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ON A STOCHASTIC DISEASE MODEL WITH VACCINATION

ELISABETTA TORNATORE - STEFANIA MARIA BUCCELLATO - PASQUALE VETRO

In memory of Benedetto Pettineo

We propose a stochastic disease model where vaccination is included and such that the immunity isn't permanent. The existence, uniqueness and positivity of the solution and the stability of disease free equilibrium is studied. The numerical simulation is done.

1. Introduction.

In this paper we consider the stochastic differential system

$$(1) \quad \begin{cases} dS(t) = (\mu - \beta S(t)I(t) - (\mu + \phi)S(t) \\ \quad + \theta V(t) + \lambda I(t))dt - \sigma S(t)I(t)dW(t) \\ dI(t) = (\beta S(t)I(t) + \rho\beta V(t)I(t) - (\lambda + \mu)I(t))dt + \sigma(S(t) \\ \quad + \rho V(t))I(t)dW(t) \\ dV(t) = (\phi S(t) - \rho\beta V(t)I(t) - (\mu + \theta)V(t))dt \\ \quad - \sigma\rho V(t)I(t)dW(t) \end{cases}$$

as a stochastically perturbed system of the ordinary deterministic system

$$(2) \quad \begin{cases} S'(t) = \mu - \beta S(t)I(t) - (\mu + \phi)S(t) + \theta V(t) + \lambda I(t) \\ I'(t) = \beta S(t)I(t) + \rho\beta V(t)I(t) - (\lambda + \mu)I(t) \\ V'(t) = \phi S(t) - \rho\beta V(t)I(t) - (\mu + \theta)V(t) \end{cases}$$

that analyzes a disease spread into a population when individuals are vaccinated.

Precisely the population can be in one of three states: susceptible, infective and vaccinated, denoted by S , I and V , respectively. In the model we suppose that the average number of contacts per infective per unit time is a constant β , that in the unit time a fraction ϕ of the susceptible class is vaccinated and

that the vaccination may reduce but not completely eliminate susceptibility to infection. For this reason in the infection rate of vaccinated members is included a factor ρ , $0 \leq \rho \leq 1$, with $\rho = 0$ meaning the vaccine is perfectly effective and $\rho = 1$ meaning the vaccine has no effect. We suppose also that the vaccination loses effect at a proportional rate θ , that the immunity isn't permanent and that a fraction λ of infective goes back in the susceptible class so the constant λ represents the recovery rate of infected people. We assume that the birth occurs in the system with the same rate constant μ of death. Consequently the total population is constant and the variables are normalized to $N = S + I + V = 1$, that is $S(t) + I(t) + V(t) = 1$ for all $t \geq 0$. Of course $\mu, \lambda, \phi, \theta, \beta \in \mathbb{R}_+$ (see [2], [3], [11]).

Recently there has been interest in the analysis of disease models where a vaccination is introduced. Attention has been given to vaccination and treatment policies both in term of different vaccine classes (all/nothing, leaky) and associated costs ([7-9]). Since the disease can go to an endemic level, control strategies to vaccination are usually implemented to eradicate the disease or at least to lower its prevalence to reasonable level. In modelings such strategies the main problem is that the vaccination is not completely efficient. Vaccines may have low efficacy and be leaky (*i.e.*, after a certain time vaccinated individual may have only partial protection from infection). Data support the fact that a vaccine usually wanes, thus providing only temporary protection. Moreover under certain parameter condition the model (2) exhibits a backward bifurcation leading to the existence of multiple endemic equilibria: a stable disease free equilibrium, an unstable endemic equilibrium and a larger stable endemic equilibrium. This result is important in designing vaccination strategies and in this case the success of the strategy depends on the number of individuals that are initially infected.

This paper is organized as follows. In Section 1 we prove the existence, uniqueness and positivity of the solution of the stochastic system and determine a sufficient condition to eradicate the disease. In Section 2 using Lyapunov functions we prove the stability of disease free equilibrium. Finally in Section 3 we use numerical simulation to compare the deterministic and stochastic behavior of the systems (1) and (2).

2. Formulation of a stochastic SIS model with vaccination and temporary immunity.

In this section we prove the existence, uniqueness and positivity of the solution. Throughout this paper, we let $W(t)$ denote a one-dimensional Brow-

nian motion defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ and we let σ be a positive constant. If A is a vector or a matrix, its transpose is denoted by A^T . If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. We also introduce the notation

$$\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_i > 0 \text{ for all } i = 1, 2, 3\}.$$

We consider the system (1) with the following initial condition

$$(3) \quad (S(0), I(0), V(0)) \in \mathbb{R}_+^3, \quad S(0) + I(0) + V(0) = 1.$$

It is observed that the total population is constant $S(t) + I(t) + V(t) = 1$ for all $t \geq 0$ almost surely (briefly a.s.), in fact summing the equations of the system (1) we obtain

$$d(S + I + V) = 0.$$

Thus, if $S(s) > 0, I(s) > 0, V(s) > 0$ for all $0 \leq s \leq t$ a.s., then

$$(4) \quad 0 < S(s) < 1, \quad 0 < I(s) < 1, \quad 0 < V(s) < 1$$

for all $0 \leq s \leq t$ a.s..

Since the coefficients of the system (1) are locally Lipschitz the following result of local existence of solutions holds.

THEOREM 2.1. (Theorem 1.1 pg 98 of [4]). *For any given initial condition $(S(0), I(0), V(0)) \in \mathbb{R}_+^3$ there exists $\tau > 0$ and a unique solution $(S(t), I(t), V(t))$ of the system (1) on $t \in [0, \tau)$.*

We consider an integer $k_0 > 0$ sufficiently large such that $(S(0), I(0), V(0)) \in \left[\frac{1}{k_0}, k_0\right]^3$. For each integer $k > k_0$ we define the stopping time

$$(5) \quad \tau_k = \inf \left\{ t \in [0, \tau) : (S(t), I(t), V(t)) \notin \left[\frac{1}{k}, k\right]^3 \right\}.$$

We shall show that the solution of (1) is positive and global using the idea exposed in [12].

