Invariant analytic orthonormalization procedure with an application to coherent states

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We discuss a general strategy which produces an orthonormal set of vectors, stable under the action of a given set of unitary operators A_j, $j=1,2,\ldots,n$, starting from a fixed normalized vector in $\mathcal{H}$ and from a set of unitary operators. We discuss several examples of this procedure and, in particular, we show how a set of *coherentlike* vectors can be produced and in which condition over the lattice spacing this can be done. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2711371]

I. INTRODUCTION

In the mathematical and physical literature many examples of complete sets of vectors in a given Hilbert space $\mathcal{H}$ are constructed starting from a single normalized element $f_0 \in \mathcal{H}$, acting on this vector several times with a given set of unitary operators. As a matter of fact, this is exactly what happens for coherent states and for wavelets, just to cite maybe the most known examples. In the first case one essentially acts several times on the vacuum of a bosonic oscillator with a modulation and a translation. In the second example, to produce a complete set of wavelets one acts on a *mother wavelet* with powers of a dilation and a translation operator. In this last situation the result of this action can be an orthonormal (o.n.) set of vectors, and this is the main result of the so-called multiresolution analysis,[1] while this is forbidden for general reasons for coherent states. Both these examples, as well as many others, can be considered as particular cases of a general procedure in which a certain set of vectors is constructed acting on a fixed element of $\mathcal{H}$, f_0, with a certain set of unitary operators, $A_1,\ldots,A_N$: $f_{k_1,\ldots,k_N} := A_1^{k_1}\cdots A_N^{k_N} f_0$, $k_j \in \mathbb{Z}$ for all $j=1,2,\ldots,n$. These vectors may or may not be orthogonal: we consider here the problem of *orthonormalizing this set*, i.e., the problem of producing a new set of vectors which shares with the original one most of its features and, moreover, is also orthonormal.

The paper is organized as follows: in the next section we state the general problem, discuss the method, and show some *prototype* examples. In Sec. III we discuss in detail the example concerning the coherent states, and we find conditions for our orthonormalization procedure to work. In particular, we show that, under certain conditions on a parameter which can be interpreted as a two-dimensional lattice spacing, a set of vectors can be obtained which shares with the coherent states a number of properties. To be explicit this new set satisfies indeed a closure condition in a certain Hilbert space, is an o.n. set of vectors, and is stable under the action of the same unitary operators which generate the set of coherent states. Moreover, each element of this new set is an eigenstate of an annihilationlike operator and saturates the Heisenberg uncertainty relation. Section IV contains our final considerations and plans for the future. The paper ends with an appendix on a generalized version of the (k,q) representation which is widely used in Sec. III.

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II. STATING THE PROBLEM AND FIRST RESULTS

Let $\mathcal{H}$ be a Hilbert space, $f_0 \in \mathcal{H}$ a fixed element of the space, and $A_1, \ldots, A_N$ N given unitary operators: $A_j^{-1}=A_j^\dagger$, $j=1,2,\ldots,N$. Let $\mathcal{H}_N$ be the closure of the linear span of the set

$$\mathcal{N}_N = \{f_{k_1,\ldots,k_N} \coloneqq A_1^{k_1}\cdots A_N^{k_N} f_0,\ k_1, \ldots, k_N \in \mathbb{Z}\}. \tag{2.1}$$

Of course, in order for this situation to be of some interest, it is necessary to assume that the vectors in $\mathcal{N}_N$, or part of them, are linearly independent: indeed, if this is not the case we may likely get a Hilbert space $\mathcal{H}_N$ which has finite dimension, and this is something not very interesting for us. Therefore in the following we will assume that all the vectors $f_{k_1,\ldots,k_N}$ are independent and it is clear, by the definition itself, that they are also complete in $\mathcal{H}_N$. In general there is no reason why the vectors in $\mathcal{N}_N$ should be mutually orthogonal. On the contrary, without a rather clever choice of both f_0 and $A_1, \ldots, A_N$, it is very unlikely to obtain an o.n. set. Our aim is to discuss some general technique which produces another vector $\varphi \in \mathcal{H}_N$ such that the set

$$\mathcal{M}_N = \{\varphi_{k_1,\ldots,k_N} \coloneqq A_1^{k_1}\cdots A_N^{k_N}\varphi, k_1, \ldots k_N \in \mathbb{Z}\} \tag{2.2}$$

is made of orthogonal vectors. Moreover, we would like this set to share as much of the original features of $\mathcal{N}_N$ as possible. For instance, if the set $\mathcal{N}_N$ is a set of coherent states, we would like the new vectors $\varphi_{k_1,\ldots,k_N}$ to be, for instance, eigenstates of an (sort of) annihilation operator, to give rise to a resolution of the identity and to saturate the Heisenberg uncertainty relation.

We will analyze this problem step by step, starting with the simplest situation which is, clearly, $N=1$. In this case the set $\mathcal{N}_1$ in Eq. (2.1) reduces to $\mathcal{N}_1=\{f_k \coloneqq A^k f_0, k \in \mathbb{Z}\}$ with $\langle f_k, f_l\rangle \neq \delta_{k,l}$ (otherwise we have already solved the problem). Since $\mathcal{N}_1$ is complete in $\mathcal{H}_1$, any element in $\mathcal{H}_1$ can be written in terms of the vectors of $\mathcal{N}_1$. Let $\varphi_0 \in \mathcal{H}_1$ be the following linear combination:

$$\varphi_0 = \sum_{k\in\mathbb{Z}} c_k f_k, \tag{2.3}$$

and let us define more vectors of $\mathcal{H}_1$ as

$$\varphi_n = A^n \varphi_0 = \sum_{k\in\mathbb{Z}} c_k f_{k+n} = X f_n, \tag{2.4}$$

where we have introduced the operator

$$X = \sum_{k\in\mathbb{Z}} c_k A^k. \tag{2.5}$$

The coefficients c_k should be fixed by the following orthogonalization requirement: $\langle \varphi_n, \varphi_0\rangle = \delta_{n,0}$. It is worth remarking that all the expansions above are, for the moment, only formal. What makes everything well defined is the asymptotic behavior of the coefficients of the expansion c_n, and we will discuss in the rest of the paper, and, in particular, in Sec. III, that there exist situations in which the series for φ_n and X do converge and other situations in which they do not.

The first useful result is that if $\langle \varphi_n, \varphi_0\rangle = \delta_{n,0}$ for all $n \in \mathbb{Z}$, then $\langle \varphi_n, \varphi_k\rangle = \delta_{n,k}$, $\forall n,k \in \mathbb{Z}$. This follows directly from the definition of φ_n since

$$\langle \varphi_n, \varphi_k\rangle = \langle A^n\varphi_0, A^k\varphi_0\rangle = \langle A^{n-k}\varphi_0, \varphi_0\rangle = \langle \varphi_{n-k}, \varphi_0\rangle = \delta_{n-k,0}.$$

For this reason, in order to fix the coefficients c_n, it is enough to require the orthogonality condition $\langle \varphi_n, \varphi_0\rangle = \delta_{n,0}$, which becomes

$$\delta_{n,0} = \langle \varphi_n, \varphi_0\rangle = \sum_{k,l\in\mathbb{Z}} \overline{c_k} c_l \langle f_{k+n}, f_l\rangle = \sum_{k,l\in\mathbb{Z}} \overline{c_k} c_l a_{k+n-l}, \tag{2.6}$$

where we have defined

$$a_j = \langle A^j f_0, f_0 \rangle. \tag{2.7}$$

If we now multiply both sides of Eq. (2.6) for e^{ipn} and sum up on $n \in \mathbb{Z}$ we get

$$|C(p)|^2 \alpha(p) = 1 \quad \text{a.e. in } [0, 2\pi[, \tag{2.8}$$

where we have introduced the following functions:

$$C(p) = \sum_{l \in \mathbb{Z}} c_l e^{ipl}, \quad \alpha(p) = \sum_{l \in \mathbb{Z}} a_l e^{ipl}. \tag{2.9}$$

Again, these series are not necessarily convergent, so that they must be considered only as formal objects at this stage.

In particular, it is an easy exercise to check that, if the following quantities all exist, then $\sum_{l \in \mathbb{Z}} |c_l|^2 = (1/2\pi) \int_0^{2\pi} |C(p)|^2 dp = (1/2\pi) \int_0^{2\pi} (dp/\alpha(p))$. This result suggests that for particular choices of f_0 and A it might happen that the series for $\alpha(p)$ is not convergent or, even if it converges to a 2π periodic and C^∞ function, this function might have in $[0, 2\pi[$ a zero which makes of $\alpha(p)^{-1}$ a nonintegrable function. If this is the case there is no reason to claim that the sequence $\{c_l\}$ belongs to $l^2(\mathbb{Z})$. On the contrary, any time that the function $\alpha(p)$ exists as a continuous function, i.e., under suitable conditions on the a_l's which are satisfied in many relevant situations, e.g., for coherent states, and if $\alpha(p)$ does not vanish in $[0, 2\pi[$, we can conclude that $\{c_l\} \in l^2(\mathbb{Z})$. But, in this case, we can do much better than this: since $\{c_l\} \in l^2(\mathbb{Z})$ then $C(p) \in \mathcal{L}^2(0, 2\pi)$ and, therefore,

$$c_l = \frac{1}{2\pi} \int_0^{2\pi} C(p) e^{-ipl} dp = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ipl} dp}{\sqrt{\alpha(p)}}, \tag{2.10}$$

with a particular choice of phase for $C(p)$. Now, due to the regularity of the function $1/\sqrt{\alpha(p)}$ and to its 2π periodicity, it is a standard exercise in Fourier series theory to check that c_l goes to zero when l diverges faster than any inverse power of l. Therefore the series in Eq. (2.3), (2.5), and (2.9) all converge, and we conclude that the set $\mathcal{M}_1 = \{\varphi_n, n \in \mathbb{Z}\}$ is an orthonormal set in $\mathcal{H}_1$. A natural question is now the following: is $\mathcal{M}_1$ complete in $\mathcal{H}_1$? To answer this question we give here the following proposition, which gives a necessary and sufficient condition for $\mathcal{M}_1$ to be complete in $\mathcal{H}_1$. In the proof of this proposition we will use the fact that, under the assumptions of the statement, X is self-adjoint and maps $\mathcal{H}_1$ into itself. The proof of this claim is a simple exercise and is left to the reader.

