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Faithfully representable topological *-algebras: some spectral properties

Abstract: A faithfully representable topological *-algebra (FR*-algebra) $\mathfrak{A}_0$ is characterized by the fact that it possesses sufficiently many *-representations. Some spectral properties are examined, by constructing a convenient quasi *-algebra $\mathfrak{A}$ over $\mathfrak{A}_0$, starting from the order bounded elements of $\mathfrak{A}_0$.

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Dedicated to the memory of Prof. Anastasios Mallios

1 Introduction

This note is devoted to a particular family of topological *-algebras that share with C*-algebras a crucial feature: they possess *sufficiently many* *-representations. Throughout this paper, we term *topological *-algebra* a *-algebra $\mathfrak{A}_0$ equipped with a locally convex topology $\mathfrak{t}$ such that, for each $a \in \mathfrak{A}_0$, the mappings $x \mapsto ax$, $a \mapsto xa$ and the involution * are continuous in $\mathfrak{A}_0[\mathfrak{t}]$. A **-representation* of $\mathfrak{A}_0$ is a *-homomorphism of $\mathfrak{A}_0$ into the *-algebra $\mathcal{L}^\dagger(\mathcal{D})$ of all weakly continuous endomorphisms of a pre-Hilbert space $\mathcal{D}$. In (16) one of us considered the case where $\mathfrak{A}_0$ is *faithfully representable*, for short an FR*-algebra, in the sense that, for every nonzero element $a \in \mathfrak{A}_0$, there exists a continuous (in a sense that will be specified later) *-representation π which does not vanish at a. The main scope of that paper was to identify elements of $\mathfrak{A}_0$ that are mapped into bounded operators by any *-representation of $\mathfrak{A}_0$ and for this reason they can deserve the name of *bounded elements* of $\mathfrak{A}_0$.

Bounded elements of topological *-algebras have a long story. This notion was first introduced by Allan (1) whose goal was to develop a spectral theory for topological *-algebras. The need of introducing bounded elements was suggested by the successful spectral analysis for closed operators in Hilbert space $\mathcal{H}$: a complex number λ is in the spectrum $\sigma(T)$ of a closed (possibly, unbounded) operator T if $T - \lambda I$ has no inverse in the *-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators. Allan's definition sounds as follows: an element x of the topological *-algebra $\mathfrak{A}_0[\mathfrak{t}]$ is *Allan bounded* if there exists $\lambda \neq 0$ such that the set $\{(\lambda^{-1}x)^n;\, n = 1, 2, \dots\}$ is a bounded subset of $\mathfrak{A}_0[\mathfrak{t}]$. Of course one would expect that Allan bounded elements of some topological *-algebra of (possibly) unbounded operators (e.g., an O*-algebra) are its bounded operators; but this is not the case as we shall see. Thus Allan bounded elements need not be realized by bounded operators in some *-representation of $\mathfrak{A}_0$ in Hilbert space, as it would be natural to expect.

Bounded elements in purely algebraic terms have been considered by Vidav (18) and Schmüdgen (11) with respect to some (positive) wedge. This purely algebraic definition can be extended by considering as strongly positive elements those belonging to the $\mathfrak{t}$-closure in $\mathfrak{A}_0$ of the, say, algebraic cone of positive elements of a *-algebra. The main result is that *order bounded* elements, as we call them, allow equivalent characterizations in terms of continuous positive linear functionals and also in terms of *-representations, that, if the positive wedge is a cone, are sufficiently many to separate points of $\mathfrak{A}_0$.

Of course, several other possibilities for defining bounded elements can be considered. For instance, one may say that x is *left* $\mathfrak{t}$-*bounded*, if there exists $\gamma_x > 0$ such that

$$p_\alpha(xy) \leq \gamma_x p_\alpha(y), \quad \forall \alpha \in \Delta;\ \forall y \in \mathfrak{A}_0,$$

where $\{p_\alpha; \alpha \in \Delta\}$ is a directed family of seminorms defining the topology $\mathfrak{t}$ of $\mathfrak{A}_0$ (6); or *spectrally bounded* if its spectrum is a bounded subset of the complex plane. Moreover some attempts to extend this notion to the larger set-up of locally convex quasi *-algebras (12; 14; 15) or locally convex partial *-algebras (4; 5) has been done. But in these cases, Allan's notion cannot be adopted, since powers of a given element x need not be defined.

The class of FR*-algebras considered here enjoys a series of nice properties. In particular, order bounded elements are exactly those realized by bounded operators by *any* continuous *-representation of $\mathfrak{A}_0$ in Hilbert space. In (16), a study of some spectral properties of an FR*-algebra was undertaken, under the additional assumption that the set of bounded elements, which is often a C*-normed algebra, is in fact a C*-algebra. This assumption is quite strong and not fulfilled in general by topological unbounded operator algebras, which are in a sense the model to which the whole study is aimed to. Removing this assumption requires to invoke the theory of partial *-algebras or quasi *-algebras (3) on which some similar studies have been performed in a recent past (2; 5; 8).

The paper is organized as follows. After some preliminaries (Section 2), we review in Section 3 the main properties of FR-algebras we will need in the sequel. In Section 4 we revisit the spectral analysis begun in (16) by considering the topological *-algebra $\mathfrak{A}_0$ where we started from as a subspace of a conveniently constructed topological quasi *-algebra $\mathfrak{A}$. With this approach, we can avoid the assumption that bounded elements of $\mathfrak{A}_0$ constitute a C*-algebra contained in $\mathfrak{A}_0$, but, as a drawback, we are forced to go beyond the original framework of topological *-algebras. This is not so surprising if we consider that a spectral theory for unbounded operator algebras cannot be successfully formulated within the same algebra, without dismissing the traditional spectral theory of closed (or closable) operators, considered as single objects.

Topological algebras have been, as well-known, the main research subject of the late Prof. Anastasios Mallios. His monography (9) remains one of the cornerstones of the literature on this topic. One of the authors of this note (C.T.) had many occasions of discussing Mathematics with him in Athens and serves both a human and scientific indelible memory of him.

2 Preliminaries and basic results

The following preliminary definitions will be needed in the sequel. For more details we refer to (3; 10).

2.1 Basics

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$, $D(X^*) \supseteq \mathcal{D}$ and by $\mathcal{L}^\dagger(\mathcal{D})$ the subspace consisting of all operators $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $X\mathcal{D} \subset \mathcal{D}$, $X^*\mathcal{D} \subset \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the ordinary operations of addition, multiplication by scalars, multiplication (defined as composition of maps) and involution $X \mapsto X^\dagger := X^* \upharpoonright_\mathcal{D}$. We denote by $I_\mathcal{D}$ the identity operator in $\mathcal{D}$. Clearly, $I_\mathcal{D}$ is the unit of $\mathcal{L}^\dagger(\mathcal{D})$. A *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$, containing the identity $I_\mathcal{D}$, is called an *O^*-algebra* (10).

