\mu'}}{\theta(s)-\theta'} + \frac{|||\Psi^{(1)} - \Psi^{(2)}|||^{(\alpha)}_{\rho',\theta',\mu(s)}}{\mu(s)-\mu'} \right] ds. \tag{9.5}$$

The fact that G_1 and G_3 are $O(1)$ is a consequence of the fact that $\mathscr{M} - \mathscr{M}_0 = O(\varepsilon)$, $1/(1+\partial_\xi X) - 1/(1+\partial_\xi X_0) = O(\varepsilon)$. The operators G_2 and G_4 are $O(1)$ because $\mathscr{R}$, inside the layer, is $O(\varepsilon)$, while the linear growth outside the layer (see Proposition 8.3) is bounded by the exponential decay of ω_0.

The contractiveness of the operators G_5, G_6 and G_7 is based on the use of the Cauchy estimate, on the higher regularity of ω_0, and on the potential estimate of u_0 and u_1 in terms of the vorticity ω_0 and ω_1 respectively.

The details can be found in the Appendix I.

9.3 Contractiveness of K_{φ_1} and K_{X_1}

The operators K_{φ_1} and K_{X_1} have the following expression:

$$K_{\varphi_1} \equiv \frac{1}{\varepsilon}\int_0^\tau [\mathscr{M}^v - \mathscr{M}_0^v + \mathscr{R}^v + \varepsilon v_1]_{Y=0}\, ds \tag{9.6}$$

$$K_{X_1} \equiv \frac{1}{\varepsilon}\int_0^\tau [\mathscr{M}^u - \mathscr{M}_0^u + \mathscr{R}^u + \varepsilon u_1]_{Y=0}\, ds \tag{9.7}$$

The contractiveness of these operators is an immediate consequence of Lemma I.2 and of Proposition 8.5 stating, besides the Lipschitz property, also the smallness needed to compensate the factor ε^{-1}.

This completes the proof of Proposition 9.2. One can therefore apply the ACK Theorem to the system (9.1) and construct the solution $[\omega_0, \omega_1, \varphi_1, X_1]^T$. Theorem 5.4 is therefore proved.

10 Convergence to Birkhoff-Rott: proof of Theorem 5.5

In this Section we shall give a proof of Theorem 5.5, i.e. we shall prove that the solution of BR equation, that we shall denote with (γ_s, φ_s), gives a good approximation of the dynamics of the vortex layer: namely the BR equation, up to $O(\varepsilon)$ terms, describes correctly both the vorticity intensity of the layer γ (i.e. the vorticity integrated across the layer), and the curve φ around which the vorticity is distributed.

To prove Theorem 5.5, given that $\gamma = \gamma_0 + \varepsilon\gamma_1$ and $\varphi = \varphi_0 + \varepsilon\varphi_1$, where, after shrinking the strip of analyticity,

$$\|\gamma_1\|^{(\alpha)}_{1,\rho_1,\beta_1,T_1} < c, \qquad \|\varphi_1\|^{(\alpha)}_{2,\rho_1,\beta_1,T_1} < c$$

it will be enough to prove that:

$$\|\gamma_s - \gamma_0\|^{(\alpha)}_{1,\rho,\beta,T} < c\varepsilon, \qquad \|\varphi_s - \varphi_0\|^{(\alpha)}_{2,\rho,\beta,T} < c\varepsilon\,.$$

Our proof will be based on the fact that the equations ruling the dynamics of γ_0 and φ_0, i.e. equations (7.2) and (7.5) respectively, when written in the Eulerian frame, coincide, up to $O(\varepsilon)$ corrections, with the BR equations (5.3)-(5.4). In fact the equations for γ_0 and φ_0 can be written as:

$$\partial_t\gamma_0 + \partial_x(\gamma_0 BR_0^u) = E_1 \tag{10.1}$$

$$\partial_t\varphi_0 + BR_0^u\partial_x\varphi_0 - BR_0^v = E_2 \tag{10.2}$$

where the terms $(E_1, E_2) = O(\varepsilon)$ have the following explicit expression:

$$E_1 = -[u^\varphi - \sigma M_0^u]_{Y=0}\,\partial_x\gamma_0 + [1-\sigma]\,\partial_x(\gamma_0 BR_0^u) \tag{10.3}$$

$$E_2 = -\{[u^\varphi - BR_0^u]\,\partial_x\varphi_0 + BR_0^v - M_0^v\}_{Y=0} \tag{10.4}$$

being the definition of σ given in (J.2). The details of the derivation of the above system can be found in Appendix J. Notice how the system ruling (γ_0, φ_0) has the same structure of the Birkhoff-Rott equations:

$$\partial_t \gamma_s + \partial_x (\gamma_s BR^u[\gamma_s, \varphi_s]) = 0 \tag{10.5}$$

$$\partial_t \varphi_s + BR^u[\gamma_s, \varphi_s]\partial_x \varphi_s - BR^v[\gamma_s, \varphi_s] = 0 \tag{10.6}$$

the only difference being: the fact that (γ_0, φ_0) are transported by the BR operator computed using γ_0 and φ_0 (we recall the notations $BR_0 = BR[\gamma_0, \varphi_0]$ while $BR = BR[\gamma_0, \varphi]$); and the presence of the terns E_i which however are $O(\varepsilon)$. In fact the following Lemma holds:

Lemma 10.1. *Suppose to hold the hypotheses of Theorem 5.5. Then $E = (E_1, E_2)$ satisfies the following bound:*

$$||E||^{(\alpha)}_{1,\rho,\beta,T'} < c\varepsilon\ .$$

The proof can be found in Appendix J.

Given that (γ_0, φ_0) and (γ^s, φ^s) satisfy systems with the same structure, the only difference being the presence of the $O(\varepsilon)$ term E, and that they have the same initial condition, to see that they remain at an $O(\varepsilon)$ distance is straightforward. The sketch of the prof goes as follows. Define $e = (\gamma_0 - \gamma_s, \varphi_0 - \varphi_s)^T$. Taking the difference between (10.1) and (10.5), and between (10.2) and (10.6), one gets:

$$\partial_t e + H(e,t) = E \tag{10.7}$$

where

$$H = \begin{pmatrix} \partial_x (e_1 BR^u[\gamma_0, \varphi_0]) + \partial_x(\gamma_s W^u) \\ W^u \partial_x \varphi_0 + BR^u[\gamma_s, \varphi_s]\partial_x e_2 - W^v \end{pmatrix} \tag{10.8}$$

being:

$$W = (W^u, W^v)^T = BR[\gamma_0, \varphi_0] - BR[\gamma_s, \varphi_s]\,.$$

The forcing term is $E = (E_1, E_2)^T$.

Equation (10.7) governs the discrepancy between between the BR solution and the vortex-layer solution. The initial condition is

$$e(x, t = 0) = 0\,.$$

Define:

$$|||e|||^{(\alpha)}_{\rho,\beta,T} = \|e_1\|^{(\alpha)}_{1,\rho,\beta,T} + \|e_2\|^{(\alpha)}_{2,\rho,\beta,T}$$

Given that the forcing term satisfies:

$$||E||^{(\alpha)}_{1,\rho,\beta,T'} < c\varepsilon\ .$$

and the fact that (see the estimate reported in Proposition 6.5 or [49, 12, 13]):

$$\|W\|^{(\alpha)}_{1,\rho,\beta,T} < c|||e|||^{(\alpha)}_{\rho,\beta,T}$$

one can easily prove, through the ACK Theorem, that (10.7) admits a unique solution in $B^{1,\alpha}_{\rho\beta,T} \times B^{2,\alpha}_{\rho\beta,T}$ and that

$$|||e|||^{(\alpha)}_{\rho,\beta,T} < c\varepsilon\,.$$

The proof of Theorem 5.5 is therefore achieved.

11 Conclusions

In fluid dynamics, configurations where vorticity is very intense and highly concentrated are ubiquitous and very relevant. They typically arise when a high Reynolds number flow, interacting with a solid boundary, generates huge amount of vorticity that eventually detaches from the boundary in the form of strong vortex cores or layers of intense vorticity. From the mathematical point of view the analysis of these configurations is challenging: the presence of strong, potentially unbounded gradients, makes classical estimates useless.

A solution that has been adopted was to tackle directly the singular data, assuming the vorticity to be concentrated on zero measure sets. This approach, for the case of a vortex sheet and with a combination of heuristics and classical potential theory, led to the derivation of the Birkhoff-Rott equation, describing the evolution of the shape of the sheet and of the vorticity intensity. More advanced mathematical arguments, pioneered by DiPerna and Majda in late eighties, led to Delort's result, establishing the existence of a solution for 2D Euler (or Navier-Stokes [37]) equations with a distinguished sign Radon measure as vorticity initial datum. The question of uniqueness was left unsolved and no indication on the possible relationship with the BR equation was given.

In this paper we have considered the 2D Euler equations, with a sequence of data consisting of vorticity layers centered around a curve; the main hypotheses are the analytic regularity of the initial vorticity distribution and the exponential decay away from the curve. The size of the layer is $O(\varepsilon)$ while the vorticity intensity is $O(\varepsilon^{-1})$. We have shown that the Euler equations, written in the comoving frame, admit a unique solution and that this solution remains exponentially decaying away from a curve that moves, to the leading order, as predicted by the BR equation; therefore giving a rigorous justification of the BR equation. In this sense analyticity seems to be unavoidable, because the BR equations are subject to ill-posedness due to Kelvin-Helmholtz instability. Alternatively, one can consider our result as a partial answer to the problems left unsolved by Delort's theory, showing uniqueness of the solution and persistence of the layer structure when the initial datum has the kind of analytic regularity we have considered, even without assuming the vorticity to be of distinguished sign.

Several problems suggest themselves as consequences of our analysis, and here we mention three of them. First, the Theorems proved in this paper are short time, meaning that after a certain time, that however does not depend on the size of the layer, our equations may develop a singularity and the layer structure is lost. It would be interesting to know whether the break-up of our construction is due to the ejection of vorticity from within the layer into the outer (i.e away from the curve) flow, maybe due to the unbounded growth of the normal velocity; or it is simply due to the development of BR curvature singularity of the curve supporting the vorticity, a possibility that however in some cases [20], is ruled out; or to the development of some other kind of instability. Second, in the present paper we have neglected the role played by the viscosity, and therefore the justification of BR equation as a good zero-viscosity approximation of the Navier-Stokes equations remains open. Usually viscosity is a regularizing phenomenon but there are cases where its presence can trigger instability: a noticeable example is encountered in boundary layer theory, where inviscid Prandtl equations are well posed [26], while the full Prandtl system is probably ill-posed [24]. Finally it is worth mentioning that the present analysis is restricted to the 2D case, where the existence of the solutions is not an issue, due to the global existence results. In 3D, however, initializing Euler equations with data of the kind we have considered here, would lead to solutions existing for a time that would shrink to zero with the size of the layer. Extending our result to the 3D case would establish the existence of the layer solutions for the Euler equations and therefore would be an interesting achievement.

Appendix A: Derivation of the Euler-VL equations in the comoving frame

Consider the change of variables $(x,y,t) \to (\xi, Y, \tau)$ defined as:

(A.1) $$x = \xi + X(\xi,\tau), \qquad y = \varepsilon Y + \psi(\xi,\tau), \qquad \tau = t$$

where

(A.2) $$X(\xi,\tau) \equiv \int_0^\tau u(x(\xi,\tau'), Y=0, \tau')d\tau'$$

We make the following Remarks:

Remark A.1. When ∂_τ acts on the (vector or scalar) quantity $\mathscr{F}$ it holds:

$$\partial_t \mathscr{F} = \partial_\tau \mathscr{F} - \frac{\partial r}{\partial \tau} \cdot \nabla \mathscr{F}$$

To prove the above Remark we denote the old variable (x,y) with r, and the new variable with r'. Therefore:

$$\partial_\tau \mathscr{F}(r(r',\tau), t(\tau)) = \frac{\partial r}{\partial \tau} \cdot \nabla \mathscr{F} + \frac{\partial t}{\partial \tau} \partial_t \mathscr{F} = \frac{\partial r}{\partial \tau} \cdot \nabla \mathscr{F} + \partial_t \mathscr{F}$$

from which the Remark follows.

Remark A.2. One has that

$$\frac{\partial r}{\partial \tau} = u^\Psi \hat{x} + v^\Psi \hat{y}$$

where $\hat{x}$ and $\hat{y}$ are the cartesian base vector. In fact by definition one has that $r = x\hat{x} + y\hat{y}$, and to prove the above Remark one has just to consider that $\partial_\tau x = u^\Psi$ because of (A.1) and that $\partial_\tau y = v^\Psi$ because of (A.1) and of the fact that, having imposed the curve to be transported by the flow, one has that $\partial_\tau \psi(\xi,\tau) = v^\Psi$.

Remark A.3. One has that, for the scalar quantity $\mathscr{F}$:

$$\nabla\mathscr{F} = \hat{x}\left[1+X_\xi(\xi,\tau)\right]^{-1}\partial_\xi\mathscr{F} + \frac{1}{\varepsilon}\left(-\left[1+X_\xi(\xi,\tau)\right]^{-1}\partial_\xi\psi\hat{x}+\hat{y}\right)\partial_Y\mathscr{F}$$

To prove the Remark we use the following expression for the gradient (see [50] pag.630, Eq.(93)):

$$\nabla\mathscr{F} = a^1\partial_\xi\mathscr{F} + a^2\partial_Y\mathscr{F}$$

where a^i can be found using $a^i \cdot a_j = \delta^i_j$ (see [50], pag.624 Eq.(67)), being:

$$a_1 = \hat{x}\partial_\xi x + \hat{y}\partial_\xi y, \qquad a_2 = \hat{x}\partial_Y x + \hat{y}\partial_Y y\,,$$

see [50] pag.628 Eq.(82). From (A.1) one gets that $\partial_\xi x = 1 + \partial_\xi X$, $\partial_\xi y = \partial_\xi \psi$, and one immediately finds:

$$a^1 = \hat{x}\left[1+X_\xi(\xi,\tau)\right]^{-1}\,, \qquad a^2 = \varepsilon^{-1}\left(-\partial_\xi\psi\left[1+X_\xi(\xi,\tau)\right]^{-1}\hat{x}+\hat{y}\right)$$

where we have introduced the notation:

$$\text{(A.3)} \quad X(\xi,\tau) = \int_0^\tau u(\xi,Y=0,\tau')d\tau' \qquad X_\xi(\xi,\tau) = \int_0^\tau \partial_\xi u(\xi,Y=0,\tau')d\tau'$$

Therefore the Remark follows.

Using all the above three Remarks, one can finally write:

$$\text{(A.4)} \quad \partial_t\mathscr{F} = \partial_\tau\mathscr{F} - \frac{u^\Psi}{\left[1+X_\xi(\xi,\tau)\right]}\partial_\xi\mathscr{F} - \frac{1}{\varepsilon}\left(v^\Psi - \frac{u^\Psi}{\left[1+X_\xi(\xi,\tau)\right]}\partial_\xi\psi\right)\partial_Y\mathscr{F}\,.$$

Using the same expression for the gradient given in Remark A.3 one can also calculate

$$\text{(A.5)} \qquad u\cdot\nabla\mathscr{F} = \frac{u}{\left[1+X_\xi(\xi,\tau)\right]}\partial_\xi\mathscr{F} + \frac{1}{\varepsilon}\left(v - \frac{u}{\left[1+X_\xi(\xi,\tau)\right]}\partial_\xi\psi\right)\partial_Y\mathscr{F}$$

Using Eqs. (A.4) and (A.5) one can write the Euler equation (2.4) in the co-moving frame:

$$\partial_\tau\omega + \frac{(u-u^\Psi)}{\left[1+X_\xi(\xi,\tau)\right]}\partial_\xi\omega + \frac{1}{\varepsilon}\left[-\partial_\xi\psi\frac{(u-u^\Psi)}{\left[1+X_\xi(\xi,\tau)\right]} + (v-v^\Psi)\right]\partial_Y\omega = 0\,.$$

If one write the above equation using the rescaled vorticity $\tilde{\omega} \equiv \omega$, one obtains (2.10).

To get the incompressibility condition we use the formula for the divergence given in [50] pag.631, before Eq.(106):

$$\nabla\cdot u = a^1\cdot\partial_\xi u + a^2\cdot\partial_Y u$$

Expressing $u = u\hat{x} + v\hat{y}$, and using the fact that $\hat{x}$ and $\hat{y}$ are constant, one immeditely gets (2.14).

