Solvable extensions of nilpotent complex Lie algebras of type $\{2n, 1, 1\}$ [*†‡]

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Abstract

We investigate derivations of nilpotent complex Lie algebras of type $\{2n, 1, 1\}$ with the aim to classify solvable complex Lie algebras the commutator ideals of which have codimension 1 and are nilpotent Lie algebras of type $\{2n, 1, 1\}$.

1. Introduction. Classification results for solvable or nilpotent complex Lie algebras are the subject of a large literature. For a nilpotent Lie algebra $\mathfrak{g}$ with descending central series $\mathfrak{g}^{(i)} = \big[\mathfrak{g}, \mathfrak{g}^{(i-1)}\big]$, M. Vergne defined in [19] the *type* $\{p_1, \ldots, p_c\}$ of $\mathfrak{g}$ by the integers $p_i := \dim\big(\mathfrak{g}^{(i-1)}/\mathfrak{g}^{(i)}\big)$. In the same paper, she defined and studied *filiform* Lie algebras, that is, having type $\{2, 1, 1, \ldots, 1\}$. Complex filiform Lie algebras became thereafter the object of several works (e. g. [3], [4], [5], [11] and [17]) aiming to a thorough classification in low dimension. Filiform n-dimensional Lie algebras $\mathfrak{g}$ are precisely those with characteristic sequence $c(\mathfrak{g}) = (n-1, 1)$, where the *characteristic sequence* $c(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ was defined in [2] as the maximum sequence, in the lexicographic ordering, of the dimensions of the Jordan blocks of ad_x. The *breadth* $b(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ has been recently introduced in [15] as the maximum of the dimensions of the images of ad_x, for all $x \in \mathfrak{g}$, in order to classify Lie algebras where this invariant is small, and the connection between the breadth and the characteristic sequence has been pointed out in [16]: if $c(\mathfrak{g}) = (c_1, \ldots, c_k, 1)$, then $b(\mathfrak{g}) = c_1 + \ldots + c_k - k$.

On the opposite side to filiform Lie algebras, the most simple nilpotent complex Lie algebras are the $(2n+1)$-dimensional Heisenberg Lie algebras, which have type $\{2n, 1\}$, characteristic sequence $c(\mathfrak{g}) = (2, 1, \ldots, 1)$, breadth $b(\mathfrak{g}) = 1$, and are defined on a $(2n+1)$-dimensional complex vector space $\mathfrak{h} = V + \langle x \rangle$ by means of a non-degenerate alternating form $F : V \times V \to \mathbb{C}$ putting $[u, v] := F(u, v)x$, for any $u, v \in V$. These algebras and their real forms, often in connection with a left invariant metric, have been widely investigated in the last thirty years (for instance in [13], [14], [18], [20]), as they offer a basic model for nilmanifolds.

As well as the type $\{2n, 1\}$ characterizes, up to a central summand, $(2n+1)$-dimensional Heisenberg Lie algebras, the dimension of nilpotent Lie algebras of type $\{2n, 1, 1\}$ determines them uniquely (see [6]). The minimal dimension for such algebras is four; more

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precisely, we have the filiform Lie algebra

$$\mathfrak{n}_4 := \langle u^1, u^2, u^3, u^4 \rangle_{\mathbb{C}}$$

defined through the relations

$$[u^2, u^4] = u^1,\ [u^3, u^4] = u^2, \tag{1}$$

corresponding to the Lie algebra $\mathfrak{g}_{1,0,1}$ with $c(\mathfrak{g}_{1,0,1}) = (3, 1)$ recently described in [12, 3.2, Example (2)] and in [16, 2.1, Prop. 6 (3)]. Manifestly $\mathfrak{n}_4$ modulo its centre $\langle u^1 \rangle_{\mathbb{C}}$ is the 3-dimensional Heisenberg Lie algebra. More generally (see [6]), for any non-negative integer n, up to a central summand the complex Lie algebra $\mathfrak{n}$ of type $\{2n + 2, 1, 1\}$ decomposes, as a vector space, into the direct sum of vector spaces