Proposition 1: *Suppose that $\{a_j\} \in l^1(\mathbb{Z})$ and that $\alpha(p) \neq 0$ for all $p \in [0, 2\pi[$. Then $\mathcal{M}_1$ is complete in $\mathcal{H}_1$ if and only if X admits a bounded inverse.*

Proof: Let $h \in \mathcal{H}_1$ be orthogonal to all the φ_n's, $n \in \mathbb{Z}$. Then, because of Eq. (2.4), we have $0 = \langle h, \varphi_n \rangle = \langle h, X f_n \rangle = \langle X h, f_n \rangle$ for all $n \in \mathbb{Z}$. But $\mathcal{N}_1$ is complete in $\mathcal{H}_1$ and $Xh \in \mathcal{H}_1$ since $h \in \mathcal{H}_1$ and $X: \mathcal{H}_1 \to \mathcal{H}_1$. Therefore $Xh = 0$. Since X is invertible, then $h = 0$ and, as a consequence, $\mathcal{M}_1$ is complete.

Let us prove the converse statement. Since $\mathcal{M}_1$ is complete in $\mathcal{H}_1$ and since $f_0 \in \mathcal{H}_1$ then we can write

$$f_0 = \sum_{l \in \mathbb{Z}} d_l \varphi_l, \tag{2.11}$$

and $\{d_l\}$ satisfies the sum rule $\sum_{l \in \mathbb{Z}} |d_l|^2 = 1$ because $\mathcal{M}_1$ is an o.n. complete set and f_0 is normalized. Moreover we have $a_j = \langle A^j f_0, f_0 \rangle = \langle f_j, f_0 \rangle = \sum_{l,k \in \mathbb{Z}} \overline{d_l} d_k \langle \varphi_{l+j}, \varphi_k \rangle = \sum_{l \in \mathbb{Z}} \overline{d_l} d_{l+j}$ which, introducing the function $D(p) = \sum_{l \in \mathbb{Z}} d_l e^{ipl} \in \mathcal{L}^2(0, 2\pi)$, becomes $|D(p)|^2 = \alpha(p)$ a.e. in $[0, 2\pi[$. Therefore we get

$$d_n = \frac{1}{2\pi}\int_0^{2\pi} D(p)\mathrm{e}^{-ipn}dp = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\alpha(p)}\mathrm{e}^{-ipn}dp, \tag{2.12}$$

with a particular choice of phase for $D(p)$. Because of our assumption on a_j it follows that the series for $\alpha(p)$ converges uniformly and define a positive C^∞ function which is also 2π periodic. These features are also shared by $\sqrt{\alpha(p)}$ and therefore d_n decreases to zero faster than any inverse power of n, as $n\to\infty$.

Now, since $f_n = A^n f_0 = A^n(\Sigma_{l\in\mathbb{Z}} d_l\varphi_l) = (\Sigma_{l\in\mathbb{Z}} d_l A^l)\varphi_n$, and since $\Sigma_{l\in\mathbb{Z}} d_l A^l$ surely converges uniformly, it is clear that this defines a new bounded operator which is exactly the inverse of X, namely, $X^{-1} = \Sigma_{l\in\mathbb{Z}} d_l A^l$. ∎

Remark: The requirement $\alpha(p)\neq 0$ for all $p\in[0,2\pi[$ is used above to ensure that the operator X exists and is bounded, as it can be deduced from the asymptotic behavior of the coefficients c_l's.

An interesting result relating the coefficients of the two expansions in Eq. (2.3) and (2.11), which may be considered as the inverse one of the other, is given by the following sum rule:

$$\sum_{n\in\mathbb{Z}} \overline{c_n} d_n = 1. \tag{2.13}$$

The proof makes use of the Poisson summation rule,[1] $\Sigma_{n\in\mathbb{Z}} \mathrm{e}^{ixan} = (2\pi/|a|)\Sigma_{n\in\mathbb{Z}}\delta(x-(2\pi/a)n)$, $a\neq 0$, and goes as follows:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \overline{c_n} d_n &= \sum_{n\in\mathbb{Z}}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{ipn}}{\sqrt{\alpha(p)}}dp\right)\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\alpha(q)}\mathrm{e}^{-iqn}\mathrm{d}q\right)\\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}p\int_0^{2\pi}\mathrm{d}q\sqrt{\frac{\alpha(q)}{\alpha(p)}}\sum_{n\in\mathbb{Z}}\mathrm{e}^{in(p-q)} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}p\int_0^{2\pi}\mathrm{d}q\sqrt{\frac{\alpha(q)}{\alpha(p)}}\sum_{n\in\mathbb{Z}}\delta(p-q-2\pi n)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}p\int_0^{2\pi}\mathrm{d}q\sqrt{\frac{\alpha(q)}{\alpha(p)}}\delta(p-q) = 1,\end{aligned}$$

because the only effective contribution arising here from $\Sigma_{n\in\mathbb{Z}}\delta(p-q-2\pi n)$ comes from $n=0$, since $p,q\in[0,2\pi[$.

A. Preliminary examples

Let $f_0(x)=\chi_{[0,a[}(x)$ be the characteristic function in the interval $[0,a[$, with $a>0$, and let A be the following translation operator: $A=\mathrm{e}^{-i\hat{p}}$. We have

$$\mathcal{N}_1 = \{f_n(x) := A^n f_0(x) = \chi_{[n,n+a[}(x), n\in\mathbb{Z}\}.$$

We want to see what our procedure produces starting with this set. For that, it is convenient to consider separately the cases $a<1$, $a=1$, and $a>1$. Let us start with the easiest case, $a=1$. In this case the set $\mathcal{N}_1$ is already made of o.n. functions, and therefore we expect that the set $\mathcal{M}_1$ coincides with $\mathcal{N}_1$. Indeed this is what happens, since $a_j=\langle f_j,f_0\rangle=\delta_{j,0}$. Therefore $\alpha(p)=1$, which is obviously never zero, and $c_l=\delta_{l,0}$, see Eq. (2.10). From Eq. (2.4) we deduce that $\varphi_n(x)=f_n(x)$ for all integer n. It is clear that both X and X^{-1} exist, and they are both equal to the identity operator.

Just a little less trivial is the situation when $a<1$. In this case, in fact, the set $\mathcal{N}_1$ is still made of orthogonal functions, since each $f_n(x)=\chi_{[n,n+a[}(x)$ does not overlap with any other $f_k(x)=\chi_{[k,k+a[}(x)$, if $k\neq n$. However, none of these functions is normalized so that we may expect that our procedure simply *cures* this feature. Indeed, we have $a_j=\langle f_j,f_0\rangle=a\delta_{j,0}$, so that $\alpha(p)=a$, which is again never zero, and $c_l=(1/\sqrt{a})\delta_{l,0}$. Therefore $\varphi_n(x)=(1/\sqrt{a})f_n(x)$ for all integer n. Of course these are now orthogonal functions with norm equal to 1. It is finally clear that both X and X^{-1} exist, and we find $X=(1/\sqrt{a})\mathbb{1}$ and $X^{-1}=\sqrt{a}\mathbb{1}$.

Surely more interesting is the case $a>1$. We restrict ourselves, for the time being, to $1<a<2$. The overlap coefficients a_j can be written as $a_j=a\delta_{j,0}+(a-1)(\delta_{j,-1}+\delta_{j,1})$, so that $\alpha(p)=a+2(a-1)\cos(p)$. This is a non-negative, real, and 2π-periodic function, as expected, and furthermore it is never zero in $[0,2\pi[$ since it has a minimum in $p=\pi$ and $\alpha(\pi)=2-a>0$. If we fix, just to be concrete, $a=3/2$, we can compute analytically $\sum_{l\in\mathbb{Z}}|c_l|^2=(1/2\pi)\int_0^{2\pi}(dp/\alpha(p))=2/\sqrt{5}$. Therefore the sequence $\{c_l\}$ belongs to $l^2(\mathbb{Z})$, as it was to be expected because of the absence of zeroes of $\alpha(p)$. As a matter of fact, it is quite easy to check also numerically that both c_l and d_l decrease very fast for increasing l: already for $|l|\geqslant 5$ we find $|c_l|\simeq 10^{-3}$ and $|d_l|\simeq 2\times 10^{-4}$. It is also easy to check that the sum rule in Eq. (2.13) is satisfied. This same analysis can be extended to $a\geqslant 2$. One can check that there are values of the parameter a for which, e.g., $\{c_l\}$ belongs to $l^2(\mathbb{Z})$, and other values of a, for which $\{c_l\}\notin l^2(\mathbb{Z})$.

For instance, if $a=2$ the overlap coefficients are the same as for $a\in]1,2[$, $a_j=a\delta_{j,0}+(a-1)\times(\delta_{j,-1}+\delta_{j,1})=2\delta_{j,0}+(\delta_{j,-1}+\delta_{j,1})$, so that $\alpha(p)=2+2\cos(p)$. This is zero for $p=\pi$ and one can check that $\sum_{l\in\mathbb{Z}}|c_l|^2=+\infty$. So the same example produces different behaviors depending on the value of a. We will recover this same feature in the next section, in the construction of the so-called *orhogonal coherent states*.

