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THE McSHANE, PU AND HENSTOCK INTEGRALS OF BANACH VALUED FUNCTIONS

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Abstract. Some relationships between the vector valued Henstock and McShane integrals are investigated. An integral for vector valued functions, defined by means of partitions of the unity (the PU-integral) is studied. In particular it is shown that a vector valued function is McShane integrable if and only if it is both Pettis and PU-integrable. Convergence theorems for the Henstock variational and the PU integrals are stated. The families of multipliers for the Henstock and the Henstock variational integrals of vector valued functions are characterized.

Keywords: Pettis, McShane, PU and Henstock integrals, variational integrals, multipliers

MSC 2000: 28B05, 26B30

1. Introduction

In this paper some integrals of functions from a real interval into a Banach space are studied; in particular the PU-integral, which is constructed by means of partitions of the unity satisfying a regularity condition. It is known (see [15], [3] and [7]) that in the case of real valued functions the PU-integral falls properly in between the Lebesgue integral and the Henstock integral. We prove that in the case of Banach valued functions the PU-integral contains properly the McShane integral (Proposition 2), while the domains of Pettis and PU-integrals are incomparable (Remark 2). Fremlin proved in [10] that a vector valued function is McShane integrable if and only if it is both Henstock and Pettis integrable. In Theorem 2 we improve Fremlin's result by showing that a vector valued function is McShane integrable if and only

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if is both PU-integrable and Pettis integrable. Our proof is different from Fremlin's one; it uses a "weak" form of Henstock's Lemma (Proposition 1). We remark that a similar characterization is no longer true for the variational McShane integral (Example of §4). In Proposition 3, Corollary 1 and Remark 5 we describe some relationships between the Henstock, Pettis, McShane and Bochner integrals.

In §5 we give some convergence theorems for the variational Henstock integral and for the PU-integral.

Finally, in the last section we prove that the family of all real valued functions of bounded essential variation characterizes the multipliers for both the Henstock and the Henstock variational integrals.

2. Preliminaries

For a subset E of the real numbers $|E|$, χ_E, $d(E)$ and $\partial(E)$ denote respectively the Lebesgue outer measure, the characteristic function, the diameter and the boundary of E. A set $E \subset \mathbb{R}$ is called *negligible* if $|E| = 0$. The word "measurable" as well as the expression "almost everywhere" (abbreviated as a.e.) always refer to the Lebesgue measure. An *interval* is a compact subinterval of $\mathbb{R}$. A collection of intervals is called *nonoverlapping* if their interiors are disjoint. The symbol $\mathcal{I}$ will denote the family of all subintervals of $[0,1]$. A *partition* $\mathcal{P}$ in $[0,1]$ is a collection $\{(I_i, t_i)\colon\ i = 1, \dots, p\}$, where $I_1, \dots, I_p$ are nonoverlapping subintervals of $[0,1]$ and $t_1, \dots, t_p \in [0,1]$. Given a set $E \subset \mathbb{R}$, we say that $\mathcal{P}$ is

(i) a partition *in* E if $\bigcup\limits_{i=1}^{p} I_i \subset E$;

(ii) a partition *of* E if $\bigcup\limits_{i=1}^{p} I_i = E$;

(iii) a partition *anchored in* E if $t_i \in E$, $i = 1, \dots, p$;

(iv) a *Perron* partition if $t_i \in I_i$, $i = 1, \dots, p$.

A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ on E a partition $P = \{(I_i, t_i)\colon\ i = 1, \dots, p\}$ in $[0,1]$ is called *δ-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$.

The usual variation of a real valued function ϑ over the interval $[0,1]$ is denoted by $V(\vartheta, [0,1])$. Let θ be a real valued function on $\mathbb{R}$ and let $S_\theta = \{x \in \mathbb{R}\colon\ \theta(x) \neq 0\}$. If $S_\theta \subset [0,1]$ we set

$$V_{\mathrm{ess}}(\theta) = \inf V(\vartheta, [0,1]),$$

where the infimum is taken over all functions ϑ such that $S_\vartheta \subset [0,1]$ and $\vartheta = \theta$ a.e. The family of all nonnegative measurable bounded functions θ on $\mathbb{R}$ for which $S_\theta \subset [0,1]$ and $V_{\mathrm{ess}}(\theta) < +\infty$ is denoted by $BV_+([0,1])$. The *regularity* of $\theta \in BV_+([0,1])$

at a point $x \in \mathbb{R}$ is the number

$$r(\theta, x) = \begin{cases} \dfrac{|\theta|_1}{d(S_\theta \cup \{x\})V_{\text{ess}}(\theta)} & \text{if } d(S_\theta \cup \{x\})V_{\text{ess}}(\theta) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\theta|_1$ denotes the L^1 norm of θ.

A *pseudopartition in* $[0,1]$ is a collection $\mathcal{Q} = \{(\theta_1, t_1), \ldots, (\theta_p, t_p)\}$ where $\theta_1, \ldots, \theta_p$ are functions from $BV_+([0,1])$ such that $\sum\limits_{i=1}^{p} \theta_i \leqslant \chi_{[0,1]}$ and $t_i \in [0,1]$ for $i = 1, \ldots, p$. Let $\mathcal{P} = \{(A_1, t_1), \ldots, (A_p, t_p)\}$ be a partition in $[0,1]$, then $\mathcal{P}^* = \{(\chi_{A_1}, t_1), \ldots, (\chi_{A_p}, t_p)\}$ is a pseudopartition in $[0,1]$, called the pseudopartition *induced* by $\mathcal{P}$.

Let $\varepsilon > 0$ and let δ be a gauge on $[0,1]$. A pseudopartition $\mathcal{Q} = \{(\theta_1, t_1), \ldots, (\theta_p, t_p)\}$ in $[0,1]$ is called:

(i) a pseudopartition *of* $[0,1]$ if $\sum\limits_{i=1}^{p} \theta_i = \chi_{[0,1]}$;

(ii) ε*-regular* if $r(\theta_i, t_i) > \varepsilon$, $i = 1, \ldots, p$;

(iii) δ*-fine* if $S_{\theta_i} \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, p$.

A partition $\mathcal{P} = \{(A_1, t_1), \ldots, (A_p, t_p)\}$ in $[0,1]$ is ε-regular whenever the pseudopartition $\mathcal{P}^*$ induced by $\mathcal{P}$ has this property.

From now on X is a real Banach space with dual X^*. Given $f\colon [0,1] \to X$, we set

$$\sigma(f, \mathcal{P}) = \sum_{i=1}^{p} |I_i| f(t_i) \quad \text{and} \quad \sigma(f, \mathcal{Q}) = \sum_{i=1}^{p} \left(\int_0^1 \theta_i \right) f(t_i)$$

for each partition $\mathcal{P} = \{(I_1, t_1), \ldots, (I_p, t_p)\}$ and each pseudopartition $\mathcal{Q} = \{(\theta_1, t_1), \ldots, (\theta_p, t_p)\}$ in $[0,1]$.

Definition 1. We recall the following classical definitions.

a) A function $f\colon [0,1] \to X$ is said to be Dunford integrable if $x^* f$ is Lebesgue integrable on $[0,1]$ for each $x^* \in X^*$. The Dunford integral of f on a measurable set $E \subset [0,1]$ is the vector $\nu(E) \in X^{**}$ such that $\langle \nu(E), x^* \rangle = \int_E x^* f(t)\,\mathrm{d}t$ for all $x^* \in X^*$.

b) A function $f\colon [0,1] \to X$ is said to be Pettis integrable if it is Dunford integrable on $[0,1]$ and $\nu(E) \in X$ for every measurable set $E \subset [0,1]$. In this case $\nu([0,1])$ is the Pettis integral of f and the map $E \to \nu(E)$ is the indefinite Pettis integral of f.

c) A function $f\colon [0,1] \to X$ is said to be McShane integrable (respectively Henstock integrable) (briefly Mc-integrable (respectively H-integrable)) on $[0,1]$, if there exists a vector $w \in X$ satisfying the following property: given $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that for each δ-fine partition (respectively Perron partition) $\mathcal{P} = \{(I_i, t_i)\colon\ i = 1,\dots,p\}$ of $[0,1]$, we have

$$\|\sigma(f,\mathcal{P}) - w\| < \varepsilon.$$

We denote by $\mathrm{Mc}([0,1],X)$ (respectively $\mathrm{H}([0,1],X)$) the family of all Mc-integrable (respectively H-integrable) functions on $[0,1]$ and we set $w = (\mathrm{Mc})\int_0^1 f$ (respectively $w = (\mathrm{H})\int_0^1 f$). For each $f \in \mathrm{Mc}([0,1],X)$ (respectively $f \in \mathrm{H}([0,1],X)$), the interval function $F(I) = (\mathrm{Mc})\int_I f$ (respectively $F(I) = (\mathrm{H})\int_I f$) is called the *primitive* of f. The function f is said to be McShane integrable on a set $E \subset [0,1]$ if the function $\chi_E f$ is McShane integrable on $[0,1]$. Then we set $(\mathrm{Mc})\int_E f = (\mathrm{Mc})\int_0^1 \chi_E f$.