Let $\mathfrak{M}$ be a †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ (i.e., an O*-family on $\mathcal{D}$), containing the identity $I_\mathcal{D}$. The space $\mathcal{D}$ endowed with the graph topology $t_\mathfrak{M}$, defined by the seminorms

$$\xi \in \mathcal{D} \to \|X\xi\|, \quad X \in \mathfrak{M},$$

will be denoted by $\mathcal{D}_\mathfrak{M}$. If $\mathfrak{M}$ is a closed O*-family, the collection of bounded subsets of $\mathcal{D}[t_\mathfrak{M}]$ is the same of that corresponding to $t_{\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})}$ and it is denoted by $\mathcal{B}$.

A (closed) O*-family $\mathfrak{M}$ on $\mathcal{D}$ can be endowed with some *traditional* operator topologies:

- the *weak operator topology* τ_w defined by the seminorms

$$X \in \mathfrak{M} \to p_{\xi,\eta}(X) := |\langle X\xi|\eta\rangle|, \quad \xi,\eta \in \mathcal{D};$$

- the *strong operator topology* τ_s defined by the seminorms

$$X \in \mathfrak{M} \to p_\xi(X) := \|X\xi\|, \quad \xi \in \mathcal{D};$$

– the *strong* operator topology* τ_{s^*} defined by the seminorms

$$X \in \mathfrak{M} \to p_\xi^*(X) := \max\{\|X\xi\|, \|X^\dagger \xi\|\}, \quad \xi \in \mathcal{D},$$

that mimick the corresponding topologies of bounded operator algebras.

The following *uniform* topologies τ_u , τ^u and τ_*^u generalize the norm of $\mathcal{B}(\mathcal{H})$.

– τ_u defined by the seminorms

$$p_{\mathcal{M}}(X) = \sup_{\xi,\eta \in \mathcal{M}} |\langle X\xi|\eta\rangle|; \quad \mathcal{M} \in \mathcal{B};$$

– τ^u defined by the seminorms

$$p^{\mathcal{M}}(X) = \sup_{\xi \in \mathcal{M}} \|X\xi\|; \quad \mathcal{M} \in \mathcal{B}$$

– τ_*^u defined by the seminorms

$$p_*^{\mathcal{M}}(X) = \max\{p^{\mathcal{M}}(X), p^{\mathcal{M}}(X^\dagger)\}; \quad \mathcal{M} \in \mathcal{B}.$$

It is easily seen that $\mathfrak{M}[\tau_w]$ and $\mathfrak{M}[\tau_u]$ are topological *-algebras.

We list here some easy properties of the topologies defined above. For every bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$, one has

– there exists $\gamma_{\mathcal{M}} > 0$ such that $p_{\mathcal{M}}(X) \le \gamma_{\mathcal{M}}\, p^{\mathcal{M}}(X), \quad \forall X \in \mathfrak{M}$.
– $p_{\mathcal{M}}(X^\dagger X) = p^{\mathcal{M}}(X)^2, \quad \forall X \in \mathfrak{M}$.

As shown in (3, Prop. 4.2.3) $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\tau_*^u]$ is complete. It is also complete when endowed with the topology τ_{s^*}.

Let $\mathfrak{A}_0$ be a *-algebra and $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$. A linear map π from $\mathfrak{A}_0$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi)$ such that:

(i) $\pi(a^*) = \pi(a)^\dagger, \quad \forall a \in \mathfrak{A}_0$,

(ii) if $a, b \in \mathfrak{A}_0$, then $\pi(ab) = \pi(a)\pi(b)$,

is called a **-representation* of $\mathfrak{A}_0$. Moreover, if $\mathfrak{A}_0$ has a unit $e \in \mathfrak{A}_0$, we assume $\pi(e) = I_{\mathcal{D}_\pi}$, the identity of $\mathcal{D}_\pi$.

A *-representation π of a topological *-algebra $\mathfrak{A}_0[\mathfrak{t}]$ is said to be a $(\mathfrak{t}, \tau_{\mathrm{w}})$-continuous if, for every $\xi, \eta \in \mathcal{D}_\pi$, there exists a $\mathfrak{t}$-continuous seminorm p on $\mathfrak{A}_0$ such that

$$|\langle \pi(a)\xi|\eta\rangle| \le p(a), \quad \forall a \in \mathfrak{A}_0.$$

A linear functional ω on $\mathfrak{A}_0$ is called positive if $\omega(a^*a) \geq 0$, for every $a \in \mathfrak{A}_0$. To every positive linear functional ω on $\mathfrak{A}_0$ there corresponds a Hilbert space $\mathcal{H}_\omega$ and a linear map λ_ω from $\mathfrak{A}_0$ into a dense subspace $\lambda_\omega(\mathfrak{A}_0) \subset \mathcal{H}_\omega$ and a *-representation π_ω acting on a dense domain $\mathcal{D}_{\pi_\omega}$ such that $\lambda_\omega(\mathfrak{A}_0) \subset \mathcal{D}_{\pi_\omega} \subset \mathcal{H}_\pi$ and

$$\omega(b^*xa) = \langle \pi_\omega(x)\lambda_\omega(a)|\lambda_\omega(b)\rangle, \quad \forall a, b, x \in \mathfrak{A}_0.$$

The representation π_ω can be taken to be closed (10). If $\mathfrak{A}_0$ has a unit e, then there exists a vector ξ_ω such that $\lambda_\omega(\mathfrak{A}_0) = \{\pi_\omega(a)\xi_\omega,\ a \in \mathfrak{A}_0\}$ and

$$\omega(x) = \langle \pi_\omega(x)\xi_\omega|\xi_\omega\rangle, \quad \forall x \in \mathfrak{A}_0.$$

We will refer to π_ω as to the GNS *-representation of $\mathfrak{A}_0$ defined by ω.

2.2 Left $\mathfrak{t}$-boundness vs. Allan boundness

It is worth comparing the two notions of boundness already considered. We have the following

Proposition 2.1. *Let $\mathfrak{A}_0[\mathfrak{t}]$ be a topological *-algebra. If $x \in \mathfrak{A}_0[\mathfrak{t}]$ is left $\mathfrak{t}$-bounded, then x is Allan bounded.*

Proof. Let $\{p_\alpha; \alpha \in \Delta\}$ be a directed family of seminorms defining the topology $\mathfrak{t}$, and let us consider a *left* $\mathfrak{t}$-*bounded* element $x \in \mathfrak{A}_0[\mathfrak{t}]$, that is:

$$\exists \gamma_x > 0 : p_\alpha(xy) \leq \gamma_x p_\alpha(y), \forall \alpha \in \Delta, \forall y \in \mathfrak{A}_0.$$

In particular, we can put $y = x^{n-1}$: in this way, we have: $p_\alpha(x^n) \leq \gamma_x p_\alpha(x^{n-1})$, then, iterating, $p_\alpha(\frac{x^n}{\gamma_x^n}) \leq \frac{1}{\gamma_x} p_\alpha(x)$. Hence, the set $\{(\lambda^{-1}x)^n;\ n = 1, 2, \dots\}$ is a bounded subset of $\mathfrak{A}_0[\mathfrak{t}]$ with $\lambda = \gamma_x$. □

The converse is not necessarily true, as shown in the following example.

