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OPTIMAL RETRACTION PROBLEM FOR PROPER K-BALL-CONTRACTIVE MAPPINGS IN $C^m[0,1]$

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Abstract. In this paper for any $\varepsilon > 0$ we construct a new proper k-ball contractive retraction of the closed unit ball of the Banach space $C^m[0,1]$ onto its boundary with $k < 1+\varepsilon$, so that the Wośko constant $W_\gamma(C^m[0,1])$ is equal to 1.

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1. Introduction and Preliminaries

Let X be an infinite-dimensional Banach space with closed unit ball $B(X)$ and unit sphere $S(X)$. After two works by Klee [22] and [23] it is known that there exists a *retraction* $R : B(X) \to S(X)$, i.e. R is a continuous mapping such that $Rx = x$, for all $x \in S(X)$. For what concerns the metric properties of such retractions Benyamini and Sternfeld ([5]), following Nowak ([24]), have obtained the remarkable result that for every Banach space X there exists a retraction of $B(X)$ onto $S(X)$ satisfying, for a constant L, the L-Lipschitz condition

$$\|Rx - Ry\| \leq L\|x - y\| \quad \text{for all } x, y \in B(X).$$

Clearly the same is not true for spaces of finite dimension due to the Brouwer's Non Retraction Theorem. The optimal retraction problem, considered for the first time in [17], consists in the evaluation of the constant

$$k_0(X) = \inf\{L : \text{there is a } L\text{-Lipschitz retraction } R : B(X) \to S(X)\}.$$

There is no space X for which the exact value of $k_0(X)$ is known, for a survey on the subject we refer to [13, 18, 19] and bibliography therein. The universal known bound from below is $k_0(X) \geq 3$; for some spaces there are better estimates, for example, $k_0(H) \geq 4.58$ for Hilbert space H (see [14]), $k_0(l^1) \geq 4$ (see [6]). From the above it is known that the supremum is finite and an attempt to give an estimate ends with $k_0(X) < 256 \cdot 10^9$, for any Banach space X (see [1]). Moreover we recall $k_0(L_1[0,1]) \leq 8$ (see [20]), $k_0(l^1) \leq 8$ (see [2]), $k_0(l^\infty) \leq 12 + 2\sqrt{30} = 22.95...$ (see [14]) and $k_0(X) \leq 4(2 + \sqrt{3}) = 14.92...$ where X is one of the space: $BC(\mathbb{R})$, $C[0,1]$, c_0, c (see [26]). At present the estimate $k_0(C_0[0,1]) \leq 2(2 + \sqrt{2}) = 6.828...$ is the minimum of upper bounds over all Banach spaces for which the upper bound is known (see [25]).

The fact of considering another metric property, namely measures of noncompactness of the above retractions leads to more useful results for applications as, for instance, applications to theorems of Birkhoff-Kellog type (see [8, 9, 11, 16, 21]). Let us recall that the *Hausdorff measure of noncompactness* $\gamma(A)$ of a bounded subset A of X is the infimum of all $\varepsilon > 0$ such that A has a finite ε-net in X. A mapping $T : dom(T) \subset X \to X$ is said to be *k-ball contractive* with constant $k \geq 0$ if it is continuous and verifies, for each bounded subset A of $dom(T)$,

$$\gamma(TA) \leq k\gamma(A).$$

In [28] Wośko has proved that in the space $X = C[0,1]$ of all real valued continuous functions defined on $[0,1]$ endowed with the maximum norm it is possible to construct for every $\varepsilon > 0$ a k-ball contractive retraction of $B(X)$ onto $S(X)$ such that $k < 1 + \varepsilon$. The optimal retraction problem for k-ball contractive mappings will now concern the evaluation of the so-called Wośko constant (see [4])

$W_\gamma(X) = \inf\{k \geq 1$: there is a k-ball contractive retraction $R : B(X) \to S(X)\}$.

Obviously, the same problem can be posed by replacing γ with an equivalent measure of noncompactness, for instance the Kuratowski or the lattice measure of noncompactness. Observe that the situation differs from the Lipschitz case. On the one hand there are good estimates for $W_\gamma(X)$ in many Banach spaces X, which are useful for applications. Concerning general results in the setting of Banach spaces, in [27] it was proved that $W_\gamma(X) \leq 6$ for any Banach space X, reaching the value 4 or 3 depending on the geometry of the space. Moreover it has been proved that $W_\gamma(X) = 1$ in some spaces of continuous functions ([7], [15]), in some classical Banach spaces of measurable functions ([12]) and in Banach spaces whose norm is monotone with respect to some basis ([3]). In [10] the problem of evaluating the Wośko constant has been considered in the setting of F-normed spaces. We recall that the problem whether there is some Banach space X in which a 1-ball contractive retraction exists has been solved positively in [12], where it is shown that it is so in Orlicz and Lorentz spaces. We observe that there is not a unified method to evaluate $W_\gamma(X)$, most of the evaluations have required individual constructions in each space X. On the other hand we point out that in opposite to the limitation $k_0(X) \geq 3$ in any Banach space X, there is no Banach space X for which it has been proved $W_\gamma(X) > 1$.