The following remarkable result was proved by Fremlin ([10], Theorem 8).

Theorem 1. *Let $f\colon [0,1] \to X$ be a function. Then f is McShane integrable if and only if it is Henstock integrable and Pettis integrable on $[0,1]$.*

3. The PU-integral and some relationships between vector valued integrals

Now we are introducing the PU-integral for a vector valued function.

Definition 2. A function $f\colon [0,1] \to X$ is said to be PU-integrable on $[0,1]$ if there is a vector $w \in X$ with the following property: given $\varepsilon > 0$, we can find a gauge δ on $[0,1]$ such that

$$\|\sigma(f,\mathcal{Q}) - w\| < \varepsilon$$

for each ε-regular δ-fine pseudopartition $\mathcal{Q}$ of $[0,1]$.

Remark 1. If $X = \mathbb{R}$ the above definition is a particular case (more precisely the case in which $G(\theta) = \int_0^1 \theta$ and pseudopartitions of $[0,1]$ are considered) of the PU-integral introduced in [15] for real functions defined on a BV set of $\mathbb{R}$. For real valued functions in $[0,1]$ the PU-integral falls properly in between Lebesgue and Henstock integrals. Moreover the ε-regularity of the pseudopartitions used guarantees the PU-integrability of each derivative (see [15] and [7]).

Remark 2. A PU-integrable function is Henstock integrable since each Perron partition $\mathcal{P}$ is ε-regular for each $\varepsilon < 1$ and $\sigma(f,\mathcal{P}) = \sigma(f,\mathcal{P}^*)$, where $\mathcal{P}^*$ is the pseudopartition induced by $\mathcal{P}$. But there is no relationship between the Pettis integral and the PU-integral. Indeed the real valued function $F(x) = x^2\cos\pi/x^2$ if $0 < x \leqslant 1$, $F(x) = 0$ if $x = 0$, is derivable everywhere and its derivative is not

Lebesgue and thus Pettis integrable, but it is PU-integrable (see [15], Theorem 4.4 or [7], Theorem 3.2). Moreover there are functions that are Pettis integrable, but are not Henstock integrable and also not PU-integrable (see Theorem 1 and [11], Example 3C).

Lemma 1. *Let $f\colon [0,1] \to \mathbb{R}$ be a measurable function, θ_i, $i = 1, \ldots, p$, be nonnegative measurable functions on $[0,1]$, c_i, $i = 1, \ldots, p$ be real constants and let S_i, $i = 1, \ldots, p$, be measurable subsets of $[0,1]$. Then*

$$\sum_{i=1}^{p} \int_{S_i} |f - c_i|\theta_i \leqslant \sum_{i=1}^{p} \int_{L'_i} |f - c_i| \sum_{j=1}^{p} \theta_j + \sum_{i=1}^{p} \int_{L''_i} |f - c_i| \sum_{j=1}^{p} \theta_j,$$

where L'_i, $i = 1, \ldots, p$ are pairwise disjoint measurable sets with $L'_i \subset \{t \in S_i\colon f(t) - c_i \geqslant 0\}$ and L''_i, $i = 1, \ldots, p$ are pairwise disjoint measurable sets with $L''_i \subset \{t \in S_i\colon f(t) - c_i < 0\}$ and $\bigcup\limits_{i=1}^{p} S_i = \bigcup\limits_{i=1}^{p} (L'_i \cup L''_i)$.

Proof. We can assume that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_p$. For $i = 1, \ldots, p$ let $S_i^+ = \{t \in S_i\colon f(t) - c_i \geqslant 0\}$ and $S_i^- = S_i \setminus S_i^+$. We have

$$\sum_{i=1}^{p} \int_{S_i} |f - c_i|\theta_i = \sum_{i=1}^{p} \int_{S_i^+} (f - c_i)\theta_i + \sum_{i=1}^{p} \int_{S_i^-} (c_i - f)\theta_i. \tag{1}$$

Set $L'_1 = S_1^+$, $L'_2 = S_2^+ \setminus S_1^+$, …, $L'_p = S_p^+ \setminus \bigcup\limits_{i=1}^{p-1} S_i^+$ and $L''_1 = S_1^- \setminus \bigcup\limits_{i=2}^{p} S_i^-$, $L''_2 = S_2^- \setminus \bigcup\limits_{i=3}^{p} S_i^-$, …, $L''_p = S_p^-$. Considering separately the two sums on the right side of the previous equality we get:

$$\begin{aligned} \sum_{i=1}^{p} \int_{S_i^+} & (f - c_i)\theta_i \\ &= \int_{L'_1} (f - c_1)\theta_1 + \int_{L'_2} (f - c_2)\theta_2 + \int_{S_2^+ \cap L'_1} (f - c_2)\theta_2 + \ldots \\ &\quad + \int_{L'_p} (f - c_p)\theta_p + \sum_{i=1}^{p-1} \int_{S_p^+ \cap L'_i} (f - c_p)\theta_p \\ &\leqslant \int_{L'_1} (f - c_1)\theta_1 + \int_{L'_2} (f - c_2)\theta_2 + \int_{L'_1} (f - c_1)\theta_2 + \ldots \\ &\quad + \int_{L'_p} (f - c_p)\theta_p + \sum_{i=1}^{p-1} \int_{L'_i} (f - c_i)\theta_p \end{aligned} \tag{2}$$

$$= \int_{L_1'} |f - c_1|(\theta_1 + \theta_2 + \ldots + \theta_p) + \int_{L_2'} |f - c_2|(\theta_2 + \ldots + \theta_p) + \ldots$$
$$+ \int_{L_p'} |f - c_p|\theta_p$$
$$\leqslant \int_{L_1'} |f - c_1| \sum_{j=1}^{p} \theta_j + \int_{L_2'} |f - c_2| \sum_{j=1}^{p} \theta_j + \ldots + \int_{L_p'} |f - c_p| \sum_{j=1}^{p} \theta_j$$
$$= \sum_{i=1}^{p} \int_{L_i'} |f - c_i| \sum_{j=1}^{p} \theta_j;$$

and

$$(3) \quad \sum_{i=1}^{p} \int_{S_i^-} (c_i - f)\theta_i$$
$$= \int_{L_1''} (c_1 - f)\theta_1 + \sum_{i=2}^{p} \int_{S_1^- \cap L_i''} (c_1 - f)\theta_1 + \int_{L_2''} (c_2 - f)\theta_2$$
$$+ \sum_{i=3}^{p} \int_{S_2^- \cap L_i''} (c_2 - f)\theta_2 + \ldots + \int_{L_p''} (c_p - f)\theta_p$$
$$\leqslant \int_{L_1''} (c_1 - f)\theta_1 + \sum_{i=2}^{p} \int_{L_i''} (c_i - f)\theta_1 + \int_{L_2''} (c_2 - f)\theta_2$$
$$+ \sum_{i=3}^{p} \int_{L_i''} (c_i - f)\theta_2 \ldots + \int_{L_p''} (c_p - f)\theta_p$$
$$= \int_{L_1''} |f - c_1|\theta_1 + \int_{L_2''} |f - c_2|(\theta_1 + \theta_2) + \int_{L_3''} |f - c_3|(\theta_1 + \theta_2 + \theta_3)$$
$$+ \sum_{i=4}^{p-1} \int_{L_i''} |f - c_i| \sum_{j=1}^{i} \theta_j + \ldots + \int_{L_p''} |f - c_p| \sum_{j=i}^{p} \theta_p$$
$$\leqslant \int_{L_1''} |f - c_1| \sum_{j=1}^{p} \theta_j + \int_{L_2''} |f - c_2| \sum_{j=1}^{p} \theta_j + \ldots + \int_{L_p''} |f - c_p| \sum_{j=1}^{p} \theta_j$$
$$= \sum_{i=1}^{p} \int_{L_i''} |f - c_i| \sum_{j=1}^{p} \theta_j.$$

From (1), (2) and (3) we infer that

$$\sum_{i=1}^{p} \int_{S_i} |f - c_i|\theta_i \leqslant \sum_{i=1}^{p} \int_{L_i'} |f - c_i| \sum_{j=1}^{p} \theta_j + \sum_{i=1}^{p} \int_{L_i''} |f - c_i| \sum_{j=1}^{p} \theta_j,$$

and the assertion follows. □

From now on we denote by $\mathcal{B}(X^*)$ the closed unit ball of X^*.