Another interesting and easy example is the following: let $f_0(x)=\chi_{[0,1[}(x)$ and let A be the following dilatation operator: $(Ah)(x)=\sqrt{2}h(2x)$, $\forall h(x)\in\mathcal{L}^2(\mathbb{R})$. Then the set $\mathcal{N}_1$ turns out to be

$$\mathcal{N}_1=\left\{f_n(x)=2^{n/2}f(2^n x)=2^{n/2}\begin{cases}1 & \text{if } 0<x\leqslant 2^{-n},\\ 0 & \text{otherwise},\end{cases}\qquad n\in\mathbb{Z}\right\}.$$

In this case all the overlap coefficients a_j are different from zero. Indeed we get $a_j=2^{-|j|/2}$, for all $j\in\mathbb{Z}$. Since $|e^{\pm ip}/\sqrt{2}|=1/\sqrt{2}<1$, it is easy to compute the analytic expression of $\alpha(p)$ and it turns out that $\alpha(p)=1/(3-2^{3/2}\cos(p))$. The minimum of $\alpha(p)$ is found again for $p=\pi$, and $\alpha(\pi)=1/(3+2^{3/2})\simeq 0.1716$, which is different from zero. Moreover we find that $\max(\alpha(p))=\alpha(0)=1/(3-2^{3/2})\simeq 5.8284$. The $\|.\|_2$ norm of the sequence $\{c_l\}$ can be computed analytically and we find $\sum_{l\in\mathbb{Z}}|c_l|^2=(1/2\pi)\int_0^{2\pi}dp/\alpha(p)=3$. Again, it is quite easy to find numerically the value of the coefficients c_l and d_l, to check that they both converge to zero quite fast, and that Eq. (2.13) is satisfied. Further, one can use these coefficients to define the *new* o.n. vectors using Eqs. (2.3) and (2.4).

III. COHERENT STATES

This section is devoted to a more interesting example involving coherent states,[2] We will see that the set of coherent states fits the general discussion of Sec. II, and we will show how and when the orthonormalization procedure works.

Let $\hat{q}$ and $\hat{p}$ be the position and momentum operators on a Hilbert space $\mathcal{H}$, $[\hat{q},\hat{p}]=i\mathbb{1}$, and let us now introduce the following unitary operators:

$$U(\underline{n})=e^{ia(n_1\hat{q}-n_2\hat{p})},\quad D(\underline{n})=e^{z_{\underline{n}}b^\dagger-\bar{z}_{\underline{n}}b},\quad T_1:=e^{ia\hat{q}},\quad T_2:=e^{-ia\hat{p}}. \tag{3.1}$$

Here a is a real constant satisfying $a^2=2\pi L$ for some $L\in\mathbb{N}$, while $z_{\underline{n}}$ and b are related to $\underline{n}=(n_1,n_2)$ and $\hat{q}$, $\hat{p}$ via the following equalities:

$$z_{\underline{n}}=\frac{a}{\sqrt{2}}(n_2+in_1),\quad b=\frac{1}{\sqrt{2}}(\hat{q}+i\hat{p}). \tag{3.2}$$

With these definitions it is clear that

$$U(\underline{n})=D(\underline{n})=(-1)^{Ln_1n_2}T_1^{n_1}T_2^{n_2}=(-1)^{Ln_1n_2}T_2^{n_2}T_1^{n_1}, \tag{3.3}$$

where we have also used the commutation rule $[T_1,T_2]=0$.

Let $\varphi_{\underline{0}}$ be the vacuum of b, $b\varphi_{\underline{0}}=0$, and let us define the following coherent states:

$$\varphi_{\underline{n}} := T_1^{n_1} T_2^{n_2} \varphi_{\underline{0}} = T_2^{n_2} T_1^{n_1} \varphi_{\underline{0}} = (-1)^{L n_1 n_2} U(\underline{n}) \varphi_{\underline{0}} = (-1)^{L n_1 n_2} D(\underline{n}) \varphi_{\underline{0}}. \tag{3.4}$$

It is very well known that the set of these vectors, $\mathcal{C}=\{\varphi_{\underline{n}}, \underline{n} \in \mathbb{Z}^2\}$, satisfies, among the others, the following properties:

1. $\mathcal{C}$ is invariant under the action of $T_j^{n_j}$, $j=1,2$;
2. each $\varphi_{\underline{n}}$ is an eigenstate of b: $b\varphi_{\underline{n}}=z_{\underline{n}}\varphi_{\underline{n}}$;
3. they satisfy the *resolution of the identity* $\Sigma_{\underline{n}\in\mathbb{Z}^2}|\varphi_{\underline{n}}\rangle\langle\varphi_{\underline{n}}|=\mathbb{1}$;
4. They saturate the Heisenberg uncertainty principle: let $(\Delta X)^2=\langle X^2\rangle-\langle X\rangle^2$ for $X=\hat{q},\hat{p}$, then $\Delta\hat{q}\Delta\hat{p}=\frac{1}{2}$.

However, it is also well known that they are not mutually orthogonal. Indeed we have

$$I_{\underline{n}} := \langle \varphi_{\underline{n}}, \varphi_{\underline{0}} \rangle = (-1)^{L n_1 n_2} \mathrm{e}^{-(\pi/2)L(n_1^2+n_2^2)}. \tag{3.5}$$

Of course, for large L the set $\mathcal{C}$ can be considered as *approximately orthogonal*, since $I_{\underline{n}} \simeq 0$ for all $\underline{n} \neq \underline{0}$. On the contrary, for small L, the overlap between nearest neighboring vectors is significantly different from zero.

Our aim is to construct a family of vectors $\mathcal{E}$ which shares with $\mathcal{C}$ most of the above features and which, moreover, is made of orthonormal vectors. We will show that this is possible, in suitable Hilbert spaces, if $L>1$, while the procedure discussed in Sec. II fails for $L=1$.

We start our analysis with some consideration concerning the set $\mathcal{C}$. For this we will make use of the results on the generalized (k,q) representation presented in the Appendix. Since most of our results will depend on the value of L, i.e., on the value of a^2, from now on we replace $\varphi_{\underline{n}}$ with $\varphi_{\underline{n}}^{(L)}$, and $\mathcal{C}$ with $\mathcal{C}^{(L)}$. However, it is important to stress that, due to its definition, $\varphi_{\underline{0}}$ does not depend on L, while all the vectors $\varphi_{\underline{n}}^{(L)}=T_1^{n_1}T_2^{n_2}\varphi_{\underline{0}}$ do. Our first result is the following.

Proposition 2: *The set $\mathcal{C}^{(L)}$ is complete in $\mathcal{H}$ if and only if $L=1$.*

Proof: The proof of this statement extends the analogous proof given in Ref. 3: let $h \in \mathcal{H}$ be a vector orthogonal to $\varphi_{\underline{n}}^{(L)}$ for all $\underline{n} \in \mathbb{Z}^2$: $\langle h, \varphi_{\underline{n}}^{(L)}\rangle=0\,\forall\,\underline{n}\in\mathbb{Z}^2$. Using the functions $\Phi_{k,q}^{(A,a)}(x)$ introduced in Eq. (A3), $A>0$ fixed and $(k,q)\in\Box^{(A)}:=[0,2\pi/A[\times[0,a[$, and their properties, we deduce that

$$0 = \langle h, \varphi_{\underline{n}}^{(L)} \rangle = \int\int_{\Box^{(A)}} \langle h, \Phi_{k,q}^{(A,a)}\rangle\langle\Phi_{k,q}^{(A,a)}, \varphi_{\underline{n}}^{(L)}\rangle dk dq$$

[see Eq. (A6)], and since

$$\langle\Phi_{k,q}^{(A,a)}, \varphi_{\underline{n}}^{(L)}\rangle = \langle\Phi_{k,q}^{(A,a)}, T_1^{n_1}T_2^{n_2}\varphi_{\underline{0}}\rangle = \mathrm{e}^{-iqan_1+ikAn_2}\langle\Phi_{k,q}^{(A,a)}, \varphi_{\underline{0}}\rangle$$

[see Eq. (A4)], we find

$$0 = \int_0^{2\pi/A} dk \mathrm{e}^{ikAn_2} \int_0^a \mathrm{d}q \mathrm{e}^{-iqan_1} C(k,q),$$

for all $\underline{n} \in \mathbb{Z}^2$. Here we have introduced the function $C(k,q):=\langle h,\Phi_{k,q}^{(A,a)}\rangle\langle\Phi_{k,q}^{(A,a)},\varphi_{\underline{0}}\rangle$. In this way the problem of the completeness of the set $\mathcal{C}^{(L)}$ has been replaced by the problem of completeness of the set $\mathcal{D}^{(L)}:=\{\mathrm{e}^{-iqan_1+ikAn_2},\underline{n}\in\mathbb{Z}^2\}$ in $\mathcal{L}^2(\Box^{(A)})$. It is now easy to prove that, if $L>1$, the function $s(k,q)=\mathrm{e}^{iqa/L}$ belongs to $\mathcal{L}^2(\Box^{(A)})$, is different from zero a.e. in $\Box^{(A)}$, and it is orthogonal to all the functions in $\mathcal{D}^{(L)}$. Therefore, if $L>1$, $\mathcal{D}^{(L)}$ is not complete and as a consequence, $\mathcal{C}^{(L)}$ is not complete either.

If $L=1$ the completeness of $\mathcal{D}^{(1)}$ is a well known fact in the theory of the Fourier series. Moreover, since $\langle\Phi_{k,q}^{(A,a)},\varphi_{\underline{0}}\rangle\neq 0$ a.e. in $\Box^{(A)}$,[3] we conclude that $h=0$: $\mathcal{C}^{(1)}$ is complete in $\mathcal{H}$. □

Let us now define, for each $L\geq 1$, the following set:

$$h_L := \text{linear span}\ \overline{\{\varphi_{\underline{n}}^{(L)}, \underline{n} \in \mathbb{Z}^2\}}^{\|.\|}. \tag{3.6}$$

It is clear that $h_1 = \mathcal{H}$, while, for $L > 1$, $h_L \subset \mathcal{H}$. It is further clear that h_L is a Hilbert space for each L, since it is a closed subspace of $\mathcal{H}$. The resolution of the indentity of the set $\{\varphi_{\underline{n}}^{(L)}\}$ stated above must be understood, clearly, in h_L. Furthermore, we can associate with h_L two different Hilbert spaces of functions, obtained by *projecting* the vectors of h_L in the coordinate or in the (k,q) representation. We have, see also the Appendix,

$$l_L^2(\mathbb{R}) := \{f(x) \in \mathcal{L}^2(\mathbb{R}) : \exists f \in h_L : f(x) = \langle \xi_x, f \rangle\},$$

and

$$l_L^2(\square^{(A)}) := \{f(k,q) \in \mathcal{L}^2(\square^{(A)}) : \exists f \in h_L : f(k,q) = \langle \Phi_{k,q}^{(A,a)}, f \rangle\}.$$

From what we have discussed above, it is clear that $l_L^2(\mathbb{R})$ and $l_L^2(\square^{(A)})$ are closed subsets of $\mathcal{L}_L^2(\mathbb{R})$ and $\mathcal{L}_L^2(\square^{(A)})$, respectively, so that they are Hilbert spaces, too.