Appendix B: Decomposition of the velocity

The decomposition formula for the velocity can be derived from the expression (3.2) through the following computations:

$$\begin{aligned}
u - iv = &\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\Omega(x',Y') - \Omega(x,Y')}{K_\psi^\varepsilon(x-x',Y-Y')}dx'dY' + \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\Omega(x,Y')}{K_\psi^\varepsilon(x-x',Y-Y')}dx'dY' = \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\Omega(x',Y') - \Omega(x,Y')}{K_\psi^\varepsilon(x-x',Y-Y')}dx'dY' + \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dY'\Omega(x,Y')\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\tilde{t}^*(x')(1+i\partial_x\psi(x'))}{K_\psi^\varepsilon(x-x',Y-Y')}dx' = \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\Omega(x',Y') - \Omega(x,Y')}{K_\psi^\varepsilon(x-x',Y-Y')}dx'dY' + \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dY'\Omega(x,Y')\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{\left(\tilde{t}^*(x') - \tilde{t}^*(x)\right)(1+i\partial_x\psi(x'))}{K_\psi^\varepsilon(x-x',Y-Y')}dx' + \\
&-\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dY'\Omega(x,Y')\tilde{t}^*(x)\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{(1+i\partial_x\psi(x'))}{K_\psi^\varepsilon(x-x',Y-Y')}dx' = \\
&\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}T(x',Y')\frac{1}{K_\psi^\varepsilon(x-x',Y-Y')}dx'dY' + \\
&-\frac{1}{2}\left[\int_{-\infty}^{Y}\Omega(x,Y')dY' - \int_{Y}^{\infty}\Omega(x,Y')dY'\right]\tilde{t}^*(x)
\end{aligned}$$

(B.1)

where we have defined

$$T(x,x',Y') = \Omega(x',Y') - \Omega(x,Y') + \Omega(x,Y')\left(\tilde{t}^*(x') - \tilde{t}^*(x)\right)(1+i\partial_x\psi(x'))$$

The first equality above is simply a consequence of the fact $\tilde{t}^*(x')(1+i\psi(x')) = 1$. The last equality can be justified as follows. Introduce the change of variable $\zeta = x' + i\psi(x')$ and define $z = x + i[\psi(x) + \varepsilon(Y - Y')]$. This allows to write the integral in x' as a path integrals in $\mathbb{C}$; the paths L_n are defined as $L_n = \{x' + i\psi(x')$ with $x' \in [x+\pi(2n-1)/2, x+\pi(2n+1)/2]\}$, and $L \equiv \bigcup_n L_n$. Therefore

$$\sum_n \frac{1}{2\pi i}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2}\frac{(1+i\partial_x\psi(x'))}{K_\psi^\varepsilon(x,x',Y-Y')}dx' = \frac{1}{2\pi i}\int_L \frac{1}{\zeta - z}d\zeta = \pm\frac{1}{2}$$

being the value of the integral $+1/2$ if the point z lies above the curve (i.e. when $Y > Y'$) and $-1/2$ if the point z lies below the curve (i.e. when $Y < Y'$).

To the expression (B.1) above we now add and subtract the following quantity:

$$\frac{1}{2\pi i}\sum_n \int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} \frac{T(x,x',Y')}{K^0_\psi(x-x')}dx'dY' .$$

Using, see [42], the following identity

$$\sum_n \frac{1}{2\pi i}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} \frac{(1+i\partial_x\psi(x'))}{K^0_\psi(x-x')}dx' = \frac{1}{2\pi i}\int_L \frac{1}{\zeta-\zeta_0}dt = 0,$$

where $\zeta_0 = x + i\psi(x)$ is a point on the path L, one can see that:

$$\frac{1}{2\pi i}\sum_n \int_{-\infty}^{\infty}\int_{x+\pi(2n-1)/2}^{x+\pi(2n+1)/2} \frac{T(x,x',Y')}{K^0_\psi(x-x')}dx'dY' = BR[\gamma,\psi]$$

Therefore, from (B.1) the decomposition (3.3) follows.

Appendix C: Details of the derivation of equations (3.18) and (3.21)

We write (2.10) introducing the asymptotic expansion:

$$\tilde{\omega} = \omega_0 + \varepsilon\omega_1 , \qquad \varphi = \varphi_0 + \varepsilon\varphi_0 , \qquad u = u_0 + \varepsilon u_1$$

where the zero-th order velocity u_0 (which is the velocity generated by ω_0 through the Biot-Savart law) can be decomposed as:

$$u_0 = \mathscr{M}(\omega_0,\varphi) + \mathscr{R}(\omega_0,\varphi);$$

and in fact we shall use the above decomposition for the terms involving the convection of the leading order vorticity ω_0. As base curve of the reference frame we shall use the *correct* φ; i.e., in (2.10) we choose $\psi = \varphi$. One gets:

$$\begin{aligned} &\partial_\tau(\omega_0+\varepsilon\omega_1)+\\ &\frac{\mathscr{M}^u+\mathscr{R}^u+\varepsilon u_1-(\mathscr{M}^u+\mathscr{R}^u+\varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi\omega_0+\varepsilon\frac{u_0+\varepsilon u_1-(u_0+\varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi\omega_1+\\ &\frac{1}{\varepsilon}\Bigg[-\partial_\xi\varphi\frac{\mathscr{M}^u+\mathscr{R}^u+\varepsilon u_1-(\mathscr{M}^u+\mathscr{R}^u+\varepsilon u_1)_{Y=0}}{1+\partial_\xi X}+\\ &\mathscr{M}^v+\mathscr{R}^v+\varepsilon u_1-(\mathscr{M}^v+\mathscr{R}^v+\varepsilon u_1)_{Y=0}\Bigg]\partial_Y\omega_0+\\ &\left[-\partial_\xi\varphi\frac{u_0+\varepsilon u_1-(u_0+\varepsilon u_1)_{Y=0}}{1+\partial_\xi X}+v_0+\varepsilon u_1-(v_0+\varepsilon u_1)_{Y=0}\right]\partial_Y\omega_1 = 0 \end{aligned} \tag{C.1}$$

Notice that in the above equation we have used the shorthand notations $\mathscr{M} = \mathscr{M}(\omega_0,\varphi)$ $\mathscr{R} = \mathscr{R}(\omega_0,\varphi)$. We also introduce the quantity $\mathscr{M}_0$ defined as follows

$$\mathscr{M}_0 \equiv \mathscr{BR}[\gamma_0,\varphi_0] + \frac{1}{2}\left[\int_Y^\infty \omega_0 dY' - \int_{-\infty}^Y \omega_0 dY'\right]$$

being the difference, with respect to $\mathscr{M} = \mathscr{M}(\omega_0, \varphi)$, as defined in section 3.1, in the fact that the BR operator is computed using φ_0 instead of φ.

Both (u_0, v_0) and (u_1, v_1) are incompressible flow fields. We write the incompressibility condition for (u_0, v_0) as

$$\frac{\partial_\xi(\mathscr{M}^u + \mathscr{R}^u)}{1+\partial_\xi X} + \frac{1}{\varepsilon}\left[-\partial_\xi \varphi \frac{\partial_Y(\mathscr{M}^u + \mathscr{R}^u)}{1+\partial_\xi X} + \partial_Y(\mathscr{M}^v + \mathscr{R}^v)\right] = 0$$

that, integrated along the normal direction from 0 to Y, gives:

$$\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi(\mathscr{M}^u + \mathscr{R}^u) dY'\right] + \frac{1}{\varepsilon}\left[-\partial_\xi \varphi \frac{\mathscr{M}^u + \mathscr{R}^u - (\mathscr{M}^u + \mathscr{R}^u)_{Y=0}}{1+\partial_\xi X} + \mathscr{M}^v + \mathscr{R}^v - (\mathscr{M}^v + \mathscr{R}^v)_{Y=0}\right] = 0. \tag{C.2}$$

One can analogously derive the incompressibility condition in the integrated form for (u_1, v_1):

$$\frac{1}{1+\partial_\xi X}\int_0^Y \partial_\xi u_1 dY' + \frac{1}{\varepsilon}\left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0}\right] = 0. \tag{C.3}$$

Using the above incompressibiity condition in the integrated form, one can rewrite (C.1):

$$\begin{aligned}
\partial_\tau \omega_0 + \frac{\mathscr{M}^u - \mathscr{M}^u_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_0 + \frac{\mathscr{R}^u + \varepsilon u_1 - (\mathscr{R}^u + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_0 - \\
\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi(\mathscr{M}^u + \mathscr{R}^u + \varepsilon u_1) dY'\right]\partial_Y \omega_0 + \\
\varepsilon \partial_\tau \omega_1 + \varepsilon \frac{u_0 + \varepsilon u_1 - (u_0 + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_1 - \\
\frac{\varepsilon}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi u_0 dY'\right]\partial_Y \omega_1 + \\
\varepsilon\left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0}\right]\partial_Y \omega_1 = 0
\end{aligned} \tag{C.4}$$

Notice that, in writing the third line above, we have also used the incompressibility condition for (u_1, v_1) while, in writing the last term, expressing how the vorticity ω_1 is convected along the normal direction by (u_1, v_1), we have not used the incompressibility condition.

In equation (C.4) the leading order vorticity ω_0 is convected by $\mathscr{M}$, which involves explicitly the curve $\varphi = \varphi_0 + \varepsilon \varphi_1$. Moreover, in the convection of ω_0 it is involved the Lagrangian factor X, which depends also on u_1 and $\mathscr{R}$. Given that our goal is to write an equation for ω_0 that involves only the leading order quantities φ_0 and $\mathscr{M}_0$, we rewrite (C.4) as follows:

$$
\begin{aligned}
&\partial_\tau \omega_0 + \frac{\mathscr{M}_0^u - \mathscr{M}_{0|Y=0}^u}{1+\partial_\xi X_0}\partial_\xi \omega_0 + \left[\frac{\mathscr{M}^u - \mathscr{M}_{Y=0}^u}{1+\partial_\xi X} - \frac{\mathscr{M}_0^u - \mathscr{M}_{0|Y=0}^u}{1+\partial_\xi X_0}\right]\partial_\xi \omega_0 + \\
&\frac{\mathscr{R}^u + \varepsilon u_1 - (\mathscr{R}^u + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_0 - \\
&\frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}_0^u dY' \partial_Y \omega_0 - \left[\frac{1}{1+\partial_\xi X}\int_0^Y \partial_\xi \mathscr{M}^u dY' - \frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}_0^u dY'\right]\partial_Y \omega_0 - \\
&\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi (\mathscr{R}^u + \varepsilon u_1) dY'\right]\partial_Y \omega_0 + \\
&\varepsilon \partial_\tau \omega_1 + \varepsilon \frac{u_0 + \varepsilon u_1 - (u_0 + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_1 - \\
&\frac{\varepsilon}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi u_0 dY'\right]\partial_Y \omega_1 + \\
&\varepsilon\left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0}\right]\partial_Y \omega_1 = 0
\end{aligned}
\tag{C.5}
$$

Finally we write the equation ruling the dynamics of ω_0 where the first two terms of the first line, and the first term of the the third line of (C.5) appear.

$$
\partial_\tau \omega_0 + \frac{\mathscr{M}_0^u - \mathscr{M}_{0|Y=0}^u}{1+\partial_\xi X_0}\partial_\xi \omega_0 - \frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}_0^u dY' \partial_Y \omega_0 = 0
$$

All the other terms appear in the equation for ω_1.

$$
\begin{aligned}
\partial_\tau \omega_1 + \frac{1}{\varepsilon}\left[\frac{\mathscr{M}^u - \mathscr{M}^u_{Y=0}}{1+\partial_\xi X} - \frac{\mathscr{M}^u_0 - \mathscr{M}^u_{0|Y=0}}{1+\partial_\xi X_0}\right]\partial_\xi \omega_0 + \\
\frac{1}{\varepsilon}\frac{\mathscr{R}^u + \varepsilon u_1 - (\mathscr{R}^u + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_0 - \\
\frac{1}{\varepsilon}\left[\frac{1}{1+\partial_\xi X}\int_0^Y \partial_\xi \mathscr{M}^u dY' - \frac{1}{1+\partial_\xi X_0}\int_0^Y \partial_\xi \mathscr{M}^u_0 dY'\right]\partial_Y \omega_0 - \\
\frac{1}{\varepsilon}\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi (\mathscr{R}^u + \varepsilon u_1) dY'\right]\partial_Y \omega_0 + \\
\frac{u_0 + \varepsilon u_1 - (u_0 + \varepsilon u_1)_{Y=0}}{1+\partial_\xi X}\partial_\xi \omega_1 - \\
\frac{1}{1+\partial_\xi X}\left[\int_0^Y \partial_\xi u_0 dY'\right]\partial_Y \omega_1 + \\
\left[-\partial_\xi \varphi \frac{u_1 - u_{1|Y=0}}{1+\partial_\xi X} + v_1 - v_{1|Y=0}\right]\partial_Y \omega_1 = 0
\end{aligned}
\tag{C.6}
$$

Appendix D: Details of the derivation of equation (3.22)

To derive the equation ruling the dynamics of the correction φ_1, we consider the equation ruling the motion of a generic curve in the adapted frame, Eq.(2.11) specifying $\varphi \equiv \varphi_0 + \varepsilon\varphi_1$, and write this equation in the reference frame with base curve $\psi = \varphi_0 + \varepsilon\varphi_1$. Therefore, given that $\varphi = \psi$, the convective term along the curve cancels, and the equation for φ assumes the simple form:

$$\partial_\tau(\varphi_0 + \varepsilon\varphi_1) = [\mathscr{M}^v(\omega_0, \varphi, X) + \mathscr{R}^v(\omega_0, \varphi, X) + \varepsilon v_1]_{Y=0}$$

where we have also used that, by definition, see (3.13), $v_0 = \mathscr{M}^v(\omega_0, \varphi, X) + \mathscr{R}^v(\omega_0, \varphi, X)$.

Given that we have set the equation for φ_0 to be (3.19), one can immediately write the following equation for φ_1:

$$\partial_\tau \varphi_1 = \frac{1}{\varepsilon}[\mathscr{M}^v(\omega_0, \varphi, X) + \mathscr{R}^v(\omega_0, \varphi, X) - \mathscr{M}^v_0(\omega_0, \varphi_0, X_0) + \varepsilon v_1]_{Y=0}$$

Appendix E: Proof of Proposition 8.1

To prove Proposition 8.1 one has to prove the estimates (8.11) and (8.12). To prove (8.11) and (8.12) we shall give an explicit estimate of the Hölder modulus of continuity of the velocity, i.e. we have to bound $|U|^{(\alpha)}_{\rho,\theta}$ and $|V|^{(\alpha)}_{\rho,\theta}$, being the estimate of $|U|_{\rho,\theta}$ and $|V|_{\rho,\theta}$ easier.

Therefore we have to prove the following bounds:

$$\text{(E.1)} \qquad \sup_{\xi,\bar{\xi}\in D(\rho),Y\in\Sigma(\theta)} \frac{|U(\xi,Y)-U(\bar{\xi},Y)|}{|\xi-\bar{\xi}|^\alpha} \le c\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}$$

$$\text{(E.2)} \qquad \sup_{\xi\in D(\rho),Y,\bar{Y}\in\Sigma(\theta)} \frac{|U(\xi,Y)-U(\xi,\bar{Y})|}{|Y-\bar{Y}|^\alpha} \le c\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}$$

and analogous estimates for V. We recall that U,V are given by the Biot-Savart law (8.3)-(8.4), with $\mathscr{H}$ given in (8.1)-(8.2) and the Jacobian J given in (8.5). We introduce the following change of variables:

$$\eta = \xi' - \xi, \qquad z = Y' - Y$$

so that the Biot-Savart law now reads as:

$$U(\xi,Y) = \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} \mathscr{H}_u(\eta,z,\xi)\Omega(\eta+\xi,z+Y)J(\eta+\xi)d\eta dz$$
$$V(\xi,Y) = \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} \mathscr{H}_v(\eta,z,\xi)\Omega(\eta+\xi,z+Y)J(\eta+\xi)d\eta dz$$

where the Biot-Savart kernel (8.1)-(8.2), in terms of the new variables (η,z), is given by

$$\mathscr{H}_u(\eta,z,\xi) = \frac{1}{8\pi^2}\frac{\sinh 2\left[\varepsilon z+\varphi(\eta+\xi)-\varphi(\xi)\right]}{\sin^2\left[\eta+X(\eta+\xi)-X(\xi)\right]+\sinh^2\left[\varepsilon z+\varphi(\eta+\xi)-\varphi(\xi)\right]}$$
$$\mathscr{H}_v(\eta,z,\xi) = -\frac{1}{8\pi^2}\frac{\sin 2\left[\eta+X(\eta+\xi)-X(\xi)\right]}{\sin^2\left[\eta+X(\eta+\xi)-X(\xi)\right]+\sinh^2\left[\varepsilon z+\varphi(\eta+\xi)-\varphi(\xi)\right]}.$$

Notice also the abuse of notation where, with $\varphi(\xi)$, we have indicated the composition $\varphi\circ(Id+X)$.

