POSITIVE SOLUTIONS FOR NONLINEAR ROBIN PROBLEMS WITH CONVECTION

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ABSTRACT. We consider a nonlinear Robin problem driven by the p-Laplacian and with a convection term $f(z, x, y)$. Without imposing any global growth condition on $f(z, \cdot, \cdot)$ and using topological methods (the Leray-Schauder alternative principle), we show the existence of a positive smooth solution.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the existence of positive solutions for the following Robin problem:

$$(1) \qquad \begin{cases} -\Delta_p u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0, & \text{on } \partial\Omega. \end{cases}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \text{div}\left(|Du|^{p-2}Du\right)$$

for all $u \in W^{1,p}(\Omega)$, $1 < p < \infty$.

The right hand side in (1) (the forcing term), has the form of a convection term, that is, f also depends on the gradient of the unknown function. This fact prevents the use of variational methods directly on problem (1). Therefore our approach is eventually topological based on the fixed point theory. In the boundary condition $\frac{\partial u}{\partial n_p}$ denotes the generalized normal derivative of u and is defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to |Du|^{p-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Problems with convection were studied in the past using a variety of methods. We mention the works of De Figueiredo-Girardi-Matzeu [3], Girardi-Matzeu [10] which deal with semilinear equations driven by the Dirichlet Laplacian. The works of Faraci-Motreanu-Puglisi [4], Huy-Quan-Khanh [12], Iturriaga-Lorca-Sanchez [13], Li-Yin-Ke [15], Lorca-Ubilla [16], Ruiz [24] consider equations driven by the Dirichlet p-Laplacian. There are also the works of Averna-Motreanu-Tornatore [2], Faria-Miyagaki-Motreanu [5], Motreanu-Tornatore

[18], Tanaka [25] on (p,q)-equations with gradient dependence (that is, equations driven by the sum of a Dirichlet p-Laplacian with a Dirichlet q-Laplacian). Finally we mention the very recent works of Gasiński-Papageorgiou [9] and Papageorgiou-Radulescu-Repovs [23] on Neumann problems with convection and a differential operator of the form $\mathrm{div}(a(u)Du)$. Moreover, in [23] the problem studied has also unilateral constraints. Here in contrast to all the aforementionated works, we do not impose any global growth condition on $f(z,\cdot,y)$. Instead we assume that $f(z,\cdot,y)$ admits a positive zero (root) and the other conditions of $f(z,x,y)$ concern its behavior for $x \in \mathbb{R}$ near zero, locally in $y \in \mathbb{R}^N$. Using the Leray-Schauder alternative principle, we prove the existence of a positive smooth solution.

2. Preliminaries-Hypotheses

In the study of problem (1) we will use the Sobolev space $W^{1,p}(\Omega)$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

In addition to $W^{1,p}(\Omega)$ we will also use the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \le q \le \infty)$. The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) \,:\, u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$intC_+ = \{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\mid_{\partial\Omega \cap u^{-1}(0)} < 0 \text{ if } \partial\Omega \cap u^{-1}(0) \ne \emptyset\}$$

and $intC_+$ contains the set

$$D_+ = \left\{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

In fact D_+ is the interior of C_+ when the latter is endowed the relative $C(\overline{\Omega})$-topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \le q \le \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to any Sobolev function (not necessarily continuous on $\overline{\Omega}$). We know that

$$\mathrm{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \text{ and } \left(\frac{1}{p} + \frac{1}{p'} = 1\right), \ \ker\gamma_0 = W_0^{1,p}(\Omega).$$

The trace map γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^q(\partial\Omega)$ for all $1 \leq q < \infty$ if $p \geq N$. Also, we have in what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. The restrictions of all Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz, \text{ for all } u, h \in W^{1,p}(\Omega).$$

For this map we have (see Motreanu-Motreanu-Papageorgiou [17], Proposition 2.72, p. 40):

Proposition 2.1. *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*
"$u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$".

In the study of (1) we will also use the first eigenvalue of the negative Robin p-Laplacian. So, we consider the following nonlinear eigenvalue problem

$$(2) \qquad \begin{cases} -\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of the negative Robin p-Laplacian, if problem (2) admits a nontrivial solution $\hat{u}$, known as an eigenfunction corresponding to eigenvalue $\hat{\lambda}$. We assume that $\beta \in L^\infty(\partial\Omega)$ and $\beta(z) \geq 0$ σ-a.e. on $\partial\Omega$. We know that (2) has a smallest eigenvalue $\hat{\lambda}_1$ given by

$$(3) \qquad \hat{\lambda}_1 = \inf\left[\frac{\|Du\|_p^p + \int_{\partial\Omega}\beta(z)|u|^p d\sigma}{\|u\|_p^p} : u \in W^{1,p}(\Omega), \ u \neq 0\right].$$

Evidently $\hat{\lambda}_1 \geq 0$ and in fact $\hat{\lambda}_1 > 0$ if $\beta \neq 0$, while $\hat{\lambda}_1 = 0$ if $\beta = 0$ (Neumann eigenvalue problem). This eigenvalue has the following properties:

- $\hat{\lambda}_1$ is simple (that is, if $\hat{u}$, $\hat{v}$ are two eigenfunctions corresponding to $\hat{\lambda}_1$, then $\hat{u} = \xi\hat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).
- $\hat{\lambda}_1$ is isolated (that is, there exists $\varepsilon > 0$ such that $(\hat{\lambda}_1, \hat{\lambda}_1 + \varepsilon) \cap \hat{\sigma}(p) = \emptyset$, where $\hat{\sigma}(p)$ denotes the spectrum of (2)).
- The iegenfunctions corresponding to $\hat{\lambda}_1$, have fixed sign.