Example 2.2. Let us consider the topological *-algebra $\mathcal{L}^\dagger(\mathcal{D})[\tau_w]$, where τ_w is the weak operator topology. It is simple to verify that if $X \in \mathcal{L}^\dagger(\mathcal{D})[\tau_w]$ is left τ_w-bounded then X is a bounded operator. In fact, if X is left τ_w-bounded, then there exists $\gamma_x > 0$: $|\langle XY\xi|\eta\rangle| \le \gamma_x|\langle Y\xi|\eta\rangle|$, for all $\xi,\eta \in \mathcal{D}$ and for all $Y \in \mathcal{L}^\dagger(\mathcal{D})$. If we choose $Y = 1$, we have

$$|\langle X\xi|\eta\rangle| \le \gamma_x|\langle \xi|\eta\rangle| \le \gamma_x\|\xi\|\|\eta\|, \quad \xi,\eta \in \mathcal{D}.$$

This implies that X is a bounded operator.

It is clear that if X is an unbounded operator of $\mathcal{L}^\dagger(\mathcal{D})$ such that $X^2 = 0$, then it is Allan bounded (as any nilpotent operator), but, X being unbounded, cannot be left τ_w-bounded. In order to construct an operator $X \in \mathcal{L}^\dagger(\mathcal{D})$ such that $X^2 = 0$ we proceed as follows. Let $\{e_k\}_{k\in\mathbb{N}^+}$ be an ortonormal basis of the separable Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be the following subspace of $\mathcal{H}$:

$$\mathcal{D} := \left\{ f = \sum_{k=1}^{\infty} f_k e_k \in \mathcal{H} : \quad \sum_{k=1}^{\infty} |f_k|^2 k^{2p} < \infty, \quad \forall p \in \mathbb{N} \right\}$$

and define

$$Xf := \sum_k k f_{2k} e_{2k+1}, \quad f \in \mathcal{D}.$$

Then X is an unbounded operator of $\mathcal{L}^\dagger(\mathcal{D})$ and $X^2 f = 0$.

3 FR*-algebras: an overlook

In this section we give a short overview to FR*-algebras, already studied in (16).

Definition 3.1. A topological *-algebra $\mathfrak{A}_0[\mathfrak{t}]$ is called *faithfully representable*, shortly an *FR*-algebra*, if for every $x \in \mathfrak{A}_0\setminus\{0\}$ there exists a $(\mathfrak{t},\tau_{\mathrm{w}})$-continuous *-representation π of $\mathfrak{A}_0$ such that $\pi(x) \neq 0$.

Let $\mathrm{Rep}_c(\mathfrak{A}_0)$ denote the family of all $(\mathfrak{t},\tau_{\mathrm{w}})$-continuous *-representation of $\mathfrak{A}_0$. Then, as shown in (16), the following statements are equivalent:

(i) $\mathfrak{A}_0$ is an FR*-algebra;

(ii) For every $x \in \mathfrak{A}_0\setminus\{0\}$, there exists a $\mathfrak{t}$-continuous positive linear functional ω such that $\omega(x^*x) > 0$.

The set of all $\mathfrak{t}$-continuous positive linear functionals on $\mathfrak{A}_0$ will be denoted by $\mathcal{P}_c(\mathfrak{A}_0)$.

Remark 3.2. It is well known that in a C*-algebra every positive linear functional is continuous. But, apart from C*-algebras, there exist nonnormed topological algebras $\mathfrak{A}_0[\mathfrak{t}]$ that share the same property. For instance, every Fréchet Q-algebra with unit has this property (7, Corollary 17.9). It is easily seen that there is equivalence between the following two statements:

a) Every *-representation is $(\mathfrak{t}, \tau_{\mathrm{w}})$-continuous.

b) Every positive linear functional is $\mathfrak{t}$-continuous.

3.1 Order structure

Let $\mathfrak{A}_0$ be a *-algebra with unit e. We denote by

$$(\mathfrak{A}_0)^+_{\mathrm{alg}} = \left\{ \sum_{k=1}^{n} x_k^* x_k,\ x_k \in \mathfrak{A}_0,\ n \in \mathbb{N} \right\}$$

the set (wedge) of positive elements of $\mathfrak{A}_0$.

If $\mathfrak{A}_0[\mathfrak{t}]$ is a topological *-algebra, *strongly positive* elements of $\mathfrak{A}_0$ are then defined as members of $\overline{(\mathfrak{A}_0)^+_{\mathrm{alg}}}^{\mathfrak{t}}$. We put $(\mathfrak{A}_0)^+ := \overline{(\mathfrak{A}_0)^+_{\mathrm{alg}}}^{\mathfrak{t}}$.

The set $(\mathfrak{A}_0)^+$ is an *m-admissible wedge* in the sense of Schmüdgen (10, Sect. 1.4); i.e.,

(1) $e \in (\mathfrak{A}_0)^+$;

(2) $x + y \in (\mathfrak{A}_0)^+, \quad \forall x, y \in (\mathfrak{A}_0)^+$;

(3) $\lambda x \in (\mathfrak{A}_0)^+, \quad \forall x \in (\mathfrak{A}_0)^+,\ \lambda \geq 0$;

(4) $a^* x a \in (\mathfrak{A}_0)^+, \quad \forall x \in (\mathfrak{A}_0)^+,\ a \in \mathfrak{A}_0$.

The wedge $(\mathfrak{A}_0)^+$ defines an order on the real vector space $(\mathfrak{A}_0)_h = \{x \in \mathfrak{A}_0 : x = x^*\}$ by $x \leq y \Leftrightarrow y - x \in (\mathfrak{A}_0)^+$.

This order can be used to introduce a further notion of *bounded* element.

Let $x \in \mathfrak{A}_0$; put $\Re(x) = \frac{1}{2}(x + x^*)$, $\Im(x) = \frac{1}{2i}(x - x^*)$. Then $\Re(x), \Im(x) \in (\mathfrak{A}_0)_h$ (the set of selfadjoint elements of $\mathfrak{A}_0$) and $x = \Re(x) + i\Im(x)$.

An element $x \in \mathfrak{A}_0$ is called *order bounded* if there exists $\gamma \geq 0$ such that

$$\pm\Re(x) \leq \gamma e; \qquad \pm\Im(x) \leq \gamma e.$$

We denote by $(\mathfrak{A}_0)_b$ the family of order bounded elements. It can be proved that $(\mathfrak{A}_0)_b$ is a *-algebra.

For $x \in (\mathfrak{A}_0)_h$, put

$$\|x\|_b := \inf\{\gamma > 0 : \, -\gamma e \leq x \leq \gamma e\}.$$

Then, $\|\cdot\|_b$ is a seminorm on the real vector space $((\mathfrak{A}_0)_b)_h$ and, if $\mathfrak{A}_0 \cap (-(\mathfrak{A}_0)^+) = \{0\}$, $\|\cdot\|_b$ is a norm on $((\mathfrak{A}_0)_b)_h$.

FR*-algebras can be characterized as follows.