THEOREM 2.2. *For any given initial condition $(S(0), I(0), V(0)) \in \mathbb{R}_+^3$ there exists a unique solution $(S(t), I(t), V(t))$ of the system (1) on $t \geq 0$ and the solution will remain in \mathbb{R}_+^3 with probability 1, namely $(S(t), I(t), V(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$ almost surely.*

For the proof of the theorem we need the following lemma.

LEMMA 2.1. *Let $(S(t), I(t), V(t))$ be the solution of the system (1), with initial condition (3); then*

$$(6) \quad E(\log^2 S(\tau_k \wedge t) + \log^2 I(\tau_k \wedge t) + \log^2 V(\tau_k \wedge t)) \leq C(t)$$

where $C(t)$ is the solution of the Cauchy problem

$$\begin{cases} y'(t) = (2 + \rho^2)\sigma^2 + 2\sqrt{3}(\beta(1 + \rho) + 3\mu + \phi + \theta + \lambda \\ \quad + \frac{3\sigma^2}{2}(1 + \rho^2))\sqrt{y(t)} \\ y(0) = \log^2 S(0) + \log^2 I(0) + \log^2 V(0). \end{cases}$$

Proof. Define a C^2 -function, $\Phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ by

$$(7) \quad \Phi(S(t), I(t), V(t)) = \log^2 S(t) + \log^2 I(t) + \log^2 V(t).$$

The Ito formula shows that

$$\begin{aligned} d(\Phi(S(t), I(t), V(t))) = & 2 \left[\left(\frac{\mu}{S} - \beta I - (\mu + \phi) + \theta \frac{V}{S} + \lambda \frac{I}{S} - \frac{\sigma^2}{2} I^2 \right) \log S \right. \\ & + (\beta S + \beta \rho V - (\lambda + \mu) - \frac{\sigma^2}{2} (S + \rho V)^2) \log I \\ & + \left(\phi \frac{S}{V} - \beta \rho I - (\mu + \theta) - \frac{\sigma^2}{2} \rho^2 I^2 \right) \log V + \frac{\sigma^2}{2} ((1 + \rho^2) I^2 + (S + \rho V)^2) \Big] dt \\ & + 2\sigma(-I \log S + (S + \rho V) \log I - \rho I \log V) dW. \end{aligned} \quad (8)$$

We observe that $(S(t), I(t), V(t)) \in \left[\frac{1}{k}, k \right]^3$ for every $t \leq \tau_k$, in virtue of stopping time defined in (5). Using (4) we have a.s.

$$(9) \quad 0 < S(t) < 1, \quad 0 < I(t) < 1, \quad 0 < V(t) < 1.$$

We set

$$Y(t) = E(\log^2 S(t \wedge \tau_k) + \log^2 I(t \wedge \tau_k) + \log^2 V(t \wedge \tau_k)).$$

Taking into account (9) and (8) we get

$$\begin{aligned} Y(t) \leq & Y(0) + (2 + \rho^2)\sigma^2 t + 2\sqrt{3}(\beta(1 + \rho) + 3\mu + \phi + \theta \\ & + \lambda + \frac{3\sigma^2}{2}(1 + \rho^2)) \int_0^t \sqrt{Y(s)} ds, \end{aligned}$$

from which it follows that

$$Y(t) \leq C(t) \quad \forall t \geq 0,$$

where $C(t)$ is the solution of the Cauchy problem

$$\begin{cases} y'(t) = (2 + \rho^2)\sigma^2 + 2\sqrt{3}(\beta(1 + \rho) + 3\mu + \phi + \theta + \lambda \\ \quad + \frac{3\sigma^2}{2}(1 + \rho^2))\sqrt{y(t)} \\ y(0) = \log^2 S(0) + \log^2 I(0) + \log^2 V(0) \end{cases}$$

and the lemma is proved. \square

Proof of the Theorem 2.2. By Theorem 2.1 there exist $\tau > 0$ and the solution $(S(t), I(t), V(t))$ of the system (1) on $t \in [0, \tau)$. To prove that this solution is global, we need to show that $\tau = \infty$ a.s.. Consider the stopping time τ_k defined in (5). Clearly (τ_k) is an increasing sequence. Set $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$, whence $\tau_\infty < \tau$, a.s.. If we can show that $\tau_\infty = \infty$ a.s. then $\tau = \infty$ a.s. and consequently the solution $(S(t), I(t), V(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$ a.s.. If this statement is false, then there are two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\omega \in \Omega : \tau \leq T\} > \varepsilon.$$

Let $k_0 > 0$ be such that $(S(0), I(0), V(0)) \in [\frac{1}{k_0}, k_0]^3$; consequently there exists an integer $k_1 \geq k_0$ such that

$$P\{\omega \in \Omega : \tau_k \leq T\} \geq \varepsilon \quad \forall k \geq k_1.$$