For a continuous mapping $T : dom(T) \subset X \to X$ we also consider the following quantitative characteristic (see, for example, [3], [11], [16]):

$$\omega(T) = \sup\{k \geq 0 : \gamma(TA) \geq k\gamma(A) \text{ for every bounded } A \subset dom(T)\},$$

called the *lower Hausdorff measure of noncompactness* of T. We observe that this characteristic is related to properness of the mapping, since from $\omega(T) > 0$ it follows that T is a *proper* mapping, i.e. $T^{-1}K$ is compact for each compact subset K of X. Let $C^m[0,1]$ be the Banach space of all real valued m-times continuously differentiable functions defined on $[0,1]$ with the norm

$$||f||_m = \max\{||f^{(s)}|| : s = 0, 1, \cdots, m\},$$

where $f^{(0)} = f$ and $|| \cdot ||$ denotes the maximum norm. The aim of this paper is to prove that $W_\gamma(C^m[0,1]) = 1$. To this end we construct a 1-ball contractive mapping Q_m from the closed unit ball $B(C^m[0,1])$ into itself such that $Q_m f = f$ for all $f \in S(C^m[0,1])$. Then for each $u > 0$ we consider a compact perturbation $Q_m + P_{u,m}$ of the mapping Q_m, by normalizing such mapping we obtain a retraction $R_{u,m}$. The retractions $R_{u,m}$ we construct satisfy $\omega(R_{u,m}) > 0$. Moreover given $\varepsilon > 0$ arbitrarily fixed we can find $u > 0$ such that the retraction $R_{u,m}$ is k-ball contractive with $k < 1 + \varepsilon$, so that $W_\gamma(C^m[0,1]) = 1$. For $m = 0$, we recover the result $W_\gamma(C[0,1]) = 1$ of Wośko ([28]).

2. The mapping Q_m

Let $C^m := C^m[0,1]$. We start by defining a mapping Q_m from $B(C^m)$ into C^m by setting, for each $f \in B(C^m)$,

$$(Q_m f)(t) = \begin{cases} \left(\frac{1+||f||_m}{2}\right)^m f\left(\frac{2}{1+||f||_m} t\right) & \text{if } t \in \left[0, \frac{1+||f||_m}{2}\right], \\ \sum_{i=0}^m \frac{1}{i!}\left(t - \frac{1+||f||_m}{2}\right)^i \left(\frac{1+||f||_m}{2}\right)^{m-i} f^{(i)}(1) & \text{if } t \in \left(\frac{1+||f||_m}{2}, 1\right]. \end{cases}$$

In this section we prove that the mapping Q_m maps $B(C^m)$ into itself and that it is a 1-ball contractive mapping, moreover we obtain $\omega(Q_m) \geq \frac{1}{2^m(m+1)}$.
For our convenience, given $f \in C^m$ and $a \in [1,2]$, we introduce the function $f_a : [0,1] \to \mathbb{R}$ defined by

$$f_a(t) = \begin{cases} \frac{1}{a^m} f(at) & \text{if } \quad t \in \left[0, \frac{1}{a}\right], \\ \sum_{i=0}^m \frac{1}{i!}\left(t - \frac{1}{a}\right)^i \frac{1}{a^{m-i}} f^{(i)}(1) & \text{if } \quad t \in \left(\frac{1}{a}, 1\right]. \end{cases}$$

We observe that for $f \in C^m$ we have $f_a \in C^m$ and, for $s = 0, 1, \cdots, m$,

$$f_a^{(s)}(t) = \begin{cases} \frac{1}{a^{m-s}} f^{(s)}(at) & \text{if } \quad t \in \left[0, \frac{1}{a}\right], \\ \sum_{i=s}^m \frac{1}{(i-s)!}\left(t - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}} f^{(i)}(1) & \text{if } \quad t \in \left(\frac{1}{a}, 1\right]. \end{cases}$$

Using the above notation, for $f \in B(C^m)$, we may write

$$Q_m f = f_{\frac{2}{1+||f||_m}}.$$

We begin with the following result.

Lemma 2.1. *Let $f \in C^m$, then for any $a \in [1,2]$ we have*

$$\frac{1}{a^m}||f||_m \leq ||f_a||_m \leq ||f||_m.$$

Proof. Let $f \in C^m$. Being the result obvious when $a = 1$, we assume $a \in (1,2]$. To obtain the right inequality we prove $||f_a^{(s)}|| \leq ||f||_m$ for each $s \in \{0, 1, \cdots, m\}$. Indeed, we have

$$\begin{aligned} ||f_a^{(s)}|| &= \max\left\{\frac{1}{a^{m-s}}||f^{(s)}||, \max_{t\in\left[\frac{1}{a},1\right]}\left|\sum_{i=s}^m \frac{1}{(i-s)!}\left(t - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}} f^{(i)}(1)\right|\right\} \\ &\leq \max\left\{\frac{1}{a^{m-s}}||f^{(s)}||, \sum_{i=s}^m \frac{1}{(i-s)!}\left(1 - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}}||f^{(i)}||\right\} \\ &= \sum_{i=s}^m \frac{1}{(i-s)!}\left(1 - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}}||f^{(i)}|| \ \leq \ ||f||_m \sum_{i=s}^m \left(1 - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}} \\ &\leq ||f||_m \left[\frac{1}{a^{m-s}} + \left(1 - \frac{1}{a}\right) \sum_{i=s+1}^m \frac{1}{a^{m-i}}\right] = ||f||_m. \end{aligned}$$

On the other hand, we obtain the left inequality by observing what follows

$$\begin{aligned} ||f_a||_m &= \max\left\{||f_a||, ||f_a^{(1)}||, \cdots, ||f_a^{(m)}||\right\} \\ &\geq \max\left\{\frac{1}{a^m}||f||, \frac{1}{a^{m-1}}||f^{(1)}||, \cdots,\ ||f^{(m)}||\right\} \\ &\geq \frac{1}{a^m}\max\left\{||f||, ||f^{(1)}||, \cdots,\ ||f^{(m)}||\right\} = \frac{1}{a^m}||f||_m. \end{aligned}$$

□

We note that by the definition of the mapping Q_m we have $Q_m f = f$ for all $f \in S(C^m)$, and by Lemma 2.1 we have indeed that Q_m maps $B(C^m)$ into itself. In the sequel we will require the following lemma.