Proposition 1. *Let $f\colon [0,1] \to X$ be a McShane integrable function. Then for each $\varepsilon > 0$ there exists a gauge δ satisfying the condition: if $E_1, \ldots, E_p$ are measurable disjoint subsets of $[0,1]$, $t_1, \ldots, t_p \in [0,1]$ and $E_i \subset \big(t_i - \delta(t_i), t_i + \delta(t_i)\big)$, $i = 1, \ldots, p$, then*

$$\sup_{x^* \in \mathcal{B}(X^*)} \sum_{i=1}^{p} \left| x^* \left[f(t_i)|E_i| - (\mathrm{Mc})\int_{E_i} f \right] \right| < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. By ([11], Lemma 2H) there exists a gauge δ such that if $A_1, \ldots, A_s$ are measurable disjoint subsets of $[0,1]$, $t_1, \ldots, t_s \in [0,1]$ and $A_i \subset \big(t_i - \delta(t_i), t_i + \delta(t_i)\big)$ for every i, then

$$\left\| \sum_{i=1}^{s} \left[|A_i| f(t_i) - (\mathrm{Mc})\int_{A_i} f \right] \right\| < \frac{\varepsilon}{4}.$$

Let now $\mathcal{D} = \{(E_i, t_i)\colon i = 1, \ldots, p\}$ where $E_1, \ldots, E_p$ are measurable disjoint subsets of $[0,1]$, $t_1, \ldots, t_p \in [0,1]$ and $E_i \subset \big(t_i - \delta(t_i), t_i + \delta(t_i)\big)$, $i = 1, \ldots, p$. Fix $x^* \in \mathcal{B}(X^*)$ and put $\mathcal{D}^+ = \{(E_i, t_i) \in \mathcal{D}\colon |E_i| x^* f(t_i) - \int_{E_i} x^* f \geqslant 0\}$ and $\mathcal{D}^- = \{(E_i, t_i) \in \mathcal{D}\colon |E_i| x^* f(t_i) - \int_{E_i} x^* f < 0\}$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{p} \left| x^* f(t_i)|E_i| - \int_{E_i} x^* f \right| \\
&\quad = \sum_{\mathcal{D}^+} \left| x^* f(t_i)|E_i| - \int_{E_i} x^* f \right| + \sum_{\mathcal{D}^-} \left| x^* f(t_i)|E_i| - \int_{E_i} x^* f \right| \\
&\quad = \left| \sum_{\mathcal{D}^+} \left[x^* f(t_i)|E_i| - \int_{E_i} x^* f \right] \right| + \left| \sum_{\mathcal{D}^-} \left[x^* f(t_i)|E_i| - \int_{E_i} x^* f \right] \right| \\
&\quad = \left| x^* \sum_{\mathcal{D}^+} \left[f(t_i)|E_i| - (\mathrm{Mc})\int_{E_i} f \right] \right| + \left| x^* \sum_{\mathcal{D}^-} \left[f(t_i)|E_i| - (\mathrm{Mc})\int_{E_i} f \right] \right| \\
&\quad \leqslant \left\| \sum_{\mathcal{D}^+} \left[|E_i| f(t_i) - (\mathrm{Mc})\int_{E_i} f \right] \right\| + \left\| \sum_{\mathcal{D}^-} \left[|E_i| f(t_i) - (\mathrm{Mc})\int_{E_i} f \right] \right\| < \frac{\varepsilon}{2}.
\end{aligned}$$

Since this is true for each $x^* \in \mathcal{B}(X^*)$ we infer that

$$\sup_{x^* \in \mathcal{B}(X^*)} \sum_{i=1}^{p} \left| x^* \left[f(t_i)|E_i| - (\mathrm{Mc})\int_{E_i} f \right] \right| < \varepsilon.$$

□

Remark 3. It is known that Henstock's Lemma no longer holds for a Banach valued function. Indeed, as it has been proved in [19], for both the McShane and the Henstock integrals this Lemma holds if and only if the space X is of finite dimension. Then Proposition 1 can be considered as a weak version of Henstock's Lemma.

Proposition 2. *Let $f\colon [0,1] \to X$ be a McShane integrable function. Then f is* PU*-integrable and the two integrals coincide.*

Proof. Fix $\varepsilon > 0$. According to Proposition 1 there is a gauge δ such that

$$\sup_{x^*\in\mathcal{B}(X^*)} \sum_{i=1}^{s} \left| x^* \left[f(t_i)|E_i| - (\mathrm{Mc})\int_{E_i} f \right] \right| < \frac{\varepsilon}{2}, \tag{4}$$

for each family $\{(E_i, t_i)\colon i = 1, \ldots, s\}$ where $E_1, \ldots, E_s$ are measurable disjoint subsets of $[0,1]$, $t_1, \ldots, t_s \in [0,1]$ and $E_i \subset \big(t_i - \delta(t_i), t_i + \delta(t_i)\big)$, $i = 1, \ldots, s$. Let $\mathcal{Q} = \{(\theta_1, t_1), \ldots, (\theta_p, t_p)\}$ be an ε-regular, δ-fine pseudopartition of $[0,1]$. Since $\theta_i \in L^1([0,1])$, $i = 1, \ldots, p$, the sets $S_i = S_{\theta_i}$ are measurable. Moreover $\sum\limits_{i=1}^{p} \theta_i = \chi_{[0,1]}$. Fix $x^* \in \mathcal{B}(X^*)$. We obtain:

$$\begin{aligned} &\left| x^* \left[(\mathrm{Mc})\int_0^1 f - \sum_{i=1}^{p} \Big(\int_0^1 \theta_i \Big) f(t_i) \right] \right| \\ &\quad = \left| \sum_{i=1}^{p} \int_0^1 x^* f(t)\theta_i(t)\,\mathrm{d}t - \sum_{i=1}^{p} \int_0^1 x^* f(t_i)\theta_i(t)\,\mathrm{d}t \right| \\ &\quad = \left| \sum_{i=1}^{p} \int_0^1 [x^* f(t) - x^* f(t_i)]\theta_i(t)\,\mathrm{d}t \right| \\ &\quad \leqslant \sum_{i=1}^{p} \int_{S_i} |x^* f(t) - x^* f(t_i)|\theta_i(t)\,\mathrm{d}t. \end{aligned} \tag{5}$$

Since $x^* f(t)$ is a real valued McShane integrable function, it is measurable. Now for $i = 1, \ldots, p$ define the sets L_i' and L_i'' as in Lemma 1. Applying the Lemma, it follows that

$$\begin{aligned} &\sum_{i=1}^{p} \int_{S_i} |x^* f(t) - x^* f(t_i)|\theta_i(t)\,\mathrm{d}t \\ &\quad \leqslant \sum_{i=1}^{p} \int_{L_i'} |x^* f(t) - x^* f(t_i)|\,\mathrm{d}t + \sum_{i=1}^{p} \int_{L_i''} |x^* f(t) - x^* f(t_i)|\,\mathrm{d}t. \end{aligned} \tag{6}$$

Since $\mathcal{Q}$ is a δ-fine pseudopartition of $[0,1]$, both L_i' and L_i'', $i=1,\ldots,p$, are measurable pairwise disjoint subsets of $\big(t_i-\delta(t_i), t_i+\delta(t_i)\big)$. Thus by (4) we have

$$
\begin{aligned}
(7)\quad & \sum_{i=1}^{p}\int_{L_i'} |x^*f(t)-x^*f(t_i)|\,\mathrm{d}t + \sum_{i=1}^{p}\int_{L_i''} |x^*f(t)-x^*f(t_i)|\,\mathrm{d}t \\
&= \sum_{i=1}^{p}\left|\int_{L_i'} [x^*f(t)-x^*f(t_i)]\,\mathrm{d}t\right| + \sum_{i=1}^{p}\left|\int_{L_i''} [x^*f(t)-x^*f(t_i)]\,\mathrm{d}t\right| \\
&= \sum_{i=1}^{p}\left|\int_{L_i'} x^*f(t)\,\mathrm{d}t - |L_i'|x^*f(t_i)\right| + \sum_{i=1}^{p}\left|\int_{L_i''} x^*f(t)\,\mathrm{d}t - |L_i''|x^*f(t_i)\right| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Then by (5), (6) and (7) we infer that

$$
\left|x^*\left[(\mathrm{Mc})\int_0^1 f - \sum_{i=1}^{p}\left(\int_0^1 \theta_i\right)f(t_i)\right]\right| < \varepsilon.
$$

Thus, since x^* is arbitrary, we get

$$
\left\|(\mathrm{Mc})\int_0^1 f - \sum_{i=1}^{p}\left(\int_0^1 \theta_i\right)f(t_i)\right\| \leqslant \varepsilon.
$$

Therefore the function f is PU-integrable and the Mc-integral and the PU-integral coincide. □

Remark 4. In the real case the previous Proposition follows directly by the definition of the Lebesgue integral (see [7]), as the McShane and the Lebesgue integrals are equivalent.