$$\mathfrak{n} = \mathfrak{n}_4 + \mathfrak{v} + \mathfrak{w}, \tag{2}$$

where

$$\mathfrak{v} = \langle v^1, v^2, \dots, v^n \rangle_{\mathbb{C}},\ \mathfrak{w} = \langle w^1, w^2, \dots, w^n \rangle_{\mathbb{C}}$$

are Abelian Lie subalgebras of dimension n which fulfill the relations

$$[v^i, w^j] = \delta^i_j u^1, \tag{3}$$

that is, modulo the centre we obtain the direct sum of the 3-dimensional Heisenberg Lie algebra by a $2n$-dimensional Abelian Lie algebra. Hence $\mathfrak{n}$ is a nilpotent Lie algebra of breadth $b(\mathfrak{n}) = 2$ with characteristic sequence $c(\mathfrak{n}) = (3, 1, \dots, 1)$, and as such it appears in [16, 2.1, Prop. 6 (3)], and as a linear deformation of the Lie algebra $\mathfrak{g}_{1,0,n-3}$ in [12, 3.2, Example (2)].

Moreover, the ideal

$$\mathfrak{h} = \langle u^1 \rangle_{\mathbb{C}} + \mathfrak{v} + \mathfrak{w},$$

is manifestly isomorphic to the $(2n + 1)$-dimensional Heisenberg algebra, hence $\mathfrak{n}$ is the direct sum with amalgamated centre of $\mathfrak{n}_4$ and $\mathfrak{h}$:

$$\mathfrak{n} = \frac{\mathfrak{n}_4 \times \mathfrak{h}}{\langle (u^1, -u^1) \rangle}$$

Solvable extensions of nilpotent Lie algebras with any given characteristic sequence have been the object of investigation in [1], whereas solvable extensions of nilpotent real Lie algebras of type $\{n, 2\}$ by a derivation contained in a compact Lie algebra have been considered in [10] and [8]. This paper, instead, is a first step in classifying solvable extensions of nilpotent complex Lie algebras of type $\{2n + 2, 1, 1\}$; more precisely, we classify in section 3 extensions $\mathfrak{s}$ of a nilpotent complex Lie algebra $\mathfrak{n}$ of type $\{2n+2, 1, 1\}$ with $\dim \mathfrak{s}/\mathfrak{n} = 1$ and $\mathfrak{s}' = \mathfrak{n}$. In order to obtain the classification we determine in section 2 the full automorphism group of $\mathfrak{n}$ and the Lie algebra of derivations of $\mathfrak{n}$. As the Heisenberg algebra appears as the 3-codimensional ideal $\mathfrak{h}$ of $\mathfrak{n}$, the automorphism group and the derivation algebra of $\mathfrak{n}$ contain the symplectic group and algebra, respectively, as the Levi complements. Thus, it was necessary to know the orbits in $\mathfrak{sp}(2n, \mathbb{C})$ under the action of $\mathrm{Sp}(2n, \mathbb{C})$. This is a particular case of a classic problem, which goes back to Weierstrass, Kronecker, and Frobenius, then continues with a series of papers by Williamson, Zassenhaus, Wall, Cikunov, Springer and Steinberg, and Milnor, and which is finally treated in a unified solution by Burgoyne and Cushman in [9], the paper from which we take the description of the action of $\mathrm{Sp}(2n, \mathbb{C})$ on $\mathfrak{sp}(2n, \mathbb{C})$.

A canonical form for a given derivation of $\mathfrak{n}$ turned out to be parametrized by a complex number and a set of eigenvalues, corresponding to the Jordan form of the derivation (cf. Theorem 3).