First we shall consider the case when the variables $\xi,\bar{\xi},Y,\bar{Y}$ are real. In subsection E.3 we shall see the necessary modifications to deal with the case when $\xi,\bar{\xi},Y,\bar{Y}$ are complex.

We introduce the notation $I\equiv[-\pi/2,\pi/2[\times\mathbb{R}$. We shall also define $\delta>0$ as

$$\frac{\delta}{2} = \left[|\xi-\bar{\xi}|^2+\varepsilon^2|Y-\bar{Y}|^2\right]^{1/2}$$

and the ball centered in (ξ,Y) as:

$$B_{\mathbf{x},\delta} \equiv \left\{(\xi',Y')\in I_\xi : |\xi-\xi'|^2+\varepsilon^2|Y-Y'|^2\le\delta^2\right\},$$

so that $(\bar{\xi},\bar{Y})\in\partial B_{\mathbf{x},\delta/2}$. Therefore $|\xi-\bar{\xi}|=O(\delta)$ while $|Y-\bar{Y}|=O(\varepsilon^{-1}\delta)$. This means that, to prove for example the estimates (E.1) and (E.2), we have to show, respectively, that:

(E.3)
$$|U(\xi,Y)-U(\bar{\xi},Y)|\le c\delta^\alpha\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}, \qquad |U(\xi,Y)-U(\xi,\bar{Y})|\le c\varepsilon^{-\alpha}\delta^\alpha\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}.$$

E.1 The case of $(\xi,Y),(\bar{\xi},\bar{Y})$ real, with $\xi=\bar{\xi}$

We shall prove that:

$$|U(\xi,Y)-U(\xi,\bar{Y})|\le c|Y-\bar{Y}|^{\alpha} \qquad |V(\xi,Y)-V(\xi,\bar{Y})|\le c|Y-\bar{Y}|^{\alpha}$$

where the constant c depends on $\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}$. We shall show how to prove the first of the above inequalities, being the second analogous.

Introducing the notations $\mathscr{H}_u=\mathscr{H}_u(\eta,z,\xi)$, $\Omega=\Omega(\eta+\xi,z+Y)$ and $\bar{\Omega}=\Omega(\eta+\xi,z+\bar{Y})$, $J=J(\eta+\xi)$, one can estimate $U(\xi,Y)-U(\xi,\bar{Y})$ as follows:

$$\begin{aligned}
|U(\xi,Y)-U(\xi,\bar{Y})|&=\\
\left|\int_I \mathscr{H}_u\left[\Omega-\bar{\Omega}\right]J\,d\eta dz\right|&\le\\
\left|\int_I \mathscr{H}_u e^{-\mu|z+Y|/2}\left[\Omega e^{\mu|z+Y|/2}-\bar{\Omega}e^{\mu|z+\bar{Y}|/2}\right]J\,d\eta dz\right|&+\\
\left|\int_I \mathscr{H}_u e^{\mu|z+\bar{Y}|/2}\bar{\Omega}\left[e^{-\mu|z+Y|/2}-e^{-\mu|z+\bar{Y}|/2}\right]J\,d\eta dz\right|&\le\\
c|Y-\bar{Y}|^{\alpha}\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}\int_I \mathscr{H}_u e^{-\mu|z+Y|/2}d\eta dz&+\\
c|Y-\bar{Y}|^{\alpha}|\Omega|_{\rho,\theta,\mu}\int_I \mathscr{H}_u e^{-\mu|z+\bar{Y}|/2}d\eta dz&\le\\
c|Y-\bar{Y}|^{\alpha}\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}&.
\end{aligned}\tag{E.4}$$

The estimate in the case of $(\xi,Y),(\bar{\xi},\bar{Y})$ real, with $\xi=\bar{\xi}$ is thus achieved. To get the last inequality we have used the following Lemma:

Lemma E.1. *Suppose to hold the hypotheses of Proposition 8.1. Then the following estimate hold:*

$$\int_I |\mathscr{H}(\eta,z,\xi)|\,e^{-\mu'|z+Y|}d\eta dz\le c,$$

where $\mathscr{H}$ denotes both $\mathscr{H}_u$ and $\mathscr{H}_v$. The above Lemma is a consequence of the following estimate on the Biot-Savart kernel:

Lemma E.2. *Suppose to hold the hypotheses of Proposition 8.1. Moreover let* $(\eta,z)\neq(0,0)$. *Then:*

$$|\mathscr{H}(\eta,z,\xi)|\le c\max\left(1,\frac{1}{(\eta^2+\varepsilon^2z^2)^{1/2}}\right).$$

The meaning of the above Lemma is that the Biot-Savart kernel has a square-root singularity in the origin while, away from the origin, it is bounded by a constant. This fact, together with the presence of the exponential decaying factor, allows to prove Lemma E.1. The proofs of the above Lemmas are postponed to Section E.4.1 and Section E.4.2.

E.2 The case of $(\xi, Y), (\bar{\xi}, \bar{Y})$ real, with $Y = \bar{Y}$

We have to prove that:

$$|U(\xi,Y) - U(\bar{\xi},Y)| \le c|\xi - \bar{\xi}|^{\alpha} \qquad |V(\xi,Y) - V(\bar{\xi},Y)| \le c|\xi - \bar{\xi}|^{\alpha}$$

where the constant c depends on $\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu}$. We shall see how to prove the first of the above inequalities, being the second analogous.

Introducing the notations $\mathscr{H}_u = \mathscr{H}_u(\eta,z,\xi)$, $\bar{\mathscr{H}_u} = \mathscr{H}_u(\eta,z,\bar{\xi})$, $\Omega = \Omega(\eta + \xi, z+Y)$, $\bar{\Omega} = \Omega(\eta + \bar{\xi}, z+Y)$, $J = J(\eta+\xi)$, $\bar{J} = J(\eta + \bar{\xi})$, one can write:

$$\begin{aligned}
&\left|U(\xi,Y) - U(\bar{\xi},Y)\right| = \\
&\left|\int_I \mathscr{H}_u \Omega J d\eta dz - \int_I \bar{\mathscr{H}_u}\bar{\Omega}\bar{J} d\eta dz\right| \le \\
&\left|\int_I \left(\mathscr{H}_u - \bar{\mathscr{H}_u}\right)\Omega J d\eta dz\right| + \left|\int_I \bar{\mathscr{H}_u}\left(\Omega - \bar{\Omega}\right) J d\eta dz\right| + \left|\int_I \bar{\mathscr{H}_u}\bar{\Omega}\left(J - \bar{J}\right) d\eta dz\right| = \\
&J_1 + J_2 + J_3.
\end{aligned} \tag{E.5}$$

To estimate the term J_1 one needs the following Lemma.

Lemma E.3. *Suppose to hold the hypotheses of Proposition 8.1. Then the following estimate hold:*

$$\int_I \left|\mathscr{H}(\eta,z,\xi) - \mathscr{H}(\eta,z,\bar{\xi})\right| e^{-\mu|z+Y|} d\eta dz \le c|\xi - \bar{\xi}|^{\alpha}$$

This Lemma is a consequence of the following estimate on the modulus of continuity of the Biot-Savart kernel:

Lemma E.4. *Suppose to hold the hypotheses of Proposition 8.1. Moreover let* $(\eta, z) \neq 0$ *and* $0 \le \beta \le 1$. *Then:*

$$|\mathscr{H}(\eta,z,\xi) - \mathscr{H}(\eta,z,\bar{\xi})| \le c|\eta|\left(|\eta| + |\xi - \bar{\xi}|\right)\max\left(1, \frac{1}{\eta^2 + \varepsilon^2 z^2}\right), \tag{E.6}$$

$$|\mathscr{H}(\eta,z,\xi) - \mathscr{H}(\eta,z,\bar{\xi})| \le c|\xi - \bar{\xi}|\left(|\eta| + |\xi - \bar{\xi}|\right)\max\left(1, \frac{1}{\eta^2 + \varepsilon^2 z^2}\right), \tag{E.7}$$

$$|\mathscr{H}(\eta,z,\xi) - \mathscr{H}(\eta,z,\bar{\xi})| \le c|\eta|^{\beta}|\xi - \bar{\xi}|^{1-\beta}\left(|\eta| + |\xi - \bar{\xi}|\right)\max\left(1, \frac{1}{\eta^2 + \varepsilon^2 z^2}\right). \tag{E.8}$$

The estimate of J_1 goes as follows:

$$J_1 = \left| \int_I \left(\mathscr{H}_u - \bar{\mathscr{H}_u} \right) \Omega J d\eta dz \right| \leq$$

$$\int_I \left| \mathscr{H}_u - \bar{\mathscr{H}_u} \right| e^{-\mu|z+Y|} |\Omega| e^{\mu|z+Y|} |J| d\eta dz \leq$$

$$c|\Omega|_{\rho,\theta,\mu} \int_I \left| \mathscr{H}_u - \bar{\mathscr{H}_u} \right| e^{-\mu|z+Y|} d\eta dz \leq$$

$$c\|\Omega\|^{(\alpha)}_{\rho,\theta,\mu} |\xi - \bar{\xi}|^{\alpha}$$

In the last inequality we have used Lemma E.3.

The term J_2 and J_3 are easily bounded, using Lemma E.1, with the same procedure adopted in Section E.1.

E.3 Complexified variables

Here we show how the above estimates on the velocity can be performed, with small variations with respect to the cases examined in the two previous Subsections, when the variable ξ and Y are complex with $\xi \in D(\rho)$ and $Y \in \Sigma(\theta)$.

The main difficulty arises in the analysis of the Hölder modulus of continuity, because the paths of integration used in evaluating $U(\bar{\xi}, \bar{Y})$ and $U(\xi, Y)$ can be in principle different. This difficulty can be more easily overcome for the paths of integration in the ξ variable because in $D(\rho)$, by contour deformation, one can always chose a path parallel to the real axis. This difficulty is more severe for the variable Y because, given the angular shape of the domain of analyticity $\Sigma(\theta)$, one cannot chose a path of integration parallel to the real axis. However, following [4], one can adopt, as integration path in Y, the path $P(Y)$ defined in (8.8), see Figure 8.1. From this choice one can see that:

a) when Y and $\bar{Y}$ have the same imaginary part, i.e. $Y_I = \bar{Y}_I$, the two paths of integration can be easily chosen to be the same. Therefore, in the evaluation of the Hölder modulus of continuity, the estimate performed in the previous two sections still holds true with integration in Y performed along $P(Y)$ rather than $\mathbb{R}$ and with obvious modifications.

b) when Y and $\bar{Y}$ have the same real part, i.e. $Y_R = \bar{Y}_R$, the two paths of integration must be different; however, after changing the integration variable so that both paths pass through the origin of $\mathbb{C}$ (where the singularity of the kernel is now located), one of the two contours can be deformed to coincide with the other. In fact (to fix the idea suppose $|\bar{Y}_I| > |Y_I|$) $U(\xi, Y)$ can be computed as:

$$U(\xi, Y) =$$

$$\int_{\xi-\frac{\pi}{2}}^{\xi+\frac{\pi}{2}} \int_{-\infty}^{\infty} \mathscr{H}_u(\xi', \xi, Y - Y') \Omega(\xi', Y') J(\xi') d\xi' dY'$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{P(Y)-Y} \mathscr{H}_u(\eta + \xi, \xi, z) \Omega(\eta + \xi, z + Y) J(\eta + \xi) d\eta dz$$

where we have defined the integration variable $\eta \equiv \xi' - \xi$ (we recall that by contour deformation we can suppose the path of integration in the ξ' variable to be parallel to the real axis, i.e. $\xi' = \xi'_R + i\xi_I$, being ξ_I the imaginary part of ξ); and the integration variable $z = Y' - Y$, with $z \in P(Y) - Y$ where $P(Y)$ is the path defined in (8.8), see also figure 8.1. Clearly the integration path can be parametrized using z_R, in the same way $P(Y)$ is parametized using Y'_R in (8.9) and (8.10).

For $U(\xi, \bar{Y})$ one can write

$$U(\xi,Y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{P(\bar{Y})-\bar{Y}} \mathscr{H}_u(\eta+\xi,\xi,z)\Omega(\eta+\xi,z+\bar{Y})J(\eta+\xi)d\eta dz.$$

Given that $\delta < 1$ one can deform the contour $P(Y) - Y$ to make it coincide with $P(\bar{Y}) - \bar{Y}$, with $z + Y$ still in Σ.

The estimates performed in the previous two Sections can therefore be carried out with obvious modifications.

c) The general case, when $Y_R \neq \bar{Y}_R$ and $Y_I \neq \bar{Y}_I$, can be easily treated using the cases a) and b) above.

E.4 Proof of the technical Lemmas

E.4.1 Proof of Lemma E.1

We shall prove the estimate for $\mathscr{H}_u$ being the estimate for $\mathscr{H}_v$ analogous. It is easy to see that

$$\int_I |\mathscr{H}_u(\eta,z,\xi)|\, e^{-\mu'|z+Y|} d\eta dz \le c \int_I |\mathscr{H}_u(\eta,z,\xi)|\, e^{-\mu'|z|} d\eta dz \le c$$

so that we can assume, without loss of generality, that $Y = 0$. Therefore:

$$\int_I |\mathscr{H}_u(\eta,z,\xi)|\, e^{-\mu'|z|} d\eta dz = \int_{B_{\mathbf{0},1}} |\mathscr{H}_u(\eta,z,\xi)|\, e^{-\mu'|z|} d\eta dz + \int_{I-B_{\mathbf{0},1}} |\mathscr{H}_u(\eta,z,\xi)|\, e^{-\mu'|z|} d\eta dz = L_1 + L_2$$

The term L_2 is easily estimated using the exponentially decaying factor and the fact that, outside $B_{\mathbf{0},1}$, the kernel $\mathscr{H}$ is bounded by a constant, see Lemma E.2.

The estimate of L_1 goes as follows:

$$L_1 \le c \int_{B_{\mathbf{0},1}} \frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}} e^{-\mu'|z|} d\eta dz = c\frac{1}{\varepsilon} \int_0^{2\pi} d\theta \int_0^1 e^{-\mu' r \sin\theta/\varepsilon} dr \tag{E.9}$$

where we have used polar coordinates $\eta = r\cos\theta$ and $\varepsilon z = r\sin\theta$. Therefore:

$$L_1 \le c\frac{1}{\varepsilon}\int_0^{2\pi}\cos\theta d\theta\int_0^1 e^{-\mu' r\sin\theta/\varepsilon}dr + c\frac{1}{\varepsilon}\int_0^{2\pi}(1-\cos\theta)d\theta\int_0^1 e^{-\mu' r\sin\theta/\varepsilon}dr = c\frac{1}{\varepsilon}\int_0^{2\pi}\frac{1-\cos\theta}{\sin\theta}d\theta\int_1^0\frac{\varepsilon}{\mu'}\frac{d}{dr}e^{-\mu' r\sin\theta/\varepsilon}dr \le c\int_0^{2\pi}\frac{1-\cos\theta}{\sin\theta}d\theta \le c$$

The proof of Lemma E.1 is thus achieved.

E.4.2 Proof of Lemma E.2

To simplify the notation we introduce the following definition:

$$E = \eta + X(\eta+\xi) - X(\xi), \quad Z = \varepsilon z + \varphi(\eta+\xi) - \varphi(\xi), \quad D = \sin^2 E + \sinh^2 Z.$$

Moreover we recall that, according to the hypotheses of Proposition 8.1, C_φ and C_X denote two sufficiently small constants (say less than $1/4$) that bound the norms of φ and X

$$\|\varphi\|_{1,\rho}^{(\alpha)} \le C_\varphi < 1/4 \qquad \|X\|_{1,\rho}^{(\alpha)} \le C_X < 1/4.$$

We can therefore make the following Remark

Remark E.5. Suppose the hypotheses of Proposition 8.1 hold. Then, $\forall(\eta, z) \in I$

$$\left(\frac{1}{2} - C_X^2\right)|\eta|^2 \le |E|^2 \le \left[\frac{5}{4}|\eta|\right]^2 \tag{E.10}$$

$$\left[\frac{|\varepsilon z|^2}{2} - C_\varphi^2\eta^2\right] \le |Z|^2 \le C\left(\eta^2 + \varepsilon^2 z^2\right) \tag{E.11}$$

We prove the bounds on Z, being the bounds on E easier.

$$Z^2 = |\varepsilon z + \varphi(\eta+\xi) - \varphi(\xi)|^2 = \left|\varepsilon z + \partial_\xi\varphi(\eta^*)\eta\right|^2 \le 2\left(\varepsilon^2 z^2 + C_\phi^2\eta^2\right) \le C\left(\eta^2 + \varepsilon^2 z^2\right)$$

$$Z^2 = |\varepsilon z + \varphi(\eta+\xi) - \varphi(\xi)|^2 \ge \left[\frac{|\varepsilon z|^2}{2} - |\varphi(\eta+\xi) - \varphi(\xi)|^2\right] \ge \left[\frac{|\varepsilon z|^2}{2} - C_\varphi^2\eta^2\right],$$

where we have used the fact that $(a+b)^2 \ge a^2/2 - b^2$. Remark E.5 is thus proved.