Theorem 2. *Let $f\colon [0,1]\to X$. Then f is McShane integrable if and only if f is Pettis integrable and* PU-*integrable on* $[0,1]$.

Proof. If f is McShane integrable, then by Proposition 2 it is PU-integrable and by ([1], Theorem 2C) it is Pettis integrable. The converse follows by Theorem 1, since each PU-integrable function is Henstock integrable. □

Proposition 3. *Let $f\colon [0,1]\to X$. If f and $\|f\|$ are Henstock integrable then f is Pettis integrable.*

Proof. Since f is Henstock integrable, for all $x^*\in\mathcal{B}(X^*)$ the real valued function x^*f is measurable. Moreover $\|f\|$ being Henstock integrable, it is also

Lebesgue integrable. For each measurable set $E \subset [0,1]$ and for each $x^* \in \mathcal{B}(X^*)$, it follows that

$$\int_E |x^* f| \leqslant \int_E \|f\| < \infty.$$

Thus f is Dunford integrable. Let $\nu(E)$ be its Dunford integral. If $[a,b] \subset [0,1]$, the Henstock integrability of f implies that $\nu([a,b]) \in X$. Fix $\varepsilon > 0$. The Lebesgue integrability of $\|f\|$ implies the existence of a positive number η such that if $|E| < \eta$ then $\int_E \|f\| < \varepsilon$. Thus if $|E| < \eta$ we have

$$\|\nu(E)\| = \sup_{x^* \in \mathcal{B}(X^*)} \left| \int_E x^* f \right| \leqslant \sup_{x^* \in \mathcal{B}(X^*)} \int_E |x^* f| \leqslant \int_E \|f\| < \varepsilon.$$

Therefore the assertion follows from ([11], Proposition 2B). □

Corollary 1. *Let $f\colon [0,1] \to X$. If f and $\|f\|$ are Henstock integrable then f is McShane integrable.*

Proof. By Proposition 3 f is Pettis integrable, thus by Theorem 1 it is Mc-integrable. □

With the symbol φ we will denote the null vector in the space X.

Remark 5. The converse of the previous Corollary is true for real valued functions but in general it is not true for a Banach valued function. In fact a McShane integrable function is Henstock integrable, but $\|f\|$ is not necessarily integrable as the following example shows. Let E be a nonmeasurable subset of $[0,1]$ and let $f\colon [0,1] \to L^\infty([0,1])$ be defined as follows:

$$f(t) = \begin{cases} \varphi & \text{if } t \notin E, \\ \chi_{\{t\}} & \text{if } t \in E, \end{cases}$$

where φ is the null function in $[0,1]$. Then f is McShane integrable (see [12], Example 14), but $\|f\| = \chi_E$ is not measurable. Even if f is a strongly measurable McShane integrable function then $\|f\|$ is not necessarily Henstock integrable. Indeed there are strongly measurable Pettis integrable functions that are not Bochner.

4. Variational integrals

We recall the definition of McShane and Henstock variational integrals.

Definition 3. A function $f\colon [0,1] \to X$ is said to be McShane (respectively Henstock) variationally integrable (briefly MV-integrable (respectively HV-integrable)) on $[0,1]$, if there exists an additive function $F\colon \mathcal{I} \to X$, satisfying the following condition: given $\varepsilon > 0$ there exists a gauge δ such that if $\mathcal{P} = \{(I_i, t_i)\colon i = 1, \dots, p\}$ is a δ-fine partition (respectively Perron partition) of $[0,1]$, we have

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\| < \varepsilon.$$

We denote by $\mathrm{MV}([0,1], X)$ (respectively $\mathrm{HV}([0,1], X)$) the family of all MV-integrable (respectively HV-integrable) functions on $[0,1]$. It follows by the definition that $\mathrm{MV}([0,1], X) \subseteq \mathrm{Mc}([0,1], X)$ (respectively $\mathrm{HV}([0,1], X) \subseteq \mathrm{H}([0,1], X)$).

Remark 6. In case of real valued functions the variational McShane (respectively Henstock) integral is equivalent to the McShane (respectively Henstock) one.

Remark 7. Each variationally integrable function is strongly measurable (see [6], Theorem 9).

Theorem 1 is no longer true for variational integrals; i.e. there exists a HV-integrable function that is Pettis integrable but not MV-integrable, as the following example shows.

From now on, if F is a function on $[0,1]$, we set $F([a,b]) = F(b) - F(a)$ for $[a,b] \subset [0,1]$.

Example. Let X be an infinitely dimensional Banach space and let $\sum\limits_{n} x_n$ be a series in X converging unconditionally but not absolutely. For each $n \in \mathbb{N}$, let $I_n = (2^{-n}, 2^{-n+1})$ and define $f\colon [0,1] \to X$ by

$$f(t) = \begin{cases} 2^n x_n & \text{if } t \in I_n,\ \ n = 1, 2, \dots, \\ \varphi & \text{otherwise.} \end{cases}$$

As f is a countably valued function, it is strongly measurable. Since $\sum\limits_{n} 2^n x_n |I_n| = \sum\limits_{n} x_n$ is unconditionally but not absolutely convergent, f is Pettis integrable, but it is not Bochner integrable (see [5], Theorem 2); hence by [9] it is not MV-integrable. Now we show that f is HV-integrable. Define:

$$F(t) = \begin{cases} 2^n\Big(t - \dfrac{1}{2^n}\Big)x_n + \sum\limits_{k=n+1}^{\infty} x_k & \text{if } t \in (2^{-n}, 2^{-n+1}], \\ \varphi & \text{if } t = 0. \end{cases}$$

Fix $0<\varepsilon<1$ and let N be a positive integer such that for each $n>N$, $\left\|\sum\limits_{k=n}^{\infty} x_k\right\| < \varepsilon/5$ and $\|x_n\|<\varepsilon/5$. Moreover let $M>1$ be such that $\|x_n\|<M$ for all n and define δ on $[0,1]$ as follows:

$$\delta(t)=\begin{cases}\operatorname{dist}(t,\partial I_n) & \text{if } t\in I_n,\\ \dfrac{\varepsilon}{5M4^n} & \text{if } t=2^{-n+1}\\ \dfrac{1}{2^N} & \text{if } t=0\end{cases}$$

where $\operatorname{dist}(t,\partial I_n)$ denotes the distance of t from the boundary of I_n. Let $\mathcal{P}=\{(J_i,t_i)\colon\ i=1,\dots,p\}$ be a δ-fine Perron partition of $[0,1]$ and let us consider the sum

$$\sum_{i=1}^{p}\|f(t_i)|J_i|-F(J_i)\|.$$

Since $\bigcup\limits_{i=1}^{p} J_i=[0,1]$ there exists $\beta>0$ such that the tagged interval $([0,\beta],0)$ belongs to $\mathcal{P}$. Moreover if $t_i\in I_n$ the tagged interval (J_i,t_i) gives no contribution to the sum. Thus we can assume that $t_1=0$ and, for $i=2,\dots,p$, $t_i=2^{-n}$ for some $n\in\mathbb{N}$. Let $J_i=[a_i,b_i]$, $i=2,\dots,p$. We have

$$\begin{aligned}(8)\quad &\|f(t_i)|J_i|-F(J_i)\|\\ &=\left\|2^n\Bigl(b_i-\frac{1}{2^n}\Bigr)x_n+\sum_{k=n+1}^{\infty}x_k-2^{n+1}\Bigl(a_i-\frac{1}{2^{n+1}}\Bigr)x_{n+1}-\sum_{k=n+2}^{\infty}x_k\right\|\\ &=\left\|2^n\Bigl(b_i-\frac{1}{2^n}\Bigr)x_n-2^{n+1}\Bigl(a_i-\frac{1}{2^n}\Bigr)x_{n+1}\right\|\\ &\leqslant\left\|2^n\Bigl(b_i-\frac{1}{2^n}\Bigr)x_n\right\|+\left\|2^{n+1}\Bigl(a_i-\frac{1}{2^n}\Bigr)x_{n+1}\right\|\\ &\leqslant 2^n\|x_n\|\frac{\varepsilon}{5M4^n}+2^{n+1}\|x_{n+1}\|\frac{\varepsilon}{5M4^n}\\ &\leqslant\frac{\varepsilon}{5\cdot2^n}+\frac{\varepsilon}{5\cdot2^{n-1}}=\frac{3\varepsilon}{5\cdot2^n}.\end{aligned}$$