LEMMA 2.2. *Let $f \in C^m$ and $\{a_n\}$ a sequence in $[1,2]$ such that $a_n \to a$. Then $||f_{a_n} - f_a||_m \to 0$.*

PROOF. We observe that for $f = 0$ the assert is immediate. For $f \neq 0$ we prove that $||f_{a_n}^{(s)} - f_a^{(s)}|| \to 0$ for any $s \in \{0, 1, \cdots, m\}$, and this will give the thesis. Let $\varepsilon > 0$ be given. Since $f^{(s)}$, for any $s \in \{0, 1, \cdots, m\}$, is uniformly continuous on $[0,1]$, we find $\delta > 0$ such that

$$|f^{(s)}(t_1) - f^{(s)}(t_2)| \leq \frac{\varepsilon}{3} \tag{2.1}$$

for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| \leq \delta$. Moreover if $s \in \{0, 1, \cdots, m-1\}$ we choose $\bar{n}$ such that for all $n \geq \bar{n}$ we have $|a_n - a| \leq \delta$ and

$$\left|\frac{1}{a_n^{m-i}} - \frac{1}{a^{m-i}}\right| \leq \frac{\varepsilon}{3(m-s)||f||_m}, \qquad \text{for} \quad i = s, \cdots, m-1. \tag{2.2}$$

Now, for any fixed $s \in \{0, 1, \cdots, m\}$, we prove $|f_{a_n}^{(s)}(t) - f_a^{(s)}(t)| \leq \varepsilon$, for every $n \geq \bar{n}$ and for all $t \in [0,1]$.
To this end, suppose first $t \in [0, 1/a] \cap [0, 1/a_n]$. Then we have

$$\begin{aligned} |f_{a_n}^{(s)}(t) - f_a^{(s)}(t)| &= \left|\frac{1}{a_n^{m-s}} f^{(s)}(a_n t) - \frac{1}{a^{m-s}} f^{(s)}(at)\right| \\ &\leq \left|f^{(s)}(a_n t)\right| \left|\frac{1}{a_n^{m-s}} - \frac{1}{a^{m-s}}\right| + \frac{1}{a^{m-s}}\left|f^{(s)}(a_n t) - f^{(s)}(at)\right| \\ &\leq ||f^{(s)}|| \left|\frac{1}{a_n^{m-s}} - \frac{1}{a^{m-s}}\right| + \left|f^{(s)}(a_n t) - f^{(s)}(at)\right| \\ &\leq ||f||_m \left|\frac{1}{a_n^{m-s}} - \frac{1}{a^{m-s}}\right| + \left|f^{(s)}(a_n t) - f^{(s)}(at)\right| \\ &\leq \frac{\varepsilon}{3(m-s)} + \frac{\varepsilon}{3} \leq \varepsilon. \end{aligned}$$

Assume now $a \leq a_n$ and $t \in [1/a_n, 1/a]$. Then since $|1 - at| \leq |a_n - a| \leq \delta$, by (2.1) we get

$$\left| f^{(s)}(at) - f^{(s)}(1) \right| \leq \frac{\varepsilon}{3}. \tag{2.3}$$

Using (2.2) and (2.3) we obtain

$$\begin{aligned}
|f^{(s)}_{a_n}(t) - f^{(s)}_a(t)| &= \left| \frac{1}{a^{m-s}} f^{(s)}(at) - \sum_{i=s}^{m} \frac{1}{(i-s)!} \left(t - \frac{1}{a_n} \right)^{i-s} \frac{1}{a_n^{m-i}} f^{(i)}(1) \right| \\
&\leq \left| \frac{1}{a^{m-s}} f^{(s)}(at) - \frac{1}{a_n^{m-s}} f^{(s)}(1) \right| + \sum_{i=s+1}^{m} ||f^{(i)}|| \left| \frac{1}{a} - \frac{1}{a_n} \right|^{i-s} \\
&\leq \left| \frac{1}{a^{m-s}} f^{(s)}(at) - \frac{1}{a_n^{m-s}} f^{(s)}(at) \right| + \left| \frac{1}{a_n^{m-s}} f^{(s)}(at) - \frac{1}{a_n^{m-s}} f^{(s)}(1) \right| \\
&\quad + ||f||_m \sum_{i=s+1}^{m} \left| \frac{1}{a} - \frac{1}{a_n} \right|^{i-s} \\
&\leq ||f||_m \left| \frac{1}{a^{m-s}} - \frac{1}{a_n^{m-s}} \right| + \left| f^{(s)}(at) - f^{(s)}(1) \right| + ||f||_m (m-s) \left| \frac{1}{a} - \frac{1}{a_n} \right| \\
&\leq \frac{\varepsilon}{3(m-s)} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