2. Automorphisms and derivations of $\mathfrak{n}$. Throughout the paper $\mathfrak{n}$ will denote the nilpotent complex Lie algebra of type $\{2n+2,1,1\}$ given in (2) and defined through the relations (1) and (3). Referring to the basis $u^1, \dots, u^4, v^1, \dots, v^n, w^1, \dots, w^n$, we notice that an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{n})$ leaves the flag of ideals

$$\mathfrak{z} = \langle u^1\rangle_{\mathbb{C}}, \quad \mathfrak{n}' = \langle u^1, u^2\rangle_{\mathbb{C}}, \quad \mathfrak{k}_1 = \langle u^1, u^2, u^3\rangle_{\mathbb{C}}, \quad \mathfrak{k}_2 = \langle u^1, u^2, u^3, v^1, \dots, v^n, w^1, \dots, w^n\rangle_{\mathbb{C}}$$

invariant, because the first is the centre of $\mathfrak{n}$, the second is the commutator ideal of $\mathfrak{n}$, the fourth is the centralizer of $\mathfrak{n}'$ and the third is its centre. Notice that, putting $\varphi(u^4) = \alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + p$, for some $p \in \langle v^1, \dots, v^n, w^1, \dots, w^n\rangle_{\mathbb{C}}$, the coefficient α_4 must be different from zero, otherwise the image under φ of an arbitrary element of $\mathfrak{n}$ would be contained in $\mathfrak{k}_2$.

Moreover, the ideal

$$\mathfrak{k}_3 := \langle u^1, u^2, v^1, v^2, \dots, v^n, w^1, w^2, \dots, w^n\rangle_{\mathbb{C}}$$

is invariant, as well. In fact, the image $\varphi(v^i)$, which belongs to the invariant ideal $\mathfrak{k}_2$, has, for some $q \in \langle v^1, \dots, v^n, w^1, \dots, w^n\rangle_{\mathbb{C}}$, the shape

$$\varphi(v^i) = \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^3 + q.$$

In view of the defining brackets

$$[u^2, u^4] = u^1, \quad [u^3, u^4] = u^2, \quad [v^i, w^j] = \delta^i_j u^1$$

of $\mathfrak{n}$, we see directly that

$$\begin{aligned} 0 = \varphi(0) = \varphi([v^i, u^4]) &= [\beta_1 u^1 + \beta_2 u^2 + \beta_3 u^3 + q, \alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + p] \\ &= \beta_2\alpha_4 u^1 + \beta_3\alpha_4 u^2 + [q,p] = \gamma u^1 + \beta_3\alpha_4 u^2, \end{aligned}$$

because $[q,p] \in \langle u^1\rangle_{\mathbb{C}}$. Since $\alpha_4 \neq 0$, it follows that $\beta_3 = 0$, that is, $\varphi(v^i) \in \mathfrak{k}_3$. The same argument holds for the image $\varphi(w^i)$. Thus, with respect to the given basis, the automorphism φ can represented by

$$\left(\begin{array}{cccc|ccc|ccc}
L^1_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
L^2_1 & L^2_2 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
L^3_1 & L^3_2 & L^3_3 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
L^4_1 & L^4_2 & L^4_3 & L^4_4 & M^4_1 & \dots & M^4_n & N^4_1 & \dots & N^4_n \\
\hline
P^1_1 & P^1_2 & 0 & 0 & Q^1_1 & \dots & Q^1_n & R^1_1 & \dots & R^1_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
P^n_1 & P^n_2 & 0 & 0 & Q^n_1 & \dots & Q^n_n & R^n_1 & \dots & R^n_n \\
\hline
S^1_1 & S^1_2 & 0 & 0 & T^1_1 & \dots & T^1_n & U^1_1 & \dots & U^1_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
S^n_1 & S^n_2 & 0 & 0 & T^n_1 & \dots & T^n_n & U^n_1 & \dots & U^n_n
\end{array}\right).$$