Now we estimate

$$\|f(0)\beta-F(\beta)+F(0)\|.$$

Let $q>N$ be such that $\beta\in(2^{-q},2^{-q+1}]$. Then

$$\begin{aligned}(9)\quad \|f(0)\beta-F(\beta)+F(0)\|&=\left\|2^q\Bigl(\beta-\frac{1}{2^q}\Bigr)x_q+\sum_{k=q+1}^{\infty}x_k\right\|\\ &\leqslant\left\|2^q\Bigl(\beta-\frac{1}{2^q}\Bigr)x_q\right\|+\left\|\sum_{k=q+1}^{\infty}x_k\right\|\leqslant\|x_q\|+\frac{\varepsilon}{5}<\frac{2\varepsilon}{5}.\end{aligned}$$

Therefore by (5) and (6) we infer that

$$\sum_{i=1}^{p} \|f(t_i)|J_i| - F(J_i)\| = \|f(0)\beta - F(\beta) + F(0)\| + \sum_{i=2}^{p} \|f(t_i)|J_i| - F(J_i)\|$$
$$< \frac{2\varepsilon}{5} + \sum_{n=1}^{\infty} \frac{3\varepsilon}{5 \cdot 2^n} = \varepsilon,$$

which gives the HV-integrability of f.

The following variational property for the primitive of a HV-integrable function is used in the next section to prove a convergence theorem for the HV-integral.

Definition 4. Let $F\colon [0,1] \to X$ be a function. F is called AC* on a subset E of $[0,1]$ whenever for each $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ such that

$$\sum_{i=1}^{p} \|F(I_i)\| < \varepsilon$$

for each δ-fine Perron partition $\mathcal{P} = \{(I_i, t_i)\colon i = 1, \ldots, p\}$ anchored in E with $\sum\limits_{i=1}^{p} |I_i| < \eta$. F is called ACG* on $[0,1]$ if there is a sequence (E_k) of measurable sets such that $[0,1] = \bigcup\limits_{k=1}^{\infty} E_k$ and F is AC* on each E_k.

Proposition 4. *Let $f\colon [0,1] \to X$ be a Henstock variationally integrable function. Then its primitive $F(t) = (\mathrm{HV})\int_0^t f$ is ACG*.*

Proof. Since the function F is strongly differentiable a.e. (see [6], Theorem 9), the proof follows as in ([4], Theorem 3.4). □

5. Convergence theorems

We will prove now some convergence theorems. We need the following definitions.

Definition 5. A family $(G_\alpha)_{\alpha \in A}$ of vector valued functions on $[0,1]$ is called *uniformly*-AC* on a subset E of $[0,1]$ whenever to each $\varepsilon > 0$ there correspond $\eta > 0$ and a gauge δ such that

$$\sup_{\alpha} \sum_{i=1}^{p} \|G_\alpha(I_i)\| < \varepsilon$$

for each δ-fine Perron partition $\mathcal{P} = \{(I_i, t_i)\colon\ i = 1, \dots, p\}$ anchored in E with $\sum_{i=1}^{p} |I_i| < \eta$. A family $\{G_\alpha\}_\alpha$ of vector valued functions on $[0,1]$ is called uniformly-ACG* on a subset E of $[0,1]$ if there is a sequence (E_k) of measurable sets such that $E = \bigcup_{k=1}^{\infty} E_k$ and $\{G_\alpha\}_\alpha$ is uniformly-AC* on each E_k.

Definition 6. A sequence $(G_n)_n$ of real valued functions on $[0,1]$ is called asymptotically-AC* on a subset E of $[0,1]$ if for each $\varepsilon > 0$ there are $\eta > 0$ and a gauge δ such that

$$\overline{\lim}_n \left| \sum_{i=1}^{p} G_n(I_i) \right| < \varepsilon,$$

for each δ-fine Perron partition $\mathcal{P} = \{(I_i, t_i)\colon\ i = 1, \dots, p\}$ anchored in E with $\sum_{i=1}^{p} |I_i| < \eta$.

A sequence $(G_n)_n$ of real valued functions on $[0,1]$ is called *asymptotically*-ACG* on a subset E of $[0,1]$ if each G_n is continuous and there is a sequence (E_k) of measurable sets such that $E = \bigcup_{k=1}^{\infty} E_k$ and $(G_n)_n$ is asymptotically-AC* on each E_k.

Let $F\colon\ [0,1] \to X$ be a function and let $E \subset [a,b]$. For each gauge δ on E set

$$V(F,\delta,E) = \sup \sum_{i=1}^{p} \|F(I_i)\|,$$

where the supremum is taken over all δ-fine partitions $\mathcal{P} = \{(I_i, t_i)\colon\ i = 1, \dots, p\}$, anchored on E. The *strong critical variation of F on E* is

$$V_* F(E) = \inf V(F,\delta,E),$$

where the infimum is taken over all gauges δ on E. It is known that the set function

$$V_* F\colon\ E \to V_* F(E)$$

is a Borel metric measure (see [20], Theorem 3.7 and Theorem 3.15).

We say that a measure ν on $[0,1]$ is absolutely continuous if $\nu(E) = 0$ for each negligible subset E of $[0,1]$. The primitives of HV-integrable functions have been characterized in [17] by means of the notion of absolute continuity of their strong critical variation:

Theorem 3 ([17], Theorem 8)**.** *Let $F\colon\ [0,1] \to X$ be a function with separable valued scalar derivative f on $[0,1]$. Then the function f is* HV*-integrable with* *primitive F if and only if the measure V_*F is absolutely continuous. In this case $F(x) = (\mathrm{HV})\int_0^x f$.*

From now on if $[a,b] \subset [0,1]$ the symbol $\mathrm{H}([a,b])$ will denote the family of all real valued Henstock integrable functions defined on $[a,b]$ and $\mathcal{H}([a,b])$ the completion of $\mathrm{H}([a,b])$ with respect to the Alexiewicz norm (i.e. the norm $\|f\|_H = \sup_t |(\mathrm{H})\int_a^t f|$).

The following theorem is a version of the Vitali convergence theorem for the Henstock variational integral. In the first part of the proof we use a technique similar to that in ([18], Theorem 1) for a convergence theorem of Pettis integrals.

Theorem 4. *Let $(f_n \in \mathrm{HV}([0,1],X))_n$ be a sequence of functions and let $F_n(t) = (\mathrm{HV})\int_0^t f_n$. If*

(a) *$f_n \to f$ weakly almost everywhere in $[0,1]$;*

(b) *the sequence $(F_n)_n$ is uniformly-ACG*;*

then $f \in \mathrm{HV}([0,1],X)$ and $(\mathrm{HV})\int_0^1 f_n \to (\mathrm{HV})\int_0^1 f$ weakly.

To prove the Theorem we need the following Lemma.

Lemma 2. *Let $(F_n)_n$ be a sequence of functions from $[0,1]$ to X weakly convergent to F and such that $F_n(0) = \varphi$ for each n. If moreover the sequence $(F_n)_n$ is uniformly-ACG* on $[0,1]$, then the strong critical variation V_*F of F is absolutely continuous.*

Proof. The sequence $(F_n)_n$ is uniformly-ACG*, then $[0,1] = \bigcup_{k=1}^{\infty} E_k$, where E_k are measurable disjoint sets and $(F_n)_n$ is uniformly-AC* on E_k for each k. Since V_*F is a measure, it is enough to prove that, for each $k \in \mathbb{N}$ and for each negligible set $E \subset E_k$, $V_*F(E) = 0$. Fix $k \in \mathbb{N}$ and $E \subset E_k$, with $|E| = 0$. Given $\varepsilon > 0$, there are a gauge δ_0 and $\eta > 0$ such that if $\{(B_i,t_i)\colon i = 1,\dots,s\}$ is a δ_0-fine Perron partition anchored in E with $\sum_{i=1}^{s} |B_i| < \eta$, then $\sum_{i=1}^{s} \|F_n(B_i)\| < \varepsilon/3$ for each $n \in \mathbb{N}$. Moreover let $O \supset E$ be an open set with $|O| < \eta$. Now for $x \in E$ define $\delta(x) = \min(\delta_0(x), \mathrm{dist}(x,\partial O))$. Let $\{(A_i,t_i)\colon i = 1,\dots,p\}$ be a δ-fine Perron partition anchored in E with $\sum_{i=1}^{p} |A_i| < \eta$. For each $i = 1,\dots,p$ there is $x_i^* \in \mathcal{B}(X^*)$ such that $\|F(A_i)\| < |x_i^* F(A_i)| + \varepsilon/3p$. Since (F_n) weakly converges to F, there exists $N \in \mathbb{N}$ such that

$$|x_i^* F(A_i) - x_i^* F_N(A_i)| < \varepsilon/3p,$$

for $i = 1, \ldots, p$. So, we obtain

$$\begin{aligned}\sum_{i=1}^{p} \|F(A_i)\| &\leqslant \sum_{i=1}^{p} |x_i^* F(A_i)| + \frac{\varepsilon}{3} \\ &< \sum_{i=1}^{p} |x_i^* F_N(A_i)| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\leqslant \sum_{i=1}^{p} \|F_N(A_i)\| + \frac{2\varepsilon}{3} < \varepsilon.\end{aligned}$$