Set $\Omega_k = \{\omega \in \Omega : \tau_k \leq T\}$ for each $k \geq k_1$, we have $P(\Omega_k) \geq \varepsilon$. Note that for every $\omega \in \Omega_k$ there is some component of $(S(\tau_k), I(\tau_k), V(\tau_k))$ equal to k or $\frac{1}{k}$ and hence by (7)

$$\Phi(S(\tau_k), I(\tau_k), V(\tau_k)) \geq \log^2 k.$$

From (6) we deduce that

$$C(T) \geq E(1_{\Omega_k} \Phi(S(\tau_k), I(\tau_k), V(\tau_k)) \geq \varepsilon \log^2 k$$

where 1_{Ω_k} is the characteristic function of Ω_k . Letting $k \rightarrow \infty$ leads to the contradiction

$$\infty = C(T) < \infty$$

so we must have $\tau = \infty$ a.s.. \square

Theorem 2.2 and Lemma 2.1 show that $Q = [0, 1] \times [0, 1] \times [0, 1]$ is the invariant set of the solutions of the system (1). We use this observation to find a sufficient condition for disease eradication. In virtue of Ito formula the second equation of system (1) is written as

$$(10) \quad d \log I(t) = \left(\beta(S(t) + \varrho V(t)) - (\lambda + \mu) - \frac{\sigma^2}{2}(S(t) + \varrho V(t))^2 \right) dt \\ + \sigma(S(t) + \varrho V(t))dW.$$

LEMMA 2.2. *Let $(S(t), I(t), V(t))$ be the solution of the system (2) with initial condition (3). If*

$$(11) \quad \frac{\beta}{\sigma \sqrt{2(\lambda + \mu)}} < 1$$

holds, then we have

$$\lim_{t \rightarrow \infty} \log I(t) = -\infty \quad a.s..$$

Proof. We observe that for every $t > 0$

$$(13) \quad \beta(S(t) + \varrho V(t)) - (\lambda + \mu) - \frac{\sigma^2}{2}(S(t) + \varrho V(t))^2 \leq \frac{\beta^2}{2\sigma^2} - (\lambda + \mu).$$

From (10) and (12), being $S(t) + \rho V(t) < 1 \quad \forall t > 0$, we have

$$d \log I(t) \leq \left(\frac{\beta^2}{2\sigma^2} - (\lambda + \mu) \right) dt + \sigma dW.$$

Using (11) and $\lim_{t \rightarrow +\infty} \frac{W(t)}{t} = 0$, we obtain as $t \rightarrow +\infty$

$$\log I(0) + \left(\frac{\beta^2}{2\sigma^2} - (\lambda + \mu) \right) t + \sigma W(t) \rightarrow -\infty \quad a.s..$$

Thus we have

$$\lim_{t \rightarrow +\infty} \log I(t) = -\infty, \quad a.s.. \quad \square$$

3. Stability of disease free equilibrium.

As in the deterministic case, the system (1) has the disease free equilibrium

$$(13) \quad E_0 = (S_0, 0, V_0) \quad \text{where} \quad S_0 = \frac{\mu + \theta}{\mu + \theta + \phi}, \quad V_0 = \frac{\phi}{\mu + \theta + \phi}.$$

We denote by

$$(14) \quad R_0 = \frac{\beta}{\mu + \lambda} \quad \text{and} \quad R_\phi = R_0 (S_0 + \rho V_0) = R_0 \left(\frac{\mu + \theta + \rho\phi}{\mu + \theta + \phi} \right)$$

the basic reproduction number (see [1]) and the basic reproduction number in a population in which a proportion ϕ has been vaccinated, respectively.

In this section we prove the stability of the disease free equilibrium using definitions and theorems introduced in [6] (see also [5]).

We recall definitions of equilibrium states of a stochastic differential equation. Consider the following n -dimensional stochastic equations system

$$(15) \quad dX(t) = f(t, X(t))dt + g(t, X(t))dW(t)$$

where $f(t, x)$ is a function in \mathbb{R}^n defined in $[t_0, +\infty[\times \mathbb{R}^n$, and $g(t, x)$ is a $n \times m$ matrix, f, g are locally Lipschitz functions in x and $W(t)$ is an m -dimensional Wiener process.

DEFINITION 3.1. *The stochastic process $X(t) = \bar{X}$ is a stationary solution of the stochastic system with initial condition $X(t_0) = \bar{X}$ if*

$$f(t, \bar{X}) = 0, \quad g(t, \bar{X}) = 0.$$

If $\bar{X} = 0$, the stationary solution is called a trivial solution.

DEFINITION 3.2. *The trivial solution $X(t) = 0$ of system (15) is said to be:*

1. *stable in probability if $\forall \varepsilon > 0$ and $s \geq t_0$*

$$\lim_{y \rightarrow 0} \mathbf{P}\left\{ \sup_{t \in [t_0, +\infty[} |X(t; s, y)| \geq \varepsilon \right\} = 0,$$

where $X(t; s, y)$ denotes the solution of system (15) with initial condition $X(s) = y$;

2. *asymptotically stable if it is stable in probability and moreover*

$$\lim_{y \rightarrow 0} \mathbf{P}\left\{ \lim_{t \rightarrow +\infty} |X(t; s, y)| = 0 \right\} = 1, \quad s \geq t_0,$$

3. *globally asymptotically stable if it is asymptotically stable and moreover*

$$\mathbf{P}\left\{ \lim_{t \rightarrow +\infty} |X(t; s, y)| = 0 \right\} = 1, \quad \forall y \in \mathbb{R}^n.$$

Let us denote by L the differential operator associated to (15), defined for a non-negative function $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ by

$$(16) \quad LV = \frac{\partial V}{\partial t} + f^T \cdot \frac{\partial V}{\partial x} + \frac{1}{2} \text{Tr}[g^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot g].$$