We notice that, being C_X and C_φ less than $1/4$, one has that

$$A^2 \equiv \frac{2}{\pi^2}\left(\frac{1}{2} - C_X^2\right) - C_\varphi^2 > 0.$$

One can therefore make the following Remark

Remark E.6. Suppose the hypotheses of Proposition 8.1 hold. Then $\forall(\eta,z)\in I$

$$D \geq A^2\eta^2 + \frac{1}{2}\varepsilon^2 z^2 \geq c\left[\eta^2+\varepsilon^2 z^2\right] \tag{E.12}$$

In fact:

$$\sin^2 E \geq \frac{2}{\pi^2}|\eta|^2\left(\frac{1}{2}-C_X^2\right)$$

which is a consequence of of (E.10) and of the elementary fact that, if $a\in[-5\pi/8,5\pi/8]$, then $\sin^2 a \geq 2a^2/\pi^2$.

One can analogously write:

$$\sinh^2 Z \geq |Z|^2 \geq \left[\frac{|\varepsilon z|^2}{2} - C_\varphi^2\eta^2\right].$$

Therefore the first of the inequality in (E.12) follows, while the second is obvious taking $c\equiv\min(A^2,1/2)$. The Remark E.6 is proven.

We can now prove Lemma E.2 distinguishing three cases:

The case $\eta^2+\varepsilon^2 z^2\leq 1$ In this case one can immediately write:

$$|\mathscr{H}_u(\eta,z,\xi)| = \left|\frac{\sinh 2Z}{\sin^2 E+\sinh^2 Z}\right| \leq c\frac{|\sinh Z|}{\sin^2 E+\sinh^2 Z} \leq$$
$$\leq c\frac{1}{\sqrt{\sin^2 E+\sinh^2 Z}} \leq c\frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}},$$

where, first, we have used that $|Z|\leq c$ (see (E.11)) and, finally, the estimate on D given in Remark E.6.

The case $\eta^2+\varepsilon^2 z^2\geq 1$ and $|Z|\leq 1/2$ In this case, using the fact that $|Z|$ is bounded and Remark E.6, one can write

$$|\mathscr{H}_u(\eta,z,\xi)| = \left|\frac{\sinh 2Z}{\sin^2 E+\sinh^2 Z}\right| \leq c\frac{1}{\sin^2 E+\sinh^2 Z} \leq c\frac{1}{\eta^2+\varepsilon^2 z^2} \leq c$$

The case $\eta^2+\varepsilon^2 z^2\geq 1$ and $|Z|\geq 1/2$ In this case one has to use the fact that the growth of the numerator is compensated by the denominator:

$$|\mathscr{H}_u(\eta,z,\xi)| = \left|\frac{\sinh 2Z}{\sin^2 E+\sinh^2 Z}\right| = \left|\frac{\sinh 2Z}{\sinh^2 Z}\right|\frac{1}{\sin^2 E/\sinh^2 Z+1} \leq c$$

One can analogously estimate $\mathscr{H}_v$. The proof of Lemma E.2 is thus achieved.

E.4.3 Proof of Lemma E.3

To prove Lemma E.3 we shall distinguish the case $|\xi-\bar{\xi}|<\varepsilon$ and the case $|\xi-\bar{\xi}|>\varepsilon$. In the first case the crucial estimate is given in (E.6) while, in the second case, one has to use (E.7).

The case $\varepsilon < |\xi - \bar{\xi}|$. We can write

$$\int_I \left|\mathscr{H}_u(\eta,z,\xi) - \mathscr{H}_u(\eta,z,\bar{\xi})\right| e^{-\mu|z+Y|} d\eta dz \leq$$
$$\int_{B_{\mathbf{0},1}} |\eta| \left(|\eta| + |\xi - \bar{\xi}|\right) \frac{1}{\eta^2+\varepsilon^2 z^2} e^{-\mu|z+Y|} d\eta dz + c\int_{I-B_{\mathbf{0},1}} e^{-\mu|z+Y|} d\eta dz =$$
$$A_1 + A_2$$

where we have used the estimate (E.6). The term A_2 is easily estimated using the exponentially decaying factor. Concerning A_1 one can write:

$$A_1 = c\int_{B_{\mathbf{0},1}} |\eta| \left(|\eta| + |\xi - \bar{\xi}|\right) \frac{1}{\eta^2+\varepsilon^2 z^2} e^{-\mu|z+Y|} d\eta dz \leq$$
$$= c\int_{B_{\mathbf{0},1}} \left(|\eta| + |\xi - \bar{\xi}|\right) \frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}} e^{-\mu|z+Y|} d\eta dz \leq$$
$$c\int_{B_{\mathbf{0},1}} e^{-\mu|z+Y|} d\eta dz + c|\xi - \bar{\xi}| \int_{B_{\mathbf{0},1}} \frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}} e^{-\mu|z+Y|} d\eta dz$$

The second of the above integrals is bounded by a constant, and this can be proved as in the estimate of the term L_1 defined in (E.9).

Concerning the first of the above integrals, it is easy to see that, without loss of generality, one can consider the case $Y = 0$. Therefore, passing to polar coordinates $\eta = r\cos\theta$, $\varepsilon z = r\sin\theta$, one can write:

$$\int_{B_{\mathbf{0},1}} e^{-\mu|z|} d\eta dz = \frac{1}{\varepsilon}\int_0^{2\pi} d\theta \int_0^1 r e^{-\mu r \sin\theta/\varepsilon} dr =$$
$$\frac{1}{\varepsilon}\int_0^{2\pi} \cos\theta d\theta \int_0^1 r e^{-\mu r\sin\theta/\varepsilon} dr + \frac{1}{\varepsilon}\int_0^{2\pi} (1-\cos\theta) d\theta \int_0^1 r e^{-\mu r\sin\theta/\varepsilon} dr =$$
$$\varepsilon \int_0^{2\pi} \frac{1-\cos\theta}{\sin^2\theta} d\theta \int_0^{\sin\theta/\varepsilon} R e^{-\mu R} dR$$
$$c\varepsilon \int_0^{2\pi} \frac{1-\cos\theta}{\sin^2\theta} d\theta \leq c\varepsilon \leq c|\xi - \bar{\xi}|$$

Therefore, in the case $\varepsilon < |\xi - \bar{\xi}|$, we have proven that

$$\int_I \left|\mathscr{H}_u(\eta,z,\xi) - \mathscr{H}_u(\eta,z,\bar{\xi})\right| e^{-\mu|z+Y|} d\eta dz \leq c|\xi - \bar{\xi}|.$$

The case $|\xi - \bar{\xi}| < \varepsilon$.

In this case we can write:

$$\int_I \left|\mathscr{H}_u(\eta,z,\xi) - \mathscr{H}_u(\eta,z,\bar{\xi})\right| e^{-\mu|z+Y|} d\eta dz \leq$$

$$\int_{B_{\mathbf{0},1}} |\xi-\bar{\xi}|^\alpha |\eta|^{1-\alpha}\left(|\eta|+|\xi-\bar{\xi}|\right) \frac{1}{\eta^2+\varepsilon^2 z^2} e^{-\mu|z+Y|} d\eta dz +$$

$$c\int_{I-B_{\mathbf{0},1}} e^{-\mu|z+Y|} d\eta dz =$$

$$B_1+B_2$$

where we have used the estimate (E.8). Again, the term B_2 is obviously bounded. Concerning the term B_1 one can write:

$$B_1 =$$

$$|\xi-\bar{\xi}|^\alpha\left[\int_{B_{\mathbf{0},1}} \frac{|\eta|^{2-\alpha}}{\eta^2+\varepsilon^2 z^2} e^{-\mu|z+Y|} d\eta dz + \int_{B_{\mathbf{0},1}} \frac{|\xi-\bar{\xi}||\eta|^{1-\alpha}}{\eta^2+\varepsilon^2 z^2} e^{-\mu|z+Y|} d\eta dz\right] \leq$$

$$c|\xi-\bar{\xi}|^\alpha\left[\int_{B_{\mathbf{0},1}} \frac{e^{-\mu|z+Y|}}{\sqrt{\eta^2+\varepsilon^2 z^2}} d\eta dz + \int_{B_{\mathbf{0},1}} \frac{|\xi-\bar{\xi}|}{(\eta^2+\varepsilon^2 z^2)^{(1+\alpha)/2}} e^{-\mu|z+Y|} d\eta dz\right]$$

The first of the above two integrals can be bounded as in the estimate of the term L_1 defined in (E.9) (again, without loss of generality one can consider the case $Y=0$). The second integral can be bounded as follows:

$$\int_{B_{\mathbf{0},1}} \frac{1}{(\eta^2+\varepsilon^2 z^2)^{(1+\alpha)/2}} e^{-\mu|z|} d\eta dz = \frac{1}{\varepsilon}\int_0^{2\pi} d\theta \int_0^1 \frac{1}{r^\alpha} e^{-\mu r\sin\theta/\varepsilon} dr =$$

$$\frac{1}{\varepsilon}\left[\int_0^{2\pi}\cos\theta d\theta\int_0^1 r e^{-\mu r\sin\theta/\varepsilon} dr + \int_0^{2\pi}(1-\cos\theta)d\theta\int_0^1 \frac{1}{r^\alpha} e^{-\mu r\sin\theta/\varepsilon} dr\right] =$$

$$\frac{1}{\varepsilon^\alpha}\int_0^{2\pi}\frac{1-\cos\theta}{(\sin\theta)^{1-\alpha}} d\theta \int_0^1 \frac{1}{R^\alpha} e^{-\mu R} dr \leq c$$

One can therefore conclude (of course, in the estimate of the modulus of continuity, one can always assume that $|\xi-\bar{\xi}|\leq 1$) that:

$$B_1 \leq c|\xi-\bar{\xi}|^\alpha.$$

Lemma E.3 is proved.

E.4.4 Proof of Lemma E.4

We introduce the following notations:

$$E = \eta + X(\eta+\xi) - X(\xi), \quad Z = \varepsilon z + \varphi(\eta+\xi) - \varphi(\xi),$$
$$\bar{E} = \eta + X(\eta+\bar{\xi}) - X(\bar{\xi}), \quad \bar{Z} = \varepsilon z + \varphi(\eta+\bar{\xi}) - \varphi(\bar{\xi}).$$

Remark E.7.

$$|E-\bar{E}| \leq c|\eta|(|\eta|+|\xi-\bar{\xi}|) \qquad |Z-\bar{Z}| \leq c|\eta|(|\eta|+|\xi-\bar{\xi}|)$$
$$|E-\bar{E}| \leq c|\xi-\bar{\xi}|(|\eta|+|\xi-\bar{\xi}|) \qquad |Z-\bar{Z}| \leq c|\xi-\bar{\xi}|(|\eta|+|\xi-\bar{\xi}|)$$

$$|E-\bar{E}| \leq c|\eta|^{\alpha}|\xi-\bar{\xi}|^{1-\alpha}(|\eta|+|\xi-\bar{\xi}|)$$
$$|Z-\bar{Z}| \leq c|\eta|^{\alpha}|\xi-\bar{\xi}|^{1-\alpha}(|\eta|+|\xi-\bar{\xi}|)$$

Assuming that $\eta > 0$, being the case $\eta < 0$ analogous, the Remark can be proven as follows:

$$\begin{aligned}|E-\bar{E}| = |X(\eta+\xi)-X(\xi)-\big(X(\eta+\bar{\xi})-X(\bar{\xi})\big)| = \\ |\eta||\partial_\xi X(\xi^*)-\partial_\xi X(\xi^{**})| = \\ |\eta||\partial_\xi^2 X(\xi^{***})|\xi^*-\xi^{**}| \leq c|\eta|(|\eta|+|\xi-\bar{\xi}|).\end{aligned}$$

In the above estimate $\xi < \xi^* < \xi+\eta$ and $\bar{\xi} < \xi^{**} < \bar{\xi}+\eta$, which implies $|\xi^*-\xi^{**}| \leq |\eta|+|\xi-\bar{\xi}|$. Clearly ξ^{***} is between ξ^* and ξ^{**}. The second derivative of X can be assumed bounded due to analyticity.

This proves the first estimate concerning $|E-\bar{E}|$. The second estimate is analogous:

$$\begin{aligned}|E-\bar{E}| = |X(\eta+\xi)-X(\eta+\bar{\xi})-\big(X(\xi)-X(\bar{\xi})\big)| = \\ |\xi-\bar{\xi}||\partial_\xi X(\xi^*)-\partial_\xi X(\xi^{**})| = \\ |\xi-\bar{\xi}||\partial_\xi^2 X(\xi^{***})|\xi^*-\xi^{**}| \leq c|\xi-\bar{\xi}|(|\eta|+|\xi-\bar{\xi}|).\end{aligned}$$

In the above estimate (assuming $\bar{\xi} > \xi$), $\xi+\eta < \xi^* < \bar{\xi}+\eta$ and $\xi < \xi^{**} < \bar{\xi}$, so that again $|\xi^*-\xi^{**}| \leq |\eta|+|\xi-\bar{\xi}|$.

The estimates for Z are analogous.

An immediate consequence of the above Remark is the following.

Remark E.8. Let $0 \leq \beta \leq 1$. Then:

$$|E-\bar{E}| \leq c|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)$$

$$|Z-\bar{Z}| \leq c|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)$$

In fact, one can simply write:

$$\begin{aligned}|E-\bar{E}| = |X(\eta+\xi)-X(\xi)-\big(X(\eta+\bar{\xi})-X(\bar{\xi})\big)| = \\ |X(\eta+\xi)-X(\xi)-\big(X(\eta+\bar{\xi})-X(\bar{\xi})\big)|^{1-\beta} \cdot \\ |X(\eta+\xi)-X(\eta+\bar{\xi})-\big(X(\xi)-X(\bar{\xi})\big)|^{\beta}\end{aligned}$$

and proceed as in the estimates of Remark E.7.

Remark E.9.

$$|\sin^2 E - \sin^2 \bar{E}| \le c\,(|\sin E| + |\sin \bar{E}|)\,|\eta|(|\eta| + |\xi - \bar{\xi}|) \tag{E.13}$$

$$|\sin^2 E - \sin^2 \bar{E}| \le c\,(|\sin E| + |\sin \bar{E}|)\,|\xi - \bar{\xi}|(|\eta| + |\xi - \bar{\xi}|) \tag{E.14}$$

$$|\sinh^2 Z - \sinh^2 \bar{Z}| \le c\,(|\sinh Z| + |\sinh \bar{Z}|)\cosh\left(\frac{Z+\bar{Z}}{2}\right)|\eta|(|\eta| + |\xi - \bar{\xi}|) \tag{E.15}$$

$$|\sinh^2 Z - \sinh^2 \bar{Z}| \le c\,(|\sinh Z| + |\sinh \bar{Z}|)\cosh\left(\frac{Z+\bar{Z}}{2}\right)|\xi - \bar{\xi}|(|\eta| + |\xi - \bar{\xi}|) \tag{E.16}$$

The proof of the above Remark can be easily achieved using elementary properties of trigonometric functions and Remark E.7.

Remark E.10. Let $0 \le \beta \le 1$. Then:

$$|\sin^2 E - \sin^2 \bar{E}| \le c\,(|\sin E| + |\sin \bar{E}|)\,|\eta|^\beta |\xi - \bar{\xi}|^{1-\beta}(|\eta| + |\xi - \bar{\xi}|),$$

$$|\sinh^2 Z - \sinh^2 \bar{Z}| \le$$
$$c\,(|\sinh Z| + |\sinh \bar{Z}|)\cosh\left(\frac{Z+\bar{Z}}{2}\right)|\eta|^\beta |\xi - \bar{\xi}|^{1-\beta}(|\eta| + |\xi - \bar{\xi}|)$$

The proof of the above Remark can be easily achieved using elementary properties of trigonometric functions and Remark E.8.