Then $V(F, \delta, E) \leqslant \varepsilon$ and $V_* F(E) = 0$. □

P r o o f of Theorem 4. By condition (b) it follows that, for each $x^* \in X^*$, the sequence $(x^* F_n(t) = (\mathrm{H})\int_0^t x^* f_n)$ is uniformly-ACG*. Then by condition (a) the real valued sequence $(x^* f_n)$ control converges to $x^* f$.[1] So $x^* f$ is Henstock integrable and

$$\lim_{n\to\infty} (\mathrm{H})\int_0^t x^* f_n = (\mathrm{H})\int_0^t x^* f, \tag{10}$$

for each $t \in [0,1]$ (see [2], Theorem 4.1). Fix $t_0 \in [0,1]$ and denote by C the weak closure of the set $((\mathrm{HV})\int_0^{t_0} f_n)_n$. Since $((\mathrm{HV})\int_0^{t_0} f_n)_n$ is a weakly Cauchy sequence, it is bounded. Moreover $C \setminus \{(\mathrm{HV})\int_0^{t_0} f_n \colon n \in \mathbb{N}\}$ contains at most one point. We want to prove that C is weakly compact. Assume by contradiction that C is not weakly compact. Then applying Theorem 1 of [14] ((1) $\longleftrightarrow$ (9)) with $T = X$ and $E = C$, there are $\theta > 0$, $(x_m) \subset C$ and a sequence (y_m^*) of equicontinuous functionals of X^* such that $\langle y_k^*, x_m\rangle = 0$ if $k > m$ and $\langle y_k^*, x_m\rangle > \theta$ if $k \leqslant m$. Thus we can find a subsequence (g_m) of (f_n) such that:

(i) $(\mathrm{H})\int_0^{t_0} y_k^* g_m = 0$ if $k > m$;
(ii) $(\mathrm{H})\int_0^{t_0} y_k^* g_m > \theta$ if $k \leqslant m$;
(iii) $\lim\limits_{m\to\infty} (\mathrm{H})\int_0^{t_0} x^* g_m = (\mathrm{H})\int_0^{t_0} x^* f$ for each $x^* \in X^*$.

Now we are going to prove that the sequence $(y_m^* f)_m$ in $\mathrm{H}([0, t_0])$ (endowed with the Alexiewicz norm) is relatively weakly compact with the weak closure contained in $\mathrm{H}([0, t_0])$. According to Theorem 16 of [1] it is enough to prove that $(y_m^* f)_m$ is $\mathcal{H}$-bounded and that $((\mathrm{H})\int_0^t y_m^* f)_m$ is equicontinuous and asymptotically-ACG* on $[0, t_0]$.

Since the sequence $(y_m^*)_m$ is equicontinuous, it is also equibounded. So by condition (b), the family $((\mathrm{H})\int_0^t y_m^* g_n \colon n, m \in \mathbb{N})$ is uniformly-ACG* on $[0, t_0]$. Moreover, by (10) for each Perron partition $\{(A_i, t_i) \colon i = 1, \ldots, p\}$ and for each $m \in \mathbb{N}$ we have

$$\sum_{i=1}^{p} \left| (\mathrm{H})\int_{A_i} y_m^* f \right| = \lim_{n\to\infty} \sum_{i=1}^{p} \left| (\mathrm{H})\int_{A_i} y_m^* g_n \right|.$$

[1] For the definition of control convergence see [2].

Then also the sequence $((\mathrm{H})\int_0^t y_m^* f)_m$ is uniformly-ACG*. Therefore it is equicontinuous and asymptotically-ACG* in $[0, t_0]$. Since $((\mathrm{H})\int_0^t y_m^* g_n\colon\ n, m \in \mathbb{N})$ is uniformly-ACG*, it is equicontinuous. Moreover $y_m^* F_n(0) = 0$ for each m and n, so $((\mathrm{H})\int_0^t y_m^* g_n\colon\ n, m \in \mathbb{N})$ is also equibounded. Therefore the same is true for the sequence $((\mathrm{H})\int_0^t y_m^* f)_m$.

Thus there exists $h \in \mathrm{H}([0, t_0])$ and a subsequence $(z_j^*) \subset (y_m^*)$ such that $\lim\limits_{j\to\infty} (\mathrm{H})\int_0^{t_0} z_j^* f g = (\mathrm{H})\int_0^{t_0} hg$, for each real function of bounded variation g. In particular,

$$\lim_{j\to\infty} (\mathrm{H})\int_0^{t_0} z_j^* f = (\mathrm{H})\int_0^{t_0} h. \tag{11}$$

By (iii) and (ii) $(\mathrm{H})\int_0^{t_0} z_j^* f = \lim\limits_{m\to\infty} (\mathrm{H})\int_0^{t_0} z_j^* g_m \geqslant \theta$ for all j; thus

$$(\mathrm{H})\int_0^{t_0} h \geqslant \theta. \tag{12}$$

Let z_0^* be a weak*-cluster point of the sequence $(z_j^*)_j$ and let $(w_s^*)_s$ be a subsequence weakly* converging to z_0^*. Then, for each n and for each $t \in [0, t_0]$, we have

$$\lim_s w_s^* g_n(t) = z_0^* g_n(t). \tag{13}$$

Moreover by condition (b) the family $((\mathrm{H})\int_0^t w_s^* g_n)_s$ is uniformly-ACG* in $[0, t_0]$, for each n, and by (13) $(w_s^* g_n)_s$ is control convergent to $z_0^* g_n$. Thus, by the controlled convergence theorem and by (i) we get

$$\lim_s (\mathrm{H})\int_0^{t_0} w_s^* g_n = (\mathrm{H})\int_0^{t_0} z_0^* g_n = 0.$$

Therefore by (iii) we infer that

$$(\mathrm{H})\int_0^{t_0} z_0^* f = 0. \tag{14}$$

As $((\mathrm{H})\int_0^t y_m^* f)_m$ is uniformly-ACG* in $[0, t_0]$, then also the family $((\mathrm{H})\int_0^t w_s^* f)_s$ is uniformly-ACG* in $[0, t_0]$. Moreover for almost each $t \in [0, t_0]$ $\lim\limits_s w_s^* f(t) = z_0^* f(t)$.

So, applying once again the controlled convergence theorem, we have

$$\lim_s (\mathrm{H})\int_0^{t_0} w_s^* f = (\mathrm{H})\int_0^{t_0} z_0^* f.$$

Thus by (11) it follows that $(\mathrm{H})\int_0^{t_0} z_0^* f = (\mathrm{H})\int_0^{t_0} h$. Hence by (12) we get

$$(\mathrm{H})\int_0^{t_0} z_0^* f \geqslant \theta,$$

in contradiction with (14). Thus the set C is weakly compact. Since t_0 is arbitrary there is $F\colon [0,1] \to X$ such that $x^*(F(t)) = \lim\limits_{n\to\infty} (\mathrm{H})\int_0^t x^* f_n = (\mathrm{H})\int_0^t x^* f$, for all $t \in [0,1]$ and for all $x^* \in X^*$. It remains to prove that $f \in \mathrm{HV}([0,1], X)$ and F is its primitive. Since each function f_n belongs to $\mathrm{HV}([0,1], X)$, it is strongly measurable (see Remark 7); so f is strongly measurable since it is the weak limit of (f_n). Hence by Pettis measurability Theorem f is essentially separably valued. Let $x^* \in X^*$ be fixed. The real valued function x^*F is the Henstock primitive of x^*f. Then $(x^*F)' = x^*f$ a.e, F is scalarly differentiable and its scalar derivative is f. Moreover, by Lemma 2 the strong critical variation V_*F of F is absolutely continuous. Thus by Theorem 3 $f \in \mathrm{H}([0,1], X)$ with primitive F and the assertion follows. □

We say that a sequence (f_n) of PU-integrable functions is *equi*-PU-*integrable* if for each $\varepsilon > 0$ there exists a gauge δ such that

$$\sup_{n\in\mathbb{N}} \left\| \sigma(f_n, \mathcal{Q}) - (\mathrm{PU})\int_0^1 f_n \right\| < \varepsilon$$

for each ε-regular δ-fine pseudopartition $\mathcal{Q}$ of $[0,1]$.