Substituting $u_1 = S - S_0, u_2 = I, u_3 = V - V_0$ into (1), we obtain the system

$$(17) \quad \begin{cases} du_1(t) = (-\beta(u_1(t) + S_0)u_2(t) - (\mu + \phi)u_1(t) + \theta u_3(t) \\ \quad + \lambda u_2(t))dt - \sigma(u_1(t) + S_0)u_2(t)dW(t) \\ du_2(t) = (\beta(u_1(t) + \rho u_3(t) + S_0 + \rho V_0) - \lambda - \mu)u_2(t)dt \\ \quad + \sigma(u_1(t) + \rho u_3(t) + S_0 + \rho V_0)u_2(t)dW(t) \\ du_3(t) = (-\beta\rho(u_3(t) + V_0)u_2(t) - (\mu + \theta)u_3(t) \\ \quad + \phi u_1(t))dt - \sigma\rho(u_3(t) + V_0)u_2(t)dW(t). \end{cases}$$

We consider the linearized system

$$(18) \quad \begin{cases} du_1(t) = (-\beta S_0 u_2(t) - (\mu + \phi)u_1(t) + \theta u_3(t) \\ \quad \quad \quad + \lambda u_2(t)dt - \sigma S_0 u_2(t)dW(t)) \\ du_2(t) = (\beta(S_0 + \rho V_0) - \lambda - \mu)u_2(t)dt \\ \quad \quad \quad + \sigma(S_0 + \rho V_0)u_2(t)dW(t) \\ du_3(t) = (-\beta \rho V_0 u_2(t) - (\mu + \theta)u_3(t) \\ \quad \quad \quad + \phi u_1(t))dt - \sigma \rho V_0 u_2(t)dW(t) \end{cases}$$

and we prove the following result:

LEMMA 3.1. *Let*

$$(19) \quad T = \frac{1}{2}\sigma^2 \frac{(S_0 + \rho V_0)^2}{\lambda + \mu}.$$

If

$$(20) \quad R_\phi < \min\{1 - T, \frac{\mu(S_0 + \rho V_0)}{\lambda + \mu}\}$$

holds, then the disease free equilibrium of (18) is globally asymptotically stable.

Proof. We denote $u = (u_1, u_2, u_3)$ and we consider the function

$$V(u) = u_1^2 + \frac{p}{q}u_2^2 + \frac{\theta}{\phi}u_3^2$$

where

$$p = (\beta + \sigma^2)(S_0^2 + \frac{\theta}{\phi}\rho^2 V_0^2) + 2\frac{\lambda^2}{\mu},$$

$$q = 2\lambda + 2\mu - 2\beta(S_0 + \rho V_0) - \sigma^2(S_0 + \rho V_0)^2.$$

Under condition (20), $V(u)$ is a Lyapunov function and moreover for all $(u_1, u_2, u_3) \neq (0, 0, 0)$, we have

$$LV = -2(\mu + \phi)u_1^2 + \sigma^2(S_0^2 + \frac{\theta}{\phi}\rho^2 V_0^2 - p)u_2^2 - 2\frac{\theta}{\phi}(\mu + \theta)u_3^2$$

$$+ 2(\lambda - \beta S_0)u_1 u_2 + 4\theta u_1 u_3 - 2\frac{\theta}{\phi}\beta \rho V_0 u_2 u_3$$

where L is the operator defined in (16). Taking into account the definition of p and using the following relations:

$$2(\lambda - \beta S_0)u_1 u_2 \leq (\beta + \mu)u_1^2 + (\beta S_0^2 + \frac{\lambda^2}{\mu})u_2^2,$$

$$4\theta u_1 u_3 \leq 2\phi u_1^2 + 2\frac{\theta^2}{\phi} u_3^2,$$

$$-2\frac{\theta}{\phi} \beta \rho V_0 u_2 u_3 \leq \frac{\theta}{\phi} \beta \rho^2 V_0^2 u_2^2 + \frac{\theta}{\phi} \beta u_3^2,$$

we have

$$LV \leq -[(\mu - \beta)u_1^2 + \frac{\lambda^2}{\mu} u_2^2 + (2\mu - \beta)u_3^2].$$

Moreover, taking

$$a(|u|) = \min \left\{ 1, \frac{p}{q}, \frac{\theta}{\phi} \right\} |u|^2,$$

$$b(|u|) = \max \left\{ 1, \frac{p}{q}, \frac{\theta}{\phi} \right\} |u|^2,$$

$$c(|u|) = \min \left\{ \frac{\lambda^2}{\mu}, \mu - \beta \right\} |u|^2,$$

the thesis follows by Theorem 5.7 of [5], p. 136. \square

For the stability of the trivial solution of the non-linear system (1), we have the following result.

THEOREM 3.1. *If*

$$R_\phi < \min \left\{ 1 - T, \frac{\mu(S_0 + \rho V_0)}{\lambda + \mu} \right\}$$

holds, then the disease free equilibrium of (1) is asymptotically stable.

Proof. The quantity

$$\sqrt{\beta^2(S - S_0)^2 I^2 + \beta^2((S - S_0) + \rho(V - V_0))^2 I^2 + \beta^2 \rho^2 (V - V_0)^2 I^2}$$

$$+ \sqrt{\sigma^2(S - S_0)^2 I^2 + \sigma^2((S - S_0) + \rho(V - V_0))^2 I^2 + \sigma^2 \rho^2 (V - V_0)^2 I^2}$$

in $]S_0 - \varepsilon, S_0 + \varepsilon[\times] - \varepsilon, \varepsilon[\times]V_0 - \varepsilon, V_0 + \varepsilon[$ is less than $\gamma|(S - S_0, I, V - V_0)|$ with

$$\gamma = (\beta + \sigma)(1 + \rho)\varepsilon.$$

The thesis follows by Theorem 1.1 of [6], p. 248, and Lemma 3.1. \square

4. Numerical simulations and concluding remarks.

In this section we use numerical simulations based on the Euler-Maruyama scheme to analyze the stochastic behaviour of the system (1) and to compare it with the deterministic behaviour of system (2).