Now, we have

$$[u^2, u^4] = u^1 \Longrightarrow L^2_2 L^4_4 = L^1_1;$$

$$[u^3, u^4] = u^2 \Longrightarrow L^3_2 L^4_4 = L^2_1 \text{ and } L^3_3 L^4_4 = L^2_2$$

and for $i = 1, 2, \dots, n$,

$$[u^4, v^i] = 0 \Longrightarrow \textstyle\sum_{j=1}^n (M^4_j R^i_j - N^4_j Q^i_j) = L^4_4 P^i_2;$$

$$[u^4, w^i] = 0 \Longrightarrow \textstyle\sum_{j=1}^n (M^4_j U^i_j - N^4_j T^i_j) = L^4_4 S^i_2;$$

whereas the fact that the ideal $\mathfrak{h} = \langle u^1\rangle_{\mathbb{C}} + \mathfrak{v} + \mathfrak{w}$ is isomorphic to the Heisenberg algebra, that is, the conditions $[v^i, v^j] = [w^i, w^j] = 0$ and $[v^i, w^j] = \delta^i_j u^1$, says that

$$\pm\frac{1}{\sqrt{L^1_1}}\left(\begin{array}{ccc|ccc} Q^1_1 & \dots & Q^1_n & R^1_1 & \dots & R^1_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Q^n_1 & \dots & Q^n_n & R^n_1 & \dots & R^n_n \\ \hline T^1_1 & \dots & T^1_n & U^1_1 & \dots & U^1_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ T^n_1 & \dots & T^n_n & U^n_1 & \dots & U^n_n \end{array}\right) \in \mathrm{Sp}(2n, \mathbb{C})$$

Summing up, we obtain a Levi decomposition of $\mathrm{Aut}(\mathfrak{n})$ by taking the matrices

$$\left(\begin{array}{cccc|ccc|ccc} (L^4_4V^3_3)^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ L^3_2L^4_4 & L^4_4(V^3_3)^2 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ L^3_1 & L^3_2 & (V^3_3)^2 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ L^4_1 & L^4_2 & L^4_3 & L^4_4 & M^4_1 & \dots & M^4_n & N^4_1 & \dots & N^4_n \\ \hline P^1_1 & -N^4_1V^3_3 & 0 & 0 & L^4_4V^3_3 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ P^n_1 & -N^4_nV^3_3 & 0 & 0 & 0 & \dots & L^4_4V^3_3 & 0 & \dots & 0 \\ \hline S^1_1 & M^4_1V^3_3 & 0 & 0 & 0 & \dots & 0 & L^4_4V^3_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S^n_1 & M^4_nV^3_3 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & L^4_4V^3_3 \end{array}\right) \tag{4}$$

as the solvable radical, and the matrices

$$\left(\begin{array}{c|c} I_4 & \\ \hline & X \end{array}\right) \text{ with } X \in \mathrm{Sp}(2n, \mathbb{C}) \tag{5}$$

as a Levi complement.

Accordingly, a Levi complement $\mathcal{L}$ of $\mathrm{Der}(\mathfrak{n}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{n}))$ is given, with respect to the same basis, by the matrices of the shape

$$\left(\begin{array}{c|c} 0 & \\ \hline & X \end{array}\right) \text{ with } X \in \mathfrak{sp}(2n, \mathbb{C}), \tag{6}$$

the nilradical $\mathcal{N}$ of $\mathrm{Der}(\mathfrak{n})$ is given by the matrices of the shape

$$\left(\begin{array}{cccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ A^3_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ A^3_1 & A^3_2 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ A^4_1 & A^4_2 & A^4_3 & 0 & G^4_1 & \dots & G^4_n & D^4_1 & \dots & D^4_n \\ \hline D^1_1 & -D^4_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D^n_1 & -D^4_n & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline G^1_1 & G^4_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G^n_1 & G^4_n & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array}\right) \tag{7}$$

(hence its nilpotency class is 3, that is, $[\mathcal{N}, [\mathcal{N}, \mathcal{N}]]$ is central) and one of its complements $\mathcal{T}$ in the solvable radical $\mathcal{R} = \mathcal{T} + \mathcal{N}$ of $\mathrm{Der}(\mathfrak{n})$ is given by the diagonal matrices

$$\mathrm{diag}(2A^3_3 + 2A^4_4, 2A^3_3 + A^4_4, 2A^3_3, A^4_4,\ A^3_3 + A^4_4, \dots, A^3_3 + A^4_4). \tag{8}$$