Theorem 5. *Let (f_n) be a sequence of real valued* PU*-integrable functions satisfying the following conditions:*
(a) $f_n \to f$ *everywhere in* $[0,1]$;
(b) (f_n) *is equi*-PU*-integrable.*
Then f is PU*-integrable and* $(\mathrm{PU})\int_0^1 f_n \to (\mathrm{PU})\int_0^1 f$.

Proof. The proof follows as in ([2], Theorem 6.1) with easy changes. □

Theorem 6. *Let (f_n) be a sequence of vector valued* PU*-integrable functions satisfying the following conditions:*
(a) $f_n \to f$ *weakly in* $[0,1]$;
(b) (f_n) *is equi*-PU*-integrable.*
Then f is PU*-integrable and* $(\mathrm{PU})\int_0^1 f_n \to (\mathrm{PU})\int_0^1 f$ *weakly.*

Proof. Condition (b) implies that for each $\varepsilon > 0$ there is a gauge δ such that

$$\sup_{n\in\mathbb{N}} \left\| \sigma(f_n, \mathcal{Q}) - (\mathrm{PU})\int_0^1 f_n \right\| < \frac{\varepsilon}{3}$$

for each ε-regular δ-fine pseudopartition $\mathcal{Q}$ of $[0,1]$. Then for each $x^* \in \mathcal{B}(X^*)$ we have

$$\sup_{n\in\mathbb{N}}\left|\sigma(x^* f_n, \mathcal{Q}) - (\mathrm{PU})\int_0^1 x^* f_n\right| < \frac{\varepsilon}{3} \tag{15}$$

for each ε-regular δ-fine pseudopartition $\mathcal{Q}$ of $[0,1]$. By the previous Theorem, for each $x^* \in X^*$, $x^* f$ is a real-valued PU-integrable function and

$$x^*(\mathrm{PU})\int_0^1 f_n = (\mathrm{PU})\int_0^1 x^* f_n \to (\mathrm{PU})\int_0^1 x^* f.$$

Therefore we can define a vector $\nu([0,1]) \in X^{**}$ such that

$$\nu([0,1])(x^*) = (\mathrm{PU})\int_0^1 x^* f.$$

We want to prove that f as function from $[0,1]$ to X^{**} is PU-integrable with integral $\nu([0,1])$.

Fix $\varepsilon > 0$ and find δ according to the equintegrability of (f_n). Let $\mathcal{Q} = \{(\theta_1, t_1), \dots, (\theta_p, t_p)\}$ be an ε-regular δ-fine pseudopartition of $[0,1]$. Now fix $x^* \in \mathcal{B}(X^*)$ and choose $k \in \mathbb{N}$ such that

$$\left|(\mathrm{PU})\int_0^1 x^* f_k - (\mathrm{PU})\int_0^1 x^* f\right| < \frac{\varepsilon}{3} \tag{16}$$

and

$$\sup_{1\leqslant i\leqslant p} |x^* f_k(t_i) - x^* f(t_i)| < \frac{\varepsilon}{3}. \tag{17}$$

Then by (17), (15) and (16) it follows that

$$\begin{aligned}
&|\sigma(x^* f, \mathcal{Q}) - \nu([0,1])(x^*)| \\
&\quad = \left|\sigma(x^* f, \mathcal{Q}) - (\mathrm{PU})\int_0^1 x^* f\right| \\
&\quad \leqslant |\sigma(x^* f, \mathcal{Q}) - \sigma(x^* f_k, \mathcal{Q})| + \left|\sigma(x^* f_k, \mathcal{Q}) - (\mathrm{PU})\int_0^1 x^* f_k\right| \\
&\qquad + \left|(\mathrm{PU})\int_0^1 x^* f_k - (\mathrm{PU})\int_0^1 x^* f\right| \\
&\quad < \frac{\varepsilon}{3}\sum_{i=1}^{p}\int_0^1 \theta_i + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

By the arbitrarity of $x^* \in \mathcal{B}(X^*)$, it follows that

$$\|\sigma(f,\mathcal{Q}) - \nu([0,1])\|_{**} \leqslant \varepsilon,$$

where $\|\cdot\|_{**}$ denotes the norm in X^{**}. Now $\sigma(f,\mathcal{Q}) \in X$, thus since X is complete, $\nu([0,1]) \in X$ and the assertion holds. □

6. Multipliers

We are going to characterize the multipliers of the HV-integral. If $F\colon [0,1] \to X$ is a continuous function and $G\colon [0,1] \to \mathbb{R}$ is a function of bounded variation, we denote by $(\mathrm{RS})\int F\,\mathrm{d}G$ the Riemann-Stieltjes integral of F with respect to G (see [13], p. 62).

We endow the space $\mathrm{HV}([0,1],X)$ with the norm

$$\|f\|_{\mathrm{HV}} = \sup_{0\leqslant t\leqslant 1}\left\|(\mathrm{HV})\int_0^t f\right\|.$$

As usual, we regard two functions f and h as identical if $f(t) = h(t)$ a.e. in $[0,1]$. If $Y \subset X$ the symbol $\overline{\mathrm{co}}(Y)$ denotes the closed convex hull of the set Y.

Proposition 5. *Let $F\colon [0,1] \to X$ be a Riemann-Stieltjes integrable function with respect to a non decreasing function G. Then for each $I \in \mathcal{I}$, one has*

$$(\mathrm{RS})\int_I F\,\mathrm{d}G \in \overline{\mathrm{co}}(\{G(I)x\colon\ x \in X \ \text{ and } \ x = F(t) \ \text{ for some } \ t \in I\}).$$

Proof. The proof follows as in ([8], Corollary 8, p. 48) after trivial changes. □

Proposition 6. *Let $f\colon [0,1] \to X$ be an* HV*-integrable function and let $F(t) = (\mathrm{HV})\int_0^t f$. If $G\colon [0,1] \to \mathbb{R}$ is a function of bounded variation, then Gf is* HV*-integrable and its primitive $H(t)$ is given by the formula*

$$H(t) = G(t)F(t) - (\mathrm{RS})\int_0^t F\,\mathrm{d}G.$$

Proof. As f is HV-integrable, its primitive $F(t) = (\mathrm{HV})\int_0^t f$ is continuous and the function H in the claim is well defined. Moreover, by the linearity of the Riemann-Stieltjes integral, we can assume that G is non decreasing on $[0,1]$. Let

M be an upper bound for G on $[0,1]$. According to Theorem 3, now we are proving that the strong critical variation V_*H of H is absolutely continuous. Let $\varepsilon > 0$ be fixed and let E be a negligible set. Since by Theorem 3 V_*F is absolutely continuous, we find a gauge δ such that

$$\sum_{i=1}^{p} \|F(A_i)\| < \frac{\varepsilon}{4(M + V(G,[0,1]))}, \tag{18}$$

for each δ-fine Perron partition $\mathcal{P} = \{(A_i,t_i)\colon\ i = 1,\dots,p\}$ anchored in E. Let $\mathcal{P} = \{(I_i,t_i)\colon\ i = 1,\dots,p\}$ be a δ-fine Perron partition anchored in E. By Proposition 5, for each $i = 1,\dots,p$ there are $x_1^{(i)},\dots,x_{n_i}^{(i)} \in I_i$ and $\lambda_1^{(i)},\dots,\lambda_{n_i}^{(i)} \in [0,1]$ with $\sum\limits_{j=1}^{n_i} \lambda_j^{(i)} = 1$, such that

$$\left\|\sum_{j=1}^{n_i} \lambda_j^{(i)} F(x_j^{(i)}) G(I_i) - (\mathrm{RS})\int_{I_i} F\,\mathrm{d}G\right\| \leqslant \frac{\varepsilon}{4pV(G,[0,1])}\, G(I_i). \tag{19}$$