Let $f(x)$ be a function in \mathbb{R}^n defined in \mathbb{R}^n , $g(x)$ be a $n \times m$ matrix and $W(t)$ be an m -dimensional Wiener process, we consider an n -dimensional system of stochastic equations

$$dX(t) = f(X(t))dt + g(X(t))dW(t);$$

the Euler-Maruyama scheme applied to this equation, on $t \in \{0, \Delta t, \dots, N\Delta t = T\}$ takes the form

$$X_{\Delta t}(t + \Delta t) = X_{\Delta t}(t) + f(X_{\Delta t}(t))\Delta t + g(X_{\Delta t}(t))\Delta W(t)$$

with initial value $X_{\Delta t}(0) = X_0$. The $\Delta W(t) = W(t + \Delta t) - W(t)$ represents N independent draws from an m -dimensional Normal distribution whose individual components have mean zero and variance Δt . To generate it using Matlab we refer to [10].

In [13] it is shown that for any finite time and a sufficiently small time step, this numerical scheme will converge to the true solution of (1) provided that a C^2 function $V : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ exists and satisfies the following conditions:

- (i) the set $\mathcal{D}(r) = \{x \in \mathbb{R}_+^n : V(x) \leq r\}$ is compact for any $r > 0$;
- (ii) $LV(x) \leq K(1 + V(x))$ with a constant $K > 0$;
- (iii) there exists a positive constant K_1 such that for all $x, y \in \mathcal{D}(r)$

$$|V(x) - V(y)| \leq K_1|x - y|;$$

$$|V_x(x) - V_x(y)| \leq K_1|x - y|;$$

$$|V_{xx}(x) - V_{xx}(y)| \leq K_1|x - y|.$$

The function $V(x)$ defined in Lemma 3.1 satisfies each of these conditions and so the Euler-Maruyama scheme may be applied to (1).

Moreover we use parameters appropriate for human disease, e.g. pertussis (see e.g. [3], [11]) taking the time unit as a day. Particularity we assume:

$\lambda = 0.04762$	a three week average disease duration
$\mu = 3.6530 * 10^{-5}$	the average lifetime to be 75 years
$\theta = 5.4794 * 10^{-4}$	the average waning time of the vaccine $\frac{1}{\theta}$ to be 5 years
$\beta = 0.4$	the average number of adequate contacts per infective per day

and we describe the behaviour as function of the degree of vaccine efficacy ρ , the vaccination rate of susceptibles ϕ and the stochastic constant σ .

Computer simulation of the deterministic model agree well with the results about the existence and stability of the disease free equilibrium and the endemic equilibria. In fact in absence of the disease there is a unique disease free equilibrium $E_0(S_0, 0, V_0) = \left(\frac{\mu + \theta}{\mu + \theta + \phi}, 0, \frac{\phi}{\mu + \theta + \phi} \right)$ that is globally asymptotically stable if $R_0 < 1$ (R_0 as (14)). If $R_0 > 1$ the endemic equilibria are the solution of (see for example [2]-[3],[11]):

$$P(I) = AI^2 + BI + C$$

where

$$A = \rho\beta, \quad B = -\rho\beta + \rho(\mu + \phi + \lambda) + \mu + \theta, \quad C = \frac{(\phi + \mu + \theta)(\lambda + \mu)}{\beta}(1 - R_\phi)$$

with R_ϕ as (14).

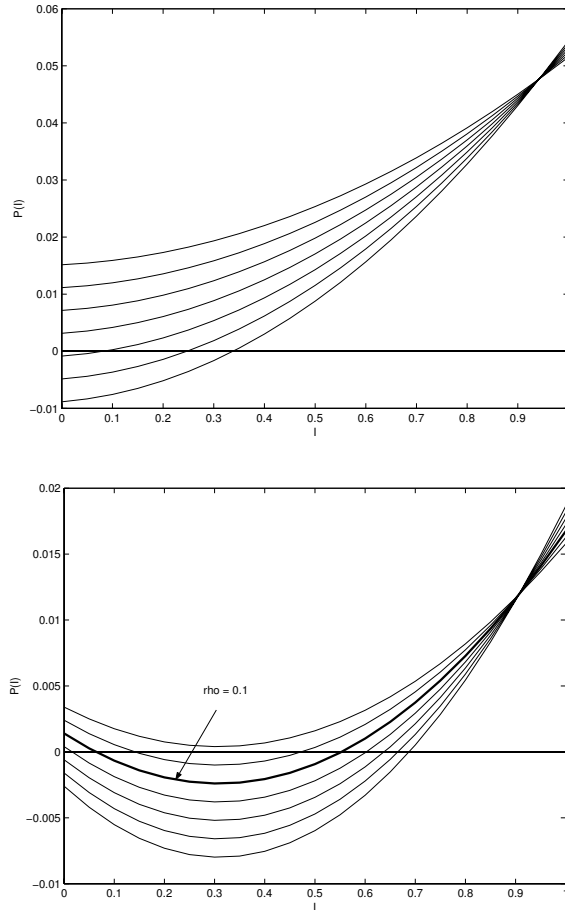


Figure 1 - Roots of $P(I)$ when $\rho = 0.08, 0.09, \dots, 0.14$ and $\phi = 0.4, \phi = 0.1$ respectively

Plotting the quadratic $P(I)$ with increasing values of ρ we visualize as the model can exhibit a backward bifurcation leading to the existence of multiple endemic equilibria and news subthreshold (Figure 1). Particular for the previous parameter values, if $\rho = 0.1$ (90 per cent protection), there is backward bifurcation for a range of ϕ values given by $0.0254 \leq \phi \leq 0.1506$ (vaccination of susceptibles on average every 1 to 8 weeks). For example, if $\phi = 0.1$ ($R_\phi = 0.8832$) we have the two positive solutions $I_1 = 0.0633$ and $I_2 = 0.5530$ and, then, two endemic equilibria $E_1(0.2370, 0.0633, 0.6997)$ and $E_2(0.0810, 0.5530, 0.3660)$. Thus to describe the behaviour of the deterministic and stochastic systems also near this equilibria, we choose $S(0) = 0.0851$, $I(0) = 0.5521$, $V(0) = 0.3628$ as initial data in the simulation. Moreover we use a broken line to represent the deterministic process and a continuous one to represent the stochastic process.