Remark E.11.

$$\frac{|\sinh 2Z|}{\sqrt{\sin^2 E + \sinh^2 Z}}\frac{1}{\sqrt{\sin^2 \bar{E} + \sinh^2 \bar{Z}}} \le c \max\left(1, \frac{1}{\sqrt{\eta^2 + \varepsilon^2 z^2}}\right)$$

To prove the above Remark we distinguish three different cases:

<u>The case $\eta^2 + \varepsilon^2 z^2 \le 1$</u> In this case, Z is bounded because of (E.11), and therefore $\sinh 2Z/\sqrt{\sin^2 E + \sinh^2 Z} < c$, while $1/\sqrt{\sin^2 \bar{E} + \sinh^2 \bar{Z}} \le 1/\sqrt{\eta^2 + \varepsilon^2 z^2}$ because of (E.12) written for $\bar{D} = \sin^2 \bar{E} + \sinh^2 \bar{Z}$.

<u>The case $\eta^2 + \varepsilon^2 z^2 \ge 1$ and $|Z| < 2$</u> In this case one can proceed in the same way as before.

<u>The case $\eta^2 + \varepsilon^2 z^2 \ge 1$ and $|Z| > 2$</u> In this case, first, one can notice that $|Z - \bar{Z}| = |\varphi(\eta + \xi) - \varphi(\xi) - \varphi(\eta + \bar{\xi}) - \varphi(\bar{\xi})| \le 1$, so that $|\sinh Z/\sinh \bar{Z}| \le c$. Then one can simply write:

$$\frac{|\sinh 2Z|}{\sqrt{\sin^2 E + \sinh^2 Z}}\frac{1}{\sqrt{\sin^2 \bar{E} + \sinh^2 \bar{Z}}} =$$
$$\left|\frac{\sinh 2Z}{\sinh Z \sinh \bar{Z}}\right| \frac{|\sinh Z|}{\sqrt{\sin^2 E + \sinh^2 Z}}\frac{|\sinh \bar{Z}|}{\sqrt{\sin^2 \bar{E} + \sinh^2 \bar{Z}}} \le c$$

This concludes the proof of Remark E.11.

Remark E.12.

$$\frac{\cosh\left[(Z+\bar{Z})/2\right]}{\sqrt{\sin^2 E+\sinh^2 Z}}, \quad \text{and} \quad \frac{\cosh\left[(Z+\bar{Z})/2\right]}{\sqrt{\sin^2 \bar{E}+\sinh^2 \bar{Z}}} \le c\max\left(1, \frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}}\right)$$

The proof of the above remark is analogous to the proof of Remark E.11.

We can now pass to the proof of Lemma E.4. We first prove estimate (E.6).

$$\begin{aligned}
|\mathscr{H}(\eta,z,\xi)-\mathscr{H}(\eta,z,\bar{\xi})| = \left|\frac{\sinh(2Z)}{\sin^2 E+\sinh^2 Z}-\frac{\sinh(2\bar{Z})}{\sin^2\bar{E}+\sinh^2\bar{Z}}\right| \le \\
\left|\frac{\sinh(2Z)\left(\sin^2 E-\sin^2\bar{E}\right)}{\left(\sin^2 E+\sinh^2 Z\right)\left(\sin^2\bar{E}+\sinh^2\bar{Z}\right)}\right| + \\
\left|\frac{\sinh(2Z)\left(\sinh^2 Z-\sinh^2\bar{Z}\right)}{\left(\sin^2 E+\sinh^2 Z\right)\left(\sin^2\bar{E}+\sinh^2\bar{Z}\right)}\right| + \\
\left|\frac{\sinh(2Z)-\sinh(2\bar{Z})}{\sin^2\bar{E}+\sinh^2\bar{Z}}\right| = \\
A_1+A_2+A_3
\end{aligned}$$

We shall see how to estimate the term A_2, being the estimate of A_1 and A_3 similar.

$$\begin{aligned}
A_2 = \left|\frac{\sinh(2Z)\left(\sinh^2 Z-\sinh^2\bar{Z}\right)}{\left(\sin^2 E+\sinh^2 Z\right)\left(\sin^2\bar{E}+\sinh^2\bar{Z}\right)}\right| \le \\
c\left|\frac{\sinh(2Z)\left(|\sinh Z|+|\sinh\bar{Z}|\right)\cosh\left[(Z+\bar{Z})/2\right]}{\left(\sin^2 E+\sinh^2 Z\right)\left(\sin^2\bar{E}+\sinh^2\bar{Z}\right)}\right| |\eta|(|\eta|+|\xi-\bar{\xi}|) \le \\
\frac{\left(|\sinh Z|+|\sinh\bar{Z}|\right)\cosh\left[(Z+\bar{Z})/2\right]}{\sqrt{\sin^2 E+\sinh^2 Z}\sqrt{\sin^2\bar{E}+\sinh^2\bar{Z}}}\max\left(1,\frac{1}{\sqrt{\eta^2+\varepsilon^2 z^2}}\right)|\eta|(|\eta|+|\xi-\bar{\xi}|) \le \\
c\max\left(1,\frac{1}{\eta^2+\varepsilon^2 z^2}\right)|\eta|(|\eta|+|\xi-\bar{\xi}|)
\end{aligned}$$

In the above estimate the first inequality is due to estimate (E.15) of Remark E.9, the second inequality to Remark E.11 while the third inequality to Remark E.12.

The estimate (E.7) can be proven in the same way, using estimate (E.16) of Remark E.9.

The estimate (E.8) can be proven in the same way, using Remark E.10.

The proof of Lemma E.4 is therefore complete.

Appendix F: Proof of the Lipschitz property of the Biot-Savart operator

The proof of Proposition 8.2 is based on the same ideas we used in the proof of Proposition 8.1. To simplify the notation we define:

$$\Omega^{(i)} = \Omega^{(i)}(\eta+\xi, z+Y), \quad \bar{\Omega}^{(i)} = \Omega^{(i)}(\eta+\bar{\xi}, z+\bar{Y})$$

$$J^{(i)} = \Omega^{(i)}(\eta+\xi), \quad \bar{J}^{(i)} = J^{(i)}(\eta+\bar{\xi})$$

$$X^{(i)} = X^{(i)}(\xi), \quad \bar{X}^{(i)} = X^{(i)}(\bar{\xi}), \quad X^{(i)\prime} = X^{(i)}(\eta+\xi), \quad \bar{X}^{(i)\prime} = X^{(i)}(\eta+\bar{\xi})$$

$$\varphi^{(i)} = \varphi^{(i)}(\xi), \quad \bar{\varphi}^{(i)} = \varphi^{(i)}(\bar{\xi}), \quad \varphi^{(i)\prime} = \varphi^{(i)}(\eta+\xi), \quad \bar{\varphi}^{(i)\prime} = \varphi^{(i)}(\eta+\bar{\xi})$$

$$\mathscr{H}_u^{(i)} \equiv \mathscr{H}_u^{(i)}(\eta,\xi,z) = \frac{1}{8\pi^2} \frac{\sinh 2\left[\varepsilon z + \varphi^{(i)\prime} - \varphi^{(i)}\right)]}{\sin^2\left(\eta + X^{(i)\prime} - X^{(i)}\right) + \sinh^2\left[\varepsilon z + \varphi^{(i)\prime} - \varphi^{(i)}\right)]}$$

$$\bar{\mathscr{H}}_u^{(i)} \equiv \bar{\mathscr{H}}_u^{(i)}(\eta,\xi,z) = \frac{1}{8\pi^2} \frac{\sinh 2\left[\varepsilon z + \bar{\varphi}^{(i)\prime} - \bar{\varphi}^{(i)}\right)]}{\sin^2\left(\eta + \bar{X}^{(i)\prime} - \bar{X}^{(i)}\right) + \sinh^2\left[\varepsilon z + \bar{\varphi}^{(i)\prime} - \bar{\varphi}^{(i)}\right)]}$$

with analogous expressions for $\mathscr{H}_v$ and $\bar{\mathscr{H}}_v$. We also define the operator δ as:

$$\delta\Omega = \Omega^{(1)} - \Omega^{(2)}, \qquad \delta\bar{\Omega} = \bar{\Omega}^{(1)} - \bar{\Omega}^{(2)}$$

$$\delta J = J^{(1)} - J^{(2)}, \qquad \delta\bar{J} = \bar{J}^{(1)} - \bar{J}^{(2)}$$

$$\delta\mathscr{H} = \mathscr{H}^{(1)} - \mathscr{H}^{(2)}, \qquad \delta\bar{\mathscr{H}} = \bar{\mathscr{H}}^{(1)} - \bar{\mathscr{H}}^{(2)}$$

$$\delta U = U[\Omega^{(1)}, \varphi^{(1)}, X^{(1)}] - U[\Omega^{(2)}, \varphi^{(2)}, X^{(2)}]$$

$$\delta V = V[\Omega^{(1)}, \varphi^{(1)}, X^{(1)}] - V[\Omega^{(2)}, \varphi^{(2)}, X^{(2)}]$$

Therefore one has:

$$\delta U(\xi,Y) = \int \delta\Omega\, \mathscr{H}_u^{(1)} J^{(1)}\, d\eta dz + \int \Omega^{(2)}\, \delta\mathscr{H}_u J^{(1)}\, d\eta dz + \int \Omega^{(2)}\, \mathscr{H}_u^{(2)} \delta J\, d\eta dz$$

One can write an analogous expression for $\delta U(\bar{\xi},\bar{Y})$, so that

$$\delta U(\xi,Y) - \delta U(\bar{\xi},\bar{Y}) = \Gamma_1 + \Gamma_2 + \Gamma_3$$

where

$$\Gamma_1 = \int \delta\Omega \mathscr{H}_u^{(1)} J^{(1)} d\eta dz - \int \delta\bar{\Omega} \bar{\mathscr{H}}_u^{(1)} \bar{J}^{(1)} d\eta dz$$

$$\Gamma_2 = \int \Omega^{(2)} \delta\mathscr{H}_u J^{(1)} d\eta dz - \int \bar{\Omega}^{(2)} \delta\bar{\mathscr{H}}_u \bar{J}^{(1)} d\eta dz$$

$$\Gamma_3 = \int \Omega^{(2)} \mathscr{H}_u^{(2)}\, \delta J d\eta dz - \int \bar{\Omega}^{(2)} \bar{\mathscr{H}}_u^{(2)}\, \delta\bar{J} d\eta dz$$

It is clear that the term Γ_1 can be estimated using exactly the same ideas used in the previous Appendix E; in fact for Γ_1 one can make the same estimate (E.4) (when $\xi = \bar{\xi}$) or use the same decomposition (E.5) (when $Y = \bar{Y}$) with $\delta\Omega$, $\mathscr{H}_u^{(1)}$ and $J^{(1)}$ instead of Ω, $\mathscr{H}_u$ and J respectively. The estimate of the term Γ_3 is analogous.

The only term that requires to be estimated is therefore Γ_2 where, instead of the kernel $\mathscr{H}_u$, appears $\delta\mathscr{H}_u$. We shall prove the two Lemmas below that are the correspective of Lemma E.2 and Lemma E.4 of the previous Appendix E.

Lemma F.1. *Suppose to hold the hypotheses of Proposition 8.2. Moreover suppose* $(\eta, z) \neq 0$. *Then:*

$$|\delta\mathscr{H}_u| \leq c \max\left(1, \frac{1}{[\eta^2+\varepsilon^2 z^2]^{1/2}}\right)\left(\|\delta X\|_{1,\rho}^{(\alpha)} + \|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

$$|\delta\mathscr{H}_v| \leq c \max\left(1, \frac{1}{[\eta^2+\varepsilon^2 z^2]^{1/2}}\right)\left(\|\delta X\|_{1,\rho}^{(\alpha)} + \|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

The proof is postponed to Section F.1

Lemma F.2. *Suppose to hold the hypotheses of Proposition 8.2. Moreover suppose* $(\eta, z) \neq 0$. *Then the following estimate holds:*

$$\left|\delta\mathscr{H}_u - \delta\bar{\mathscr{H}_u}\right| \leq c \max\left\{1, \frac{1}{\eta^2+\varepsilon^2 z^2}\right\} \cdot |\eta|^{1-\beta}|\xi-\bar{\xi}|^\beta(|\eta|+|\xi-\bar{\xi}|)$$
$$\left(\|\delta X\|_{1,\rho}^{(\alpha)} + \|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

Analogously:

$$\left|\delta\mathscr{H}_v - \delta\bar{\mathscr{H}_v}\right| \leq c \max\left\{1, \frac{1}{\eta^2+\varepsilon^2 z^2}\right\} \cdot |\eta|^{1-\beta}|\xi-\bar{\xi}|^\beta(|\eta|+|\xi-\bar{\xi}|)$$
$$\left(\|\delta X\|_{1,\rho}^{(\alpha)} + \|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

The proof is postponed to Section F.2.

Using the two above Lemmas the proof of the estimate of the term Γ_2 can now proceed exactly as in the previous Appendix E: the details are omitted.

F.1 Proof of Lemma F.1

We shall show the estimate for $\delta\mathscr{H}_u$, being the estimate for $\delta\mathscr{H}_v$ similar. Moreover we shall focus on the case $\eta^2+\varepsilon^2 z^2 \leq 1$, which is the most difficult to handle. At the end of this section we shall briefly mention how to handle the case $\eta^2+\varepsilon^2 z^2 > 1$. To simplify the notation we define

$$N^{(i)} = \frac{1}{8\pi^2}\sinh 2\left[\varepsilon z + \varphi^{(i)} - \varphi^{(i)\prime})\right]$$
$$D^{(i)} = \sin^2\left(\eta + X^{(i)} - X^{(i)\prime}\right) + \sinh^2\left[\varepsilon z + \varphi^{(i)} - \varphi^{(i)\prime})\right],$$

so that

(F.1)
$$\delta\mathscr{H}_u = \frac{N^{(1)}}{D^{(1)}} - \frac{N^{(2)}}{D^{(2)}} = \frac{N^{(1)}-N^{(2)}}{D^{(1)}} + N^{(2)}\left(\frac{1}{D^{(1)}} - \frac{1}{D^{(2)}}\right) \equiv (\delta\mathscr{H}_u)_A + (\delta\mathscr{H}_u)_B$$

We make the following Remarks:

Remark F.3. Let $\eta^2+\varepsilon^2 z^2 \le 1$ with $(\eta,z)\ne 0$. Then:

$$|N^{(1)}-N^{(2)}| \le c\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)|\eta|$$

This can be proved simply recalling that $\sinh 2a^{(1)}-\sinh 2a^{(2)} = 2\sinh\left(a^{(1)}-a^{(2)}\right)\cosh\left(a^{(1}\right.$ moreover, being $\eta^2+\varepsilon^2 z^2 \le 1$, then the cosh term is bounded by a constant and the sinh term is bounded by its argument times a constant.

Remark F.4. Let $\eta^2+\varepsilon^2 z^2 \le 1$ with $(\eta,z)\ne 0$. Then:

$$\frac{1}{D^{(i)}} \le c\frac{1}{\eta^2+\varepsilon^2 z^2}$$

This Remark is simply a consequence of Remark E.6

The above two Remarks immediately give the desired estimate for $(\delta\mathscr{H}_u)_A$ when $\eta^2+\varepsilon^2 z^2 \le 1$, i.e.:

$$(\delta\mathscr{H}_u)_A \le \frac{c}{\sqrt{\eta^2+\varepsilon^2 z^2}}\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

To prove the same estimate for $(\delta\mathscr{H}_u)_B$ we need the following two additional Remarks

Remark F.5. Let $\eta^2+\varepsilon^2 z^2 \le 1$ with $(\eta,z)\ne 0$. Then:

$$|D^{(2)}-D^{(1)}| \le c\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)\left[\eta^2+\varepsilon^2 z^2\right]$$

If one defines

$$Z^{(i)} = \varepsilon z+\varphi^{(i)\prime}-\varphi^{(i)}$$

it is obvious that

$$|Z^{(i)}| \le c\left[\eta^2+\varepsilon^2 z^2\right]^{1/2}$$

$$|Z^{(2)}-Z^{(1)}| \le c\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)|\eta|$$

The Remark then follows from the identity

$$\sinh^2 Z^{(2)}-\sinh^2 Z^{(1)} = \sinh\left[(Z^{(2)}-Z^{(1)})/2\right]\cosh\left[(Z^{(2)}-Z^{(1)})/2\right](\sinh Z^{(2)}+\sinh Z^{(1)})$$

and from a similar indentity involving $\sin^2$.

Remark F.6. Let $\eta^2+\varepsilon^2 z^2 \le 1$ with $(\eta,z)\ne 0$. Then:

$$\left|\frac{N^{(2)}}{D^{(2)}}\right| \le \frac{1}{\left[D^{(2)}\right]^{1/2}}$$

This can be proved simply recalling that $\sinh 2a = 2\sinh a\cosh a$ and again using the boundedness of the cosh term.