If $a_n \leq a$ and $t \in [1/a, 1/a_n]$, the assert follows as in the previous case.
Finally, considering the case $t \in [\max\{1/a, 1/a_n\}, 1]$, we have

$$\begin{aligned}
|f^{(s)}_{a_n}(t) - f^{(s)}_a(t)| &= \left| \sum_{i=s}^{m} \frac{f^{(i)}(1)}{(i-s)!} \left[\left(t - \frac{1}{a_n} \right)^{i-s} \frac{1}{a_n^{m-i}} - \left(t - \frac{1}{a} \right)^{i-s} \frac{1}{a^{m-i}} \right] \right| \\
&\leq ||f||_m \sum_{i=s}^{m} \frac{1}{(i-s)!} \left| \left(t - \frac{1}{a_n} \right)^{i-s} \frac{1}{a_n^{m-i}} - \left(t - \frac{1}{a} \right)^{i-s} \frac{1}{a^{m-i}} \right|
\end{aligned}$$

where, for $i \in \{s, \cdots, m\}$,

$$\frac{1}{(i-s)!}\left|\left(t-\frac{1}{a_n}\right)^{i-s}\frac{1}{a_n^{m-i}}-\left(t-\frac{1}{a}\right)^{i-s}\frac{1}{a^{m-i}}\right|$$

$$\leq \frac{1}{(i-s)!}\left|\left(t-\frac{1}{a_n}\right)^{i-s}\frac{1}{a_n^{m-i}}-\left(t-\frac{1}{a}\right)^{i-s}\frac{1}{a_n^{m-i}}\right|$$

$$+\frac{1}{(i-s)!}\left|\left(t-\frac{1}{a}\right)^{i-s}\frac{1}{a_n^{m-i}}-\left(t-\frac{1}{a}\right)^{i-s}\frac{1}{a^{m-i}}\right|$$

$$\leq \frac{1}{(i-s)!}\left|\left(t-\frac{1}{a_n}\right)^{i-s}-\left(t-\frac{1}{a}\right)^{i-s}\right|+\frac{1}{(i-s)!}\left|\frac{1}{a_n^{m-i}}-\frac{1}{a^{m-i}}\right|$$

$$\leq \frac{1}{(i-s)!}\left|\frac{1}{a_n}-\frac{1}{a}\right|\left|\left(t-\frac{1}{a_n}\right)^{i-s-1}+\cdots+\left(t-\frac{1}{a}\right)^{i-s-1}\right|+\left|\frac{1}{a_n^{m-i}}-\frac{1}{a^{m-i}}\right|$$

$$\leq \frac{i-s}{(i-s)!}\left|\frac{1}{a_n}-\frac{1}{a}\right|+\left|\frac{1}{a_n^{m-i}}-\frac{1}{a^{m-i}}\right|.$$

Consequently, using (2.2) we obtain

$$|f_{a_n}^{(s)}(t)-f_a^{(s)}(t)| \leq ||f||_m\left[\sum_{i=s+1}^{m}\left|\frac{1}{a_n}-\frac{1}{a}\right|+\sum_{i=s}^{m-1}\left|\frac{1}{a_n^{m-i}}-\frac{1}{a^{m-i}}\right|\right]$$

$$\leq ||f||_m\left[(m-s)\left|\frac{1}{a_n}-\frac{1}{a}\right|+(m-s)\left|\frac{1}{a_n^{m-i}}-\frac{1}{a^{m-i}}\right|\right]$$

$$\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}<\varepsilon.$$

The proof is completed. □

Proposition 2.3. *The mapping Q_m is 1-ball contractive.*

Proof. Let $\{f_n\}$ be a sequence in $B(C^m)$ and f a function in $B(C^m)$ such that $||f_n - f||_m \to 0$. Set $a_n = \frac{2}{1+\|f_n\|_m}$ and $a = \frac{2}{1+\|f\|_m}$, then $a_n \in [1,2]$ for each $n \in \mathbb{N}$, $a \in [1,2]$ and $a_n \to a$. Moreover,

$$\|Q_m f_n - Q_m f\|_m = ||(f_n)_{\frac{2}{1+\|f_n\|_m}} - f_{\frac{2}{1+\|f\|_m}}||_m = ||(f_n)_{a_n} - f_a||_m.$$

Since by the hypothesis and Lemma 2.2 we have

$$||(f_n)_{a_n} - f_a||_m \leq ||(f_n)_{a_n} - f_{a_n}||_m + ||f_{a_n} - f_a||_m$$

$$= ||(f_n - f)_{a_n}||_m + ||f_{a_n} - f_a||_m$$

$$\leq ||f_n - f||_m + ||f_{a_n} - f_a||_m \to 0,$$

we conclude that the mapping Q_m is continuous.
Now to complete the proof we show that for $M \subseteq B(C^m)$ we have

$$\gamma(Q_m M) \leq \gamma(M).$$

First we observe that for $f \in C^m$ the set $A_f = \{f_a : a \in [1,2]\}$ is compact. Indeed, if $\{f_{a_n}\}$ is a sequence of elements in A_f and $\{a_{n_k}\}$ a subsequence of $\{a_n\}$ which is convergent, say to a, then by Lemma 2.2 we have $||f_{a_{n_k}} - f_a||_m \to 0$.
Now let $\alpha > \gamma(M)$. Let $\{\varphi_1, \cdots, \varphi_l\}$ an α-net for M in C^m. Then the set $A := \bigcup_{i=1}^{l} A_{\varphi_i}$ is compact. Hence given $\delta > 0$ we can choose a δ-net $\{\psi_1, \cdots, \psi_p\}$ for A in C^m.