Let $\hat{u}_1$ be the L^p-normalized (that is, $\|\hat{u}_1\|_p = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1$. The infimum in (3) is realized on the corresponding one-dimensional eigenspace $\mathbb{R}\hat{u}_1$. If we strengthen the regularity of the boundary coefficient $\beta(\cdot)$, then we can improve our conclusions. More precisely, suppose that $\beta \in C^{0,\mu}(\partial\Omega)$ with $\mu \in (0, 1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. Then the nonlinear regularity theory of

Lieberman [14] (Theorem 2) implies that every eigenfunction $\hat{u} \in C^1(\overline{\Omega})$. In particular, using the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [8], Theorem 6.2.8, p.738), we have $\hat{u}_1 \in D_+$. On the other hand every eigenfunction $\hat{u}$ corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ is nodal (that is, sign changing). Since the spectrum $\hat{\sigma}(p)$ is closed and $\hat{\lambda}_1$ is isolated, the second eigenvalue $\hat{\lambda}_2$ is well-defined by $\hat{\lambda}_2 = \min \left[\hat{\lambda} \in \hat{\sigma}(p) : \hat{\lambda} > \hat{\lambda}_1 \right]$. Additional eigenvalues can be produced using the Ljusternik-Schnirelmann minimax scheme, which generates a whole nondecreasing sequence $\{\hat{\lambda}_k\}_{k \in \mathbb{N}}$ of eigenvalues (known as " variational eigenvalues") such that $\hat{\lambda}_k \to +\infty$ as $k \to +\infty$. We do not know if these eigenvalues exhaust $\hat{\sigma}(p)$. For further details we refer to Papageorgiou-Radulescu [19].

We will also need a strong comparison theorem, which can be found in Papageorgiou-Radulescu-Repovs [23] and is a variant of an earlier result of Fragnelli-Mugnai-Papageorgiou [6].

Proposition 2.2. *If $\hat{\xi} \in L^\infty(\Omega)$, $\hat{\xi}(z) \geq 0$ for a.a. $z \in \Omega$, g_1, $g_2 \in L^\infty(\Omega)$ with $0 < c_0 \leq g_2(z) - g_1(z)$ for a.a $z \in \Omega$ and $u, v \in C^1(\overline{\Omega})$ satisfy $u \leq v$ and*

$$-\Delta_p u(z) + \hat{\xi}(z)|u(z)|^{p-2}u(z) = g_1(z) \text{ for a.a. } z \in \Omega,$$

$$-\Delta_p v(z) + \hat{\xi}(z)|v(z)|^{p-2}v(z) = g_2(z) \text{ for a.a. } z \in \Omega.$$

Then $v - u \in int C_+$.

As we already mentioned in the Introduction our approach is topological and uses the Leray-Schauder Alternative Principle, which we recall here (see Gasiński-Papageorgiou [7], Theorem 7.2.16, p. 827).

Proposition 2.3. *If X is a Banach space, $C \subset X$ is nonempty, convex with $0 \in C$, $k : C \to C$ is compact and $E(k) = \{u \in C : u = tk(u) \text{ for some } t \in (0,1)\}$, then either $E(k)$ is unbounded or k has a fixed point.*

For $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then for all $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

For a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ we say that it is Caratheodory, if

- for all $x \in \mathbb{R}$, $y \in \mathbb{R}^N$ $z \to f(z, x, y)$ is measurable;
- for a.a. $z \in \Omega$, $(x, y) \to f(z, x, y)$ is continuous.

We know that such a function is jointly measurable (see Hu-Papageorgiou [11], Proposition 1.6, p.142) and so for every $u \in \Omega \to \mathbb{R}$ and $v \in \mathbb{R}^N$ measurable, we have that $z \to f(z, u(z), v(z))$ is measurable.

Now we are ready to introduce the hypotheses on the convection term $f(z, x, y)$.

H(f) : $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0, y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

(i) there exist $\eta > 0$ and a function $a_\eta \in L^\infty(\Omega)$ such that

$$f(z, \eta, y) = 0 \text{ for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N,$$

$$f(z, x, y) \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \eta, \text{ all } y \in \mathbb{R}^N,$$

$$f(z, x, y) \leq a_\eta(z)(1 + |y|^{p-1}) \text{ for a.a. } z \in \Omega, , \text{ all } 0 \leq x \leq \eta, \text{ all } y \in \mathbb{R}^N;$$

(ii) for every $M > 0$, there exists $\eta_M \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1 \leq \eta_M(z) \text{ for a.a. } z \in \Omega, \ \eta_M \neq \hat{\lambda}_1,$$

$$\eta_M(z) \leq \liminf_{x \to 0^+} \frac{f(z, x, y)}{x^{p-1}} \text{ uniformly for a.a. } z \in \Omega, \text{ all } |y| \leq M;$$

(iii) there exists $\xi_\eta > 0$ such that for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ the function

$$x \to f(z, x, y) + \xi_\eta x^{p-1}$$

is nondecreasing for $0 \leq x \leq \eta$, moreover for a.a. $z \in \Omega$, all $0 \leq x \leq \eta$, all $y \in \mathbb{R}^N$ and for every $\lambda \in (0, 1)$, we have

$$f\left(z, \frac{1}{\lambda}x, y\right) \leq \frac{1}{\lambda^{p-1}} f(z, x, y).$$

Remark 2.1. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

(4) $$f(z, x, y) = 0 \text{ for a.a } z \in \Omega, \text{ all } x \leq 0, \text{ all } y \in \mathbb{R}^N.$$

The last part of hypothesis H(f)(iii) is satisfied, if for example for a.a. $z \in \Omega$ and for all $y \in \mathbb{R}^N$, the map $x \to \frac{f(z, x, y)}{x^{p-1}}$ is nonincreasing on $(0, +\infty)$.

Example 2.1. *The following function satisfies the above hypotheses* H*(f). For the sake of simplicity, we drop the z-dependence:*

$$f(x, y) = \begin{cases} \eta(x^{p-1} - x^{r-1})(1 + |y|^{p-1}) & \text{if } x \in [0, 1] \\ g(x, y) & \text{if } 1 < x, \end{cases} \quad \text{with } \eta > \hat{\lambda}_1, \ p < r$$

and $g(\cdot, \cdot)$ is any continuous function on $\mathbb{R} \times \mathbb{R}^N$ such that $g(1, y) = 0$ for all $y \in \mathbb{R}^N$.

The hypotheses on the boundary coefficient $\beta(z)$ are the following:
H(β): $\beta \in C^{1,\mu}(\partial\Omega)$ with $\mu \in (0, 1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

3. Auxiliary Results

Let $\xi_\eta > 0$ be as postulated by hypothesis H(f) (iii). We introduce the following truncation-perturbation of the convection term $f(z, \cdot, y)$:

$$(5) \qquad \hat{f}(z, x, y) = \begin{cases} f(z, x, y) + \xi_\eta (x^+)^{p-1} & \text{if } x \leq \eta \\[2mm] \xi_\eta \eta^{p-1} & \text{if } \eta < x. \end{cases}$$

Evidently this is a Carathéodory function (see hypotesis H(f)(i)).