With the help of the above Remarks one can easily estimate $(\delta\mathscr{H}_u)_B$ as follows:

$$|(\delta\mathscr{H}_u)_B| \le c\left|\frac{N^{(2)}}{D^{(1)}D^{(2)}}\right|\left(\|\delta X\|^{(\alpha)}_{1,\rho}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}\right)\left[\eta^2+\varepsilon^2z^2\right] \le$$

$$c\left|\frac{N^{(2)}}{D^{(2)}}\right|\left(\|\delta X\|^{(\alpha)}_{1,\rho}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}\right) \le c\frac{1}{\left[D^{(2)}\right]^{1/2}}\left(\|\delta X\|^{(\alpha)}_{1,\rho}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}\right) \le$$

$$\frac{c}{[\eta^2+\varepsilon^2z^2]^{1/2}}\left(\|\delta X\|^{(\alpha)}_{1,\rho}+\|\delta\varphi\|^{(\alpha)}_{1,\rho}\right)$$

This concludes the proof of Lemma F.1 when $\eta^2+\varepsilon^2z^2\le 1$.

To handle the case $\eta^2+\varepsilon^2z^2>1$ one can: first handle the case when $Z^{(i)}<C$ are bounded, so that $\sinh Z^{(i)}$ and $\cosh Z^{(i)}$ are bounded; second, when $Z^{(i)}>C$, with C sufficiently large, one does not have any singularity of the kernels and one has to make sure of the boundedness for large $Z^{(i)}$, which is easily accomplished.

F.2 Proof of Lemma F.2

We prove the estimate involving $\delta\mathscr{H}_u$, being the estimate involving $\delta\mathscr{H}_v$ similar. The difference between $\delta\mathscr{H}_u$ and $\delta\bar{\mathscr{H}}_u$

$$\delta\mathscr{H}_u-\delta\bar{\mathscr{H}}_u=\left[(\delta\mathscr{H}_u)_A-(\delta\bar{\mathscr{H}}_u)_A\right]+\left[(\delta\mathscr{H}_u)_B-(\delta\bar{\mathscr{H}}_u)_B\right]$$

where we have used the decomposition of $\delta\mathscr{H}_u$ given in section F.1. We shall focus on the estimate of $\left[(\delta\mathscr{H}_u)_B-(\delta\bar{\mathscr{H}}_u)_B\right]$ being the estimate of $\left[(\delta\mathscr{H}_u)_A-(\delta\bar{\mathscr{H}}_u)_A\right]$ easier.

$$\left[(\delta\mathscr{H}_u)_B-(\delta\bar{\mathscr{H}}_u)_B\right]=N^{(2)}\left[\frac{1}{D^{(1)}}-\frac{1}{D^{(2)}}\right]-\bar{N}^{(2)}\left[\frac{1}{\bar{D}^{(1)}}-\frac{1}{\bar{D}^{(2)}}\right]=$$

$$\left(N^{(2)}-\bar{N}^{(2)}\right)\left[\frac{1}{D^{(1)}}-\frac{1}{D^{(2)}}\right]+\bar{N}^{(2)}\left\{\left[\frac{1}{D^{(1)}}-\frac{1}{D^{(2)}}\right]-\left[\frac{1}{\bar{D}^{(1)}}-\frac{1}{\bar{D}^{(2)}}\right]\right\}=$$

$$\left(N^{(2)}-\bar{N}^{(2)}\right)\left[\frac{1}{D^{(1)}}-\frac{1}{D^{(2)}}\right]+\bar{N}^{(2)}\left\{\left[\frac{D^{(2)}-D^{(1)}}{D^{(1)}D^{(2)}}-\frac{D^{(2)}-D^{(1)}}{\bar{D}^{(1)}D^{(2)}}\right]+\right.$$

$$\left.\left[\frac{D^{(2)}-D^{(1)}}{\bar{D}^{(1)}D^{(2)}}-\frac{D^{(2)}-D^{(1)}}{\bar{D}^{(1)}\bar{D}^{(2)}}\right]+\frac{1}{\bar{D}^{(1)}\bar{D}^{(2)}}\left[D^{(2)}-D^{(1)}-\left(\bar{D}^{(2)}-\bar{D}^{(1)}\right)\right]\right\} \tag{F.2}$$

The estimate of the above quantity is immediately achieved thanks to the following Remarks.

Remark F.7. Let $(\eta,z)\neq 0$ and $0\le\beta\le 1$. Then:

$$\left|\frac{N^{(2)}-\bar{N}^{(2)}}{D^{(i)}}\right| \le c\max\left\{1,\frac{1}{\eta^2+\varepsilon^2z^2}\right\}|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)$$

The proof of the above Remark can be achieved with the help of Remark E.10 of the previous Section.

Remark F.8. Let $(\eta,z)\neq 0$. Then:

$$\left|\frac{D^{(2)}-D^{(1)}}{D^{(i)}}\right| \leq c\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right), \qquad i=1,2$$

The proof of the above Remark is based on the same considerations used to prove Remark F.5.

With the use of the above two Remarks one immediately bounds the first term in (F.2).

It is now useful to introduce the notations:

$$\bar{Z}^{(i)} = \varepsilon z + \bar{\varphi}^{(i)\prime} - \varphi^{(i)},$$

as well as

$$E^{(i)} = \eta + X^{(i)\prime} - X^{(i)}$$
$$\bar{E}^{(i)} = \eta + \bar{X}^{(i)\prime} - X^{(i)}$$

Remark F.9. Let $(\eta,z)\neq 0$ and $0\leq\beta\leq 1$. Then:

$$\left|\frac{D^{(i)}-\bar{D}^{(i)}}{D^{(j)}}\right| \leq$$
$$c\max\left\{1,\frac{1}{[\eta^2+\varepsilon^2 z^2]^{1/2}}\right\}\cdot|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)$$

The proof of the above Remark follows from Remark E.10.

With the help of the above Remark and of Remark F.6, one can easily bound the second and the third term in (F.2).

Remark F.10. Let $(\eta,z)\neq 0$ and $0\leq\beta\leq 1$. Then:

$$\frac{1}{[\bar{D}^{(1)}]}\left|D^{(1)}-D^{(2)}-\left(\bar{D}^{(1)}-\bar{D}^{(2)}\right)\right| \leq$$
$$c|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)\left(\|\delta X\|_{1,\rho}^{(\alpha)}+\|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)\max\left\{1,\frac{1}{[\eta^2+\varepsilon^2 z^2]^{1/2}}\right\}$$

The proof is based on Remark E.10, which gives (focusing on the case $\eta^2+\varepsilon^2 z^2\leq 1$) that

$$\left|D^{(1)}-D^{(2)}-\left(\bar{D}^{(1)}-\bar{D}^{(2)}\right)\right| = \left|D^{(1)}-\bar{D}^{(1)}-\left(D^{(2)}-\bar{D}^{(2)}\right)\right| \leq$$
$$c|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)\left|\sin E^{(1)}+\sin\bar{E}^{(1)}-(\sin E^{(2)}+\sin\bar{E}^{(2)})\right| +$$
$$c|\eta|^{1-\beta}|\xi-\bar{\xi}|^{\beta}(|\eta|+|\xi-\bar{\xi}|)\left|\sinh Z^{(1)}+\sin\bar{Z}^{(1)}-(\sin Z^{(2)}+\sin\bar{Z}^{(2)})\right|$$

The proof then follows from the fact that:

$$\left|\sin E^{(1)} - \sin E^{(2)}\right| \le c|\eta| \left(\|\delta X\|_{1,\rho}^{(\alpha)} + \|\delta\varphi\|_{1,\rho}^{(\alpha)}\right)$$

and similar estimates involving $\bar{E}$, Z and $\bar{Z}$. This concludes the proof of the above Remark.

The above Remark together with Remark F.6 give the bound of the fourth term in (F.2), which concludes the proof of Lemma F.2.

Appendix G: Proof of the estimate on $\mathscr{R}$ and of the far field approximation

G.1 Proof of Proposition 8.3

We consider the expression (3.11) for $\mathscr{R}$ and notice that we can rewrite it as:

$$\begin{aligned} &\mathscr{R}(\Omega,\psi,X) = \\ &\sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \mathscr{T}(\xi,\xi',Y')\left[\frac{1}{\mathscr{K}_\psi^\varepsilon} - \frac{1}{\mathscr{K}_\psi^0}\right] J(\xi')d\xi'dY' = \\ &\sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \mathscr{T}(\xi,\xi',Y')\cdot \\ &\varepsilon(Y-Y')\left[\frac{1}{D^\varepsilon} - \frac{\varepsilon(Y-Y')(\varphi-\varphi')}{D^\varepsilon D^0} - 2\frac{(\varphi-\varphi')^2}{D^\varepsilon D^0}\right] J(\xi')d\xi'dY' = \\ &A^1 + A^2 + A^3 \end{aligned} \tag{G.1}$$

where we have defined

$$D^0 \equiv (\xi-\xi')^2 + (\varphi-\varphi')^2$$

$$D^\varepsilon \equiv (\xi-\xi')^2 + \left[(\varphi-\varphi') + \varepsilon(Y-Y')\right]^2$$

The term $\varepsilon(Y-Y')$, together with the exponential decay in Y' of the function $\mathscr{T}(\xi,\xi',Y')$, gives the $\varepsilon(Y+1)$ behavior expressed in the thesis of the Proposition. The difficulty in the estimate of the remaining term is in their singular character at the origin: the fact that $\mathscr{T}(\xi,\xi,Y') = 0$, together with the regularity of $\mathscr{T}$, gives a $O(\xi-\xi')$ behavior which, compared with the quadratic behavior of D^ε and D^0, would lead to logarithmic singularities in each of the A^i. However, using cancellation properties, one can estimate each of A^i. In what follows by $A^i_{n=0}$ we shall denote, in the n-series in (G.1), the terms with $n=0$, which are the singular ones:

$$A^1_{n=0} \equiv \int_{-\infty}^{\infty}\int_{\xi-\pi/2}^{\xi+\pi/2} \mathscr{T}(\xi,\xi',Y')\varepsilon(Y-Y')\frac{1}{D^\varepsilon}J(\xi')d\xi' dY' \approx$$

$$\int_{-\infty}^{\infty}\int_{\xi-\pi/2}^{\xi+\pi/2} \partial_\xi \mathscr{T}(\xi,\xi,Y')(\xi-\xi')\varepsilon(Y-Y')\frac{1}{D^\varepsilon}J(\xi')d\xi' dY' \approx$$

$$\int_{-\infty}^{\infty} \varepsilon(Y-Y')\partial_\xi \mathscr{T}(\xi,\xi,Y')\int_{\xi-\pi/2}^{\xi+\pi/2} \frac{(\xi-\xi')}{(\xi-\xi')^2+\left[\varepsilon(Y-Y')+\partial_\xi\varphi(\xi)(\xi-\xi')\right]^2}J(\xi)d\xi' dY'$$

where by the simbol $\approx$ we mean equal up to nonsingular terms. The cancellation property can be shown as follows. Define ξ'' as:

$$\xi-\xi' = a(\xi-\xi'')-b \quad \text{where} \quad a\equiv\frac{1}{\sqrt{1+[\partial_\xi\varphi(\xi)]^2}},\ b\equiv\varepsilon(Y-Y')\partial_\xi\varphi(\xi)a^2$$

and rewrite $A^1_{n=0}$ as

$$A^1_{n=0} \equiv \int_{-\infty}^{\infty} \varepsilon(Y-Y')\partial_\xi \mathscr{T}(\xi,\xi,Y')J(\xi)\cdot$$

$$\int_{\xi-\pi/(2a)-b/a}^{\xi+\pi/(2a)-b/a}\left[\frac{(\xi-\xi'')}{(\xi-\xi'')^2+\varepsilon^2(Y-Y')^2}-\frac{b}{(\xi-\xi'')^2+\varepsilon^2(Y-Y')^2}\right]d\xi'' dY'$$

Both terms inside the integration are singular; however, for the first term, the singularity cancels because the function is odd with respect to the singularity $\xi''=\xi$. For the second term, introducing $z\equiv(\xi-\xi'')/(\varepsilon(Y-Y'))$ one immediately recognizes that, after integration in dz, one obtains a finite constant (notice that $b\sim\varepsilon(Y-Y')$). The term A^1 is thus estimated.

The estimate of A^2 and A^3 can be easily achieved using the same ideas.

G.2 Proof of Proposition 8.4

Looking at the expression for $\mathscr{T}$ (3.7), and inserting $\Omega=\omega_0$, $\psi=\varphi_0+\varepsilon\varphi_1$ and $X=X_0+\varepsilon X_1$ one recognizes that $\mathscr{T}$ can be decomposed as $\mathscr{T}=\mathscr{T}_0+\varepsilon\mathscr{T}_1$, where in the leading order part $\mathscr{T}_0$ appear only ω_0, φ_0 and X_0. Shrinking the strip of analyticity of ω_0, φ_0 and X_0 one can get the regularity properties required to use the same procedure used in the proof of Proposition 8.3 reported above. Concerning the term $\varepsilon\mathscr{T}_1$, being already $O(\varepsilon)$, one does not need to use the fact the difference $1/\mathscr{K}^\varepsilon_\psi-1/\mathscr{K}^0_\psi$ is $O(\varepsilon)$, and the desired estimate follows estimating separately the terms involving $1/\mathscr{K}^\varepsilon_\psi$ and $1/\mathscr{K}^0_\psi$ separately. These estimates are easily achieved using standard arguments. This concludes the proof of Proposition 8.4.

G.3 Proof of Proposition 8.5

To simplify the notation we use the complex variable and define:

$$\mathscr{K}^{\varepsilon(i)}=(\xi-\xi')+X_0+\varepsilon X_1^{(i)}-(X_0'+\varepsilon X_1^{(i)'})+i\left[\varepsilon(Y-Y')+\varphi_0+\varepsilon\varphi_1^{(i)}-({\varphi_0}'+\varepsilon{\varphi_1}^{(i)'})\right]$$

$$\mathscr{K}^{0(i)} = (\xi-\xi') + X_0 + \varepsilon X_1^{(i)} - (X_0' + \varepsilon X_1^{(i)'}) + i\left[\varphi_0 + \varepsilon\varphi_1^{(i)} - ({\varphi_0}' + \varepsilon{\varphi_1}^{(i)'})\right]$$

where, as usual, the primed quantities are computed in ξ'.

Looking at the expression of $\mathscr{R}$ one sees that $\delta\mathscr{R}$ can be conveniently written as:

$$\delta\mathscr{R} = B_1 + B_2 + B_3$$

where

$$B_1 = \sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \delta\mathscr{T}(\xi,\xi',Y')\left[\frac{1}{\mathscr{K}^{\varepsilon(1)}} - \frac{1}{\mathscr{K}^{0(1)}}\right] J^{(1)}(\xi')d\xi' dY'$$

$$B_2 = \sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \mathscr{T}^{(2)}(\xi,\xi',Y')\delta\left[\frac{1}{\mathscr{K}^{\varepsilon}} - \frac{1}{\mathscr{K}^{0}}\right] J^{(1)}(\xi')d\xi' dY'$$

$$B_3 = \sum_n \int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \mathscr{T}^{(2)}(\xi,\xi',Y')\left[\frac{1}{\mathscr{K}^{\varepsilon(2)}} - \frac{1}{\mathscr{K}^{0(2)}}\right] \delta J(\xi')d\xi' dY'$$

The term B_3 can be easily estimated given that $J = 1+X$, so that $\delta J = \varepsilon\delta X_1$. To estimate the term B_1 one has to use again, as we did in Section G.2 above, the decomposition $\mathscr{T} = \mathscr{T}_0 + \varepsilon\mathscr{T}_1$, and write:

$$\delta\mathscr{T}_0(\xi,\xi') = \delta\left[\omega_0(\xi') - \omega_0(\xi)\right] + \delta\omega_0(\xi)\left[\tilde{t}_0^{*(1)}(\xi') - \tilde{t}_0^{*(1)}(\xi)\right](1 + i\partial_\xi{\varphi_0}^{(1)}) +$$
$$\omega_0^{(2)}(\xi)\delta\left[\tilde{t}_0^{*}(\xi') - \tilde{t}_0^{*}(\xi)\right](1 + i\partial_\xi{\varphi_0}^{(1)}) + \omega_0^{(2)}(\xi)\left[\tilde{t}_0^{*(2)}(\xi') - \tilde{t}_0^{*(2)}(\xi)\right] i\delta\partial_\xi\varphi_0$$

Using the fact that $\delta\mathscr{T}_0(\xi,\xi')_{\xi=\xi'} = 0$, and the fact that $\delta\mathscr{T}_0$ depends only on ω_0, φ_0 and X_0, i.e. on quantities that can be supposed to have the necessary regularity, one can adopt the same procedure of Section G.1, and get the desired estimate. The term $\varepsilon\delta\mathscr{T}_1$, being already $O(\varepsilon)$, can be estimated using standard arguments.