A direct computation shows that the only element in $\mathcal{N}$ which commutes with all the elements in $\mathcal{T}$ is the trivial one, that is, the 2-dimensional toral subalgebra $\mathcal{T}$ is a Cartan subalgebra of $\mathcal{R}$. Moreover, $[\mathcal{T}, \mathcal{L}] = 0$, hence $\mathcal{T} \oplus \mathcal{L}$ is the centralizer of $\mathcal{T}$ in $\mathrm{Der}(\mathfrak{n}) = \mathcal{R} + \mathcal{L}$, and, since $\mathcal{L}$ is isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$, a maximal toral subalgebra $\mathcal{M}$ of $\mathrm{Der}(\mathfrak{n})$ is $\mathcal{M} = \mathcal{T} \oplus \mathcal{D}$, where

$$\mathcal{D} = \{\mathrm{diag}(0, 0, 0, 0, z_1, \dots, z_n, -z_1, \dots, -z_n) : z_k \in \mathbb{C}\}$$

is the subalgebra of diagonal matrices of $\mathcal{L}$. Consequently, any element y in the centralizer of $\mathcal{M}$ in $\mathrm{Der}(\mathfrak{n})$ is the sum $y = y_0 + y_1$ with $y_0 \in \mathcal{T}$ and y_1 in the centralizer of $\mathcal{M}$ in $\mathcal{L}$, hence, *a fortiori*, in the centralizer of $\mathcal{D}$ in $\mathcal{L}$. Since $\mathcal{L}$ is simple, the element y_1 has to be contained in $\mathcal{D}$, thus $\mathcal{M}$ is also a Cartan subalgebra of $\mathrm{Der}(\mathfrak{n})$, and the rank of $\mathrm{Der}(\mathfrak{n})$ is $n + 2$.

3. A solvable extension of $\mathfrak{n}$. Note that, as $\mathcal{L}$ is simple, the largest extension of $\mathfrak{n}$, whose solvable radical is $\mathfrak{n}$ and which has no semisimple ideal, is the non-splitting extension $\mathfrak{g} = \mathfrak{n} + \mathcal{L}$. On the other hand, a solvable extension $\mathfrak{s}$, for which $\mathfrak{n}$ is the nilpotent radical, has an Abelian quotient algebra $\mathfrak{s}/\mathfrak{n}$, and in the following we show that the case where $\mathfrak{n}$ is maximal can be thoroughly described.

Assume $\mathfrak{s}$ is a solvable complex Lie algebra the commutator ideal of which is a nilpotent Lie algebra $\mathfrak{n}$ of type $\{2n + 2, 1, 1\}$ having codimension one in $\mathfrak{s}$; more precisely, we have

$$\mathfrak{s} = \langle s \rangle_{\mathbb{C}} + \mathfrak{n}$$

with s defining a derivation $\delta : x \mapsto [s, x]$ of $\mathfrak{n}$ that we may represent, with respect to the given basis, by the sum of the matrices in (6), (7), and (8). Also, up to the inner derivation

$$x \mapsto \big[- A_1^4 u^2 - A_2^4 u^3 + A_2^3 u^4 - \textstyle\sum_{i=1}^n G_1^i v^i + \sum_{i=1}^n D_1^i w^i,\ x \big],$$

we may take in (7) $A_1^4 = A_2^4 = A_2^3 = G_1^i = D_1^i = 0$ $(i = 1, 2, \dots, n)$. Since we are assuming $\mathfrak{s}' = \mathfrak{n}$, the square sub-matrix determined by the last $2n + 2$ rows and columns must be non-singular and this requires that the eigenvalues $2A_2^2 - A_1^1$ and $A_1^1 - A_2^2$ are both $\neq 0$: thus, we may take $A_1^1 = 2A_2^2 - 1$ up to replacing s by $\frac{s}{2A_2^2 - A_1^1}$, and we can represent δ by the matrix