For $g \in Q_m M$ arbitrarily fixed, let $f \in M$ such that $Q_m f = g$. Then choose $i \in \{1, \cdots, l\}$ such that $\|f - \varphi_i||_m \leq \alpha$ and $j \in \{1, \cdots, p\}$ such that

$$\|(\varphi_i)_{\frac{2}{1+\|f||_m}} - \psi_j||_m \leq \delta.$$

Then we obtain

$$\begin{aligned} ||g - \psi_j||_m &= \|Q_m f - \psi_j||_m = \|f_{\frac{2}{1+\|f||_m}} - \psi_j||_m \\ &\leq \|f_{\frac{2}{1+\|f||_m}} - (\varphi_i)_{\frac{2}{1+\|f||_m}}\| + \|(\varphi_i)_{\frac{2}{1+\|f||_m}} - \psi_j||_m \\ &\leq \|f - \varphi_i||_m + \delta \leq \alpha + \delta. \end{aligned}$$

We have proved $\gamma(Q_m M) \leq \alpha + \delta$, by the arbitrariness of δ we have the desired result $\gamma(Q_m M) \leq \gamma(M)$. □

Now for $f \in C^m$ and $a \in [1,2]$, we set

$$(f^{\frac{1}{a}})(t) = a^m f\left(\frac{1}{a}t\right), \quad \text{if } t \in [0,1].$$

We need the following two lemmas. The proof of the first lemma is similar to the first case we have considered in Lemma 2.2, hence it is omitted.

Lemma 2.4. *Let $f \in C^m$ and $\{a_n\}$ a sequence in $[1,2]$ such that $a_n \to a$. Then $||f^{\frac{1}{a_n}} - f^{\frac{1}{a}}||_m \to 0$.*

Lemma 2.5. *Let $f \in B(C^m)$, $g \in C^m$ and $a \in [1,2]$. Then*

$$\left\|f_a - (g^{\frac{1}{a}})_a\right\|_m \leq (m+1)||f_a - g||_m.$$

Proof. Let $f \in B(C^m)$, $g \in C^m$ and $a \in [1,2]$. To prove the thesis we will show that, for $s = 0, 1, \cdots, m$, we have

$$\left\|f_a^{(s)} - \left(\left(g^{\frac{1}{a}}\right)_a\right)^{(s)}\right\| \leq (m+1)||f_a - g||_m.$$

Clearly $g^{\frac{1}{a}} \in C^m$, where for each $s = 0, 1, \cdots, m$ we have $(g^{\frac{1}{a}})^{(s)}(t) = a^{m-s} g^{(s)}\left(\frac{1}{a}t\right)$ $(t \in [0,1])$. Hence we can consider $(g^{\frac{1}{a}})_a$, and we have

$$(g^{\frac{1}{a}})_a(t) = \begin{cases} g(t) & \text{if} \quad t \in \left[0, \frac{1}{a}\right], \\ \sum_{i=0}^{m} \frac{1}{i!}\left(t - \frac{1}{a}\right)^i g^{(i)}\left(\frac{1}{a}\right) & \text{if} \quad t \in \left(\frac{1}{a}, 1\right]. \end{cases}$$

Then we have

$$\left\| f_a^{(s)} - \left((g^{\frac{1}{a}})_a\right)^{(s)} \right\| = \max\Bigg\{ \max_{t\in\left[0,\frac{1}{a}\right]} \left| f_a^{(s)}(t) - g^{(s)}(t) \right|,$$

$$\max_{t\in\left(\frac{1}{a},1\right]} \left| \sum_{i=s}^{m} \frac{1}{(i-s)!}\left(t - \frac{1}{a}\right)^{i-s} \frac{1}{a^{m-i}} f^{(i)}(1) - \sum_{i=s}^{m} \frac{1}{(i-s)!}\left(t - \frac{1}{a}\right)^{i-s} g^{(i)}\left(\frac{1}{a}\right) \right| \Bigg\}$$

$$\leq \max\left\{ ||f_a^{(s)} - g^{(s)}||,\ \max_{t\in\left[\frac{1}{a},1\right]} \sum_{i=s}^{m} \frac{1}{(i-s)!}\left(t - \frac{1}{a}\right)^{i-s} \left| \frac{1}{a^{m-i}} f^{(i)}(1) - g^{(i)}\left(\frac{1}{a}\right) \right| \right\}$$

$$\leq \max\left\{ ||f_a^{(s)} - g^{(s)}||,\ \sum_{i=s}^{m} \left| \frac{1}{a^{m-i}} f^{(i)}(1) - g^{(i)}\left(\frac{1}{a}\right) \right| \right\}$$

$$= \max\left\{ ||f_a^{(s)} - g^{(s)}||,\ \sum_{i=s}^{m} \left| f_a^{(i)}\left(\frac{1}{a}\right) - g^{(i)}\left(\frac{1}{a}\right) \right| \right\}$$

$$= \max\left\{ ||f_a^{(s)} - g^{(s)}||,\ \sum_{i=s}^{m} ||f_a^{(i)} - g^{(i)}|| \right\}$$

$$= ||f_a^{(s)} - g^{(s)}|| + \cdots + ||f_a^{(m)} - g^{(m)}|| \leq (m - s + 1)||f_a - g||_m$$

$$\leq (m+1)||f_a - g||_m,$$

which completes the proof. □

Proposition 2.6. *For the mapping Q_m the following estimate of its lower Hausdorff measure of noncompactness holds:*

$$\omega(Q_m) \geq \frac{1}{2^m(m+1)}.$$

Proof. It is enough to show that for $M \subseteq B(C^m)$ we have

$$\frac{1}{2^m(m+1)}\gamma(M) \leq \gamma(Q_m M). \tag{2.4}$$

If $f \in C^m$, using Lemma 2.4, it follows that the set $A^f := \{f^{\frac{1}{a}} : a \in [1,2]\}$ is compact. Now let $\eta > \gamma(Q_m M)$. Fix an η-net $\{\lambda_1, \cdots, \lambda_q\}$ for $Q_m M$ in C^m. Then the set $K := \bigcup_{i=1}^{q} A^{\lambda_i}$ is also compact in C^m.