To apply the Leray-Schauder alternative principle (see Proposition 2.3), we need to produce the compact map on which the fixed point principle will be used. To this end, we freeze the third variable of f. So, given $v \in C^1(\overline{\Omega})$, we consider the following nonlinear Robin problem

$$(6) \qquad \begin{cases} -\Delta_p u(z) + \xi_\eta |u(z)|^{p-2} u(z) = \hat{f}(z, u(z), Dv(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

In what follows $[0, \eta]$ denotes the order interval

$$[0, \eta] = \{u \in W^{1,p}(\Omega) \ : \ 0 \leq u(z) \leq \eta \text{ for a.a. } z \in \Omega\}.$$

Proposition 3.1. *If hypotheses* H*(f),* H*(β) hold, then problem* (6) *admits a positive solution* $u_v \in [0, \eta] \cap D_+$

Proof. Let $\hat{F}_v(z, x) = \int_0^x \hat{f}(z, s, v(z))ds$ and consider the C^1-functional

$$\hat{\varphi}_v : W^{1,p}(\Omega) \to \mathbb{R}$$

defined by

$$\hat{\varphi}_v(u) = \frac{1}{p}\|Du\|_p^p + \frac{\xi_\eta}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \hat{F}_v(z, u)dz.$$

for all $u \in W^{1,p}(\Omega)$.

From (5) and hypothesis H(β), we see that $\hat{\varphi}_v$ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\hat{\varphi}_v$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_v \in W^{1,p}(\Omega)$ such that

$$(7) \qquad \hat{\varphi}_v(u_v) = \inf \left[\hat{\varphi}_v(u) \ : \ u \in W^{1,p}(\Omega)\right].$$

Let $M \geq \|v\|_{C^1(\overline{\Omega})}$. On account of hypothesis H(f)(ii), given $\varepsilon > 0$ we can find $0 < \delta \leq \eta$ such that

$$f(z, x, y) \geq (\eta_M(z) - \varepsilon)x^{p-1} \text{ for a.a. } z \in \Omega \text{ all } 0 \leq x \leq \delta, \text{ all } |y| \leq M,$$

which implies, taking into account (5)

$$\hat{f}(z, x, Dv(z)) \geq (\eta_M(z) - \varepsilon)x^{p-1} + x^{p-1} \text{ for a.a. } z \in \Omega \text{ all } 0 \leq x \leq \delta,$$

so, we have

(8) $$\hat{F}_v(z,x) \geq \frac{1}{p}(\eta_M(z) - \varepsilon)x^p + \frac{1}{p}x^p \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta.$$

Since $\hat{u}_1 \in D_+$, we can find $t \in (0,1)$ small such that

(9) $$t\hat{u}_1(z) \in (0,\delta] \text{ for all } z \in \overline{\Omega}.$$

By using (5), (8), (9) and recalling that $\|\hat{u}_1\|_p = 1$ and $\|D\hat{u}_1\|_p = \hat{\lambda}_1$ we have

(10)
$$\hat{\varphi}_v(t\hat{u}_1) \leq \frac{t^p}{p}\hat{\lambda}_1 - \frac{t^p}{p}\int_\Omega \left[\eta_M(z) - \varepsilon\right]\hat{u}_1^p dz$$

$$\leq \frac{t^p}{p}\left[\int_\Omega \left(\hat{\lambda}_1 - \eta_M(z)\right)\hat{u}_1^p dz + \varepsilon\right].$$

Hypothesis H(f)(ii) and the fact that $\hat{u}_1 \in D_+$ imply

$$r_0 = \int_\Omega \left(\eta_M(z) - \hat{\lambda}_1\right)\hat{u}_1^p dz > 0.$$

From (10) we have

$$\hat{\varphi}_v(t\hat{u}_1) \leq \frac{t^p}{p}[-r_0 + \varepsilon].$$

Choosing $\varepsilon \in (0, r_0)$, we obtain

$$\hat{\varphi}_v(t\hat{u}_1) < 0,$$

using (7), we have

$$\hat{\varphi}_v(u_v) < 0 = \hat{\varphi}_v(0),$$

hence

$$u_v \neq 0.$$

From (7) we have

$$\hat{\varphi}_v'(u_v) = 0$$

then

(11)
$$\langle A(u_v), h \rangle + \xi_\eta \int_\Omega |u_v|^{p-2}u_v h dz + \int_{\partial\Omega} \beta(z)|u_v|^{p-2}u_v h d\sigma$$

$$= \int_\Omega \hat{f}(z, u_v, Dv) h dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

In (11) we choose $h = -u_v^- \in W^{1,p}(\Omega)$. Then, taking into account (4), (5) and hypothesis H(β),

$$\|Du_v^-\|_p^p + \xi_\eta\|u_v^-\|_p^p \leq 0$$

which implies

$$u_v \geq 0, \ u_v \neq 0.$$

Next in (9) we choose $h = (u_v - \eta)^+ \in W^{1,p}(\Omega)$. Then

$$\langle A(u_v), (u_v - \eta)^+ \rangle + \xi_\eta \int_\Omega u_v^{p-1}(u_v - \eta)^+ dz + \int_{\partial\Omega} \beta(z)u_v^{p-1}(u_v - \eta)^+ d\sigma = \int_\Omega \xi_\eta \eta^{p-1}(u_v - \eta)^+ dz,$$

using hypothesis H(β) and noting that $A(\eta) = 0$, we have

$$\langle A(u_v) - A(\eta), (u_v - \eta)^+ \rangle + \xi_\eta \int_\Omega (u_v^{p-1} - \eta^{p-1})(u_v - \eta)^+ dz \leq 0$$

which implies

$$u_v \leq \eta.$$

So, we have proved

(12) $u_v \in [0, \eta] = \{u \in W^{1,p}(\Omega) \ : \ 0 \leq u(z) \leq \eta \text{ for a. a. } z \in \Omega\}.$