The problem we want to discuss here is the following: *is it possible to produce, starting from $\mathcal{C}^{(L)}$, a set of vectors which is still coherent (to a certain extent) and which is mutually orthogonal*? It is clear that this last requirement is not compatible with what one usually calls coherent states.[4] However, we will see that adopting here the procedure discussed in Sec. II a rather nontrivial structure emerges.

We start extending formula (2.4) to the present settings:

$$\Psi_{\underline{n}}^{(L)} := \sum_{\underline{k} \in \mathbb{Z}^2} c_{\underline{k}}^{(L)} \varphi_{\underline{k}+\underline{n}}^{(L)}. \tag{3.7}$$

Of course this means that $\Psi_{\underline{0}}^{(L)} := \sum_{\underline{k} \in \mathbb{Z}^2} c_{\underline{k}}^{(L)} \varphi_{\underline{k}}^{(L)}$ and, because of the commutativity of T_1 and T_2, that

$$\Psi_{\underline{n}}^{(L)} = T_1^{n_1} T_2^{n_2} \Psi_{\underline{0}}^{(L)}. \tag{3.8}$$

Therefore the new set constructed in this way, $\mathcal{E}^{(L)} := \{\Psi_{\underline{n}}^{(L)}, \underline{n} \in \mathbb{Z}^2\}$, is invariant under the action of T_1 and T_2, exactly as the set $\mathcal{C}^{(L)}$, independently of the choice of the coefficients of the expansion $c_{\underline{k}}^{(L)}$. Useless to say, in order to have a converging expansion in Eq. (3.7), the following inequality must be satisfied:

$$\sum_{\underline{k},\underline{s} \in \mathbb{Z}^2} (-1)^{L(k_1-s_1)(k_2-s_2)} e^{-(\pi/2)L((k_1-s_1)^2+(k_2-s_2)^2)} \overline{c_{\underline{k}}^{(L)}} c_{\underline{s}}^{(L)} < \infty, \tag{3.9}$$

which is equivalent to require that $\|\Psi_{\underline{n}}^{(L)}\| = \|\Psi_{\underline{0}}^{(L)}\| < \infty$. It is clear then, because of the Schwarz inequality, that if Eq. (3.9) holds then all the scalar products $\langle \Psi_{\underline{n}}^{(L)}, \Psi_{\underline{s}}^{(L)} \rangle$ are well defined. Of course the coefficients $c_{\underline{s}}^{(L)}$ must not be chosen freely: they are fixed by requiring that the vectors in the set $\mathcal{E}^{(L)}$ are orthonormal: $\langle \Psi_{\underline{n}}^{(L)}, \Psi_{\underline{s}}^{(L)} \rangle = \delta_{\underline{n},\underline{s}}$. This will fix (not uniquely) the value of the $c_{\underline{s}}^{(L)}$'s, with a procedure which extends what we have discussed in the previous section and which is also close to the one used in Ref. 5 in a different context. We will also check that the set $\mathcal{E}^{(L)}$ is complete in h_L.

In order to deduce the expression for $c_{\underline{s}}^{(L)}$ we start observing that in order to have orthogonality among all the $\Psi_{\underline{n}}^{(L)}$, it is enough to require that $\langle \Psi_{\underline{n}}^{(L)}, \Psi_{\underline{0}}^{(L)} \rangle = \delta_{\underline{n},\underline{0}}$ for all $\underline{n} \in \mathbb{Z}^2$. Indeed, if this is satisfied, then the invariance under translations of the set $\mathcal{E}^{(L)}$ implies also that $\langle \Psi_{\underline{n}}^{(L)}, \Psi_{\underline{s}}^{(L)} \rangle = \delta_{\underline{n},\underline{s}}$ for all $\underline{n}, \underline{s} \in \mathbb{Z}^2$. Using expansion (3.7) we find that

$$\langle \Psi_{\underline{n}}^{(L)}, \Psi_{\underline{0}}^{(L)} \rangle = \sum_{\underline{k},\underline{s} \in \mathbb{Z}^2} \overline{c_{\underline{l}}^{(L)}} c_{\underline{s}}^{(L)} I_{\underline{l}+\underline{n}-\underline{s}} = \delta_{\underline{n},\underline{0}}, \tag{3.10}$$

which is equivalent to the following equation:

$$F_L(\underline{P})|C_L(\underline{P})|^2 = 1 \quad \text{a.e. in } [0,2\pi[\times [0,2\pi[, \tag{3.11}$$

where

$$F_L(\underline{P}) \coloneqq \sum_{\underline{k}\in\mathbb{Z}^2} I_{\underline{k}} e^{i\underline{P}\cdot\underline{k}} \text{ and } C_L(\underline{P}) \coloneqq \sum_{\underline{k}\in\mathbb{Z}^2} c_{\underline{k}}^{(L)} e^{i\underline{P}\cdot\underline{k}}. \tag{3.12}$$

Here we are not explicitating the dependence of $I_{\underline{k}}$ on L. It is clear now that the coefficients can be recovered via the formula

$$c_{\underline{k}}^{(L)} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{-i\underline{P}\cdot\underline{k}}}{\sqrt{F_L(\underline{P})}} d\underline{P}, \tag{3.13}$$

which corresponds to a special choice of the phase of the function $C_L(\underline{P})$. We will show in a moment that this integral does not need to exist in general and, even if it exists, there is no reason *a priori* to ensure that the coefficients $c_{\underline{k}}^{(L)}$'s satisfy condition (3.9). This is a consequence of the nonorthogonality of the set $\mathcal{C}^{(L)}$ and of the procedure we are adopting. However, under simple conditions, it is possible to analyze the asymptotic behavior of the $c_{\underline{k}}^{(L)}$'s for $\underline{k}$ diverging using more or less standard techniques which relates this behavior to the analytic features of $F_L(\underline{P})$. First we see that, since

$$F_L(\underline{P}) = \sum_{\underline{m}\in\mathbb{Z}^2} (-1)^{Lm_1m_2} e^{-\pi/2L(m_1^2+m_2^2)} e^{i\underline{P}\cdot\underline{m}}, \tag{3.14}$$

F_L can be rewritten in terms of the Jacobi θ_3 function as follows:

$$F_L(\underline{P}) = \theta_3\left(\frac{P_1}{2}, e^{-\pi/2L}\right)\theta_3(P_2, e^{-2\pi L}) + e^{iP_2-\pi/2L}\theta_3\left(\frac{P_1+\pi L}{2}, e^{-\pi/2L}\right)\theta_3(P_2+i\pi L, e^{-2\pi L}). \tag{3.15}$$

We have also found a different expression for $F_L(\underline{P})$, again in terms of θ_3, which we report here just for completeness:

$$F_L(\underline{P}) = e^{i\pi LD}\theta_3\left(\frac{P_1}{2}, e^{-\pi/2L}\right)\theta_3\left(\frac{P_2}{2}, e^{-\pi/2L}\right), \tag{3.16}$$

where D is the differential operator defined as $D=(-i\partial/\partial P_1)(-i\partial/\partial P_2)$. A nice feature of formula (3.16), when compared to Eq. (3.15), is that Eq. (3.16) is manifestly invariant under the exchange $P_1 \leftrightarrow P_2$, as the original expression in Eq. (3.14), while the other is not.

The function $F_L(P_1,P_2)$ is surely non-negative, since it has to satisfy Eq. (3.11), and 2π periodic: $F_L(P_1+2\pi, P_2+2\pi)=F_L(P_1,P_2)$ a.e. It is also infinitely differentiable, for all $L\geqslant 1$. However, since $F_1(\pi,\pi)=0$, there is no reason *a priori* for the integral in Eq. (3.13) to be convergent if $L=1$ and, even if this happens, there is no reason for the related $\{c_{\underline{k}}^{(L)}\}$ to satisfy condition (3.9). For this reason it is more convenient to consider separately the two situations $L=1$ and $L>1$.

A. What if $L>1$?

If $L>1$ it is possible to prove that the function $F_L(\underline{P})$ has no zero at all. Indeed, if we write $F_L(\underline{P})=1+F_L^o(\underline{P})$, $F_L^o(\underline{P})=\sum_{\underline{m}\in\mathbb{Z}^2\setminus(0,0)}(-1)^{Lm_1m_2}e^{-\pi/2L(m_1^2+m_2^2)}e^{i\underline{P}\cdot\underline{m}}$, we deduce that

$$|F_L^o(\underline{P})| \leqslant \sum_{\underline{m}\in\mathbb{Z}^2\setminus(0,0)} e^{-(\pi/2)L(m_1^2+m_2^2)} = \sum_{\underline{m}\in\mathbb{Z}^2} e^{-(\pi/2)L(m_1^2+m_2^2)} - 1 = (\theta_3(0,e^{-(\pi/2)L}))^2 - 1,$$

for all $\underline{P}\in[0,2\pi[\times[0,2\pi[$. The right-hand side can be easily computed for different values of L. We get $|F_1^o(\underline{P})|\leqslant 1.014\,97$, while $|F_2^o(\underline{P})|\leqslant 0.180\,341$, $|F_3^o(\underline{P})|\leqslant 0.036\,256$, and so on. As we can see, $F_L(\underline{P})$ can only be zero for some $\underline{P}$ if $L=1$, and this is exactly what happens for $\underline{P}=(\pi,\pi)$, while for $L\geq 2$ $F_L(\underline{P})$ is strictly positive.