To estimate B_2 it is enough to write that:

$$\delta\frac{1}{\mathscr{K}^{\varepsilon}} = \varepsilon\frac{\delta\left[X_1(\xi) - X_1(\xi')\right] + \delta\left[\varphi_1(\xi) - \varphi_1(\xi')\right]}{\mathscr{K}^{\varepsilon(1)}\mathscr{K}^{\varepsilon(2)}}$$

with a similar expression for $\delta(1/\mathscr{K}^0)$. In the above expression, given that ${\varphi_1}^{(i)}, X_1^{(i)} \in B^{1,\alpha}_{\rho/2}$, one can use the fact that the numerator is zero when $\xi = \xi'$ to lower the singularity of the denominator. The order one singularity of the denominator can therefore be estimated as done in Section G.1. This completes the proof of Proposition 8.5.

G.4 Proof of Proposition 8.6

Given that $Y > \varepsilon^{-1}$ we can approximate the velocity as:

$$u - iv = \frac{1}{2\pi i}\sum_n \int_{-Y/3}^{Y/3}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\Omega(\xi',Y')}{\mathscr{K}_\psi^{\varepsilon}(\xi,\xi',Y-Y')} J d\xi' dY' + O(e^{-\mu/(3\varepsilon)}),$$

In the rest of our analysis we shall neglect the exponentially small term. We now make the following remark:

Remark G.1. Given that $\|X\|_{1,\rho}^{(\alpha)} < 1/4$ and $\|\varphi\|_{1,\rho}^{(\alpha)} < 1/4$, then:

$$\left|\xi-\xi'+X-X'+i\left[\psi(\xi)-\psi(\xi')+\varepsilon Y\right]\right| \geq \frac{1}{\sqrt{2}}\sqrt{|\xi-\xi'|^2+\varepsilon^2Y^2}$$

The estimate above can be obtained as follows:

$$\begin{aligned} \left|\xi-\xi'+X-X'+i\left[\psi(\xi)-\psi(\xi')+\varepsilon Y\right]\right| \geq \\ \left[3(\xi-\xi')^2/4-3(X-X')^2+(\varepsilon Y)^2/2-(\psi-\psi')^2\right]^{1/2} \geq \\ \frac{1}{\sqrt{2}}\sqrt{|\xi-\xi'|^2+\varepsilon^2Y^2} \end{aligned} \tag{G.2}$$

The first of the above inequalities follows from the elementary facts: $2AB \geq -(A^2/4+4B^2)$ and $2AB \geq -(A^2/2+2B^2)$. The second inequality in (G.2) is an obvious consequence of $\sup_\xi |\partial_\xi \psi(\xi)| < 1/4$ and $\sup_\xi |X(\xi)| < 1/4$.

We now define

$$a = \frac{i\varepsilon Y'}{\xi-\xi'+X-X'+i\left[\psi(\xi+X)-\psi(\xi'+X')+\varepsilon Y\right]},$$

and, given that $|Y'| \leq Y/3$ and the above Remark, one immediately recognizes that $|a| < 1$.

Using $|a| < 1$ we can write

$$\begin{aligned} \frac{1}{\xi+X-\xi'-X'+i\left[\psi-\psi'+\varepsilon(Y-Y')\right]} = \\ \frac{1}{\xi+X-\xi'-X'+i\left[\psi-\psi'+\varepsilon Y\right]}\frac{1}{1-a} \\ \frac{1}{\xi+X-\xi'-X'+i\left[\psi-\psi'+\varepsilon Y\right]}\sum_{k=0}^{\infty} a^k. \end{aligned}$$

Therefore, in computing the difference between the velocity and the far field, the term with $k=0$ cancels:

$$\begin{aligned} u+iv-u^f-iv^f = \\ \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{-Y/3}^{Y/3}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2}\sum_{k=1}^{\infty}\frac{\Omega(\xi',Y')\,(i\varepsilon Y')^k\,J d\xi' dY'}{\left\{\xi-\xi'+X-X'+i\left[\psi-\psi'+\varepsilon Y\right]\right\}^{k+1}} \end{aligned} \tag{G.3}$$

We can now give the following estimate:

$$\left| \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{-Y/3}^{Y/3} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \sum_{k=1}^{\infty} \frac{\Omega(\xi',Y')\,(i\varepsilon Y')^k J d\xi' dY'}{\{\xi-\xi'+X-X'+i[\psi-\psi'+\varepsilon Y]\}^{k+1}} \right| \leq$$

$$c\,\|\Omega\|_{0,\rho,\sigma,\mu} \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \int_{-Y/3}^{Y/3} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{e^{-\mu|Y'|}\,|\varepsilon Y'|^k d\xi' dY'}{|\xi-\xi'+X-X'+i[\psi-\psi'+\varepsilon Y]|^{k+1}} \leq$$

$$c \sum_{k=1}^{\infty} \int_{-Y/3}^{Y/3} dY' e^{-\mu|Y'|} \left|\varepsilon Y'\right|^k \sum_{n=-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{2^{(k+1)/2} d\xi'}{[|\xi-\xi'|^2+\varepsilon^2 Y^2]^{(k+1)/2}} \leq$$

$$c \sum_{k=1}^{\infty} \int_{0}^{Y/3} dY' e^{-\mu Y'} \left|\frac{Y'}{Y}\right|^k \int_{-\infty}^{\infty} \frac{2^{(k+1)/2} dz}{[z^2+1]^{(k+1)/2}} \leq$$

$$c \sum_{k=1}^{\infty} 2^{(k+1)/2} \int_{0}^{Y/3} dY' e^{-\mu Y'} \left|\frac{Y'}{Y}\right|^k$$

Looking, in the above series, at the term with $k=1$, one immediately recognizes the boundedness an the $1/Y$ behavior in Y. We now consider the integral in $[0,Y]$ of the generic term of the series and show: 1) the boundedness of the integral when $Y \to \infty$; and 2) the convergence of the series.

$$2^{(k+1)/2} \int_0^Y dY' \int_0^{Y'/3} dY'' e^{-\mu Y''} \left[\frac{Y''}{Y'}\right]^k =$$

$$2^{(k+1)/2} \int_0^{Y/3} dY'' \int_{3Y''}^{Y} dY' e^{-\mu Y''} \left[\frac{Y''}{Y'}\right]^k \leq$$

$$c \frac{2^{(k+1)/2}}{(k-1)3^{k-1}} \int_0^{Y/3} dY'' e^{-\mu Y''} Y'' \leq$$

$$c \frac{1}{2^{k/2}}$$

The proof of the far field estimate Proposition 8.6 is thus achieved.

Appendix H: Proof of Proposition 9.3

Remark H.1. From Remark 7.3 we know that $\gamma_0 \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}$, $\gamma_0^+ - \gamma_0^- \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}$, $\varphi_0 \in B^{3,\alpha}_{\rho_0,\beta_0,T_0}$ $X_0 \in B^{2,\alpha}_{\rho_0,\beta_0,T_0}$ are known quantities. Taking T_0 small enough, one can suppose that $1+\partial_\xi X_0 \geq c > 0$.

The above Remark is an obvious consequence of the fact that $X_0(\xi, t=0) = 0$.

Notice that in Remark 7.3 we have increased the regularity of γ_0, φ_0 and X_0 because these quantities in the equation for ω_0 appear through their derivatives: directly the stretching factor X_0 and through the BR operator the vorticity intensity γ_0 and the curve φ_0. We can now pass to the proof of Proposition 9.3.

We first estimate the operator F_2. We write:

$$F_2 \equiv \int_0^Y \partial_\xi \left[\mathscr{B}\mathscr{R}_0 + \frac{1}{2}\left(\int_{Y'}^\infty \omega_0 dY'' - \int_{-\infty}^{Y'} \omega_0 dY'' \right) \right] dY' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} =$$

$$\partial_\xi \mathscr{B}\mathscr{R}_0 \frac{Y\partial_Y \omega_0}{1+\partial_\xi X_0} + \frac{1}{2}\int_0^Y \left(\int_{Y'}^\infty \partial_\xi \omega_0 dY'' - \int_{-\infty}^{Y'} \partial_\xi \omega_0 dY'' \right) dY' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} \tag{H.1}$$

The first of these terms is easily estimated knowing that γ_0 and φ_0 are bounded in the appropriate function spaces, using the properties of the BR operator, and using the Cauchy estimate given in (6.4) to bound the linear growth of Y and the Y-derivative.

We now show how to estimate the second of the three terms in the above expression for F_2. We have to estimate this operator in the norm $\|\cdot\|_{1,\rho,\theta,\mu}^{(\alpha)}$ and therefore we show separately how to estimate the Hölder norm of the ξ and of the Y derivatives. The Y derivative is more easily estimated because, when it hits the integral term, it is compensated by the integration in Y while, when it hits $\partial_Y \omega_0$, it is bounded using the Cauchy estimate. We are therefore left with estimating the $|\cdot|^{(\alpha)}$ norm of the ξ-derivative. We can estimate the above norm estimating separately $|\cdot|^{(\alpha,\xi)}$ and $|\cdot|^{(\alpha,Y)}$. We notice that the Hölder norm along the Y-direction is more easily estimated because of the presence of the integration in Y and therefore we focus on the estimate of the Hölder norm along the ξ-direction. We show explicitly how to estimate the case when ∂_ξ hits the integral term:

$$\left\| \int_0^Y \left(\int_{Y'}^\infty \partial_\xi^2 \omega_0 dY'' \right) dY' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} \right\|_{\rho,\theta,\mu}^{(\alpha,\xi)} =$$

$$\sup_{Y>0} \left\| e^{\mu Y} \int_0^Y dY' \int_{Y'}^\infty \partial_\xi^2 \omega_0 dY'' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} \right\|_\rho^{(\alpha,\xi)} +$$

$$\sup_{Y<0} \left\| e^{-\mu Y} \int_0^Y dY' \int_{Y'}^\infty \partial_\xi^2 \omega_0 dY'' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} \right\|_\rho^{(\alpha,\xi)} \leq$$

$$c \left\| \partial_\xi^2 \omega_0 \right\|_{\rho,\theta,\mu}^{(\alpha,\xi)} \|\partial_Y \omega_0\|_{\rho,\theta,\mu}^{(\alpha,\xi)} \cdot \sup_{Y>0} \left| \int_0^Y dY' \int_{Y'}^\infty e^{-\mu Y''} dY'' \right| +$$

$$c\|\partial_\xi^2 \gamma_0\|_\rho^{(\alpha,\xi)} \sup_{Y<0} \|e^{-\mu Y} Y \partial_Y \omega_0\|_\rho^{(\alpha,\xi)} + c \sup_{Y<0} \left\| e^{-\mu Y} \int_0^Y dY' \int_{-\infty}^{Y'} \partial_\xi^2 \omega_0 dY'' \frac{\partial_Y \omega_0}{1+\partial_\xi X_0} \right\|_\rho^{(\alpha,\xi)} \leq$$

$$c \left[\frac{\|\omega_0\|_{1,\rho',\theta,\mu}^{(\alpha)}}{\rho'-\rho} + \frac{\|\omega_0\|_{1,\rho,\theta,\mu'}^{(\alpha)}}{\mu'-\mu} \right]$$

Notice how we have used the estimate on X_0 given in Remark H.1. Moreover, in the estimate above we have supposed Y to be real. When Y is complex the same estimate can be carried out with minor modifications. The case when ∂_ξ hits $\partial_Y \omega_0$ is more easily handled through a direct use of the Cauchy estimate.

The third term in the expression (H.1) for F_2 can be handled analogously.

Using the linearity of the derivative and of the integral, the proof of the quasi-contractiveness of F_2 is now straightforward.

The quasi–contractiveness of the operator F_1 is easily achieved through a Cauchy estimate of the ξ-derivative and using the fact that the operator $\mathscr{M}_0 \equiv \mathscr{M}(\omega_0, \varphi_0, X_0)$ is bounded in terms of the norm of φ_0 and X_0 (which we know, from Proposition 7.2, to be *a priori* bounded in the appropriate analytic space) and in terms of the norm of ω_0.

Appendix I: Proof of Proposition 9.4

We first state a series of technical Lemmas that will be useful in the estimate of the various G_j.

Lemma I.1. *Suppose* $\omega_0 \in B^{1,\alpha}_{\rho,\theta,\mu}$, $\varphi_0 \in B^{2,\alpha}_{\rho}$, $X_0 \in B^{2,\alpha}_{\rho}$, $\varphi_1 \in B^{1,\alpha}_{\rho}$, $X_1 \in B^{1,\alpha}_{\rho}$,. *Then the following estimate holds:*

$$\|\mathscr{M} - \mathscr{M}_0\|^{(\alpha)}_{\rho,\theta} \leq c\varepsilon \left(\|\varphi_1\|^{(\alpha)}_{1,\rho} + \|X_1\|^{(\alpha)}_{1,\rho} \right) \tag{I.1}$$

where the constant c depends only on $\|\omega_0\|^{(\alpha)}_{1,\rho,\theta,\mu}$, $\|\varphi_0\|^{(\alpha)}_{2,\rho}$ *and* $\|X_0\|^{(\alpha)}_{2,\rho}$.

To prove the above Lemma it is enough to recall that, by definition:

$$\mathscr{M} \equiv \mathscr{B}\mathscr{R}[\gamma_0, \varphi_0 + \varepsilon\varphi_1, X_0 + \varepsilon X_1] + \frac{1}{2}\left[\int_Y^\infty \omega_0 dY' - \int_{-\infty}^Y \omega_0 dY'\right]\tilde{t}^*$$

$$\mathscr{M}_0 \equiv \mathscr{B}\mathscr{R}[\gamma_0, \varphi_0, X_0] + \frac{1}{2}\left[\int_Y^\infty \omega_0 dY' - \int_{-\infty}^Y \omega_0 dY'\right]\tilde{t}_0^* .$$

Looking at the definition of $\tilde{t}$ and $\tilde{t}_0$ (see (3.6) with $\psi = \varphi$ and (3.15)) it is obvious that $\|\tilde{t} - \tilde{t}_0\|^{(\alpha)}_{\rho} \leq c\varepsilon\|\varphi_1\|^{(\alpha)}_{1,\rho}$.

Moreover, from the properties of the BR operator (see Proposition 6.5) one immediately recognizes that:

$$\|\mathscr{B}\mathscr{R}[\gamma_0, \varphi_0 + \varepsilon\varphi_1, X_0 + \varepsilon X_1] - \mathscr{B}\mathscr{R}[\gamma_0, \varphi_0, X_0]\|^{(\alpha)}_{\rho,\theta} \leq c\varepsilon\left(\|\varphi_1\|^{(\alpha)}_{1,\rho} + \|X_1\|^{(\alpha)}_{1,\rho}\right) .$$

Lemma I.1 is therefore proved.

Lemma I.2. *Suppose the hypotheses of Proposition 9.4 are satisfied. Then the following estimate holds:*

$$\|\delta(\mathscr{M} - \mathscr{M}_0)\|^{(\alpha)}_{\rho,\theta,\beta,T} \leq c\varepsilon\left(\|\delta\omega_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} + \|\delta\varphi_1\|^{(\alpha)}_{\rho,\beta,T} + \|\delta X_1\|^{(\alpha)}_{\rho,\beta,T}\right)$$

To prove the Lemma it is enough to recall that, by definition

$$\begin{aligned}\delta(\mathscr{M} - \mathscr{M}_0) = \\ \left(\mathscr{M}[\omega_0^{(1)}, \varphi_0 + \varepsilon{\varphi_1}^{(1)}, X_0 + \varepsilon X_1^{(1)}] - \mathscr{M}[\omega_0^{(1)}, \varphi_0, X_0]\right) - \\ \left(\mathscr{M}[\omega_0^{(2)}, \varphi_0 + \varepsilon{\varphi_1}^{(2)}, X_0 + \varepsilon X_1^{(2)}] - \mathscr{M}[\omega_0^{(2)}, \varphi_0, X_0]\right)\end{aligned}$$

and then use the estimate on $\mathscr{M}$ given in Proposition 6.6.

The following Lemma is an obvious consequence of the fact that $\|X_1^{(i)}\|_{1,\rho}^{(\alpha)} < 1/4 < 1$.