From (5), (11), (12) we have

$$\langle A(u_v), h \rangle + \int_{\partial\Omega} \beta(z) u_v^{p-1} h d\sigma = \int_\Omega f(z, u_v, Dv) h dz \text{ for a.a. } h \in W^{1,p}(\Omega),$$

which implies (see Papageorgiou-Radulescu [19])

(13) $$\begin{cases} -\Delta_p u_v(z) = f(z, u_v(z), Dv(z)) & \text{in } \Omega, \\ \frac{\partial u_v}{\partial n_p} + \beta(z) u_v^{p-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

From (13) and Papageorgiou-Radulescu [20], we have

$$u_v \in L^\infty(\Omega).$$

Invoking Theorem 2 of Lieberman [14], we infer that

$$u_v \in C_+ \setminus \{0\}.$$

Hypothesis H(f)(iii) and (12) imply that

$$f(z, u_v(z), Dv(z)) + \xi_\eta u_v(z)^{p-1} \geq 0 \text{ for a.a. } z \in \Omega.$$

Using this in (13), we obtain

$$\Delta_p u_v(z) \leq \xi_\eta u_v(z)^{p-1} \text{ for a.a. } z \in \Omega.$$

The nonlinear strong maximum principle (see, for example, Gasiński-Papageorgiou [8], Theorem 6.2.8, p.738) implies that

$$u_v \in D_+.$$

$\square$

We will show that problem (6) has a smallest positive solution in the order interval $[0, \eta]$. So, we define the set

$$S_v = \{u \in W^{1,p}(\Omega) \ : \ u \neq 0, \ u \in [0, \eta], \ u \text{ is a solution of (6)}\}.$$

From Proposition 3.1 we know that

$$\emptyset \neq S_v \subset [0, \eta] \cup D_+.$$

Given $\epsilon > 0$ and $r \in (p, p^*)$ (recall that $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$, the critical Sobolev exponent), hypotheses H(f)(i), (ii) imply that we can find $c_1 = c_1(\varepsilon, r, M) > 0$ (recall $M \geq \|v\|_{C^1(\overline{\Omega})}$) such that

$$(14) \qquad f(z, x, Dv(z)) \geq (\eta_M(z) - \varepsilon) x^{p-1} - c_1 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \eta.$$

This unilateral growth estimate on $f(z, \cdot, Dv(z))$, leads to the following auxiliary nonlinear Robin problem

$$(15) \qquad \begin{cases} -\Delta_p u(z) = [\eta_M(z) - \varepsilon]u(z)^{p-1} - c_1 u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0, & \text{on } \partial\Omega, \\ u > 0. \end{cases}$$

Proposition 3.2. *If hypothesis* H(β) *holds, then for all* $\varepsilon > 0$ *small problem* (15) *has a unique positive solution* $u^* \in D_+$.

Proof. First we show the existence of a positive solution.

To this end, let $\Psi : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\Psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \frac{1}{p}\int_\Omega [\eta_M(z) - \varepsilon](u^+)^p dz + \frac{c_1}{r}\|u^+\|_r^r$$

for all $u \in W^{1,p}(\Omega)$. We have

$$\Psi(u) \geq \frac{1}{p}\|Du^+\|_p^p + \frac{c_1}{r}\|u^+\|_r^r - \frac{1}{p}\int_\Omega [\eta_M(z) - \varepsilon](u^+)^p dz + \frac{1}{p}\|Du^-\|_p^p + \frac{1}{p}\|u^-\|_p^p.$$

Since $\eta_M \in L^\infty(\Omega)$ and $r > p$, choosing $c_1 > 0$ even bigger if necessary (see (14)), we have

$$\Psi(u) \geq c_2\|u\|^p - c_3 \text{ for some } c_2, c_3 > 0,$$

which implies that Ψ is coercive. Also as before, we have that $\Psi(\cdot)$ is sequentially weakly lower semicontinuos. So, we can find $u^* \in W^{1,p}(\Omega)$ such that

$$(16) \qquad \Psi(u^*) = \inf\left[\Psi(u) \ : \ u \in W^{1,p}(\Omega)\right].$$

As in the proof of Proposition 3.1, using the hypothesis on $\eta_M(\cdot)$ (see hypothesis H(f)(ii)), we show that for $t > 0$ and $\varepsilon > 0$ small

$$\Psi(t\hat{u}_1) < 0,$$

which implies (see (16))

$$\Psi(u^*) < 0 = \Psi(0),$$

hence $u^* \neq 0$.

From (16) we have $\Psi'(u^*) = 0$ we obtain

(17)
$$\langle A(u^*), h \rangle + \int_\Omega ((u^*)^-)^{p-1} h dz + \int_{\partial\Omega} \beta(z)((u^*)^+)^{p-1} h d\sigma$$
$$= \int_\Omega [\eta_M(z) - \varepsilon]((u^*)^+)^{p-1} h dz - c_1 \int_\Omega ((u^*)^+)^{r-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

In (17) we choose $h = -(u^*)^- \in W^{1,p}(\Omega)$. Then

$$\|D(u^*)^-\|_p^p + \|(u^*)^-\|_p^p = 0,$$

hence

$$u^* \geq 0, \ u^* \neq 0.$$

So, (17) becomes

$$\langle A(u^*), h \rangle + \int_{\partial\Omega} \beta(z)(u^*)^{p-1} h d\sigma$$

$$= \int_\Omega [\eta_M(z) - \varepsilon](u^*)^{p-1} h dz - c_1 \int_\Omega (u^*)^{r-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

hence

(18)
$$\begin{cases} -\Delta_p u^*(z) = [\eta_M(z) - \varepsilon] u^*(z)^{p-1} - c_1 u^*(z)^{r-1} & \text{for a. a. } z \in \Omega, \\ \frac{\partial u^*}{\partial n_p} + \beta(z)(u^*)^{p-1} = 0, & \text{on } \partial\Omega. \end{cases}$$

As before (18) and the nonlinear regularity theory imply that $u^* \in C_+ \setminus \{0\}$.
From (18), we have

$$\Delta_p u^*(z) \leq c_1 u^*(z)^{r-1} + (\|\eta\|_\infty + \varepsilon) u^*(z)^{p-1} \leq c_4 (\|u^*\|_\infty^{r-p} + 1) u^*(z)^{p-1}$$

for a.a. $z \in \Omega$, for some $c_4 > 0$, then $u^* \in D_+$ (from the nonlinear strong maximum principle).