With this in mind we conclude that for $L>1$ the function $C_L(\underline{P})=1/\sqrt{F_L(\underline{P})}$ is always well defined, belongs to C^∞, and is $(2\pi,2\pi)$ periodic together with all its derivatives. A standard argument allows us to conclude therefore that the coefficients $c_{\underline{k}}^{(L)}$ in Eq. (3.13) go to zero faster than any inverse power of $\|\underline{k}\|=\sqrt{k_1^2+k_2^2}$. Let us now put, for $N\in\mathbb{N}, \Psi_{\underline{0},N}^{(L)}=\Sigma_{\|\underline{k}\|\leq N}c_{\underline{k}}^{(L)}\varphi_{\underline{k}}^{(L)}$, and let $N>M$. Then we have

$$\|\Psi_{\underline{0},N}^{(L)}-\Psi_{\underline{0},M}^{(L)}\|\leq \sum_{M<\|\underline{k}\|\leq N}|c_{\underline{k}}^{(L)}|\|\varphi_{\underline{k}}^{(L)}\|=\sum_{M<\|\underline{k}\|\leq N}|c_{\underline{k}}^{(L)}|\to 0,$$

when $M,N\to\infty$, due to the asymptotic behavior of $c_{\underline{k}}^{(L)}$. Since h_L is complete, the sequence $\{\Psi_{\underline{0},N}^{(L)}\}$ is convergent to an element of h_L, which is clearly $\Psi_{\underline{0}}^{(L)}$. The same argument can be repeated to check that $\Psi_{\underline{n}}^{(L)}$ is well defined and belongs to h_L. Alternatively, we can simply observe that since $\Psi_{\underline{0}}^{(L)}$ belongs to h_L, and since h_L is invariant under the action of T_1 and T_2, also $\Psi_{\underline{n}}^{(L)}=T_1^{n_1}T_2^{n_2}\Psi_{\underline{0}}^{(L)}$ belongs to h_L.

Going back to Eq. (3.7), if we introduce an operator X_L as in Eq. (2.5),

$$X_L=\sum_{\underline{k}\in\mathbb{Z}^2}c_{\underline{k}}^{(L)}T_1^{k_1}T_2^{k_2}, \tag{3.17}$$

this can be rewritten as

$$\Psi_{\underline{k}}^{(L)}=X_L\varphi_{\underline{k}}^{(L)} \tag{3.18}$$

for all $\underline{k}\in\mathbb{Z}^2$. This is exactly the analogous of Eq. (2.4). The operator X_L is, for $L>1$, bounded and self-adjoint. Indeed we have

$$\|X_L\|\leq \sum_{\underline{k}\in\mathbb{Z}^2}|c_{\underline{k}}^{(L)}|\|T_1^{k_1}\|\|T_2^{k_2}\|=\sum_{\underline{k}\in\mathbb{Z}^2}|c_{\underline{k}}^{(L)}|<\infty$$

again because of the asymptotic behavior of $c_{\underline{k}}^{(L)}$. Moreover we have, since formula (3.13) implies that $\overline{c_{\underline{k}}^{(L)}}=c_{-\underline{k}}^{(L)}$,

$$X_L^\dagger=\sum_{\underline{k}\in\mathbb{Z}^2}\overline{c_{\underline{k}}^{(L)}}T_1^{k_1\dagger}T_2^{k_2\dagger}=\sum_{\underline{k}\in\mathbb{Z}^2}c_{-\underline{k}}^{(L)}T_1^{-k_1}T_2^{-k_2}=\sum_{\underline{n}\in\mathbb{Z}^2}c_{\underline{n}}^{(L)}T_1^{n_1}T_2^{n_2}=X_L.$$

We will show in the last part of this subsection that X_L admits a bounded inverse, as soon as $L>1$. At this stage we simply assume that this is so: X_L^{-1} exists and belongs to $B(h_L)$, the set of all the bounded operators on h_L. This assumption allows us to prove that the set $\mathcal{E}^{(L)}$ is complete in h_L, just extending the same argument of the previous section. Indeed, let $g\in h_L$ be such that $\langle g,\Psi_{\underline{n}}^{(L)}\rangle=0$ for all $\underline{n}\in\mathbb{Z}^2$. Then we have, $\forall\underline{n}\in\mathbb{Z}^2$, $0=\langle g,X_L\varphi_{\underline{n}}^{(L)}\rangle=\langle X_Lg,\varphi_{\underline{n}}^{(L)}\rangle$. Since the set $\mathcal{C}^{(L)}$ is complete in h_L by construction, then we must have $X_Lg=0$ or, applying X_L^{-1}, $g=0$.

Remark: of course it is necessary to check that $X_Lg\in h_L$ for any $g\in h_L$, but this is a simple exercise and is left to the reader. It is also easy to reverse this statement and to check that, under additional conditions that remind those of Proposition 1, if $\mathcal{E}^{(L)}$ is complete in h_L then the operator X_L must admit a bounded inverse.

Once we have proven that the set $\mathcal{E}^{(L)}$ is complete in h_L we can expand each vector $\varphi_{\underline{n}}^{(L)}$ in terms of the $\Psi_{\underline{n}}^{(L)}$ in a translationally invariant way:

$$\varphi_{\underline{n}}^{(L)}=\sum_{\underline{k}\in\mathbb{Z}^2}\alpha_{\underline{k}}^{(L)}\Psi_{\underline{n}+\underline{k}}^{(L)}. \tag{3.19}$$

As we have already seen in Sec. II, the analysis of these coefficients is, in a sense, much simpler than that of the $c_{\underline{k}}^{(L)}$, since we can here use the Parseval equality because of the orthonormality of the set $\mathcal{E}^{(L)}$. For instance, we have $1=\|\varphi_{\underline{n}}^{(L)}\|^2=\Sigma_{\underline{k}\in\mathbb{Z}^2}|\alpha_{\underline{k}}^{(L)}|^2$, which proves that $\{\alpha_{\underline{k}}^{(L)}\}$ belongs to $l^2(\mathbb{Z}^2)$ for all $L>1$. Moreover, using Eqs. (3.19) and (3.5) (and replacing $I_{\underline{n}}$ with $I_{\underline{n}}^{(L)}$), we find that $I_{\underline{n}}^{(L)}=\langle\varphi_{\underline{n}}^{(L)},\varphi_{\underline{0}}^{(L)}\rangle=\Sigma_{\underline{k},\underline{s}\in\mathbb{Z}^2}\overline{\alpha_{\underline{k}}^{(L)}}\alpha_{\underline{s}}^{(L)}\langle\Psi_{\underline{n}+\underline{k}}^{(L)},\Psi_{\underline{s}}^{(L)}\rangle=\Sigma_{\underline{k}\in\mathbb{Z}^2}\overline{\alpha_{\underline{k}}^{(L)}}\alpha_{\underline{k}+\underline{n}}^{(L)}$. If we now multiply both sides of this equality for $e^{i\underline{P}\cdot\underline{n}}$ and sum up on $\underline{n}\in\mathbb{Z}^2$, we get

$$F_L(\underline{P})=|G_L(\underline{P})|^2, \quad \text{a.e. in } [0,2\pi[\times[0,2\pi[, \tag{3.20}$$

where $F_L(\underline{P})$ has been defined in Eq. (3.12), while

$$G_L(\underline{P})=\sum_{\underline{k}\in\mathbb{Z}^2}\alpha_{\underline{k}}^{(L)}e^{i\underline{P}\cdot\underline{k}}. \tag{3.21}$$

Since $\{\alpha_{\underline{k}}^{(L)}\}\in l^2(\mathbb{Z}^2)$ for all $L>1$, and since $(1/(2\pi)^2)\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2|G_L(\underline{P})|^2=\Sigma_{\underline{k}\in\mathbb{Z}^2}|\alpha_{\underline{k}}^{(L)}|^2$, we see that $G_L(\underline{P})\in\mathcal{L}^2([0,2\pi[\times[0,2\pi[)$. For this reason there is no problem in recovering the coefficients $\alpha_{\underline{k}}^{(L)}$ as usual:

$$\alpha_{\underline{k}}^{(L)}=\frac{1}{(2\pi)^2}\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2G_L(\underline{P})e^{-i\underline{P}\cdot\underline{k}}=\frac{1}{(2\pi)^2}\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2\sqrt{F_L(\underline{P})}e^{-i\underline{P}\cdot\underline{k}},$$

with a particular choice of phase. Of course we can repeat our analysis of the asymptotic behavior of the $\alpha_{\underline{k}}^{(L)}$'s even now: what we get, using the same arguments, is that also the sequence $\{\alpha_{\underline{k}}^{(L)}\}$ decreases to zero for $\|\underline{k}\|$ diverging faster than any inverse power.

Moreover we can also check that the following sum rule is satisfied:

$$\sum_{\underline{k}\in\mathbb{Z}^2}\overline{\alpha_{\underline{k}}^{(L)}}c_{\underline{k}}^{(L)}=1, \tag{3.22}$$

for any $L>1$. The proof of this equation makes use twice of the Poisson summation rule. We have

$$\begin{aligned}\sum_{\underline{k}\in\mathbb{Z}^2}\overline{\alpha_{\underline{k}}^{(L)}}c_{\underline{k}}^{(L)}&=\frac{1}{(2\pi)^4}\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2\int_0^{2\pi}dQ_1\int_0^{2\pi}dQ_2\sqrt{\frac{F_L(\underline{P})}{F_L(\underline{Q})}}\sum_{\underline{k}\in\mathbb{Z}^2}e^{i(\underline{P}-\underline{Q})\cdot\underline{l}}\\&=\frac{1}{(2\pi)^2}\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2\int_0^{2\pi}dQ_1\int_0^{2pi}dQ_2\sqrt{\frac{F_L(\underline{P})}{F_L(\underline{Q})}}\\&\quad\times\sum_{\underline{k}\in\mathbb{Z}^2}\delta(P_1-Q_1-2\pi l_1)\delta(P_2-Q_2-2\pi l_2).\end{aligned}$$

Now, since P_j, $Q_j\in[0,2\pi[$, the two delta functions reduce to $\delta(P_j-Q_j-2\pi l_j)=\delta(P_j-Q_j)\delta_{l_j,0}$, $j=1,2$. Therefore we get

$$\sum_{\underline{k}\in\mathbb{Z}^2}\overline{\alpha_{\underline{k}}^{(L)}}c_{\underline{k}}^{(L)}=\frac{1}{(2\pi)^2}\int_0^{2\pi}dP_1\int_0^{2\pi}dP_2\int_0^{2\pi}dQ_1\int_0^{2i}dQ_2\sqrt{\frac{F_L(\underline{P})}{F_L(\underline{Q})}}\delta(P_1-Q_1)\delta(P_2-Q_2)=1,$$

as we had to prove.