The diagram of bifurcation (Figure 2) shows the roots as function of the parameter β and then of R_ϕ . We observe that R_c is the value of R_ϕ at the saddle node bifurcation point where the two values of I coincide with $I = I_c = -\frac{B}{2A}$.

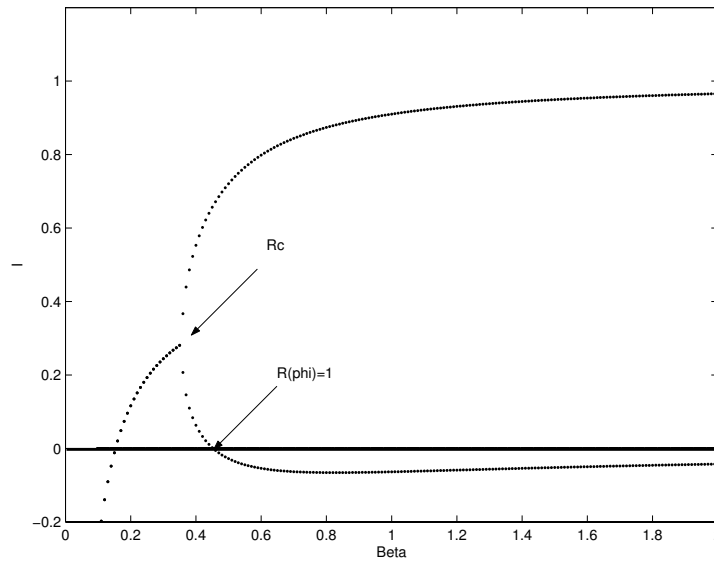


Figure 2 - Roots of $P(I)$ as function of the parameter β and then of R_ϕ when $\rho = 0.1$ and $\phi = 0.1$.

About the stability of this equilibria we observe that being the quantity R_ϕ the number of secondary infectious cases produced by one primary case introduced into a population in which a proportion ϕ has been vaccinated, the

condition $R_\phi < 1$ is a necessary condition for diseases eradication. However the local stability of E_0 does not necessarily imply its global stability and this condition is not sufficient.

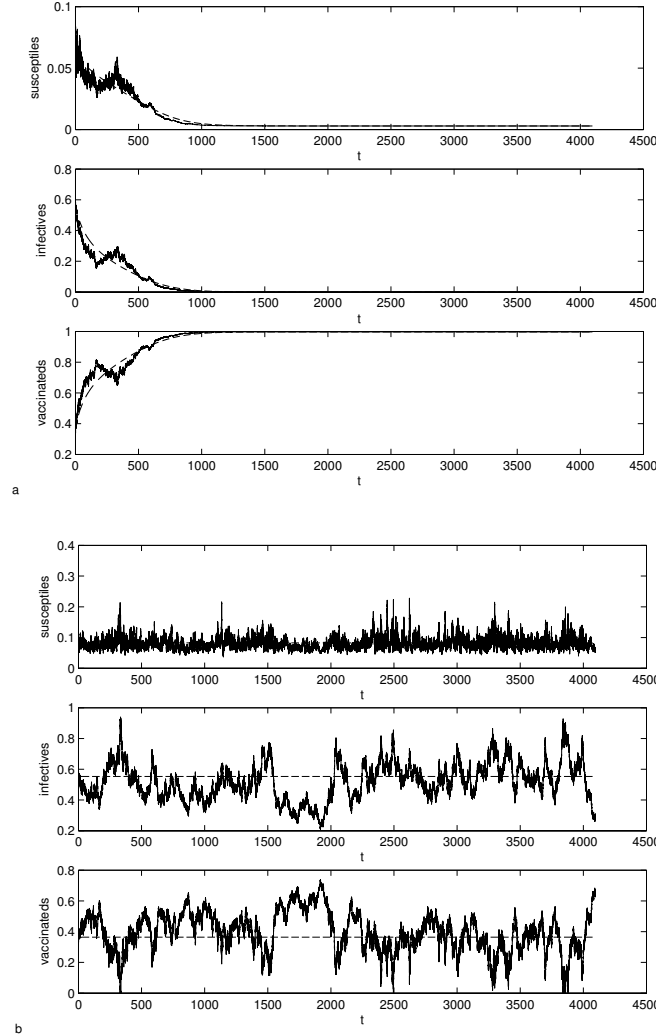


Figure 3 - Number of susceptible, infective and vaccinated population when $\sigma = 0.3$, $\rho = 0.1$ and $\phi = 0.2000$ ($R_\phi = 0.8614 < 1 - T = 0.9901$), $\phi = 0.1000$ ($R_\phi = 0.8832 < 1 - T = 0.9895$) respectively.

In fact it is known that for the deterministic system (2) with $V(0) \geq 0$, $I(0) > 0$, if $R_\phi < R_c < 1$ then the disease dies out (Figure 3a, Figure 4a and Figure 5a), if $R_c < R_\phi < 1$, then the endemic equilibria E_2 with larger I is locally asymptotically stable and the endemic equilibria E_1 with

smaller I is unstable (Figure 3b, Figure 4b and Figure 5b), if $R_\phi > 1$, then the unique endemic equilibria E_+ is globally asymptotically stable in $[0, 1]^3 \setminus \{(S, I, V) : I = 0\}$ (see Theorem 3.1 of [3], p. 487) (Figure 6).