Let $\delta > 0$ be given, and choose a δ-net $\{\xi_1, \cdots, \xi_r\}$ for K in C^m. Let $f \in M$.

Fix $i \in \{1, \cdots, q\}$ such that $\|Q_m f - \lambda_i||_m \leq \eta$. Since $(\lambda_i)^{\frac{1+\|f||_m}{2}} \in K$ we can choose $j \in \{1, \cdots, r\}$ such that $\|(\lambda_i)^{\frac{1+\|f||_m}{2}} - \xi_j||_m \leq \delta$. Then

$$\|f - \xi_j||_m \leq \|f - (\lambda_i)^{\frac{1+\|f||_m}{2}}||_m + \|(\lambda_i)^{\frac{1+\|f||_m}{2}} - \xi_j||_m$$

$$\leq 2^m \|f_{\frac{2}{1+\|f||_m}} - ((\lambda_i)^{\frac{1+\|f||_m}{2}})_{\frac{2}{1+\|f||_m}}||_m + \delta.$$

Now by Lemma 2.5 we have

$$\|f_{\frac{2}{1+\|f||_m}} - ((\lambda_i)^{\frac{1+\|f||_m}{2}})_{\frac{2}{1+\|f||_m}}||_m \leq (m+1)\|f_{\frac{2}{1+\|f||_m}} - \lambda_i||_m,$$

hence we obtain

$$\begin{aligned} \|f - \xi_j||_m &\leq 2^m(m+1)\|f_{\frac{2}{1+\|f||_m}} - \lambda_i||_m + \delta \\ &= 2^m(m+1)\|Q_m f - \lambda_i||_m + \delta \leq 2^m(m+1)\eta + \delta. \end{aligned}$$

Therefore

$$\frac{1}{2^m(m+1)}\gamma(M) \leq \eta + \delta.$$

By the arbitrariness of δ we obtain (2.4), and the proof is completed. □

3. The mapping $P_{u,m}$

For $u > 0$, we define $P_{u,m} : B(C^m) \to C^m$ by setting

$$(P_{u,m}f)(t) = \begin{cases} 0 & \text{if} \quad t \in \left[0, \frac{1+||f||_m}{2}\right] \\ \frac{u}{(m+1)!}\left(t - \frac{1+||f||_m}{2}\right)^{m+1} & \text{if} \quad t \in \left(\frac{1+||f||_m}{2}, 1\right]. \end{cases}$$

We observe that if f and $g \in B(C^m)$ and $\|f||_m = \|g||_m$ we have $P_{u,m}f = P_{u,m}g$, in particular $P_{u,m}f$ coincides with the null function if $\|f||_m = 1$.
Clearly $P_{u,m}f \in C^m$, and for $s = 0, 1, \cdots, m$ we have

$$(P_{u,m}f)^{(s)}(t) = \begin{cases} 0 & \text{if } t \in \left[0, \frac{1+||f||_m}{2}\right] \\ \frac{u}{(m+1-s)!}\left(t - \frac{1+||f||_m}{2}\right)^{m+1-s} & \text{if } t \in \left(\frac{1+||f||_m}{2}, 1\right]. \end{cases}$$

In particular, we have $(P_{u,m}f)^{(m)} = P_{u,0}f$.

Lemma 3.1. *Let $u > 0$. Let $\{f_n\}$ be a sequence in $B(C^m)$ and $f \in B(C^m)$ such that $||f_n||_m \to ||f||_m$, then*

$$||P_{u,m}f_n - P_{u,m}f||_m \to 0.$$

Proof. We will show, that for each $s = 0, 1, \cdots, m$ we have

$$||(P_{u,m}f_n)^{(s)} - (P_{u,m}f)^{(s)}|| \to 0, \tag{3.1}$$

To this end, fix $s \in \{0, 1, \cdots, m\}$ and $\varepsilon > 0$. Find $\overline{n}$ such that for all $n \geq \overline{n}$ we have $| \, ||f_n||_m - ||f||_m| \leq \varepsilon/u$. We will prove that for every $n \geq \overline{n}$

$$\left|(P_{u,m} f_n)^{(s)}(t) - (P_{u,m} f)^{(s)}(t)\right| \leq \varepsilon, \quad \text{for all } t \in [0,1]. \tag{3.2}$$

If $t \in \left[0, \frac{1+||f||_m}{2}\right] \cap \left[0, \frac{1+||f_n||_m}{2}\right]$, then

$$\left|(P_{u,m} f_n)^{(s)}(t) - (P_{u,m} f)^{(s)}(t)\right| = 0.$$

Assume now $||f||_m \leq ||f_n||_m$ and $t \in \left[\frac{1+||f||_m}{2}, \frac{1+||f_n||_m}{2}\right]$, then

$$\begin{aligned} \left|(P_{u,m} f_n)^{(s)}(t) - (P_{u,m} f)^{(s)}(t)\right| &= \tfrac{u}{(m+1-s)!}\left(t - \tfrac{1+||f||_m}{2}\right)^{m+1-s} \\ &\leq \tfrac{u}{(m+1-s)!}\left|\tfrac{1+||f_n||_m}{2} - \tfrac{1+||f||_m}{2}\right|^{m+1-s} \\ &\leq u| \, ||f_n||_m - ||f||_m| \leq \varepsilon. \end{aligned}$$