Fix i and let $I_i = [a_i,b_i]$. By (19) we obtain

$$\begin{aligned}
(20)\quad & \|H(b_i) - H(a_i)\| \\
&= \left\|G(b_i)F(b_i) - G(a_i)F(a_i) - (\mathrm{RS})\int_{a_i}^{b_i} F\,\mathrm{d}G\right\| \\
&= \left\|G(b_i)[F(b_i) - F(a_i)] + [G(b_i) - G(a_i)]\left[F(a_i) - \sum_{j=1}^{n_i} \lambda_j^{(i)} F(x_j^{(i)})\right]\right. \\
&\qquad \left. + [G(b_i) - G(a_i)]\sum_{j=1}^{n_i} \lambda_j^{(i)} F(x_j^{(i)}) - (\mathrm{RS})\int_{a_i}^{b_i} F\,\mathrm{d}G\right\| \\
&\leqslant |G(b_i)|\,\|F(b_i) - F(a_i)\| + [G(b_i) - G(a_i)]\left\|F(a_i) - \sum_{j=1}^{n_i} \lambda_j^{(i)} F(x_j^{(i)})\right\| \\
&\qquad + \left\|[G(b_i) - G(a_i)]\sum_{j=1}^{n_i} \lambda_j^{(i)} F(x_j^{(i)}) - (\mathrm{RS})\int_{a_i}^{b_i} F\,\mathrm{d}G\right\| \\
&\leqslant M\|F(b_i) - F(a_i)\| + [G(b_i) - G(a_i)]\left\|\sum_{j=1}^{n_i} \lambda_j^{(i)} [F(a_i) - F(x_j^{(i)})]\right\| \\
&\qquad + \frac{\varepsilon}{4pV(G,[0,1])}\, G(I_i) \\
&\leqslant M\|F(b_i) - F(a_i)\| + V(G,[0,1]) \sum_{j=1}^{n_i} \lambda_j^{(i)} \|F(a_i) - F(x_j^{(i)})\| \\
&\qquad + \frac{\varepsilon}{4pV(G,[0,1])}\, G(I_i).
\end{aligned}$$

Assume that $t_i \in [a_i, x_j^{(i)}]$ for $j = 1, \ldots, l$ and that $t_i \in (x_j^{(i)}, b_i]$ for $j = l+1, \ldots, n_i$. Then we infer that

$$\begin{aligned}(21)\quad & M\|F(b_i) - F(a_i)\| + V(G,[0,1]) \sum_{j=1}^{n_i} \lambda_j^{(i)} \|F(a_i) - F(x_j^{(i)})\| \\ &\leqslant M\|F(b_i) - F(a_i)\| + V(G,[0,1]) \Big[\sum_{j=1}^{l} \lambda_j^{(i)} \|F(a_i) - F(x_j^{(i)})\| \\ &\quad + \sum_{j=l+1}^{n_i} \lambda_j^{(i)} \|F(b_i) - F(x_j^{(i)})\| + \sum_{j=l+1}^{n_i} \lambda_j^{(i)} \|F(b_i) - F(a_i)\|\Big] \\ &\leqslant [M + V(G,[0,1])]\, \|F(b_i) - F(a_i)\| \\ &\quad + V(G,[0,1]) \Big[\sum_{j=1}^{l} \lambda_j^{(i)} \|F(a_i) - F(x_j^{(i)})\| + \sum_{j=l+1}^{n_i} \lambda_j^{(i)} \|F(b_i) - F(x_j^{(i)})\|\Big].\end{aligned}$$

Denote by x_i' the vector among $x_1^{(i)}, \ldots, x_l^{(i)}$ for which the norm $\|F(a_i) - F(x_j^{(i)})\|$ attains its maximum value and by x_i'' the vector among $x_{l+1}^{(i)}, \ldots, x_{n_i}^{(i)}$ for which also the norm $\|F(b_i) - F(x_j^{(i)})\|$ attains its maximum value. We have

$$\begin{aligned}(22)\quad & V(G,[0,1]) \Big[\sum_{j=1}^{l} \lambda_j^{(i)} \|F(a_i) - F(x_j^{(i)})\| + \sum_{j=l+1}^{n_i} \lambda_j^{(i)} \|F(b_i) - F(x_j^{(i)})\|\Big] \\ &\leqslant V(G,[0,1]) [\|F(a_i) - F(x_i')\| + \|F(b_i) - F(x_i'')\|].\end{aligned}$$

We observe that $\{([a_i, x_i'], t_i) \colon\ i = 1, \ldots, p\}$ and $\{([x_i'', b_i], t_i) \colon\ i = 1, \ldots, p\}$ are δ-fine Perron partitions anchored in E. So by (20), (21), (22), (19) and (18) we get

$$\begin{aligned}& \sum_{i=1}^{p} \|H(b_i) - H(a_i)\| \\ &\qquad \leqslant [M + V(G,[0,1])] \sum_{i=1}^{p} \|F(b_i) - F(a_i)\| \\ &\qquad\quad + V(G,[0,1]) \Big[\sum_{i=1}^{p} \|F(a_i) - F(x_i') + \sum_{i=1}^{p} \|F(b_i) - F(x_i'')\|\Big] \\ &\qquad\quad + \frac{\varepsilon}{4pV(G,[0,1])} \sum_{i=1}^{p} G(I_i) \\ &\qquad \leqslant [M + V(G,[0,1])] \frac{\varepsilon}{4(M + V(G,[0,1]))} \\ &\qquad\quad + 2V(G,[0,1]) \frac{\varepsilon}{4(M + V(G,[0,1]))} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

Since this is true for every δ-fine Perron partition $\mathcal{P}$ anchored in E and since ε is arbitrary we obtain $V_*H(E)=0$. So the strong critical variation of H is absolutely continuous. Besides, by Theorem 3 f is the scalar derivative of F; so for each $x^*\in X^*$, we have

$$\begin{aligned}(x^*H)' &= \Big(x^*(GF) - x^*(\mathrm{RS})\int F\,\mathrm{d}G\Big)' \\ &= (x^*F)'G + (x^*F)G' - (x^*F)G' = (x^*F)'G = (x^*f)G = x^*(Gf),\end{aligned}$$

a.e. in $[0,1]$. Hence the scalar derivative of H is Gf. Moreover, since G is measurable and f is strongly measurable, Gf is strongly measurable and then essentially separably valued. Thus all the hypotheses of Theorem 3 are fulfilled for Gf and the assertion follows. □

Proposition 7. *If $G\colon [0,1]\to\mathbb{R}$ is a multiplier for* $\mathrm{HV}([0,1],X)$ *then G is equivalent to a function of bounded variation.*

Proof. Let x be a non null vector in X and let $h\in \mathrm{H}([0,1])$ with primitive $H(t)=(\mathrm{H})\int_0^t h$. The function hx is HV-integrable. Indeed fix $\varepsilon>0$ and find a gauge δ such that

$$\sum_{i=1}^{p}\big|h(t_i)|A_i| - H(A_i)\big| < \frac{\varepsilon}{\|x\|}, \tag{23}$$

for each δ-fine Perron partition $\mathcal{P}=\{(A_i,t_i)\colon\ i=1,\dots,p\}$.

Then, by (23)

$$\sum_{i=1}^{p}\|h(t_i)|A_i|x - H(A_i)x\| < \varepsilon,$$

for every δ-fine Perron partition $\mathcal{P}=\{(A_i,t_i)\colon\ i=1,\dots,p\}$.

Since G is a multiplier for $\mathrm{HV}([0,1],X)$, the function $G(hx)=(Gh)x$ belongs to $\mathrm{HV}([0,1],X)$ and also to $\mathrm{H}([0,1],X)$. So for each $\varepsilon>0$ there is a gauge δ such that

$$\|\sigma(Ghx,\mathcal{P}_1)-\sigma(Ghx,\mathcal{P}_2)\| < \varepsilon\|x\|, \tag{24}$$

for each pair $\mathcal{P}_1$ and $\mathcal{P}_2$ of δ-fine Perron partitions. Note that

$$\|\sigma(Ghx,\mathcal{P}_1)-\sigma(Ghx,\mathcal{P}_2)\| = \|x\|\,|\sigma(Gh,\mathcal{P}_1)-\sigma(Gh,\mathcal{P}_2)|.$$

Thus, by (24) we have

$$|\sigma(Gh,\mathcal{P}_1)-\sigma(Gh,\mathcal{P}_2)| < \varepsilon.$$

Therefore $Gh\in\mathrm{H}([0,1])$, for each $h\in\mathrm{H}([0,1])$ and G is a multiplier for the family $\mathrm{H}([0,1])$. Thus G is equivalent to a function of bounded variation (see [16], Theorem 12.9, p. 78) and the assertion is true. □

Proposition 6 and Proposition 7 give the following

Theorem 7. *The family of multipliers for the* HV*-integral coincides with the family of all functions of bounded essential variation.*

Remark. The previous Theorem holds also for the Henstock integral. Indeed by using Proposition 5 and the fact that a Henstock primitive is continuous, Proposition 6 can be proved as ([16], Theorem 12.1, p. 72) after trivial changes.

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