$$\left(\begin{array}{cccc|ccc|ccc}
2A_2^2 - 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & A_2^2 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
A_1^3 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & A_3^4 & A_2^2 - 1 & G_2^1 & \dots & G_2^n & -D_2^1 & \dots & -D_2^n \\
\hline
0 & D_2^1 & 0 & 0 & A_2^2 - \frac{1}{2} + B_1^1 & & B_n^1 & F_1^1 & \dots & F_n^1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & D_2^n & 0 & 0 & B_1^n & & A_2^2 - \frac{1}{2} + B_n^n & F_n^1 & \dots & F_n^n \\
\hline
0 & G_2^1 & 0 & 0 & H_1^1 & \dots & H_n^1 & A_2^2 - \frac{1}{2} - B_1^1 & & -B_1^n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & G_2^n & 0 & 0 & H_n^1 & \dots & H_n^n & -B_n^1 & & A_2^2 - \frac{1}{2} - B_n^n
\end{array}\right). \qquad (9)$$

Using automorphisms of the solvable radical (4) of $\mathrm{Aut}(\mathfrak{n})$ having null entries M_i^4 and N_i^4, $i = 1, 2, \dots, n$, we have

Proposition 1 . *Up to automorphisms of $\mathfrak{n}$, the upper-left 4×4 minor A in (9) is one of the following:*

$$i)\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & -1 \end{pmatrix}, \quad b = 1 \textit{ or } b = 0;$$

$$ii)\ \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{pmatrix}, \quad c = 1 \textit{ or } c = 0;$$

$$iii\ \begin{pmatrix} 2a-1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix}, \quad a \neq 0, 1, 2.$$

Proof. Automorphisms

$$\left(\begin{array}{c|c} L & 0 \\ \hline 0 & L_4^4 V_3^3 I_{2n} \end{array}\right),$$

of the solvable radical of $\mathrm{Aut}(\mathfrak{n})$ move A to LAL^{-1}. We list, for each item in the statement, the conditions on the entries of L we need to achieve the claimed canonical form for A.

i) Let $A_1^1 = -1$ and $A_2^2 = 0$. Take $L_1^3 = \frac{A_1^3 L_3^3}{2}$, $L_2^3 = L_2^4 = 0$, $L_3^4 = -\frac{A_3^4 L_4^4}{2}$ and, if $A_1^3 A_3^4 \neq 0$, $L_3^3 = -\frac{A_1^3 A_3^4}{L_4^4}$.

ii) Let $A_1^1 = 3$ and $A_2^2 = 2$. Take $L_1^3 = -\frac{A_1^3 L_3^3}{2}$, $L_2^3 = L_2^4 = 0$, $L_1^4 = -\frac{A_1^3(2L_3^4 + A_3^4 L_4^4)}{4}$ and, if $A_3^4 \neq 0$, $L_3^3 = A_3^4 L_4^4$.

iii) Let the eigenvalues of A be distinct. Take $L_1^3 = -\frac{A_1^3 L_3^3}{2(A_2^2 - 1)}$, $L_2^3 = L_2^4 = 0$, $L_1^4 = -\frac{A_1^3 A_3^4 L_4^4}{(A_2^2 - 2) A_2^2}$, $L_3^4 = \frac{A_3^4 L_4^4}{(A_2^2 - 2)}$. □

Since the Levi complement of the automorphism group of $\mathfrak{n}$, isomorphic to $\mathrm{Sp}(2n, \mathbb{C})$, operates on the subalgebra of derivations in (6) with $A = D = G = 0$, isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$, looking at the Table II in [9] we see that

Proposition 2 . *Conjugations by automorphisms in the Levi complement (5) allow one to put the submatrix (B_j^i) in (9) into Jordan canonical form, as well as make H zero and F diagonal with diagonal entries 0 and 1 where, if $F_i^i = 1$, the corresponding row B^i of (B_j^i) is zero. Moreover, coordinates can be chosen in such a way (B_j^i) and $-(B_j^i)^{\mathrm{T}}$ do not share nonzero eigenvalues.* □

In order to get the desired canonical representation for δ, we have only to arrange the columns D_2 and G_2 of the sub-matrices D and G in (9) by using automorphisms which preserve the results obtained by Propositions 1 and 2, automorphisms such as the ones belonging to the solvable radical (4) of $\mathrm{Aut}(\mathfrak{n})$ with $L_j^i = \delta_j^i$.