Let us now continue the analysis of the consequences of our orthonormalization procedure considering more in detail the special features of a set of coherent states: which properties of the set $\mathcal{C}^{(L)}$ can still be proven for $\mathcal{E}^{(L)}$?

The first obvious result is that both these sets produce a resolution of the identity: $\Sigma_{\underline{k}\in\mathbb{Z}^2}|\varphi_{\underline{k}}^{(L)}\rangle\langle\varphi_{\underline{k}}^{(L)}|=\Sigma_{\underline{k}\in\mathbb{Z}^2}|\Psi_{\underline{k}}^{(L)}\rangle\langle\Psi_{\underline{k}}^{(L)}|=\mathbb{1}_{h_L}$, where $\mathbb{1}_{h_L}$ is the identity operator on h_L.

Further, let us define the operator $B_L:=X_LbX_L^{-1}$. It is not hard to check that each $\Psi_{\underline{n}}^{(L)}$ belongs to the domain of B_L. More than this, we can check that $\Psi_{\underline{n}}^{(L)}$ is an eigenstate of B_L with eigenvalue $z_{\underline{n}}$. Indeed we have

$$B_L\Psi_{\underline{n}}^{(L)} = (X_L b X_L^{-1})(X_L\varphi_{\underline{n}}^{(L)}) = X_L b\varphi_{\underline{n}}^{(L)} = z_{\underline{n}} X_L\varphi_{\underline{n}}^{(L)} = z_{\underline{n}}\Psi_{\underline{n}}^{(L)}. \tag{3.23}$$

It is easy to compute the commutation rule between B_L and its adjoint. We get $[B_L,B_L^\dagger]=X_L b X_L^{-2} b^\dagger X_L - X_L^{-1} b^\dagger X_L^2 b X_L^{-1}$, which shows that in general B_L is not an annihilation operator. This is not surprising and, actually, cannot be avoided since, if B_L were a bosonic annihilation operator, its eigenstates $\{\Psi_{\underline{n}}^{(L)}\}$ should have surely been not mutually orthogonal.

It is a well known fact that coherent states saturate the Heisenberg uncertainty relation $(\Delta\hat{q})\times(\Delta\hat{p})=\frac{1}{2}$. Indeed we easily find $\Delta\hat{q}=\Delta\hat{p}=\frac{1}{\sqrt{2}}$. We ask here if the same is true also for the vectors $\Psi_{\underline{n}}^{(L)}$. The computation, say, of $\Delta\hat{q}$ is not very hard but surely requires some care and one can check that $(\Delta\hat{q})(\Delta\hat{p})=\frac{1}{2}$ does not hold. This is not surprising, since the position and momentum operators do not play such a central role here as for the *canonical* coherent states. For this reason, it is surely more interesting to introduce a new operator Q_L which mimics $\hat{q}$ in the following sense: since $\hat{q}=(b+b^\dagger)/\sqrt{2}$, and since b has been replaced by B_L in Eq. (3.23), then we put $Q_L=(B_L+B_L^\dagger)/\sqrt{2}$. It is now a trivial computation to check that

$$(\Delta Q_L)^2 = \langle\Psi_{\underline{n}}^{(L)}, Q_L^2\Psi_{\underline{n}}^{(L)}\rangle - \langle\Psi_{\underline{n}}^{(L)}, Q_L\Psi_{\underline{n}}^{(L)}\rangle^2 = \frac{1}{2}(\|B_L^\dagger\Psi_{\underline{n}}^{(L)}\|^2 - |z_{\underline{n}}|^2),$$

which would give 1/2, as in the standard situation, if we had $X_L=\mathbb{1}$. In the same way, putting $P_L=i(B_L^\dagger-B_L)/\sqrt{2}$ in analogy to $\hat{p}=i(b^\dagger-b)/\sqrt{2}$, we find that

$$(\Delta P_L)^2 = \langle\Psi_{\underline{n}}^{(L)}, P_L^2\Psi_{\underline{n}}^{(L)}\rangle - \langle\Psi_{\underline{n}}^{(L)}, P_L\Psi_{\underline{n}}^{(L)}\rangle^2 = \frac{1}{2}(\|B_L^\dagger\Psi_{\underline{n}}^{(L)}\|^2 - |z_{\underline{n}}|^2).$$

Therefore $(\Delta Q_L)(\Delta P_L)=\frac{1}{2}(\|B_L^\dagger\Psi_{\underline{n}}^{(L)}\|^2-|z_{\underline{n}}|^2)$, which is equal to 1/2 if $X_L=\mathbb{1}$ but not in general. This is in agreement with the fact that $[Q_L,P_L]\neq i\mathbb{1}$. Moreover, it is not difficult to check that each $\Psi_{\underline{n}}^{(L)}$ saturates again the Heisenberg uncertainty relation in the sense that, using $[Q_L,P_L]=i[B_L,B_L^\dagger]$, the following equality holds: $\Delta Q_L\cdot\Delta P_L=\frac{1}{2}\langle\Psi_{\underline{n}}^{(L)},[B_L,B_L^\dagger]\Psi_{\underline{n}}^{(L)}\rangle$.

It is now interesting to use our generalized (k,q) representation to deduce, in analogy with Ref. 3, how should a function look like in order to produce, together with its translated, an orthonormal set. In other words, let $\Psi_{\underline{n}}^{(L)}$ be our o.n. set: $\langle\Psi_{\underline{n}}^{(L)},\Psi_{\underline{0}}^{(L)}\rangle=\delta_{\underline{n},\underline{0}}$. Then we have, inserting the identity operator in Eq. (A6), and in analogy with what has been done in Proposition 1,

$$\delta_{\underline{n},\underline{0}} = \int\int_{\square^{(A)}}\langle\Psi_{\underline{n}}^{(L)},\Phi_{k,q}^{(A,a)}\rangle\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle dkdq = \int\int_{\square^{(A)}} \mathrm{e}^{iqan_1-ikAn_2}|\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle|^2 dkdq,$$

which has L different solutions, i.e., all the functions

$$|\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle|^2 = \begin{cases}\dfrac{AL}{2\pi aj} & \text{a.e. for } (k,q)\in\left[0,\dfrac{2\pi}{A}\right[\times\left[0,\dfrac{aj}{L}\right[\\ 0 & \text{otherwise in } \square^{(A)},\end{cases} \tag{3.24}$$

where $j=1,2,\ldots,L$. In particular, if $L=1$, then $j=1$ and if $a=A=\sqrt{2\pi}$ we recover the same result as in Ref. 3: in this case $\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle$ must be a constant times a phase.

Of course, once we fix the form of $\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle$, we can recover the expression of $\Psi_{\underline{0}}^{(L)}$ as a vector in h_L using the following reconstruction formula $\Psi_{\underline{0}}^{(L)}=\int\int_{\square(A)}dkdq\langle\Phi_{k,q}^{(A,a)},\Psi_{\underline{0}}^{(L)}\rangle\Phi_{k,q}^{(A,a)}$. A natural question would be to relate the above solutions of the *ortogonality requirement* as obtained directly using the (k,q) representation with the particular $\Psi_{\underline{0}}^{(L)}$ we have constructed in Eq. (3.7). This will be done elsewhere.

We dedicate the last part of this subsection to some perturbative results concerning our problem starting with an approximated expression for the coefficients $c_{\underline{n}}^{(L)}$ of the expansion (3.7). Since $F_L(\underline{P})=1+F_L^o(\underline{P})$, with $F_L^o(\underline{P})=\sum_{\underline{m}\in\mathbb{Z}^2\setminus(0,0)}(-1)^{Lm_1m_2}\mathrm{e}^{-(\pi/2)L(m_1^2+m_2^2)}\mathrm{e}^{i\underline{P}\cdot\underline{m}}$, Eq. (3.13) can be rewritten as follows:

$$c_{\underline{k}}^{(L)} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{-i\underline{P}\cdot\underline{k}}}{\sqrt{1+F_L^o(\underline{P})}}d\underline{P} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\underline{P}\cdot\underline{k}}\left(1-\frac{1}{2}F_L^o(\underline{P})+\frac{3}{8}F_L^o(\underline{P})^2+\cdots\right)d\underline{P}.$$

Considering only the first two contributions of this expansion we easily get

$$c_{\underline{k}}^{(L)} \simeq \delta_{\underline{k},\underline{0}} - \frac{1}{2}(1-\delta_{\underline{k},\underline{0}})(-1)^{Lk_1k_2}\mathrm{e}^{-(\pi/2)L(k_1^2+k_2^2)}. \tag{3.25}$$

Of course, in order for this approximation to be meaningful, we further need to restrict ourselves to those $\underline{k}$ such that $\underline{k}=(\pm 1,0),(0,\pm 1)$. In fact, a contribution like $\underline{k}=(\pm 1,\pm 1)$ can only be considered in the expansion above if we also keep into account those contributions arising from $\frac{3}{8}F_L^o(\underline{P})^2$, which contains terms of the same order. On the contrary, all these contributions will be neglected here. Nevertheless we will see that this apparently rude approximation already produces very good results. If we introduce the following subset of $\mathbb{Z}^2$, $\Gamma:=\{(1,0),(-1,0),(0,1),(0,-1)\}$, then we get the following expression for $\Psi_{\underline{n}}^{(L)}$:

$$\Psi_{\underline{n}}^{(L)} \simeq \varphi_{\underline{n}}^{(L)} - \frac{1}{2}\mathrm{e}^{-\pi/2L}\sum_{\underline{s}\in\Gamma}\varphi_{\underline{n}+\underline{s}}^{(L)}. \tag{3.26}$$

It is easy to check now that the set of the approximated vectors $\Psi_{\underline{n}}^{(L)}$ obtained in this way are mutually orthogonal and normalized with a very good approximation already for $L=2$. Indeed we find, first of all,

$$\|\Psi_{\underline{0}}^{(L)}\|^2 \simeq 1-3\mathrm{e}^{-\pi L} = \begin{cases} 0.994\,40 & \text{if } L=2\\ 0.999\,76 & \text{if } L=3\\ 0.999\,99 & \text{if } L=4,\end{cases}$$

and so on. Of course, since $\Psi_{\underline{n}}^{(L)}=T_1^{n_1}T_2^{n_2}\Psi_{\underline{0}}^{(L)}$, the same norms are obtained for $\|\Psi_{\underline{n}}^{(L)}\|^2$, $\forall\underline{n}\in\mathbb{Z}^2$. Moreover, if we compute the overlap between two neighboring vectors, for instance, between $\Psi_{\underline{0}}^{(L)}$ and $\Psi_{(1,0)}^{(L)}$, we find that

$$|\langle\Psi_{\underline{0}}^{(L)},\Psi_{(1,0)}^{(L)}\rangle| \simeq \begin{cases} 0.000\,16 & \text{if } L=2\\ 0.000\,001 & \text{if } L=3,\end{cases}$$

and so on. We see that the approximation considered here, which as we have already remarked looks quite rude, allows to recover normalization and orthogonalization of the vectors with a meaningless error already for $L=2$, i.e., for $a^2=4\pi$. Therefore, we can safely claim that in this way we get a rather good approximation.