If we assume $||f_n||_m \leq ||f||_m$ and $t \in \left[\frac{1+||f_n||_m}{2}, \frac{1+||f||_m}{2}\right]$, then similarly to the previous case we have

$$\left|(P_{u,m} f_n)^{(s)}(t) - (P_{u,m} f)^{(s)}(t)\right| = \frac{u}{(m+1-s)!}\left(t - \frac{1+||f_n||_m}{2}\right)^{m+1-s} \leq \varepsilon.$$

Last we assume $t \in \left[\max\left\{\frac{1+||f_n||_m}{2}, \ \frac{1+||f||_m}{2}\right\}, \ 1\right]$, then

$$\begin{aligned} &\left|(P_{u,m} f_n)^{(s)}(t) - (P_{u,m} f)^{(s)}(t)\right| \\ \leq\ & \tfrac{u}{(m+1-s)!}\left|\left(t - \tfrac{1+||f_n||_m}{2}\right)^{m+1-s} - \left(t - \tfrac{1+||f||_m}{2}\right)^{m+1-s}\right| \\ \leq\ & \tfrac{u}{(m+1-s)!}\left|\tfrac{||f_n||_m - ||f||_m}{2}\right|\left[\left(t - \tfrac{1+||f_n||_m}{2}\right)^{m-s} + \cdots + \left(t - \tfrac{1+||f||_m}{2}\right)^{m-s}\right] \\ \leq\ & \tfrac{u}{(m+1-s)!}\left|\tfrac{||f_n||_m - ||f||_m}{2}\right|(m+1-s) \leq u| \, ||f_n||_m - ||f||_m| \leq \varepsilon. \end{aligned}$$

□

Proposition 3.2. *Let $u > 0$. The mapping $P_{u,m}$ is compact.*

Proof. Let $\{f_n\}$ be a sequence in $B(C^m)$ and $f \in B(C^m)$ such that $||f_n - f||_m \to 0$. Then $||f_n||_m \to ||f||_m$, and Lemma 3.1 implies that $P_{u,m}$ is continuous.

Now we prove that the mapping $P_{u,m}$ is sequentially-compact. To this end let $\{g_n\}$ be a sequence in $P_{u,m}(B(C^m))$. For each $n \in \mathbb{N}$ fix $h_n \in B(C^m)$ such that $g_n = P_{u,m} h_n$. Passing, if necessary, to a subsequence, we may assume without loss of generality that $||h_n||_m \to c \in [0,1]$. Now we choose $h \in B(C^m)$ such that $||h||_m = c$ so that $||h_n||_m \to ||h||_m$. Set $g := P_{u,m} h$. Since $||g_n - g||_m = ||P_{u,m} h_n - P_{u,m} h||_m$, Lemma 3.1 implies $||g_n - g||_m \to 0$, as desired. □

4. The retraction $R_{u,m}$

Let $u > 0$ be arbitrarily fixed. We define $T_{u,m} : B(C^m) \to C^m$, by setting

$$T_{u,m} = Q_m + P_{u,m}.$$

Clearly $T_{u,m}$ is a 1-ball-contractive mapping. Our first step is that of proving that $\inf_{f \in B(C^m)} ||T_{u,m} f||_m > 0$. It will require the following lemma.

Lemma 4.1. *Let $u > 0$ and $f \in B(C^m)$. If $0 \le ||f||_m \le \frac{u}{u+4}$, then we have*

$$\max\left\{||f^{(m)}||, -||f^{(m)}|| + \frac{u}{2}(1 - ||f||_m)\right\} \ge \frac{u}{u+4}.$$

Proof. For every $c \in [0,1]$, we define the auxiliary function $\varphi_c : [0,c] \to \mathbb{R}$ by setting

$$\varphi_c(x) := -x + \frac{u}{2}(1-c), \quad \text{for } x \in [0,c],$$

and we denote by φ the function $\varphi(x) = x$ for $x \in [0,1]$. Then we set

$$c_u := \max\{c : \ c \in [0,1] \text{ and } \varphi_c(x) \ge \varphi(x) \quad \text{for } x \in [0,c]\}.$$

A straightforward calculation shows that $c_u = \frac{u}{u+4}$. Then for every $c \in [0, c_u]$ the function $\psi_c : [0,1] \to \mathbb{R}$ defined by

$$\psi_c(x) = \max\{\varphi(x), \varphi_c(x)\} = \max\{x, \ -x + \frac{u}{2}(1-c)\}$$

satisfies

$$\min_{x \in [0,c]} \psi_c(x) \ge \frac{u}{u+4}. \tag{4.1}$$

Now if $f \in B(C^m)$ and $0 \le ||f||_m \le \frac{u}{u+4}$, the result follows by (4.1) considering $c = \|f\|_m$ and letting $x = ||f^{(m)}||$. □

Proposition 4.2. *Let $u > 0$ and $f \in B(C^m)$. Then*

$$||T_{u,m} f||_m \ge \frac{1}{2^m}\left(1 + \frac{u}{u+4}\right)^m \frac{u}{u+4}.$$