Next we show that this positive solution is in fact unique. So, suppose that $\tilde{u}^* \in W^{1,p}(\Omega)$ is another positive solution of (15). As above we show that $\tilde{u}^* \in D_+$.

Let $\varrho = \max\{\|u^*\|_\infty, \|\tilde{u}^*\|_\infty\}$. We can find $\hat{\xi}_\varrho > 0$ such that for a.a $z \in \Omega$

$$x \to (\eta_M(z) - \varepsilon) x^{p-1} - c_1 x^{r-1} + \hat{\xi}_\varrho x^{p-1}$$

is nondecreasing on $[0, \varrho]$ (recall $\eta_M \in L^\infty(\Omega)$ and $r > \varrho$).
Let $t > 0$ be the biggest positive real such that

(19)
$$t\tilde{u}^*(z) \leq u^*(z) \text{ for all } z \in \overline{\Omega}.$$

It exists since $\tilde{u}^*$, $u^* \in D_+$. Assume that $t \in (0,1)$. Using (19), we have

$$-\Delta_p(t\tilde{u}^*)(z) + \hat{\xi}_\varrho(t\tilde{u}^*)(z)^{p-1} = t^{p-1}\left[-\Delta_p\tilde{u}^*(z) + \hat{\xi}_\varrho\tilde{u}^*(z)^{p-1}\right]$$

$$= t^{p-1}\left[\left(\eta_M(z) - \varepsilon\right)\tilde{u}^*(z)^{p-1} - c_1\tilde{u}^*(z)^{r-1} + \hat{\xi}_\varrho\tilde{u}^*(z)^{p-1}\right]$$

$$< \left(\eta_M(z) - \varepsilon\right)(t\tilde{u}^*)(z)^{p-1} - c_1(t\tilde{u}^*)(z)^{r-1} + \hat{\xi}_\varrho(t\tilde{u}^*)(z)^{p-1}$$

$$\leq \left(\eta_M(z) - \varepsilon\right)u^*(z)^{p-1} - c_1u^*(z)^{r-1} + \hat{\xi}_\varrho u^*(z)^{p-1} = -\Delta_p u^* + \hat{\xi}_\varrho u^*(z)^{p-1} \text{ for a.a. } z \in \Omega,$$

which implies (see Proposition 2.2)

$$u^* - t\tilde{u}^* \in intC_+.$$

This contradicts the maximality of $t > 0$. Therefore $t \geq 1$ and so

$$\tilde{u}^* \leq u^*.$$

Interchanging the roles of $\tilde{u}^*$ and u^* in the above argument we also have $u^* \leq \tilde{u}^*$, hence $u^* = \tilde{u}^*$.

So, the positive solution $u^* \in D_+$ of (15) is unique. $\qquad\square$

Remark 3.1. The minimal solution $u^* \in D_+$ depends only on $M > 0$.

Proposition 3.3. *If hypotheses* H(β), H(f) *hold and* $u \in S_v$, *then* $u^* \leq u$ *(for* $M \geq \|v\|_{C^1(\overline{\Omega})}$*).*

Proof. We consider the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$(20) \qquad g(z,x) = \begin{cases} (\eta_M(z) - \varepsilon)(x^+)^{p-1} - c_1(x^+)^{r-1} + \xi_\eta(x^+)^{p-1} & \text{if } x \leq u(z) \\ (\eta_M(z) - \varepsilon)u(z)^{p-1} - c_1u(z)^{r-1} + \xi_\eta u(z)^{p-1} & \text{if } u(z) < x \end{cases}$$

We set $G(z,x) = \int_0^x g(z,s)ds$ and introduce the C^1-functional $e : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$e(u) = \frac{1}{p}\|Du\|_p^p + \frac{\xi_\eta}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^pd\sigma - \int_\Omega G(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (20) it is clear that $e(\cdot)$ is coercive. Also, it is sequentially weakly lower semi-continuous. So, by the Weierstrass-Tonelli theorem, we can find $\tilde{u}^* \in W^{1,p}(\Omega)$ such that

$$(21) \qquad\qquad e(\tilde{u}^*) = \inf\left[e(u) \,:\, u \in W^{1,p}(\Omega)\right].$$

As before the condition on $\eta_M(\cdot)$ (see hypothesis H(f)(ii)) and the fact that $r > p$, imply that $e(\tilde{u}^*) < 0 = e(0)$, then $\tilde{u}^* \neq 0$.

From (21) we have $e'(\tilde{u}^*) = 0$, so we obtain

$$\langle A(\tilde{u}^*), h \rangle + \xi_\eta \int_\Omega |\tilde{u}^*|^{p-2}\tilde{u}^* h dz + \int_{\partial\Omega} \beta(z)|\tilde{u}^*|^{p-2}\tilde{u}^* h d\sigma$$

(22)

$$= \int_\Omega g(z, \tilde{u}^*) h dz \quad \text{for all} \ \ h \in W^{1,p}(\Omega).$$

In (22) first we choose $h = -(\tilde{u}^*)^- \in W^{1,p}(\Omega)$. Using (20) and hyphothesis H(β) we have

$$\|D(\tilde{u}^*)^-\|_p^p + \xi_\eta\|(\tilde{u}^*)^-\|_p^p \leq 0,$$

hence,

$$\tilde{u}^* \geq 0, \quad \tilde{u}^* \neq 0.$$

Next (22) we choose $h = (\tilde{u}^* - u)^+ \in W^{1,p}(\Omega)$. From (14), (20) and since $u \in S_v$ we obtain

$$\langle A(\tilde{u}^*), (\tilde{u}^* - u)^+ \rangle + \xi_\eta \int_\Omega (\tilde{u}^*)^{p-1}(\tilde{u}^* - u)^+ dz + \int_{\partial\Omega} \beta(z)(\tilde{u}^*)^{p-1}(\tilde{u}^* - u)^+ d\sigma$$

$$= \int_\Omega \left[(\eta_M(z) - \varepsilon)u^{p-1} - c_1 u^{r-1}\xi_\eta u^{p-1} \right](\tilde{u}^* - u)^+ dz$$

$$\leq \int_\Omega [f(z, u, Dv) + \xi_\eta u^{p-1}](\tilde{u}^* - u)^+ dz$$

$$= \langle A(u), (\tilde{u}^* - u)^+ \rangle + \xi_\eta \int_\Omega u^{p-1}(\tilde{u}^* - u)^+ dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\tilde{u}^* - u)^+ d\sigma$$

then

$$\langle A(\tilde{u}^*) - A(u), (\tilde{u}^* - u)^+ \rangle + \xi_\eta \int_\Omega ((\tilde{u}^*)^{p-1} - u^{p-1})(\tilde{u}^* - u)^+ dz$$

$$+ \int_{\partial\Omega} \beta(z)((\tilde{u}^*)^{p-1} - u^{p-1})(\tilde{u}^* - u)^+ d\sigma \leq 0,$$

so, we have $\tilde{u}^* \leq u$.