As for the operators X_L and X_L^{-1} we find that $X_L \simeq \mathbb{1}-\frac{1}{2}\mathrm{e}^{-\pi/2L}\Sigma_{\underline{s}\in\Gamma}T_1^{s_1}T_2^{s_2}$ and $X_L^{-1}\simeq\mathbb{1}+\frac{1}{2}\mathrm{e}^{-\pi/2L}\Sigma_{\underline{s}\in\Gamma}T_1^{s_1}T_2^{s_2}$ or, more explicitly,

$$X_L \simeq \mathbb{1} - \frac{1}{2}\mathrm{e}^{-\pi/2L}K_L,$$
$$X_L^{-1} \simeq \mathbb{1} + \frac{1}{2}\mathrm{e}^{-\pi/2L}K_L, \tag{3.27}$$

where

$$K_L = T_1 + T_1^{-1} + T_2 + T_2^{-1}.$$

In order to check that X_L^{-1} above is a good approximation of the inverse of X_L it is enough to observe that

$$\|X_LX_L^{-1}-\mathbb{1}\|=\|X_L^{-1}X_L-\mathbb{1}\|\leqslant 4\mathrm{e}^{-\pi L}=\begin{cases}0.00747 & \text{if } L=2\\ 0.00032 & \text{if } L=3,\end{cases}$$

and so on.

Remark: From the above estimates it is clear that the only *dangerous* case is $L=1$, which, in fact, has not even be considered. Just as an example, if $L=1$ then we can only prove that $\|X_1X_1^{-1}-\mathbb{1}\|\leqslant 0.172\,85$, which is surely not enough to claim that X_1^{-1} as given in Eq. (3.27) can really be interpreted as the inverse of X_1. We will go back on the situation for $L=1$ shortly.

Using the expansion (3.27) it is finally possible to derive an approximated version for B_L, which looks now as $B_L\simeq b-\frac{1}{2}\mathrm{e}^{-\pi/2L}[K_L,b]$, so that we get

$$[B_L,B_L^\dagger]\simeq \mathbb{1}-\frac{1}{2}\mathrm{e}^{-\pi/2L}([[K_L,b],b^\dagger]+[b,[b^\dagger,K_L]]),$$

which *converges* toward the identity operator as L diverges, as expected.

Remark: These o.n. vectors can be used to define to define certain traces on the von Neumann algebra $\mathcal{M}_L=\mathcal{B}(h_L)$. Let $\mathcal{M}_L^+$ be the positive part of $\mathcal{M}_L$. Then, if we put $\omega_L(X)=\sum_{\underline{n}\in\mathbb{Z}^2}\langle\Psi_{\underline{n}}^{(L)},X\Psi_{\underline{n}}^{(L)}\rangle$, for $X\in\mathcal{M}_L^+$ and $L>1$, this is a faithful normal trace on $\mathcal{M}_L^+$.

To prove this claim we start noticing that ω_L is linear. Moreover, since $\omega_L(X)$ is a sum of only non-negative terms, the summation and the supremum can be interchanged so that the normality of ω_L follows from that of each $\langle\Psi_{\underline{n}}^{(L)},\Psi_{\underline{n}}^{(L)}\rangle$.

We now prove that $\omega_L(X^*X)=\omega_L(XX^*)$, $X\in\mathcal{M}_L^+$. Indeed we have

$$\omega_L(X^*X)=\sum_{\underline{n}\in\mathbb{Z}^2}\langle\Psi_{\underline{n}}^{(L)},X^*X\Psi_{\underline{n}}^{(L)}\rangle=\sum_{\underline{n}\in\mathbb{Z}^2}\|X\Psi_{\underline{n}}^{(L)}\|^2=\sum_{\underline{k}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}|\langle X\Psi_{\underline{n}}^{(L)},\Psi_{\underline{k}}^{(L)}\rangle|$$
$$=\sum_{\underline{k}\in\mathbb{Z}^2}\sum_{\underline{n}\in\mathbb{Z}^2}|\langle X^*\Psi_{\underline{k}}^{(L)},\Psi_{\underline{n}}^{(L)}\rangle|=\sum_{\underline{k}\in\mathbb{Z}^2}\|X^*\Psi_{\underline{k}}^{(L)}\|^2=\sum_{\underline{k}\in\mathbb{Z}^2}\langle\Psi_{\underline{k}}^{(L)},XX^*\Psi_{\underline{k}}^{(L)}\rangle=\omega_L(XX^*).$$

Moreover, let us suppose that $0=\omega_L(X)=\sum_{\underline{n}\in\mathbb{Z}^2}\|X^{1/2}\Psi_{\underline{n}}^{(L)}\|^2$, $X\in\mathcal{M}_L^+$. Therefore $X=0$, which implies that ω_L is faithful.

It is finally clear that these considerations can be extended with no particular difficulty to the general settings introduced in Sec. II, but this extension will not be repeated here.

B. The case $L=1$

We have already noticed that, if $L=1$, the perturbation results stated above are likely not to work as we would like. This claim can be actually proven by the following *reductio ad absurdum* argument. Suppose that the same procedure discussed previously also works for $L=1$, so that an o.n. set $\{\Psi_{\underline{n}}^{(1)}\}$ can be constructed in $h_1=\mathcal{H}$. Let S be the following operator: $Sf=\sum_{\underline{n}\in\mathbb{Z}^2}\langle\Psi_{\underline{n}}^{(1)},f\rangle\Psi_{\underline{n}}^{(1)}$. It is possible to check that $S=\sum_{\underline{l},\underline{s}\in\mathbb{Z}^2}c_{\underline{l}}^{(1)}\overline{c_{\underline{s}}^{(1)}}T_1^{l_1-s_1}T_2^{l_2-s_2}=X_1^2$, see Eq. (3.17). Indeed using definition (3.7), since S is bounded and therefore continuous we have, $\forall f,g\in\mathcal{H}$,

$$\langle f,Sg\rangle=\sum_{\underline{n}\in\mathbb{Z}^2}\langle f,\Psi_{\underline{n}}^{(1)}\rangle\langle\Psi_{\underline{n}}^{(1)},g\rangle=\sum_{\underline{n},\underline{l},\underline{s}\in\mathbb{Z}^2}c_{\underline{l}}^{(1)}\overline{c_{\underline{s}}^{(1)}}\langle f,\varphi_{\underline{l}+\underline{n}}^{(1)}\rangle\langle\varphi_{\underline{s}+\underline{n}}^{(1)},g\rangle$$
$$=\sum_{\underline{n},\underline{l},\underline{s}\in\mathbb{Z}^2}c_{\underline{l}}^{(1)}\overline{c_{\underline{s}}^{(1)}}\langle T_1^{-l_1}T_2^{-l_2}f,\varphi_{\underline{n}}^{(1)}\rangle\langle\varphi_{\underline{n}}^{(1)},T_1^{s_1}T_2^{s_2}g\rangle=\sum_{\underline{l},\underline{s}\in\mathbb{Z}^2}c_{\underline{l}}^{(1)}\overline{c_{\underline{s}}^{(1)}}\langle T_1^{-l_1}T_2^{-l_2}f,T_1^{s_1}T_2^{s_2}g\rangle,$$

since $\{\varphi_{\underline{n}}^{(1)}\}$ is complete in $\mathcal{H}$. Therefore $S=\sum_{\underline{l},\underline{s}\in\mathbb{Z}^2}c_{\underline{l}}^{(1)}\overline{c_{\underline{s}}^{(1)}}T_1^{l_1-s_1}T_2^{l_2-s_2}$ and, due to Eq. (3.17), $S=X_1^2$. Now, since $\Psi_{\underline{n}}^{(1)}=X_1\varphi_{\underline{n}}^{(1)}$, we have

$$\delta_{\underline{n},\underline{0}} = \langle \Psi_{\underline{n}}^{(1)}, \Psi_{\underline{0}}^{(1)} \rangle = \langle X_1 \varphi_{\underline{n}}^{(1)}, X_1 \varphi_{\underline{0}}^{(1)} \rangle = \langle S \varphi_{\underline{n}}^{(1)}, \varphi_{\underline{0}}^{(1)} \rangle.$$

Of course, if the set $\mathcal{E}^{(1)}$ were complete, then we should have $S=\mathbb{1}$, which, as the above equality shows, would also imply that $\langle \varphi_{\underline{n}}^{(1)}, \varphi_{\underline{0}}^{(1)} \rangle = \delta_{\underline{n},\underline{0}}$, which is false. Therefore the same procedure developed for $L>1$ cannot work for $L=1$.

IV. MORE DIFFICULTIES AND OUTCOME

It is very easy to imagine how to extend the procedure described so far to $\mathcal{N}_N$ for $N>2$, at least if the different unitary operators commute as for coherent states. More difficult and still under consideration is the situation when the various A_j's do not commute. In this case, which is a relevant case, there is still work to do. We want to close the paper with a couple of such examples and the difficulties which arise in this case.