Choose coordinates in such a way that, if (B_j^i) has eigenvalue $\frac{1}{2}$ the Jordan sub-matrix of (B_j^i) corresponding to such an eigenvalue is confined in the last $n-m$ rows and columns $(0 \leq m < n)$. As (B_j^i) and $-(B_j^i)^{\mathrm{T}}$ do not share nonzero eigenvalues (Proposition 2), we obtain non-sigular matrices, say Y and Z, taking the first m rows and columns of

$\frac{1}{2}I_n + (B^i_j)$ and $\frac{1}{2}I_n - (B^i_j)^{\mathrm{T}}$. Then it is well-defined the automorphism in the solvable radical (4) with $L^r_s = \delta^r_s$, $P^j_1 = S^j_1 = 0$ for all j, $M^4_k = N^4_k = 0$ for $m+1 \le k \le n$ and

$$(M^4_1, \dots, M^4_m) = -(G^1_2, \dots, G^m_2)Y^{-1};$$
$$(N^4_1, \dots, N^4_m) = -\big((D^1_2, \dots, D^m_2) + (M^4_1, \dots, M^4_m)F\big)Z^{-1}.$$

Such an automorphism turns $(G^1_2, \dots, G^m_2)$ and $(D^1_2, \dots, D^m_2)$ into zero vectors: thus, we may assume that $\frac{1}{2}$ is the unique eigenvalue of (B^i_j).

As above, we can make zero all the entries G^i_2 $(i = 1, \dots, n)$ as well as all the entries D^i_2, provided the corresponding column of $\frac{1}{2}I_n - (B^i_j)^{\mathrm{T}}$ is not zero. Let $D^{i_1}_2, D^{i_2}_2, \dots, D^{i_m}_2$ be the entries corresponding to the zero columns of $\frac{1}{2}I_n - (B^i_j)^{\mathrm{T}}$: clearly $i_m = n$ and we can arrange coordinates in such a way $i_1 \le i_r - i_{r-1} \le i_{r+1} - i_r$ for all $1 < r < m$.

Let

$$(B^i_j) = J_{i_1}(\tfrac{1}{2}) \oplus J_{i_2-i_1}(\tfrac{1}{2}) \oplus \dots \oplus J_{i_m-i_{m-1}}(\tfrac{1}{2})$$

be the decomposition of (B^i_j) into Jordan blocks. Since the centralizer in $\mathrm{GL}(l,\mathbb{C})$ of the $l \times l$ Jordan block $J_l(\lambda)$ of eigenvalue λ is spanned by the powers $J_l(0)^k$ for $k = 0, \dots, l-1$, the centralizer of $(B^i_j) \oplus -(B^i_j)^{\mathrm{T}}$ in $\mathrm{Sp}(2n,\mathbb{C})$ consists of the matrices $C \oplus C^{-\mathrm{T}}$ with

$$C := \begin{pmatrix} C11 & C12 & \dots & C1m \\ C21 & C22 & & C2m \\ \vdots & & \ddots & \vdots \\ Cm1 & Cm2 & \dots & Cmm \end{pmatrix}, \tag{10}$$

where each block Crs is either of the form $K = \sum a_k(r,s)J_l(0)^k$, with $a_k(r,s) \in \mathbb{C}$, if $i_r - i_{r-1} = i_s - i_{s-1} = l$, or of the form $\left(\frac{K}{0}\right)$, if $i_r - i_{r-1} > i_s - i_{s-1} = l$, or of the form $(0|K)$, if $l = i_r - i_{r-1} < i_s - i_{s-1}$.