Proposition 3.3. *Let $\mathfrak{A}_0[\mathfrak{t}]$ be a topological *-algebra with unit e. The following statements are equivalent.*

(i) $(\mathfrak{A}_0)^+ \cap (-(\mathfrak{A}_0)^+) = \{0\}$*, i.e.* $(\mathfrak{A}_0)^+$ *is a cone.*

(ii) $\mathcal{P}_c(\mathfrak{A}_0)$ *is sufficient; i.e., for every* $a \in \mathfrak{A}_0 \setminus \{0\}$*, there exists* $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$ *such that* $\omega(a^*a) > 0$.

(iii) $\mathfrak{A}_0[\mathfrak{t}]$ *is an FR*-algebra.*

Remark 3.4. The notion of FR*-algebra looks very much close to that of *-semisimple topological *-algebra. It is convenient to compare the two definitions. We remind that, if $\mathfrak{A}_0[\mathfrak{t}]$ is a topological *-algebra, one can consider the set $R'(\mathfrak{A}_0)$ of all *bounded* topologically irreducible *-representations π on a Hilbert space $\mathcal{H}$ that are continuous from $\mathfrak{A}_0[\mathfrak{t}]$ into $\mathcal{B}(\mathcal{H})[\|\cdot\|]$. The *-radical of $\mathfrak{A}_0$ is then defined by

$$R^*_{\mathfrak{A}_0} := \bigcap\{\mathrm{Ker}(\pi);\, \pi \in R'(\mathfrak{A}_0)\}.$$

A topological *-algebra $\mathfrak{A}_0[\mathfrak{t}]$ is called **-semisimple* if $R^*_{\mathfrak{A}_0} = \{0\}$.

Following a similar path we can consider the set

$$\mathfrak{R}^*_{\mathfrak{A}_0} := \bigcap\{\mathrm{Ker}(\pi);\, \pi \in \mathrm{Rep}_c(\mathfrak{A}_0)\}.$$

Clearly a topological *-algebra is an FR*-algebra if, and only if, $\mathfrak{R}^*_{\mathfrak{A}_0} = \{0\}$.

Since $R'(\mathfrak{A}_0) \subseteq \mathrm{Rep}_c(\mathfrak{A}_0)$, then $\mathfrak{R}^*_{\mathfrak{A}_0} \subseteq R^*_{\mathfrak{A}_0}$. Thus every *-semisimple topological *-algebra is an FR*-algebra. The converse is false in general. Indeed, let $\mathfrak{M}$ be an O*-algebra on a domain $\mathcal{D}$ of Hilbert space $\mathcal{H}$, endowed with the weak operator topology. The identical *-representation $\iota_{\mathfrak{M}}$, $\iota_{\mathfrak{M}} : X \in \mathfrak{M} \to X \in \mathfrak{M}$, is clearly faithful, so that if $\mathfrak{M}$ has no nontrivial invariant subspaces, then $\mathfrak{R}^*_{\mathfrak{M}}$ reduces to $\{0\}$. Thus $\mathfrak{M}$ is an FR*-algebra. This applies in particular to the O*-algebra $\mathfrak{M}(q,p)$ on the Schwartz space $\mathcal{S}(\mathbb{R})$, generated by the operator q of multiplication by the real variable x and the operator $p = -i\frac{d}{dx}$. This algebra does not admit any *bounded* representation; thus the notion of *-semisimplicity becomes meaningless in this case.

Proposition 3.5. *Let $\mathfrak{A}_0[\mathfrak{t}]$ be an* FR**-algebra with unit e.*

Assume that the following condition (P) holds

(P) *If $y \in \mathfrak{A}_0$ and $\omega(y) \geq 0$, for every $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$, then $y \in (\mathfrak{A}_0)^+$.*

Then, for an element $x \in \mathfrak{A}_0$, the following statements are equivalent.
(i) $x \in (\mathfrak{A}_0)^+$;
(ii) $\omega(x) \geq 0$, *for every* $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$
(iii) $\pi(x) \geq 0$, *for every* $\pi \in \mathrm{Rep}_c(\mathfrak{A}_0)$.

Theorem 3.6. *Let $\mathfrak{A}_0[\mathfrak{t}]$ be a topological *-algebra with unit e and assume that condition (P) holds. For $x \in \mathfrak{A}_0$, the following statements are equivalent.*
(i) x is order bounded.
(ii) There exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \quad \forall \omega \in \mathcal{P}_c(\mathfrak{A}_0),\ a \in \mathfrak{A}_0.$$

(iii) There exists $\gamma_x > 0$ such that

$$|\omega(b^*xa)| \leq \gamma_x \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}, \quad \forall \omega \in \mathcal{P}_c(\mathfrak{A}_0),\ a, b \in \mathfrak{A}_0.$$

(iv) $\pi(x)$ is a bounded operator, for every $\pi \in \mathrm{Rep}_c(\mathfrak{A}_0)$, and

$$\sup\{\|\overline{\pi(x)}\|,\ \pi \in \mathrm{Rep}_c(\mathfrak{A}_0)\} < \infty.$$

Let x be order bounded and define

$$\mathsf{q}(x) = \sup\{|\omega(b^*xa)|;\ \omega \in \mathcal{P}_c(\mathfrak{A}_0), a, b \in \mathfrak{A}_0; \omega(a^*a) = \omega(b^*b) = 1\}.$$

Then, $\mathsf{q}(x) = \|x\|_b$, for every $x = x^* \in (\mathfrak{A}_0)_b$. Since q extends $\|\cdot\|_b$, we adopt the notation $\|\cdot\|_b$ for both. By (iv) it follows easily that, for every $x \in (\mathfrak{A}_0)_b$,

$$\|x\|_b = \sup\{\|\overline{\pi(x)}\|,\ \pi \in \mathrm{Rep}_c(\mathfrak{A}_0)\}. \tag{1}$$

It is easy to see that $\|\cdot\|_b$ is a norm on $(\mathfrak{A}_0)_b$ such that, for every $x, y \in (\mathfrak{A}_0)_b$,

(i) $\|x^*\|_b = \|x\|_b$;

(ii) $\|xy\|_b \leq \|x\|_b \, \|y\|_b$;

(iii) $\|x^*x\|_b = \|x\|_b^2$.

4 Spectral properties revisited

In what follows a crucial role will be played by an *auxiliary* topology which stems from the family $\mathcal{P}_c(\mathfrak{A}_0)$ of $\mathfrak{t}$-continuous positive linear functionals. We define in fact the *strong** topology $\mathfrak{t}_{s^*}$, on $\mathfrak{A}_0$ by the family of seminorms

$$x \in \mathfrak{A}_0 \to \omega(x^*x)^{1/2} + \omega(xx^*)^{1/2}, \quad \omega \in \mathcal{P}_c(\mathfrak{A}_0).$$

From the definition itself it follows that $\mathfrak{t}_{s^*}$ is coarser than $\mathfrak{t}$ and that every $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$ is also $\mathfrak{t}_{s^*}$-continuous. Moreover, by an easy application of the Cauchy-Schwarz inequality and by the definition itself of the topology $\mathfrak{t}_{s^*}$, the sesquilinear form φ_ω defined by

$$\varphi_\omega(x, y) = \omega(y^*x), \quad x, y \in \mathfrak{A}_0. \tag{2}$$

is *jointly* continuous for $\mathfrak{t}_{s^*}$.