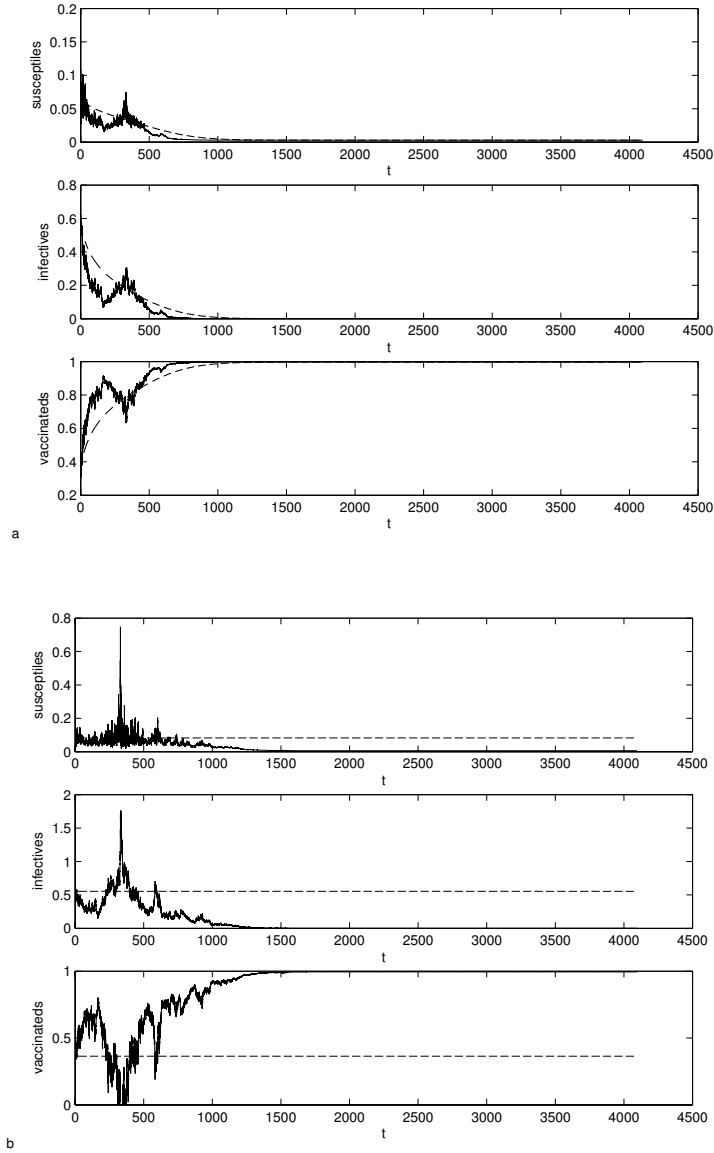


Figure 4 - Number of susceptible, infective and vaccinated population when $\sigma = 0.6$, $\rho = 0.1$ and $\phi = 0.2000$ ($R_\phi = 0.8614 < 1 - T = 0.9582$), $\phi = 0.1000$ ($R_\phi = 0.8832 < 1 - T = 0.9256$) respectively.

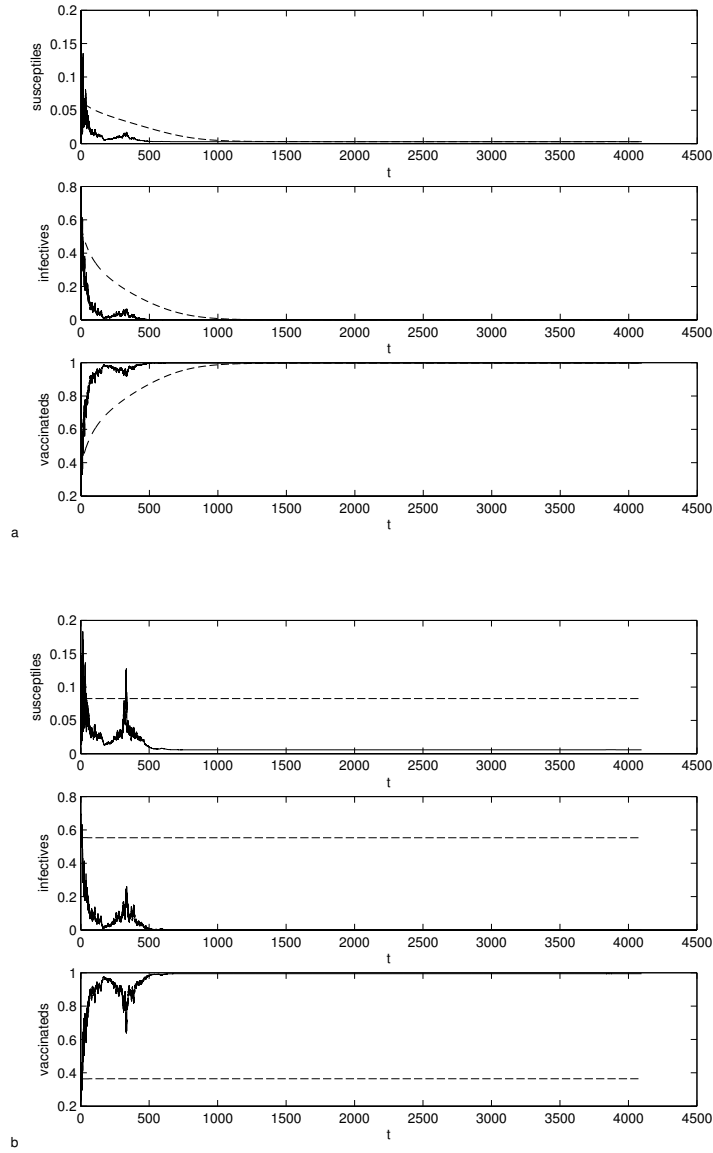


Figure 5 - Number of susceptible, infective and vaccinated population when $\sigma = 1.2$, $\rho = 0.1$ and $\phi = 0.2000$ ($1 - T = 0.8409 < R_\phi = 0.8614$), $\phi = 0.1000$ ($1 - T = 0.8594 < R_\phi = 0.8832$) respectively.