In order to make zero more entries D^i_2, we have to change coordinates using the automorphism of the Levi complement (5) of $\mathrm{Aut}(\mathfrak{n})$ defined through (10), where we take $C_{rs} = 0$ for $r > s$, $b_{rs} = \dots = c_{rs} = 0$ for all r, s and $a_{rr} \neq 0$ for all r with $m-1$ of the vectors $(a_{11}, \dots, a_{1m})$, $(0, a_{22}, \dots, a_{2m})$, $\dots \dots$, $(0, \dots, 0, a_{m-1\,m-1}, a_{m-1\,m})$, $(0, \dots, 0, a_{mm})$ belonging to the vector hyperplane $\sum_{r=1}^m D^{i_r}_2 x_r = 0$, and the remaining one in the affine hyperplane $\sum_{r=1}^m D^{i_r}_2 x_r = 1$: such an automorphism moves the column vector $D_2 = (0, \dots, 0, D^{i_1}_2, \dots \dots, 0, \dots, 0, D^{i_m}_2)$ to a vector with $n-1$ entries zero and the remaining one equal to 1 for some of the row indexes $\{i_1, i_2, \dots, i_m\}$. Thus, we can state

Theorem 3 . *Let $\mathfrak{s} = \langle s \rangle_{\mathbb{C}} + \mathfrak{n}$ be a solvable extension of a complex nilpotent Lie algebra $\mathfrak{n}$ of type $\{2n+2, 1, 1\}$. $\mathfrak{n}$ is the commutator ideal of $\mathfrak{s}$ precisely if the derivation $x \mapsto [s, x]$ of $\mathfrak{n}$ can be represented by means of a matrix*

$$\begin{pmatrix} A & 0 & C \\ D & E & F \\ 0 & 0 & K \end{pmatrix}$$

with

– $A := \begin{pmatrix} 2a-1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & c & a-1 \end{pmatrix}$, *where $a \neq 1$ and either $b = c = 0$, or $b = 1$ if $a = 0$, or $c = 1$ if $a = 2$;*

– $E \in \mathrm{GL}_n$ *the direct sum* $E(e_1) \oplus E(e_2) \oplus \ldots \oplus E(e_t)$ *of Jordan matrices of distinct nonzero eigenvalues* $e_1, \ldots, e_t$ *with*

$$E(e_i) = J_{i_1}(e_i) \oplus J_{i_2-i_1}(e_i) \oplus \ldots \oplus J_{i_m-i_{m-1}}(e_i) \quad (i = 1, 2, \ldots, t)$$

the decomposition of $E(e_i)$ *into Jordan blocks,* $i_1 \leq i_r - i_{r-1} \leq i_{r+1} - i_r$ *for all* $1 < r < m$*;*

– $K := (2a-1)I_n - E^T$*;*

– $F \in \mathrm{M}_{n\times n}$ *a diagonal matrix, where the only non-zero entries* F_j^j *can possibly occur if some* e_i *is equal to* $a - \frac{1}{2}$ *and* $j \in \{i_1, \ldots, i_m\}$*, and in this case* $F_j^j = 1$*;*

– $D \in \mathrm{M}_{n\times 4}$*, where at most one non-zero entries* D_2^j *can possibly occur if some* e_i *is equal to* a *and* $j \in \{i_1, \ldots, i_m\}$*, and in this case* $D_2^j = 1$*;*

– $C \in \mathrm{M}_{4\times n}$ *with entries* $C_s^r = 0$*, excepting* $C_s^4 = -D_2^s$*.* □

Remark 4 . From the proof of Theorem 3 it turns out that the parameter a and the Jordan canonical form of E are invariants. Furthermore, if $e_i = a - \frac{1}{2}$ for some $i \in \{1, \ldots, t\}$, the number of entries $F_j^j = 1$, $j \in \{i_1, \ldots, i_m\}$, is also an invariant, as well as the size of the row index $j \in \{i_1, \ldots, i_m\}$ of the entry $D_2^j = 1$ in case $e_i = a$ for some $i \in \{1, \ldots, t\}$.

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