Proposition 4.1. *Let $\mathfrak{A}_0$ be a FR*-algebra with unit e. Then $\|\cdot\|_b$ is a C^*-norm on $(\mathfrak{A}_0)_b$. Moreover if $\mathfrak{A}_0$ is $\mathfrak{t}_{s^*}$-complete, then $(\mathfrak{A}_0)_b$ is a C*-algebra with norm $\|\cdot\|_b$.*

4.1 Constructing a locally convex quasi *-algebra over $(\mathfrak{A}_0)_b$

In (16) a notion of spectrum was proposed, under the assumption that $(\mathfrak{A}_0)_b$ is a C*-algebra contained in $\mathfrak{A}_0$. We remove this condition, but we suppose that $(\mathfrak{A}_0)_b$ is $\mathfrak{t}_{s^*}$-dense in $\mathfrak{A}_0$. For doing this, as announced in the Introduction, we need to build up a larger structure (actually, a quasi*-algebra (3)), having $\mathfrak{A}_0$ as a subspace. The construction runs as follows.

Lemma 4.2. *For every $x \in (\mathfrak{A}_0)_b$, the multiplications $a \mapsto ax$, $a \mapsto xa$, $a \in \mathfrak{A}_0$, are $\mathfrak{t}_{s^*}$-continuous.*

Proof. We first observe that, if $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$ and $y \in \mathfrak{A}_0$, the linear functional ω_y on $\mathfrak{A}_0$ defined by $\omega_y(b) = \omega(y^*by)$, $b \in \mathfrak{A}_0$, is $\mathfrak{t}$-continuous, since the multiplication in $\mathfrak{A}_0$ is separately continuous; i.e. $\omega_y \in \mathcal{P}_c(\mathfrak{A}_0)$. If $x \in (\mathfrak{A}_0)_b$ we have, using (ii) of Theorem 3.6,

$$\omega(x^*a^*ax)^{1/2} + \omega(axx^*a)^{1/2} = \omega_x(a^*a)^{1/2} + \omega(axx^*a^*)^{1/2}$$
$$\leq (\omega_x(a^*a)^{1/2} + \omega_x(aa^*)^{1/2}) + \gamma_{x^*x}(\omega(a^*a)^{1/2} + \omega(aa^*)^{1/2})$$

which proves that the map $a \mapsto ax$ is $\mathfrak{t}_{s^*}$-continuous in $\mathfrak{A}_0$. The continuity of the map $a \mapsto xa$ follows easily by taking adjoints.

□

Let us denote by $\mathfrak{A} := \widetilde{\mathfrak{A}_0}[\mathfrak{t}_{s^*}]$, the completion of $\mathfrak{A}_0[\mathfrak{t}_{s^*}]$. The assumption that $(\mathfrak{A}_0)_b$ is $\mathfrak{t}_{s^*}$-dense in $\mathfrak{A}_0$, implies, obviously, that $\mathfrak{A} := \widetilde{(\mathfrak{A}_0)_b}[\mathfrak{t}_{s^*}]$. Then, if $a \in \mathfrak{A}$ there exists a net (a_α), of elements of $(\mathfrak{A}_0)_b$, $\mathfrak{t}_{s^*}$-converging to a. Hence, taking into account Lemma 4.2, we can define multiplications as follows

$$a \cdot x := \lim_\alpha a_\alpha x, \qquad x \cdot a := \lim_\alpha x a_\alpha, \quad x \in (\mathfrak{A}_0)_b, a \in \mathfrak{A}$$

and the involution by

$$a^* := \lim_\alpha a_\alpha^*.$$

It is easily seen that these maps are well-defined (i.e., they do not depend on the particular net chosen for approximating a).

Proposition 4.3. *Let $\mathfrak{A}_0$ be an* FR**-algebra, such that $(\mathfrak{A}_0)_b$ is $\mathfrak{t}_{s^*}$-dense in $\mathfrak{A}_0$. Then, $\mathfrak{A}[\mathfrak{t}_{s^*}]$ is a locally convex quasi *-algebra over $(\mathfrak{A}_0)_b$.*

Proof. By the definition itself of the topology of the completion $\mathfrak{A}$, it follows that both the continuity properties stated in Lemma 4.2 are preserved when a runs over $\mathfrak{A}$. Similarly, the involution $a \mapsto a^*$ is continuous. □

Remark 4.4. As a quasi *-algebra $\mathfrak{A}$, over $(\mathfrak{A}_0)_b$, $\mathfrak{A}$ is a particular *partial *-algebra* (3). The latter is characterized by the two lattices $\mathcal{F}^L$, $\mathcal{F}^R$ of, respectively, left- and right-multipliers, respectively $L(S)$, $R(S)$, of a subset

$S \in \mathfrak{A}$. For the quasi *-algebra $\mathfrak{A}$ over $(\mathfrak{A}_0)_b$ constructed here, we simply have, for $a \in \mathfrak{A}$

$$L(a) = R(a) = \begin{cases} \mathfrak{A} & \text{if } a \in (\mathfrak{A}_0)_b \\ (\mathfrak{A}_0)_b & \text{if } a \in \mathfrak{A} \setminus (\mathfrak{A}_0)_b. \end{cases}$$

4.2 Ips-forms

In order to explore spectral properties of $\mathfrak{A}_0[\mathfrak{t}]$ we need to introduce a particular class of positive sesquilinear forms on $\mathfrak{A} = \widetilde{\mathfrak{A}_0}[\mathfrak{t}_{s^*}]$.

Let φ be a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$. Then we have

$$\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall\, x, y \in \mathfrak{A}, \tag{3}$$

$$|\varphi(x, y)|^2 \leqslant \varphi(x, x)\varphi(y, y), \quad \forall\, x, y \in \mathfrak{A}. \tag{4}$$

We put

$$N_\varphi = \{x \in \mathfrak{A} : \varphi(x, x) = 0\}.$$

By (4), we have

$$N_\varphi = \{x \in \mathfrak{A} : \varphi(x, y) = 0, \quad \forall\, y \in \mathfrak{A}\},$$

and so N_φ is a subspace of $\mathfrak{A}$ and the quotient space $\mathfrak{A}/N_\varphi := \{\lambda_\varphi(x) \equiv x + N_\varphi; x \in \mathfrak{A}\}$ is a pre-Hilbert space with respect to the inner product $\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y), x, y \in \mathfrak{A}$. We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by completion of $\mathfrak{A}/N_\varphi$.

Let us consider an *ips-form* φ with core $(\mathfrak{A}_0)_b$, that is φ is a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ satisfying
(ips_1) $\lambda_\varphi((\mathfrak{A}_0)_b)$ is dense in $\mathcal{H}_\varphi$;
(ips_2) $\varphi(ax, y) = \varphi(x, a^* y), \forall\, a \in \mathfrak{A}, \forall\, x, y \in (\mathfrak{A}_0)_b$;
(ips_3) $\varphi(a^* x, by) = \varphi(x, (ab)y), \forall\, a \in L(b), \forall\, x, y \in (\mathfrak{A}_0)_b$.