The first example we want to mention generalizes that of coherent states in the following way: the two unitary operators $T_1=\mathrm{e}^{ia\hat{q}}$ and $T_2=\mathrm{e}^{-ia\hat{p}}$ in Eq. (3.1) are now supposed to satisfy $a^2 \neq 2\pi L$, for any $L \in \mathbb{Z}$, so that $[T_1,T_2] \neq 0$. However, the two operators can be commuted paying the price of adding a phase: $T_1T_2=T_2T_1\mathrm{e}^{ia^2}$, and therefore $T_1^{n_1}T_2^{n_2}=T_2^{n_2}T_1^{n_1}\mathrm{e}^{ia^2 n_1 n_2}$ for all integers n_1 and n_2. We can think of repeating the same procedure, so that we put $f_{\underline{n}}(x)=T_1^{n_1}T_2^{n_2}f_{\underline{0}}(x)$, for a fixed function $f_{\underline{0}}(x)$ in $\mathcal{L}^2(\mathbb{R})$, and then, if $\langle f_{\underline{n}}, f_{\underline{k}} \rangle \neq \delta_{\underline{n},\underline{k}}$, we define a new function $\varphi_{\underline{0}}(x)$ as the usual linear combination of the $f_{\underline{n}}(x)$: $\varphi_{\underline{0}}(x)=\sum_{\underline{k}\in\mathbb{Z}^2} c_{\underline{k}} f_{\underline{k}}(x)$. We try to fix the expression of the coefficients $c_{\underline{n}}$ by the usual orthonormalization requirement: $\langle \varphi_{\underline{n}}, \varphi_{\underline{0}} \rangle = \delta_{\underline{n},\underline{0}}$, where $\varphi_{\underline{n}}=T_1^{n_1}T_2^{n_2}\varphi_{\underline{0}}$. The difficulty now arises: Eq. (3.10) must now be replaced by the following equation:

$$\langle \varphi_{\underline{n}}, \varphi_{\underline{0}} \rangle = \sum_{\underline{k},\underline{l}\in\mathbb{Z}^2} \overline{c_{\underline{k}}} c_{\underline{l}} I_{\underline{n}+\underline{k}-\underline{l}} \mathrm{e}^{ia^2((n_1-l_1)l_2-(n_2-l_2)k_1)} = \delta_{\underline{n},\underline{0}}.$$

This is a system of equations, one for each value of $\underline{n} \in \mathbb{Z}^2$, which cannot be solved with the same strategy adopted to solve Eq. (3.10) because of the presence of the phase $\mathrm{e}^{ia^2(\cdots)}$ which makes it impossible to separate the contributions arising from the $c_{\underline{n}}$ from those arising from $I_{\underline{n}}$.

The same difficulties also arise in a different context, i.e., when applying this procedure to a family of nonorthogonal wavelets. More in detail, let T and D be the usual translation and dilation operators acting on a general function $f(x) \in \mathcal{L}^2(\mathbb{R})$ as follows: $(Tf)(x)=f(x-1)$, $(Df)(x)=\sqrt{2}f(2x)$. This means, first of all, that $TD=DT^2$. Let now $f_{\underline{0}}(x)$ be a fixed function normalized in $\mathcal{L}^2(\mathbb{R})$ and suppose that the various functions $f_{\underline{l}}(x)=D^{l_1}T^{l_2}f_{\underline{0}}(x)$ are not mutually orthogonal. We can define a new square integrable function $\varphi_{\underline{0}}(x)=\sum_{k\in\mathbb{Z}} c_{\underline{k}} f_{\underline{k}}(x)$ and, from this, $\varphi_{\underline{n}}(x)=D^{n_1}T^{n_2}\varphi_{\underline{0}}(x)$, $\underline{n}\in\mathbb{Z}^2$. The main idea is the usual one: we try to fix the coefficients of the expansion, $c_{\underline{n}}$, by requiring that $\langle \varphi_{\underline{n}}, \varphi_{\underline{0}} \rangle = \delta_{\underline{n},\underline{0}}$. Again, this procedure does not seem to work properly since, even if we can find an infinite number of equations involving the $c_{\underline{n}}$'s, again we are not able to solve *easily* this system.

The conclusion of this short analysis suggests that our procedure, which works very well when the unitary operators in Eq. (2.1) commute, should be properly generalized when these operators do not commute. This is exactly our future task and we hope to be able to solve this problem shortly.

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APPENDIX: GENERALIZED kq REPRESENTATION

The relevance of the kq representation in many-body physics has been established since its first appearances.[6] What was originally a physical tool has became, during the years, also a mathematical interesting object, widely analyzed in the literature, see Refs. 7 and 8 for instance. We give here only few definitions and refer to Refs. 6, 8, 9, and 3 for further reading and for applications.

The origin of the kq representation consists in the well known possibility of a simultaneous diagonalization of two commuting operators. In Ref. 9 it is shown that the following distributions

$$\psi_{kq}(x) = \sqrt{\frac{2\pi}{a}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{ikna}\delta(x-q-na), \quad k\in[0\ ,a[\ , \quad q\in\left[0,\frac{2\pi}{a}\right[\tag{A1}$$

are (generalized) eigenstates of both $T(a)=\mathrm{e}^{ipa}$ and $\tau(2\pi/a)=\mathrm{e}^{ix2\pi/a}$. Here a is a positive real number which plays the role of a lattice spacing.

As discussed in Ref. 9, these $\psi_{kq}(x)$ are Bloch-like functions corresponding to infinitely localized Wannier functions. They also satisfy orthogonality and closure properties. This implies that, roughly speaking, they can be used to define a new representation of the wave functions by means of the integral transform $Z:\mathcal{L}^2(\mathbb{R})\to\mathcal{L}^2(\square)$, where $\square=[0,a[\times[0,2\pi/a[$, defined as follows:

$$h(k,q) \coloneqq (ZH)(k,q) \coloneqq \int_{\mathbb{R}} d\omega\overline{\psi_{kq}(\omega)}H(\omega), \tag{A2}$$

for all functions $H(\omega)\in\mathcal{L}^2(\mathbb{R})$. The result is a function $h(k,q)\in\mathcal{L}^2(\square)$.

To be more rigorous, Z should be defined first on the functions of $\mathcal{C}_o^\infty(\mathbb{R})$ and then extended to $\mathcal{L}^2(\mathbb{R})$ using its continuity,[8] In this way it is possible to give a rigorous meaning to formula (A2) above. In most applications,[10] the lattice spacing a is chosen as $a^2=2\pi$. Here we are interested in a more general situation: we need to consider a different lattice with rectangular lattice cells with surface $2\pi L$, $L=1,2,3,\ldots$.

Let therefore $T(a)=\mathrm{e}^{i\hat{p}a}$ and $\tau(b)=\mathrm{e}^{i\hat{q}b}$, with $ab=2\pi L$, for some natural L. It is clear that for all possible $L\in\mathbb{N}$ the two operators still commute: $[T(a),\tau(b)]=0$. For each given $A>0$ let us define the set of (generalized) functions,

$$\Phi_{k,q}^{(A,a)}(x) = \sqrt{\frac{A}{2\pi}}\sum_{l\in\mathbb{Z}} \mathrm{e}^{iklA}\delta(x-q-la), \tag{A3}$$

where $(k,q)\in\square^{(A)}\coloneqq[0,2\pi/A[\times[0,a[$. If ξ_x is the generalized eigenvector of the position operator $\hat{q}$, $\hat{q}\xi_x=x\xi_x$, we write $\Phi_{k,q}^{(A,a)}(x)$ as $\Phi_{k,q}^{(A,a)}(x)=\langle\xi_x,\Phi_{k,q}^{(A,a)}\rangle$.

It is not hard to prove the following statements:

$$T(a)\Phi_{k,q}^{(A,a)}(x) = \mathrm{e}^{ikA}\Phi_{k,q}^{(A,a)}(x), \quad \tau(b)\Phi_{k,q}^{(A,a)}(x) = \mathrm{e}^{iqb}\Phi_{k,q}^{(A,a)}(x), \tag{A4}$$

$$\int\int_{\square^{(A)}} \overline{\Phi_{k,q}^{(A,a)}(x)}\Phi_{k,q}^{(A,a)}(x')dkdq = \delta(x-x'), \tag{A5}$$

$$\int\int_{\square^{(A)}} |\Phi_{k,q}^{(A,a)}\rangle\langle\Phi_{k,q}^{(A,a)}| = \mathbb{1}, \tag{A6}$$

where the usual Dirac bra-ket notation has been adopted;

$$\int_{\mathbb{R}} \overline{\Phi_{k,q}^{(A,a)}(x)}\Phi_{k',q'}^{(A,a)}(x)dx = \delta(k-k')\delta(q-q'). \tag{A7}$$

The proof of these statements does not differ significantly from the standard one, and will be omitted here. We just want to remark that, for general a and a', we find that $T(a)\Phi_{k,q}^{(A,a')} \neq \mathrm{e}^{ikA}\Phi_{k,q}^{(A,a')}(x)$. In other words, in general $\Phi_{k,q}^{(A,a')}(x)$ is not an eigenstate of $T(a)$ if $a\neq a'$.

Also, it should be noticed that the value of the parameter b entering in the definition of $\tau(b)$ is fixed by requiring that T and τ commute but play no role in the definition of the lattice cell $\square^{(A)}$, which on the other way is defined by an extra positive parameter, A, which needs not to be related to b itself. However, quite often in applications A coincides with a and with b.

[1] I. Daubechies, *Ten Lectures on Wavelets* (Society for Industrial and Applied Mathematics, Philadelphia, 1992).
[2] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations* (Springer-Verlag, New York, 2000).
[3] H. Bacry, A. Grossmann, and J. Zak, Phys. Rev. B **12**, 1118 (1975).
[4] J. R. Klauder and B. S. Skagerstam, *Coherent States: Applications to Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[5] F. Bagarello, G. Morchio, and F. Strocchi, Phys. Rev. B **48**, 5306 (1993).
[6] J. Zak, Phys. Rev. **168**, 686 (1968).
[7] I. Daubechies, A. Grossmann, and Y. Meyer, J. Math. Phys. **27**, 1271 (1986).
[8] A. J. E. M. Janssen, J. Math. Phys. **23**, 720 (1982).
[9] J. Zak, *The kq-representation in the Dynamics of Electrons in Solids*, Solid State Physics Vol. 27, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1972).
[10] J. P. Antoine and F. Bagarello, in *Advances in Gabor Analysis*, edited by H. G. Feichtinger and T. Strohmer (Birkhäuser, Boston, 2003), pp. 223–258.