Proof. Fix $u > 0$ and $f \in B(C^m)$. Assume first that $0 \le ||f||_m \le \frac{u}{u+4}$. We have

$$\begin{aligned}
||T_{u,m} f||_m &\ge ||(T_{u,m} f)^{(m)}|| = \max_{t \in [0,1]} \left|(T_{u,m} f)^{(m)}(t)\right| \\
&= \max\Big\{ \max_{t \in \left[0, \frac{1+\|f\|_m}{2}\right]} \left| f^{(m)}\left(\tfrac{2}{1+\|f\|_m} t\right)\right|, \\
&\qquad \max_{t \in \left[\frac{1+\|f\|_m}{2}, 1\right]} \left| f^{(m)}(1) + u\left(t - \tfrac{1+||f||_m}{2} t\right)\right| \Big\} \\
&= \max\left\{||f^{(m)}||, \ \left|f^{(m)}(1) + \tfrac{u}{2}(1 - ||f||_m)\right|\right\} \\
&\ge \max\left\{||f^{(m)}||, f^{(m)}(1) + \tfrac{u}{2}(1 - ||f||_m)\right\} \\
&\ge \max\left\{||f^{(m)}||, -||f^{(m)}|| + \tfrac{u}{2}(1 - ||f||_m)\right\}
\end{aligned}$$

In view of Lemma 4.1 we obtain

$$||T_{u,m}f||_m \geq \frac{u}{u+4}.$$

Now assume $\frac{u}{u+4} \leq ||f||_m \leq 1$. We have

$$\begin{aligned} ||T_{u,m}f||_m &\geq \max\left\{\left(\tfrac{1+||f||_m}{2}\right)^m ||f||, \left(\tfrac{1+||f||_m}{2}\right)^{m-1} ||f^{(1)}||, \cdots, ||f^{(m)}||\right\} \\ &\geq \left(\tfrac{1+||f||_m}{2}\right)^m \max\left\{||f||, ||f^{(1)}||, \cdots, ||f^{(m)}||\right\} \\ &= \left(\tfrac{1+||f||_m}{2}\right)^m ||f||_m \geq \tfrac{1}{2^m}\left(1+\tfrac{u}{u+4}\right)^m \tfrac{u}{u+4}. \end{aligned}$$

The proof is completed. □

We recall, for bounded sets $A, B \subset X$, the following properties of the measure γ, which we will tacitly use in the proof of our main result:
1. $\gamma(A) = 0$ if and only if A is precompact,
2. $\gamma(A) \leq \gamma(B)$ for $A \subseteq B$,
3. $\gamma(\overline{co}A) = \gamma(A)$ where $\overline{co}A$ denotes the closed convex hull of A,
4. $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$,
5. $\gamma(A + B) \leq \gamma(A) + \gamma(B)$,
6. $\gamma(\lambda A) = |\lambda|\gamma(A)$ for all $\lambda \in \mathbb{R}$,
7. $\gamma([0,1]A) = \gamma(A)$.

THEOREM 4.3. *For any $\varepsilon > 0$ there exists a proper k-ball contractive retraction of the closed unit ball $B(C^m)$ onto $S(C^m)$ with $k < 1 + \varepsilon$, so that $W_\gamma(C^m[0,1]) = 1$.*

PROOF. Given $u > 0$, in view of Proposition 4.2, we can define a retraction $R_{u,m} : B(C^m) \to S(C^m)$ by setting

$$R_{u,m}f = \frac{1}{||T_{u,m}f||_m} T_{u,m}f.$$

Let now $M \subseteq B(C^m)$. Since $P_{u,m}$ is a compact mapping, from Proposition 2.3 and Proposition 2.6 it follows that

$$\frac{1}{2^m(m+1)}\gamma(M) \leq \gamma(T_{u,m}M) \leq \gamma(M). \tag{4.2}$$

Moreover by the definition of $R_{u,m}$ and by Proposition 4.2 we get

$$R_{u,m}M \subseteq \left[0, \left(\frac{1}{2^m}\left(1+\frac{u}{u+4}\right)^m \frac{u}{u+4}\right)^{-1}\right] T_{u,m}M.$$

Therefore using the properties of γ, from (4.2) it follows

$$\gamma(R_{u,m}M) \leq \left(\frac{1}{2^m}\left(1+\frac{u}{u+4}\right)^m \frac{u}{u+4}\right)^{-1} \gamma(M),$$

this means that the retraction $R_{u,m}$ is k_u- ball contractive with

$$k_u = \left(\frac{1}{2^m} \left(1 + \frac{u}{u+4}\right)^m \frac{u}{u+4} \right)^{-1}.$$

On the other hand, an easy calculation shows that

$$||T_{u,m}f||_m \leq ||Q_m f||_m + ||P_{u,m}f||_m \leq 1 + \frac{u}{2}$$

for all $f \in B(C^m)$, and so we have

$$T_{u,m}M \subseteq \left[0,\ 1 + \frac{u}{2}\right] R_{u,m}M.$$

Therefore we get

$$\gamma(T_{u,m}M) \leq \left(1 + \frac{u}{2}\right) \gamma(R_{u,m}M),$$

and from (4.2)

$$\frac{1}{2^m(m+1)} \left(1 + \frac{u}{2}\right)^{-1} \gamma(M) \leq \gamma(R_{u,m}M).$$

The latter inequality implies

$$\omega(R_{u,m}) \geq \frac{1}{2^m(m+1)} \left(1 + \frac{u}{2}\right)^{-1},$$

consequently $\omega(R_{u,m}) > 0$ for every $u > 0$, so that $R_{u,m}$ is a proper retraction. Now given $\varepsilon > 0$, since

$$\lim_{u \to \infty} \frac{1}{2^m} \left(1 + \frac{u}{u+4}\right)^m \frac{u}{u+4} = 1,$$

we can find $\bar{u} > 0$ such that $k_{\bar{u}} < 1 + \varepsilon$. Then letting $k = k_{\bar{u}}$ we have that $R_{\bar{u},m}$ is the desired proper k-ball contractive retraction. $\square$

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