In other words, an ips-form is an *everywhere defined* biweight with core $(\mathfrak{A}_0)_b$, in the sense of (3).

To every ips-form φ on $\mathfrak{A}$, with core $(\mathfrak{A}_0)_b$, there corresponds a triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$, where $\mathcal{H}_\varphi$ is a Hilbert space, λ_φ is a linear map from $(\mathfrak{A}_0)_b$ into

$\mathcal{H}_\varphi$ and π_φ is a *-representation on $\mathfrak{A}$ in the Hilbert space $\mathcal{H}_\varphi$. We refer to (3) for more details on this celebrated GNS construction.

If $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$, the positive sesquilinear form φ_ω on $\mathfrak{A}_0 \times \mathfrak{A}_0$, defined as in (2), is, as we already remarked, jointly continuous with respect to $\mathfrak{t}_{s^*}$. Then φ_ω extends by continuity to $\mathfrak{A} \times \mathfrak{A}$. Let us call $\widetilde{\varphi}_\omega$ this extension.

Lemma 4.5. *If $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$, then $\widetilde{\varphi}_\omega$ is an ips-form on $\mathfrak{A} \times \mathfrak{A}$ with core $\mathfrak{A}_0$.*

Proof. It is easy to check that the conditions (ips_2) and (ips_3) are satisfied and since each φ_ω, $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$ is jointly $\mathfrak{t}_{s^*}$-continuous, then (ips_1) is also satisfied, so that every $\widetilde{\varphi}_\omega$, $\omega \in \mathcal{P}_c(\mathfrak{A}_0)$ is an ips-form (see (4, Sect.4)).

□

Let us denote by $\mathcal{M}(\mathfrak{A})$ the family of ips-forms on $\mathfrak{A} \times \mathfrak{A}$ defined as above starting from $\mathcal{P}_c(\mathfrak{A}_0)$; i.e.,

$$\mathcal{M}(\mathfrak{A}) = \{\varphi = \widetilde{\varphi}_\omega; \, \omega \in \mathcal{P}_c(\mathfrak{A}_0)\}.$$

As in (4), we extend to families of ips-forms, the notion of *sufficiency* in the following way.

Definition 4.6. A family $\mathcal{M}$ of ips-forms on $\mathfrak{A} \times \mathfrak{A}$ is *sufficient* if $x \in \mathfrak{A}$ and $\varphi(x,x) = 0$, for every $\varphi \in \mathcal{M}$, imply $x = 0$.

By Proposition 3.3, we know that, for a FR-*-algebra $\mathfrak{A}_0$, $\mathcal{P}_c(\mathfrak{A}_0)$ is sufficient. However the corresponding family of ips-forms $\mathcal{M}(\mathfrak{A})$ need not be sufficient in the sense of Definition 4.6.

Definition 4.7. We say that the FR-*-algebra $\mathfrak{A}_0$ is *full* if the family $\mathcal{M}(\mathfrak{A})$ is sufficient.

It is not always simple to determine the whole set $\mathcal{M}(\mathfrak{A})$. But it is clear that if some subset $\mathcal{N}$ of $\mathcal{M}(\mathfrak{A})$ is sufficient so is $\mathcal{M}(\mathfrak{A})$. So that in what follows we suppose that $\mathcal{M}(\mathfrak{A})$ has a sufficient subset $\mathcal{N}$ which is *balanced* in the following sense (17; 13): if $\varphi \in \mathcal{N}$ then $\varphi_a \in \mathcal{N}$, for every $a \in (\mathfrak{A}_0)_b$ where $\varphi_a(x,y) = \varphi(xa, ya)$, $x, y \in \mathfrak{A}$. This choice reveals to be more flexible for examining examples. The term *balanced* is borrowed by Yood, who gave a similar definition for positive functionals (19).

Remark 4.8. Let $\mathcal{N}$ be a balanced subset of $\mathcal{M}(\mathfrak{A})$. The sesquilinear forms of $\mathcal{N}$ can be used to define on $\mathfrak{A}$ several topologies. We will only consider the topology $\mathsf{t}_{s^*}^{\mathcal{N}}$ defined by the following family of seminorms:
$\mathsf{t}_{s^*}^{\mathcal{N}}$: $\quad x \mapsto \varphi(x,x)^{1/2} + \varphi(x^*,x^*)^{1/2}, \quad \varphi \in \mathcal{N}.$
It is easily seen that $\mathsf{t}_{s^*}^{\mathcal{N}}$ induces on $\mathfrak{A}_0$ the topology t_{s^*}.

For full FR*-algebras we can define *extensions* of the multiplication in the following way. We prevent the raeder that what follows is an application of results obtained in (5) to the present situation. Proofs which are simple adaptations of those given in the quoted paper are therefore omitted.

Definition 4.9. Let $\mathfrak{A}_0$ be a full FR*-algebra with unit e and $x, y \in \mathfrak{A}$ and $\mathcal{N}$ a sufficient balanced subset of $\mathcal{M}(\mathfrak{A})$.

- We say that the *weak* multiplication $x \square y$ is well-defined if there exists $z \in \mathfrak{A}$ such that:

  $$\varphi(ya, x^*b) = \varphi(za, b), \ \forall\, a, b \in (\mathfrak{A}_0)_b, \forall\, \varphi \in \mathcal{N}.$$

  In this case, we put $x \square y := z$.
- We say that the *strong* multiplication $x \bullet y$ is well-defined (and $x \in L^s(y)$ or $y \in R^s(x)$) if there exists $w \in \mathfrak{A}$ such that:

  $$\varphi(wa, z^*b) = \varphi(ya, (x^*z^*)b) \ \text{ whenever } \ z \in L(x), \ \forall\, \varphi \in \mathcal{N}, \ \forall\, a, b \in (\mathfrak{A}_0)_b,$$

  and

  $$\varphi(w^*a, vb) = \varphi(x^*a, (yv)b) \ \text{ whenever } \ v \in R(y), \ \forall\, \varphi \in \mathcal{N}, \ \forall\, a, b \in \mathfrak{A}_0.$$

  In this case, we put $x \bullet y := w$.

It is obvious that the these defifinitions depend on the choice of $\mathcal{N}$. The following result is immediate.

Proposition 4.10. *If the* FR-*-*algebra* $\mathfrak{A}_0$ *is full, then* $\mathfrak{A}$ *is a partial* *-*algebra with respect to the weak multiplication defined by any sufficient balanced subset* $\mathcal{N}$ *of* $\mathcal{M}(\mathfrak{A})$.

Remark 4.11. The uniqueness of each element z, w in the previous proposition results from the sufficiency of the family $\mathcal{N}$. Moreover, it is clear that if $x \bullet y$ is well-defined, then $x \,\Box\, y$ is well-defined too. In particular, if $x, y \in (\mathfrak{A}_0)_b$ then, $x \bullet y$ and $x \,\Box\, y$ are both well defined and $x \bullet y = x \,\Box\, y = xy$.