So, we have proved that

(23) $\tilde{u}^* \in [0, u] = \{v \in W^{1,p}(\Omega) \ : \ 0 \leq v(z) \leq u(z) \text{ for a.a. } z \in \Omega\}, \quad \tilde{u}^* \neq 0.$

Using (23), (20), we see that (22) becomes

$$\langle A(\tilde{u}^*), h \rangle + \int_{\partial\Omega} \beta(z)(\tilde{u}^*)^{r-1} h d\sigma$$

$$= \int_\Omega [(\eta_M(z) - \varepsilon)(\tilde{u}^*)^{p-1} - c_1(\tilde{u}^*)^{p-1}] h dz \quad \text{for all} \ \ h \in W^{1,p}(\Omega).$$

hence $\tilde{u}^* = u^* \in D_+$ (see Proposition 3.2),

Therefore we conclude that $u^* \leq u$ for all $u \in S_v$. $\square$

Now we can generate the smallest positive solution for problem (6).

Proposition 3.4. *If hypotheses* H(β), H*(f) hold, then problem* (6) *admits a smallest positive solution* $\hat{u}_v \in D_+$.

Proof. From Papageorgiou-Radulescu-Repovs [21] (Proposition 7), we know that S_v is downward directed (that is, if u_1, $u_2 \in S_v$, then there exists $\tilde{u} \in S_v$ such that $\tilde{u} \le u_1$, $\tilde{u} \le u_2$). So, invoking Lemma 3.10, p 178, of Hu-Papageorgiou [11], we can find a decreasing sequence $\{u_n\}_{n \ge 1} \subseteq S_v$ such that

$$(24) \qquad \inf S_v = \inf_{n \ge 1} u_n.$$

For every $n \in \mathbb{N}$, we have

$$(25) \qquad \langle A(u_n), h \rangle + \int_{\partial \Omega} \beta(z) u_n^{p-1} h d\sigma = \int_{\Omega} f(z, u_n, Dv) h dz$$

for all $h \in W^{1,p}(\Omega)$, which implies

$$(26) \qquad u^* \le u_n \le \eta$$

(with $M \ge \|v\|_{C^1(\overline{\Omega})}$, see Proposition 3.3).

From (26) and hypotheses H(f)(i), H(β), it follows that $\{u_n\}_{n \ge 1} \subset W^{1,p}(\Omega)$ is bounded.

So, by passing to a subsequence if necessary, we can say that

$$(27) \qquad u_n \rightharpoonup \hat{u}_v \quad \text{in } W^{1,p}(\Omega) \text{ and } u_n \to \hat{u}_v \text{ in } L^p(\Omega) \text{ and } L^p(\partial \Omega).$$

In (25) we choose $h = u_n - \hat{u}_v \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (27), then

$$\lim_{n \to \infty} \langle A(u_n), u_n - \hat{u}_v \rangle = 0,$$

which implies (see Proposition 2.1

$$(28) \qquad u_n \to \hat{u}_v \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

If in (25) we pass to the limit as, $n \to \infty$ and use (28), then

$$\langle A(\hat{u}_v), h \rangle + \int_{\partial \Omega} \beta(z) \hat{u}_v^{p-1} h d\sigma = \int_{\Omega} f(z, \hat{u}_v, Dv) h dz$$

for all $h \in W^{1,p}(\Omega)$ and $u^* \le \hat{u}_v \le \eta$ (see (26)).

It follows that $\hat{u}_v \in S_v$ and $\hat{u}_v = \inf S_v$ (see (24)).

$\square$

Let $C = \{u \in C^1(\overline{\Omega}) : 0 \le u(z) \le \eta \text{ for all } z \in \overline{\Omega}\}$ and consider the map $\theta : C \to C$ defined by

$$\theta(v) = \hat{u}_v \text{ for all } v \in C.$$

A fixed point of this map will be a positive solution of problem (1). To generate a fixed point for $\theta(\cdot)$ we will use Proposition 2.3 (the Leray-Schauder Alternative Principle). According to that result, we need to show that θ is compact (that is, continuous and maps bounded set to relatively compact sets). The following lemma will be helpful in this direction.

Lemma 3.1. *If hypotheses* H(f), H(β) *hold*, $\{v_n\}_{n \geq 1} \subset C$, $v_n \to v$ *in* $C^1(\overline{\Omega})$ *and* $u \in S_v$, *then we can find* $u_n \in S_{v_n}$ $(n \in \mathbb{N})$, *such that* $u_n \to u$ *in* $C^1(\overline{\Omega})$.

Proof. We consider the following Robin problem

(29)
$$
\begin{cases}
-\Delta_p w(z) + \xi_\eta |w(z)|^{p-2} w(z) = \hat{f}(z, u(z), Dv_n(z)) & \text{in } \Omega, \\
\frac{\partial w}{\partial n_p} + \beta(z)|w|^{p-2} w = 0 & \text{on } \partial\Omega, \ n \in \mathbb{N}.
\end{cases}
$$

Since $u \in S_v \subset [0, \eta] \cap D_+$, from (5) and hypothesis H(f)(i), we have

$$\hat{f}(\cdot, u(\cdot), Dv_n(\cdot)) \in L^\infty(\Omega) \setminus \{0\} \text{ and } \hat{f}(z, u(z), Dv_n(z)) \geq 0 \text{ for a.a. } z \in \Omega.$$