Numerical simulations of the stochastic model shows that, from the biological point of view, the introduction of a noise in the deterministic model modifies the deterministic threshold $R_\phi = 1$ of the disease free equilibrium E_0 giving rise to new threshold $1 + T$ (T as (19)) for R_ϕ . In fact, differently

that the deterministic model, the disease free equilibrium E_0 is asymptotically stable for the stochastic model also under the following condition (Figure 6a):

$$1 < R_\phi < 1 + T.$$

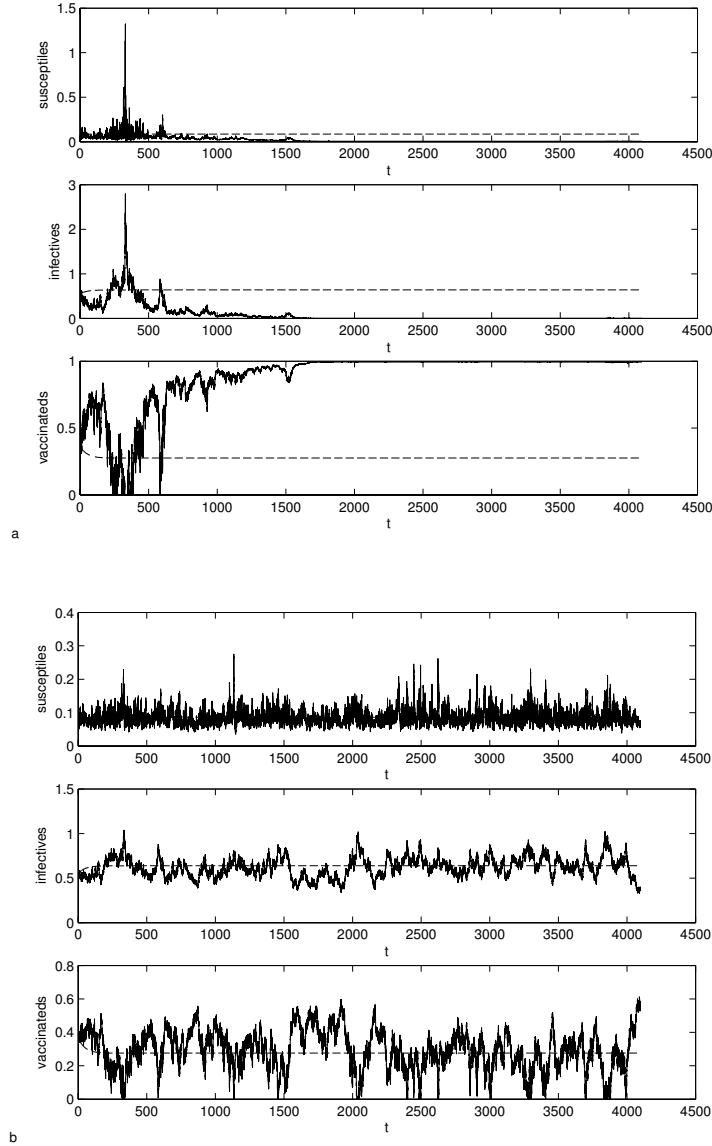


Figure 6 - Number of susceptible, infective and vaccinated population when $\phi = 0.1000$, $\rho = 0.1200$ and $\sigma = 0.8000$ ($1 < R_\phi = 1.0501 < 1 + T = 1.1051$), $\sigma = 0.3000$ ($1 < 1 + T = 1.0148 < R_\phi = 1.0501$) respectively.

While if $R_\phi > 1 + T$ then, as the deterministic model, the disease free equilibrium E_0 is unstable and the solution of system (1) fluctuates around the endemic equilibrium E_+ (Figure 6b).

Finally if $R_\phi < 1$ the computer simulation shows that the solution of system (1) converges to the disease free equilibrium if there is not a backward bifurcation for the deterministic model (Figure 3a, Figure 4a and Figure 5a) or if there is a backward bifurcation for the deterministic model but $1 - T < R_\phi < 1$ (Figure 5b). While, if $R_\phi < 1 - T$ and if there is a backward bifurcation for the deterministic model, we conjecture that there exists a σ_c value (with the previous parameters value $0.46 < \sigma_c < 0.47$) such that, if $\sigma < \sigma_c$ the solution of system (1) fluctuates around the biggest endemic equilibrium (Figure 3b) moreover if $\sigma_c < \sigma$ the solution of system (1) converges to the disease free equilibrium (Figure 4b).

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E. Tornatore
*Dipartimento di Matematica ed Applicazioni,
Università di Palermo,
Via Archirafi, 34,
90123 Palermo, Italia
e-mail: elisa@math.unipa.it*

Stefania M. Buccellato
*Dipartimento di Matematica ed Applicazioni,
Università di Palermo,
Via Archirafi, 34,
90123 Palermo, Italia
e-mail: bucci@math.unipa.it*

P. Vetro
*Dipartimento di Matematica ed Applicazioni,
Università di Palermo,
Via Archirafi, 34,
90123 Palermo, Italia
e-mail: vetro@math.unipa.it*