An easy consequence of the previous definitions is the following form of the *associative law.*

Proposition 4.12. *Let* $x, y, z \in \mathfrak{A}$. *Assume that* $x \square y$, $(x \square y) \square z$ *and* $y \bullet z$ *are all well-defined. Then* $x \in L(y \bullet z)$ *and*

$$x \square (y \bullet z) = (x \square y) \square z.$$

An element x has a *strong inverse* if there exists $x^{-1} \in \mathfrak{A}$ such that $x \bullet x^{-1} = x^{-1} \bullet x = e$. The mixed associativity of Proposition 4.12 implies that, if a strong inverse of x exists, then it is unique.

Definition 4.13.

An element $x \in \mathfrak{A}$ is called $\mathcal{N}$-*bounded* if there exists $\gamma > 0$ such that:

$$|\varphi(xa, b)| \leq \gamma\, \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \ \forall \varphi \in \mathcal{N},\ a, b \in \mathfrak{A}_0 \,.$$

Remark 4.14. It is clear that if, $x \in \mathfrak{A}_0$, then x is $\mathcal{M}(\mathfrak{A})$-bounded, if and only if it is order bounded, by Theorem 3.6.

Let us now define:

$$\begin{aligned} q_{\mathcal{N}}(x) &:= \inf\{\gamma > 0 : \varphi(xa, xa) \leq \gamma^2 \varphi(a, a),\ \forall \varphi \in \mathcal{N},\ \forall a \in (\mathfrak{A}_0)_b\} \\ &= \sup\{\varphi(xa, xa)^{1/2} : \varphi \in \mathcal{N},\ a \in (\mathfrak{A}_0)_b,\ \varphi(a, a)^{1/2} = 1\} \end{aligned}$$

and

$$\mathcal{D}(q_{\mathcal{N}}) := \{x \in \mathfrak{A} : x \text{ is } \mathcal{N}\text{-bounded}\}.$$

It is clear that $(\mathfrak{A}_0)_b \subseteq \mathcal{D}(q_{\mathcal{N}})$.

Then the following holds:

Proposition 4.15. *Let x, y be $\mathcal{N}$-bounded elements of $\mathfrak{A}$. The following statements hold:*
(i) x^* is $\mathcal{N}$-bounded also, and $q_{\mathcal{N}}(x) = q_{\mathcal{N}}(x^*)$;
(ii) If $x \square y$ is well-defined, then $x \square y$ is $\mathcal{N}$-bounded and

$$q_{\mathcal{N}}(x \square y) \leq q_{\mathcal{N}}(x)\, q_{\mathcal{N}}(y).$$

The proof of (ii) is a consequence of the following inequality (5).

Let $x, y \in \mathcal{D}(q_{\mathcal{N}})$ then, for every $\varphi \in \mathcal{N}$ and for every $a, b \in (\mathfrak{A}_0)_b$ we get

$$\begin{aligned} |\varphi(ya, x^*b)| &\leq \varphi(ya, ya)^{1/2}\varphi(x^*b, x^*b)^{1/2} \\ &\leq q_{\mathcal{N}}(x)^{1/2} q_{\mathcal{N}}(y)^{1/2} \varphi(a, a)^{1/2}\varphi(b, b)^{1/2}. \end{aligned} \tag{5}$$

As we have seen at the beginning of this section, to every $\varphi \in \mathcal{N}$ there corresponds a Hilbert space $\mathcal{H}_\varphi$ constructed from cosets $\lambda_\varphi(a), a \in \mathfrak{A}$. Then (5) can be read as follows. Let us define, for $x, y \in \mathfrak{A}$ fixed, the sesquilinear form on $\lambda_\varphi((\mathfrak{A}_0)_b)$

$$\Theta_{x,y}(\lambda_\varphi(a), \lambda_\varphi(b)) := \varphi(ya, x^*b), \quad a, b \in (\mathfrak{A}_0)_b.$$

Then, by (5), $\Theta_{x,y}$ is a bounded sesquilinear form on $\lambda_\varphi((\mathfrak{A}_0)_b) \times \lambda_\varphi((\mathfrak{A}_0)_b)$ and it extends to $\mathcal{H}_\varphi \times \mathcal{H}_\varphi$. Hence, there exist a bounded operator $B_{\varphi,x,y}$ such that

$$\Theta_{x,y}(\lambda_\varphi(a), \lambda_\varphi(b)) = \varphi(ya, x^*b) = \langle B_{\varphi,x,y}\lambda_\varphi(a)|\lambda_\varphi(b)\rangle, \quad \forall a, b \in (\mathfrak{A}_0)_b).$$

Assume now that the following condition hold

(wb)there exists $z \in \mathfrak{A}$ such that, for every $\varphi \in \mathcal{N}$, $B_{\varphi,x,y}\lambda_\varphi(a) = \lambda_\varphi(za)$ for every $a \in (\mathfrak{A}_0)_b$.

In this case, $x \square y$ is well defined and equals z.

If the condition (wb) is satisfied for every $x, y \in \mathcal{D}(q_{\mathcal{N}})$, then we say that $\mathcal{N}$ is *well-behaved*. In this case, $\mathcal{D}(q_{\mathcal{N}})$ is a C*-algebra with the weak or strong multiplication and the norm $q_{\mathcal{N}}$. This has been proved in (5), under a stronger condition.

The above discussion makes clear that requiring that $\mathcal{D}(q_{\mathcal{N}})$ is a C*-algebra, as we did in (5), is a quite strong condition, rarely fulfilled in concrete examples. But in any case we have:

Proposition 4.16. *$\mathcal{D}(q_{\mathcal{N}})$ is complete under the norm $q_{\mathcal{N}}$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence with respect to the norm $q_{\mathcal{N}}$. Then $\{x_n^*\}$ is Cauchy too. Since, for every $\varphi \in \mathcal{N}$ and $a \in (\mathfrak{A}_0)_b$ the sesquilinear form φ_a, with $\varphi_a(x, y) = \varphi(xa, ya)$, belongs to $\mathcal{N}$, we have

$$\varphi((x_n - x_m)a, (x_n - x_m)a) \to 0, \text{ as } n, m \to \infty$$

and

$$\varphi((x_n^* - x_m^*)a, (x_n^* - x_m^*)a) \to 0, \text{ as } n, m \to \infty.$$

Therefore, $\{x_n\}$ is Cauchy also with respect to $\mathfrak{t}_{s^*}^{\mathcal{N}}$. Then, there exists $x \in \mathfrak{A}$ such that $x_n \overset{\mathfrak{t}_{s^*}^{\mathcal{N}}}{\to} x$. Since

$$\varphi(xa, xa) = \lim_{n\to\infty} \varphi(x_n a, x_n a) \leq \limsup_{n\to\infty} q_{\mathcal{M}}(x_n)^2 \varphi(a, a)$$

and $\limsup_{n\to\infty} q_{\mathcal{M}}(x_n)^2 < \infty$ (by the boundedness of the sequence $\{q_{\mathcal{M}}(x_n)\}$), we conclude that x is $\mathcal{N}$-bounded. Finally, by the Cauchy condition, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that, for every $n, m > n_\epsilon$, $q_{\mathcal{M}}(x_n - x_m) < \epsilon$. This implies that

$$\varphi((x_n - x_m)a, (x_n - x_m)a) < \epsilon\varphi(a, a), \quad \forall \varphi \in \mathcal{N},\ a \in \mathfrak{A}_0.$$

Then, for fixed $n > n_\epsilon$ and $m \to \infty$, we obtain

$$\varphi((x_n - x)a, (x_n - x)a) \leq \epsilon\varphi(a, a), \quad \forall \varphi \in \mathcal{N},\ a \in \mathfrak{A}_0.$$

This implies that $q_{\mathcal{N}}(x_n - x) \leq \epsilon$. □

Since $q_{\mathcal{N}}(a) = \|a\|_b$, for every $a \in (\mathfrak{A}_0)_b$, for making notations lighter, we adopt the notation $\|\cdot\|_b$ for both of them.