Hence, for every $n \in \mathbb{N}$, problem (29) has a unique nontrivial solution $u_n^0 \in W^{1,p}(\Omega)$ and the nonlinear regularity theory and the nonlinear maximum principle imply that $u_n^0 \in D_+$. Also, taking into account hypotheses H(f)(i),(iii), H(β) and since that $A(\eta) = 0$, we have

$$\langle A(u_n^0, (u_n^0 - \eta)^+\rangle + \xi_\eta \int_\Omega (u_n^0)^{p-1}(u_n^0 - \eta)^+ dz + \int_{\partial\Omega} \beta(z)(u_n^0)^{p-1}(u_n^0 - \eta)^+ d\sigma$$

$$= \int_\Omega \left[f(z, u, Dv_n) + \xi_\eta u^{p-1} \right](u_n^0 - \eta)^+ dz$$

$$\leq \int_\Omega \xi_\eta \eta^{p-1}(u_n^0 - \eta)^+ dz$$

then

$$\langle A(u_n^0) - A(\eta), (u_n^0 - \eta)^+\rangle + \xi_\eta \int_\Omega ((u_n^0)^{p-1} - \eta^{p-1})(u_n^0 - \eta)^+ dz \leq 0,$$

then, we have $u_n^0 \leq \eta$ for all $n \in \mathbb{N}$.

So, we have

(30)
$$u_n^0 \in [0, \eta] \cap D_+ \text{ for all } n \in \mathbb{N}.$$

Therefore from (5) and (30), we see that problem (29) becomes

(31)
$$
\begin{cases}
-\Delta_p u_n^o(z) = f(z, u(z), Dv_n(z)) & \text{in } \Omega, \\
\frac{\partial u_n^0}{\partial n_p} + \beta(z)(u_n^0)^{p-1} = 0, & \text{on } \partial\Omega, \ n \in \mathbb{N}.
\end{cases}
$$

From (30), (31) and hypothesis H(f)(i) we see that $\{u_n^0\}_{n \in \mathbb{N}} \subset W^{1,p}(\Omega)$ is bounded.

Then Theorem 2 of Lieberman [14] implies that we can find $\alpha \in (0, 1)$ and $c_5 > 0$ such that $u_n^0 \in C^{1,\alpha}(\overline{\Omega})$ and $\|u_n^0\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_5$ for all $\in \mathbb{N}$.

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ implies that we can find a subsequence $\{u_{n_k}^0\}_{k \in \mathbb{N}}$ of $\{u_n^0\}_{n \in \mathbb{N}}$ such that $u_{n_k}^0 \to \tilde{u}^0$ in $C^1(\overline{\Omega})$ as $k \to \infty$.

From (31) we have

(32)
$$
\begin{cases}
-\Delta_p \tilde{u}^o(z) = f(z, u(z), Dv(z)) & \text{for a.a } z \in \Omega, \\
\frac{\partial \tilde{u}^0}{\partial n_p} + \beta(z)(\tilde{u}^0)^{p-1} = 0, & \text{on } \partial\Omega.
\end{cases}
$$

Since (32) has a unique solution and $u \in S_v$ solve it, we have $\tilde{u}^o = u \in S_v$.

Therefore for the original sequence $\{u_n^0\}_{n\in\mathbb{N}}$ we have $u_n^0 \to u$ in $C^1(\overline{\Omega})$ as $n \to \infty$.
Next we consider the following Robin problem:

$$(33) \qquad \begin{cases} -\Delta_p w(z) + \xi_\eta |w(z)|^{p-2} w(z) = \hat{f}(z, u_n^0(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial w}{\partial n_p} + \beta(z)|w|^{p-2}w = 0 & \text{on } \partial\Omega, \ n \in \mathbb{N}. \end{cases}$$

This problem too has a unique solution $u_n^1 \in [0, \eta]$, $n \in \mathbb{N}$.

As before from (33) and the nonlinear regularity we have $u_n^1 \to u$ in $C^1(\overline{\Omega})$ as $n \to \infty$.

Continuing this way we generate a sequence $\{u_n^k\}_{k,n\in\mathbb{N}}$ such that

$$(34) \qquad \begin{cases} -\Delta_p u_n^k(z) + \xi_\eta u_n^k(z)^{p-1} = \hat{f}(z, u_n^{k-1}(z), Dv_n(z)) & \text{in } \Omega, \\ \frac{\partial u_n^k}{\partial n_p} + \beta(z)(u_n^k)^{p-1} = 0, & \text{on } \partial\Omega, \ k, n \in \mathbb{N}. \end{cases}$$

$$(35) \qquad\qquad u_n^k \in [0, \eta] \cap D_+ \text{ for all } k, n \in \mathbb{N},$$

$$(36) \qquad\qquad u_n^k \to u \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty \text{ for all } k \in \mathbb{N}.$$

We fix $n \in \mathbb{N}$. As before, via Theorem 2 of Lieberman [14], we have that $\{u_n^k\}_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ is relatively compact.

So, we can find a subsequence $\{u_n^{k_m}\}_{m\in\mathbb{N}}$ of $\{u_n^k\}_{k\in\mathbb{N}}$ such that $u_n^{k_m} \to \tilde{u}_n$ in $C^1(\overline{\Omega})$ as $m \to \infty$,

then

$$(37) \qquad \begin{cases} -\Delta_p \tilde{u}_n(z) + \xi_\eta \tilde{u}_n(z)^{p-1} = \hat{f}(z, \tilde{u}_n(z), Dv_n(z)) & \text{for a.a } z \in \Omega, \\ \frac{\partial \tilde{u}_n}{\partial n_p} + \beta(z)(\tilde{u}_n)^{p-1} = 0, & \text{on } \partial\Omega, \ n \in \mathbb{N}. \end{cases}$$

Problem (37) has a unique solution. Therefore by Urysohn criterion for convergence of sequences (see Gasiński-Papageorgiou [9], p.33), for the original sequence $\{u_n^k\}_{k\in\mathbb{N}}$, we have

$$u_n^k \to \tilde{u}_n \text{ in } C^1(\overline{\Omega}) \text{ as } k \to \infty.$$

From (35) we have

$$\tilde{u}_n \in [0, \eta] \cap C^1(\overline{\Omega}) \text{ for all } n \in \mathbb{N}.$$

From the double limit lemma (see Aubin [1], p.221 and Gasiński-Papageorgiou [9], p.61), we have

$$\tilde{u}_n \in [0, \eta] \cap D_+ \text{ for all } n \geq n_0, \ \tilde{u}_n \to u \text{ in } C^1(\overline{\Omega}),$$

hence

$$\tilde{u}_n \in S_{v_n} \text{ for all } n \geq n_0, \ \tilde{u}_n \to u \text{ in } C^1(\overline{\Omega}).$$

$\square$

4. Existence of Positive Solutions

In this section using Proposition 2.3 (the Leray-Schauder Alternative Principle), we produce a positive solution for problem (1).