Lemma I.3. *Suppose the hypotheses of Proposition 9.4 are satisfied. Then the following estimates hold:*

$$\left\| \frac{1}{1+\partial_\xi\left(X_0+\varepsilon X_1^{(i)}\right)} \right\|_\rho^{(\alpha)} \leq c$$

$$\left\| \frac{1}{1+\partial_\xi\left(X_0+\varepsilon X_1^{(i)}\right)} - \frac{1}{1+\partial_\xi X_0} \right\|_\rho^{(\alpha)} \leq c\varepsilon \|X_1^{(i)}\|_\rho^{(\alpha)},$$

$$\left\| \delta \frac{1}{1+\partial_\xi\left(X_0+\varepsilon X_1\right)} \right\|_\rho^{(\alpha)} \leq c\varepsilon \|\delta X_1\|_{1,\rho}^{(\alpha)}$$

I.1 Estimate on G_1

The desired bound on G_1 is an immediate consequence of the following estimate:

$$\left\| \delta \left[\left(\frac{\mathscr{M}}{1+\partial_\xi X} - \frac{\mathscr{M}_0}{1+\partial_\xi X_0} \right) \partial_\xi \omega_0 \right] \right\|_{\rho',\theta',\mu'/2}^{(\alpha)} \leq$$

$$\left\| \delta \left[\left(\frac{\mathscr{M}-\mathscr{M}_0}{1+\partial_\xi X} \right) \partial_\xi \omega_0 \right] \right\|_{\rho',\theta',\mu'/2}^{(\alpha)} + \left\| \delta \left[\mathscr{M}_0 \left(\frac{1}{1+\partial_\xi X} - \frac{1}{1+\partial_\xi X_0} \right) \partial_\xi \omega_0 \right] \right\|_{\rho',\theta',\mu'/2}^{(\alpha)} \leq$$

$$c\varepsilon \left(\|\delta\omega_0\|_{1,\rho',\theta',\mu'}^{(\alpha)} + \frac{\|\delta X_1\|_{1,\rho}^{(\alpha)}}{\rho-\rho'} + \frac{\|\delta\varphi_1\|_{1,\rho}^{(\alpha)}}{\rho-\rho'} \right)$$

where to get the last inequality, we have used the Leibnitz properties of the δ operator, Lemmas I.1, I.2 and I.3, and the higher regularity of ω_0 to bound $\partial_\xi \omega_0$.

I.2 Estimate on G_2

To estimate G_2 one has to consider separately the terms $\delta(\mathscr{R}^u \partial_\xi \omega_0)$ and $\delta(\varepsilon u_1 \partial_\xi \omega_0)$. The latter term is easily estimated using the potential estimates given in Propositions 8.1 and 8.2 to bound u_1 in terms of ω_1. The estimate on $\delta(\mathscr{R}^u \partial_\xi \omega_0)$ instead, derives from Propositions 8.4 and 8.5. Here we show how to estimate $\delta\mathscr{R}^u \partial_\xi \omega_0^{(1)}$, being the estimate on $\mathscr{R}^{u(2)}\delta\partial_\xi\omega_0$ similar.

$$
\begin{aligned}
&\|\delta\mathscr{R}^u\partial_\xi\omega_0^{(1)}\|^{(\alpha)}_{\rho',\theta',\mu'/2} \leq \\
&\sup_{Y\in\Sigma(\theta')} e^{\mu'|Y|/2}\|\delta\mathscr{R}^u(\cdot,Y)\|^{(\alpha)}_{\rho'}\|\partial_\xi\omega_0(^{(1)}\cdot,Y)\|^{(\alpha)}_{\rho'} \leq \\
c\varepsilon &\sup_{Y\in\Sigma(\theta')} e^{\mu'|Y|/2}\left[(1+Y)\|\delta\omega_0\|_{1,\rho',\theta',\mu'}+\|\delta\varphi_1\|^{(\alpha)}_{1,\rho'}+\|\delta X_1\|^{(\alpha)}_{1,\rho'}\right]\|\partial_\xi\omega_0^{(1)}(\cdot,Y)\|^{(\alpha)}_{\rho'} \leq \\
&c\varepsilon\left[\|\delta\omega_0\|_{1,\rho',\theta',\mu'}+\|\delta\varphi_1\|^{(\alpha)}_{1,\rho'}+\|\delta X_1\|^{(\alpha)}_{1,\rho'}\right]\|\omega_0^{(1)}\|^{(\alpha)}_{1,\rho',\theta',\mu'} \leq \\
&c\varepsilon\left[\|\delta\omega_0\|_{1,\rho',\theta',\mu'}+\|\delta\varphi_1\|^{(\alpha)}_{1,\rho'}+\|\delta X_1\|^{(\alpha)}_{1,\rho'}\right]
\end{aligned}
$$

The second inequality above is an application of Proposition 8.5, while the third inequality is a consequence of the higher x-regularity of ω_0 that allows to bound the ξ-derivative, and of the stronger exponential decay rate in Y of ω_0, that allows to bound the linearly growing term $(1+Y)$.

I.3 Estimate on G_3

The estimate of $\|\delta G_3\|^{(\alpha)}_{\rho',\theta',\mu'/2}$ is based on the same ideas we used to estimate G_1, the only difference being that integration in Y gives rise to a linearly growing factor. Such linear growth, however, is easily tamed by the exponential decay of ω_0, which in fact decays at rate μ'.

I.4 Estimate on G_4

To show how to estimate δG_4 we first consider the term involving $\mathscr{R}^u$. Given the Leibnitz properties of the operator δ we consider the case when δ acts on $\mathscr{R}^u$, being the other cases similar.

$$\left\| \frac{1}{1+\partial_\xi\left(X_0+\varepsilon X_1^{(2)}\right)} \int_0^Y \delta\left(\partial_\xi \mathscr{R}^u\right) dY' \partial_Y \omega_0^{(1)} \right\|_{\rho',\theta',\mu'/}^{(\alpha)}$$

$$\left\| \frac{1}{1+\partial_\xi\left(X_0+\varepsilon X_1^{(2)}\right)} \right\|_{\rho'}^{(\alpha)} \sup_{Y\in\Sigma(\theta')} e^{\mu'|Y|/2} \left(\int_0^Y \left\|\delta\left(\partial_\xi \mathscr{R}^u(\cdot,Y')\right)\right\|_{\rho'}^{(\alpha)} dY' \left\|\partial_Y \omega_0^{(1)}(\cdot,Y)\right\|_{\rho'}^{(\alpha)} \right)$$

$$c \sup_{Y\in\Sigma(\theta')} e^{\mu'|Y|/2} \left(\int_0^Y \frac{\left\|\delta\left(\mathscr{R}^u(\cdot,Y')\right)\right\|_{\rho}^{(\alpha)}}{\rho-\rho'} dY' \left\|\partial_Y \omega_0^{(1)}(\cdot,Y)\right\|_{\rho'}^{(\alpha)} \right)$$

$$c\varepsilon \sup_{Y\in\Sigma(\theta')} e^{\mu'|Y|/2} \left(\int_0^Y \frac{\left[(1+Y')\|\delta\omega_0\|_{1,\rho,\theta',\mu'} + \|\delta\varphi_1\|_{1,\rho}^{(\alpha)} + \|\delta X_1\|_{1,\rho}^{(\alpha)}\right]}{\rho-\rho'} dY' \left\|\partial_Y \omega_0^{(1)}(\cdot,Y)\right\|_{\rho'}^{(\alpha)} \right)$$

$$c\varepsilon \frac{\left[\|\delta\omega_0\|_{1,\rho,\theta',\mu'} + \|\delta\varphi_1\|_{1,\rho}^{(\alpha)} + \|\delta X_1\|_{1,\rho}^{(\alpha)}\right]}{\rho-\rho'} \left\|\omega_0^{(1)}\right\|_{1,\rho',\theta',\mu}^{(\alpha)}$$

$$c\varepsilon \frac{\left[\|\delta\omega_0\|_{1,\rho,\theta',\mu'} + \|\delta\varphi_1\|_{1,\rho}^{(\alpha)} + \|\delta X_1\|\right.}{\rho-\rho'}$$

In the above estimate, the second inequality has been obtained using the Cauchy estimate on $\partial_\xi \delta\mathscr{R}^u$; the third inequality derives from the estimate on $\delta\mathscr{R}$ given in Proposition 8.5; to get the fourth inequality we have used the higher exponential decay rate in Y of ω_0 to bound the quadratically growing term in Y deriving from the integration.

I.5 Estimate on G_5

The estimate on the term G_5 is straighforward, and is obtained using a Cauchy estimate to bound $\partial_\xi\omega_1$, and using the potential estimates given in Section 8.1 to bound u_0 and u_1 in terms of ω_0 and ω_1 respectively.

I.6 Estimate on G_6

The estimate on the term G_6 is similar to the estimate of G_5, the only difference being the fact that, to bound $\partial_\xi u_0$, one has to use, combined, the potential estimate to get a bound in terms of ω_0, and the higher regularity of ω_0; finally, the linear growth given by the integration in Y, is bounded using the Cauchy estimate given in (6.4).

I.7 Estimate on G_7

The estimate of the term G_7 does not present any difficulty, and can be easily achieved using the potential estimates to bound u_1 and v_1 in terms of ω_1, and using the Cauchy estimate to bound $\partial_Y\omega_1$.

Appendix J: Details on the convergence to Birkhoff-Rott

J.1 Derivations of equations 10.1 and 10.2

From Remarks A.1, A.2 and A.3 one has that, on the curve φ:

$$\partial_t = \partial_\tau - u^\varphi \frac{\partial_\xi}{1+\partial_\xi X}, \qquad \partial_x = \frac{1}{1+\partial_\xi X}\partial_\xi \tag{J.1}$$

where X is the Lagrangian factor

$$X(\xi,\tau) = \int_0^\tau u^\varphi(\xi,\tau')d\tau',$$

having, as usual, indicated with (u^φ, v^φ) the velocity computed on φ, i.e.

$$u^\varphi = u(\xi, Y=0,\tau) = [u_0+\varepsilon u_1]_{Y=0} = [\mathscr{M}^u[\omega_0,\varphi] + \mathscr{R}^u[\omega_0,\varphi] + \varepsilon u_1]_{Y=0}.$$

$$v^\varphi = v(\xi, Y=0,\tau) = [v_0+\varepsilon v_1]_{Y=0} = [\mathscr{M}^v[\omega_0,\varphi] + \mathscr{R}^v[\omega_0,\varphi] + \varepsilon v_1]_{Y=0}.$$

Equation (7.2), after some rearrangement, can be written as:

$$\partial_\tau \gamma_0 - \frac{u^\varphi}{1+\partial_\xi X}\partial_\xi \gamma_0 + \frac{1}{1+\partial_\xi X}\partial_\xi\left(\gamma_0 \mathscr{B}\mathscr{R}_0^u\right) = \mathscr{E}_1(\xi,\tau)$$

where

$$\mathscr{E}_1(\xi,\tau) = -\left[\frac{u^\varphi}{1+\partial_\xi X} - \frac{\mathscr{M}^u_{0Y=0}}{1+\partial_\xi X_0}\right]\partial_\xi \gamma_0 + \left[\frac{1}{1+\partial_\xi X} - \frac{1}{1+\partial_\xi X_0}\right]\partial_\xi\left(\gamma_0 \mathscr{B}\mathscr{R}_0^u\right)$$

Given the expressions (J.1), the equation for γ_0, in the Eulerian reference frame, can therefore be written:

$$\partial_t \gamma_0 + \partial_x\left(\gamma_0 BR_0^u\right) = E_1(x,t)$$

where $E_1(x,t) = \mathscr{E}_1(\xi(x,t),t)$ has explicit expression:

$$E_1 = -\left[u^\varphi - \sigma M_0^u\right]_{Y=0}\partial_x \gamma_0 + [1-\sigma]\,\partial_x(\gamma_0 BR_0^u)$$

where

$$\sigma \equiv \frac{1+\partial_\xi X}{1+\partial_\xi X_0}. \tag{J.2}$$

We now pass to consider the equation satisfied by the curve φ_0 which,written in the Lagrangian frame is $\partial_\tau \varphi_0 = \mathscr{M}_0^v$. This equation can be rewritten as:

$$\partial_\tau \varphi_0 - u^\varphi \frac{1}{1+\partial_\xi X}\partial_\xi \varphi_0 + \mathscr{B}\mathscr{R}_0^u \frac{1}{1+\partial_\xi X}\partial_\xi \varphi_0 - \mathscr{B}\mathscr{R}_0^v = \mathscr{E}_2$$

where

$$\mathscr{E}_2 = -\left\{\left[u^\varphi - \mathscr{B}\mathscr{R}_0^u\right]\frac{1}{1+\partial_\xi X}\partial_\xi \varphi_0 + \mathscr{B}\mathscr{R}_0^v - \mathscr{M}_0^v\right\}_{Y=0}$$

Written in the Eulerian frame, using (J.1), the equation for φ_0 takes the form:

$$\partial_t \varphi_0 + BR_0^u \partial_x \varphi_0 - BR_0^v = E_2$$

where $E_2(x,t) = \mathscr{E}_2(\xi(x,t),t)$ can be written as:

$$E_2 = -\{[u^\varphi - BR_0^u]\,\partial_x\varphi_0 + BR_0^v - M_0^v\}_{Y=0}$$

J.2 Proof of Lemma 10.1

We prove that $\|\mathscr{E}_i\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$; this bound, given that E_i is related to $\mathscr{E}_i$ through an analytic change of coordinate, will provide the proof of Lemma 10.1.

The bound $\|\mathscr{E}_1\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$ is an obvious consequence of the following estimates:

(1) $\|\mathscr{M}[\omega_0,\varphi] - \mathscr{M}_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$, that is simply a version of Lemma I.1 with higher regularity;

(2) $\|\mathscr{R}_{Y=0}\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$, that is simply a version of Proposition 8.3, with higher regularity, and with the operator $\mathscr{R}$ computed on the curve, so that no linear growth in Y is present;

(3) $\|\varepsilon u_1\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$, that is obvious;

(4) $\|\sigma - 1\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$, which is a consequence of the obvious fact that $\|X - X_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$.

(5) $\|\mathscr{BR} - \mathscr{BR}_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$ which derives from the definitions $\mathscr{BR} = \mathscr{BR}[\gamma_0,\varphi]$, $\mathscr{BR}_0 = \mathscr{BR}[\gamma_0,\varphi_0]$ and from Proposition 6.5.

To get the bound $\|\mathscr{E}_2\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$ first we use the expression $u^\varphi = [\mathscr{M}^u + \mathscr{R}^u + \varepsilon u_1]_{Y=0}$, second we recall that:

$$\mathscr{M}_{Y=0} = \mathscr{BR} + \frac{1}{2}\left[\gamma_0^+ - \gamma_0^-\right]\tilde{t}^*$$

$$\mathscr{M}_{0Y=0} = \mathscr{BR}_0 + \frac{1}{2}\left[\gamma_0^+ - \gamma_0^-\right]\tilde{t}_0^*$$

and finally rearrange the expression for $\mathscr{E}_2$ in the following way:

$$\begin{aligned}\mathscr{E}_2 = -\Bigg\{\left[\mathscr{BR}^u - \mathscr{BR}_0^u + \frac{1}{2}\left(\gamma_0^+ - \gamma_0^-\right)\tilde{t}^u + \mathscr{R}^u + \varepsilon u_1\right]\frac{1}{1+X}\partial_\xi\varphi_0 - \frac{1}{2}\left(\gamma_0^+ - \gamma_0^-\right)\tilde{t}_0^v\Bigg\} = \\ -\Bigg\{\left[\mathscr{BR}^u - \mathscr{BR}_0^u + \frac{1}{2}\left(\gamma_0^+ - \gamma_0^-\right)(\tilde{t}^u - \tilde{t}_0^u) + \mathscr{R}^u + \varepsilon u_1\right]\frac{1}{1+\partial_\xi X}\partial_\xi\varphi_0 + \\ \frac{1}{2}\left(\gamma_0^+ - \gamma_0^-\right)\left[\tilde{t}_0^u\left(\frac{1}{1+\partial_\xi X} - \frac{1}{1+\partial_\xi X_0}\right) + \tilde{t}_0^u\frac{1}{1+X_0}\partial_\xi\varphi_0 - \tilde{t}_0^v\right]\Bigg\}\end{aligned}$$

To see that $\mathscr{E}_2$ satisfy the bound of Lemma 10.1 one has to recall, besides the already mentioned properties of $\mathscr{R}$, $\mathscr{BR} - \mathscr{BR}_0$ and $X - X_0$, also the following facts:

(1) $\|\tilde{t} - \tilde{t}_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$ which derives from the definitions (3.6) ot $\tilde{t}$ with $\psi = \varphi$ and definition (3.15) of $\tilde{t}_0$, and from the obvious $\|\varphi - \varphi_0\|^{(\alpha)}_{1,\rho,\theta,\beta,T} \le c\varepsilon$,

(2) the (approximate) tangent vector $\tilde{t}_0$ is orthogonal to the (approximate) normal vector $(\partial_\xi \varphi_0/(1+\partial_\xi X_0), -1)$, so that the last two terms in the last line of the above computation cancels exactly.

This concludes the proof of Lemma 10.1.

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