Corollary 4.17. *If $(\mathfrak{A}_0)_b$ is dense in $\mathcal{D}(q_{\mathcal{N}})[\|\cdot\|_b]$, then $\mathcal{D}(q_{\mathcal{N}})[\|\cdot\|_b]$ is a C*-algebra, possibly not contained in $\mathfrak{A}_0$.*

4.3 The spectrum

Let now $\mathfrak{A}_0$ be an FR*-algebra with unit e and such that $(\mathfrak{A}_0)_b$ is t_{s^*}-dense in $\mathfrak{A}_0$. We assume, in addition that $\mathcal{D}(q_{\mathcal{N}})[\|\cdot\|_b]$ is a C*-algebra.

Then we define the *resolvent* $\rho_\circ(x)$ of $x \in \mathfrak{A}_0$ is defined by

$$\rho_\circ(x) = \left\{\lambda \in \mathbb{C} : \text{ the strong inverse } (x - \lambda e)^{-1} \text{exists in } \mathcal{D}(q_{\mathcal{N}})\right\}.$$

The *spectrum* of x is defined as $\sigma_\circ(x) := \mathbb{C} \setminus \rho_\circ(x)$.

In similar way as in (12) it can be proved that: (a) $\rho_\circ(x)$ is an open subset of the complex plane; (b) the map $\lambda \in \rho_\circ(x) \mapsto (x - \lambda e)^{-1} \in (\mathfrak{A}_0)_b$ is analytic in each connected component of $\rho_\circ(x)$.

As usual, we define the *spectral radius* of $x \in \mathfrak{A}_0$ by

$$r_\circ(x) := \sup\{|\lambda| : \lambda \in \sigma_\circ(x)\}.$$

Now we conclude by proving that bounded elements of $\mathfrak{A}_0$ can be characterized in terms of the spectrum. We have, in fact

Theorem 4.18. *Let $\mathfrak{A}_0$ be an* FR**-algebra with unit e and such that $(\mathfrak{A}_0)_b$ is t_{s^*}-dense in $\mathfrak{A}_0$. Assume, in addition, that $\mathcal{D}(q_{\mathcal{N}})[\|\cdot\|_b]$ is a C*-algebra. Let $x \in \mathfrak{A}_0$. Then, $r_\circ(x) < \infty$ if and only if $x \in (\mathfrak{A}_0)_b$.*

Proof. If $x \in (\mathfrak{A}_0)_b$, then $\sigma_\circ(x)$ coincides with the spectrum of x as an element of the C*-algebra $\mathcal{D}(q_{\mathcal{N}})[\|\cdot\|_b]$ and so $\sigma_\circ(x)$ is compact. Conversely, assume that $r_\circ(x) < \infty$. Then the function $\lambda \mapsto (x - \lambda e)^{-1}$ is $\|\cdot\|_b$-analytic in the region $|\lambda| > r_\circ(x)$. Therefore it can be expanded in a $\|\cdot\|_b$-convergent Laurent series

$$(x - \lambda e)^{-1} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \quad |\lambda| > r^{\mathcal{M}}(x),$$

with $a_k \in \mathcal{D}(q_{\mathcal{N}})$ for each $k \in \mathbb{N}$. As usual

$$a_k = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda^{-k+1}} d\lambda, \qquad k \in \mathbb{N},$$

where $\gamma := \{\lambda \in \mathbb{C} : |\lambda| = R : R > r_\circ(x)\}$ and the integral on the r.h.s. is meant to converge with respect to $\|\cdot\|_b$.

Using the previous integral representation and the continuity, for every $\varphi \in \mathcal{N}$ and $b, b' \in (\mathfrak{A}_0)_b$, we have

$$\varphi(xa_k b, b') = \varphi(a_{k+1} b, b').$$

This implies that $xa_k = a_{k+1}$.

Similarly, one shows that $xa_1 = -x$. Thus, in conclusion, $x = -a_2 \in \mathcal{D}(q_{\mathcal{N}}) \cap \mathfrak{A}_0 = (\mathfrak{A}_0)_b$. □

Corollary 4.19. *Let $\mathfrak{A}_0$ be an* FR**-algebra $\mathfrak{A}_0$, with unit e. Then*

$$\begin{cases} r_\circ(x) \leq \|x\|_b & \text{if } x \in (\mathfrak{A}_0)_b \\ r_\circ(x) = +\infty & \text{if } x \notin (\mathfrak{A}_0)_b. \end{cases}$$

Example 4.20. The maximal O*-algebra $\mathcal{L}^\dagger(\mathcal{D})$ on a domain $\mathcal{D}$, endowed the topology $\mathfrak{t} = \tau_u$ is an FR-algebra. For every $\xi \in \mathcal{D}$ the positive linear functional ω_ξ defined by $\omega_\xi(X) = \langle X\xi|\xi\rangle$ is τ_u-continuous. Let $\mathcal{N}$ be the family of extensions of the set of forms $\{\varphi_{\omega_\xi}, \xi \in \mathcal{D}\}$, defined as in (2). The topology $\mathfrak{t}_{s^*}^{\mathcal{N}}$ coincides with τ_{s^*}. The *-algebra $(\mathcal{L}^\dagger(\mathcal{D}))_b$ is exactly the *-algebra $\mathcal{L}_b^\dagger(\mathcal{D}) = \{X \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{X} \in \mathcal{B}(\mathcal{H})\}$ of all elements of $\mathcal{L}^\dagger(\mathcal{D})$ that are bounded operators. This *-algebra is $\tau_{s^*}-$*dense* in $\mathcal{L}^\dagger(\mathcal{D})$. The completion of $\mathcal{L}_b^\dagger(\mathcal{D})$ is the partial *-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ whose bounded part consists of the restrictions to $\mathcal{D}$ of the elements of $\mathcal{B}(\mathcal{H})$. The weak- and strong multiplications defined here are nothing but the weak- and strong multiplications of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ (see, e.g. (3; 5)). Finally, as shown in (4), the spectrum of an element $Y \in \mathcal{L}^\dagger(\mathcal{D})$ as defined in this section coincides with the ordinary spectrum of the closed operator $\overline{Y}$.

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