First, using Lemma 3.1, we show that the map $\theta : C \to C$ is compact.

Proposition 4.1. *If hypotheses* H(f), H(β) *hold, then the map* $\theta : C \to C$ *is compact.*

Proof. First we show the continuity of $\theta(\cdot)$.

So, suppose that $v_n \to v$ in $C^1(\overline{\Omega})$ and let $\hat{u}_n = \theta(v_n)$, $n \in \mathbb{N}$. We have

$$(38) \qquad \begin{cases} -\Delta_p \hat{u}_n(z) = f(z, \hat{u}_n(z), Dv_n(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \hat{u}_n}{\partial n_p} + \beta(z)\hat{u}_n^{p-1} = 0, & \text{on } \partial\Omega, \ n \in \mathbb{N}. \end{cases}$$

We know that $\hat{u}_n \in [0, \eta] \cap D_+$ for all $n \in \mathbb{N}$. This fact and (38) imply that $\{\hat{u}_n\}_{n \in \mathbb{N}} \subset W^{1,p}(\Omega)$ is bounded.

Then Theorem 2 of Lieberman [14] gives $\alpha \in (0, 1)$ and $c_6 > 0$ such that $\hat{u}_n \in C^{1,\alpha}(\overline{\Omega})$ and $\|\hat{u}_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_6$ for all $n \in \mathbb{N}$.

So, by passing to a subsequence if necessary we may assume that

$$(39) \qquad \hat{u}_n \to \hat{u} \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty.$$

From (38) and (39), we obtain

$$(40) \qquad \begin{cases} -\Delta_p \hat{u}(z) = f(z, \hat{u}(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial \hat{u}}{\partial n_p} + \beta(z)\hat{u}^{p-1} = 0, & \text{on } \partial\Omega. \end{cases}$$

From Proposition 3.3, we have $u^* \leq u_n$ for all $n \in \mathbb{N}$ (with $M \geq \sup_{n \in \mathbb{N}} \|v_n\|_{C^1(\overline{\Omega})}$), then $u^* \leq \hat{u}$, so we have

$$(41) \qquad \hat{u} \in S_v.$$

We claim that $\hat{u} = \theta(v)$. Since $\theta(v) \in S_v$, Lemma 3.1 implies that we can find $u_n \in S_{v_n}$, $n \in \mathbb{N}$, such that

$$(42) \qquad u_n \to \theta(v) \text{ in } C^1(\overline{\Omega}).$$

Recall that $\hat{u}_n = \theta(v_n)$. So, we have

$$\hat{u}_n \leq u_n \text{ for all } n \in \mathbb{N},$$

then (see (39), (42)), $\hat{u} \leq \theta(v)$, which implies

$$\hat{u} = \theta(v)$$

hence $\theta(\cdot)$ is continuous.

Next we show that $\theta(\cdot)$ maps bounded sets into relatively compact sets. So, let $B \subset C$ be bounded. Using hypothesis H(f)(i), we see that

$$\theta(B) \subset W^{1,p}(\Omega) \text{ is bounded.}$$

As before, using Theorem 2 of Lieberman [14] and recalling that $C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ is embedded compactly in $C^1(\overline{\Omega})$, we infer that

$$\theta(B) \subset C^1(\overline{\Omega}) \text{ is relatively compact.}$$

We conclude that the map $\theta(\cdot)$ is compact. $\qquad\square$

Now we are ready for the existence theorem.

Theorem 4.1. *If hypotheses* H(f)*,* H(β) *hold, then problem* (1) *has a solution* $\hat{u} \in [0, \eta] \cap D_+$.

Proof. As we already indicate, the positive solution will be produced as a fixed point of the minimal solution map $\theta(\cdot)$. To generate such a fixed point, we will use Proposition 2.3 (the Leray-Schauder Alternative Principle). So, we consider the set

$$B = \{u \in C \ : \ u = \lambda\theta(u) \, , \, 0 < \lambda < 1\}.$$

If $u \in B$, then

$$\frac{1}{\lambda}u = \theta(u),$$

which implies

$$(43) \quad \langle A(u), h \rangle + \int_{\partial\Omega} \beta(z)u^{p-1}hd\sigma = \lambda^{p-1}\int_{\Omega} f(z, \frac{1}{\lambda}u, Du)hdz \text{ for all } h \in W^{1,p}(\Omega).$$

Hypotheses H(f)(iii) implies

$$(44) \quad \lambda^{p-1}f(z, \frac{1}{\lambda}u(z), Du(z)) \leq f(z, u(z), Du(z)) \text{ for a.a. } z \in \Omega.$$

From (43) with $h = u \in W^{1,p}(\Omega)$, (see also, (44) and hypotheses H(f)(i), H(β)), we have

$$\|Du\|_p^p \leq \int_{\Omega} f(z, u, Du)udz \leq c_7(1 + \|Du\|_p^{p-1})$$

for some $c_7 > 0$, then we obtain

$$\{Du\}_{u\in B} \subset L^p(\Omega, \mathbb{R}^N) \text{ is bounded,}$$

hence, recall that $B \subset [0, \eta]$

$$(45) \qquad\qquad \{u\}_{u\in B} \subset W^{1,p}(\Omega) \text{ is bounded.}$$

From (43) we have

$$(46) \quad \begin{cases} -\Delta_p u(z) = \lambda^{p-1}f(z, \frac{1}{\lambda}u(z), Dv(z)) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0, & \text{on } \partial\Omega. \end{cases}$$

Then (44), (45), (46), hypothesis H(f)(i) and Theorem 2 of Lieberman [14] imply that

$$B \subset C^1(\overline{\Omega}) \text{ is bounded.}$$

Applying Proposition 2.3, we can find $\hat{u} \in C$ such that $\hat{u} = \theta(\hat{u})$, then

$$\hat{u} \in [0, \eta] \cap D_+ \text{ is a solution of (1).}$$

$\square$

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